A Novel Approach to Transforming Piecewise Functions

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Abstract

We present a unified algebraic framework for representing and transforming arbitrary piecewise-defined functions into smooth, differentiable expressions suitable for analysis and optimization. We introduce the Kronecker naught operator, an analytic analog of the Kronecker delta, to encode indicator-style discontinuities via Fourier-series and trigonometric membership conditions. Our approach systematically replaces hard conditional branches with "soft" approximations, parameterized smooth maxima, Heaviside steps, and differentiable logical operators, that converge to the original piecewise form in the sharp-limit. We further extend these techniques to number-theoretic and combinatorial domains, showing how divisibility and integer-membership tests can be expressed through sine-squared filters and gamma-function identities. We also introduce the core concept of a **conditional function**, which we will denote with $\mathfrak{C}(\mathcal{S})$ for some mathematical statement $\mathcal{S}$, which will be key to converting these piecewise functions into a more straightforward entity. Finally, we apply our algebraic toolkit to a continuous reformulation of the 3x + 1 (Collatz) mapping, yielding a novel smooth dynamical system that retains the integer-orbital structure while admitting gradient-based analysis. Throughout, we illustrate key constructions with explicit formulas, discuss convergence properties, and highlight connections to modern machine-learning architectures that rely on differentiable decision boundaries. This work lays a foundation for both theoretical investigations of piecewise phenomena and practical implementations in differentiable programming environments.

Keywords: Piecewise function, Laws of Algebraic-Piecewise Transformation, Conditional function, Binary logic, Smooth representations, Optimization, Differentiable programming

1 A Novel Approach to Transforming Piecewise Functions

Piecewise-defined functions play a central role in mathematics, engineering, computer science, and applied sciences, modeling phenomena that switch regimes according to their inputs. However, the usual notation

$$f(x) = \begin{cases} a_1 & \text{if } \mathcal{S}_1 \\ a_2 & \text{if } \mathcal{S}_2 \\ \vdots & \\ a_n & \text{if } \mathcal{S}_n \end{cases}$$

while compact, often yields discontinuous or non-differentiable expressions at the boundaries of each region. These sharp transitions pose obstacles in optimization, numerical simulation, and modern machine-learning architectures, all of which benefit from smoothness and well-behaved gradients.

In this work, we introduce a unified algebraic-analytic transform that systematically replaces hard conditional branches with smooth, parameterized approximations, yet exactly recovers the original piecewise definition in the limit. At its core lies a conditional function, denoted $\mathfrak{C}(\mathcal{S})$, for some mathematical statement $\mathcal{S}$, which algebraically encodes the test $\mathcal{S}$ into a real-analytic function. By composing and combining these functions with standard operations, sums, products, trigonometric filters, and generalized "soft-max" constructs, we obtain the following closed-form expressions:

- Converging to each branch a_i on its designated domain (statement) $\mathcal{S}$ as the sharpness parameter increases,
- We remain infinitely differentiable for every finite value of the sharpness parameter,
- Extending seamlessly to logical combinations of conditions (AND, OR, NOT) and even number-theoretic tests such as divisibility or integer membership,

- Natural interpretations are adopted as smoothed indicator functions in Fourier-analytic and optimization contexts.

After developing the theoretical foundations of the conditional function and its algebraic properties, we demonstrate its versatility through three detailed case studies: a smooth encoding of classical piecewise laws (e.g., minimum/maximum, sign functions, and finitely many discrete cases); trigonometric and gammafunction filters that detect divisibility or integer equality via $\sin^2(\frac{\pi x}{n})$ and related identities; and a continuous reformulation of the 3x+1 (Collatz) map, yielding a smooth dynamical system whose orbits coincide with the integer Collatz trajectories in the sharp limit.

Throughout, we provide explicit formulas, convergence analyses, and implementation notes for embedding these smooth piecewise representations into differentiable programming frameworks and gradient-based solvers. Our framework opens new pathways for both theoretical exploration of piecewise phenomena and practical integration into applications that demand analytic regularity.

2 Theory and Formalism

In this section, we define and develop the algebraic representation of piecewise functions. We begin by introducing the foundational theory behind it, presenting a systematic transformation from traditional piecewise forms to algebraic expressions. This transformation will be explored through the formal declaration of key definitions, supporting lemmas, and conjectures that underpin the framework.

2.1 Binary Piecewise Forms

We start by examining piecewise functions of the simplest form, namely *binary piecewise forms*, where we address piecewise functions in only two possible scenarios.

2.1.1 The Kronecker Naught

The Kronecker delta function is a function of two variables that equal 1 if the variables are equal, and 0 otherwise:

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

In particular, we will frequently use the special case $\delta(x, 0)$ throughout the paper, so for simplicity, we define the **Kronecker naught** function/operator as

$$\mathfrak{d}(x) = \delta(x, 0) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Let us try to derive some approximations for the Kronecker naught function that we may be able to utilize later.

We start off with some approximations that assume x to be an integer. We begin by considering the complex exponential function $e^{2\pi\iota x}$, where ι denotes the imaginary unit. This function represents a unit vector in the complex plane, and exponentiating it by a real parameter $\alpha \in [0, 1]$ yields a family of unit vectors with varying angles.

If we integrate these vectors over the interval $[0, 1]$, we are effectively summing them along a full rotation on the complex unit circle (for $x \neq 0$). Because the complex exponentials are orthogonal over this interval, they cancel out due to symmetry, unless $x = 0$, in which case the exponential becomes identically 1 and the integral is evaluated as 1. Formally,

$$\int_0^1 e^{2\pi\iota\alpha x}\ \mathrm{d}\alpha = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} = \mathfrak{d}(x)$$

We can justify this more rigorously by evaluating the integral explicitly. When $x = 0$, the integrand is constant:

$$\int_0^1 e^{2\pi\iota\alpha\cdot 0}\ \mathrm{d}\alpha = \int_0^1 1\ \mathrm{d}\alpha = 1 - 0 = 1$$

When $x \neq 0$, the integrand is a nontrivial complex exponential:

$$\int_0^1 e^{2\pi\iota\alpha x}\ \mathrm{d}\alpha = \left[\frac{1}{2\pi\iota x} e^{2\pi\iota\alpha x}\right]_0^1 = \frac{1}{2\pi\iota x}\left(e^{2\pi\iota x} - 1\right) = 0$$

since $e^{2\pi\iota x} = 1$ for any integer x.

Therefore, we arrive at the exact identity:

$$\mathfrak{d}(x) = \int_0^1 \exp(2\pi\iota\alpha x)\ \mathrm{d}\alpha \qquad \forall x \in \mathbb{Z} \tag{2}$$

Now, let us look at a more general case, not restricted to just the integers. Let

$$0^+ = \lim_{\alpha \to 0^+} \alpha$$

If we have a real number $x \in \mathbb{R}$, then, if x is nonzero, then

$$\min \left[0^+, \underset{\neq 0}{|x|}\right] = 0^+$$

Now, if x is equal to 0, then

$$\min \left[0^+, |0|\right] = 0$$

This means that

$$\lim_{\alpha \to 0^+} \min [\alpha, |x|] = \begin{cases} 0 & \text{if } x = 0 \\ 0^+ & \text{otherwise} \end{cases}$$

$$\implies \lim_{\alpha \to \infty} \min \left[\frac{1}{\alpha}, |x|\right] = \begin{cases} 0 & \text{if } x = 0 \\ 0^+ & \text{otherwise} \end{cases}$$

If we take this quantity inside the limit and multiply it by α, then, we get $\lim_{\alpha\to\infty} \alpha \min \left[\frac{1}{\alpha}, |x|\right]$. If x is nonzero, then $\alpha \min \left[\frac{1}{\alpha}, |x|\right] = \alpha \frac{1}{\alpha} = 1$, and if x is zero, then $\alpha \min \left[\frac{1}{\alpha}, |x|\right] = \alpha \times 0 = 0$. Therefore,

$$\lim_{\alpha \to \infty} \alpha \min \left[\frac{1}{\alpha}, |x|\right] = \lim_{\alpha \to \infty} \min (1, \alpha|x|) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases} = 1 - \mathfrak{d}(x) \qquad (3)$$

Now, according to the **soft-maximum approximation** (here, it is an equation, not an approximation, as we use a limit) for the maximum function,

$$\max(x_1, x_2, ..., x_n) = \lim_{\beta \to \infty} \frac{1}{\beta} \ln \sum_{k=1}^{n} \exp \beta x_k$$

Since $-\min(x_1, x_2, ..., x_n) = \max(-x_1, -x_2, ..., -x_n)$, we can write

$$\min(x_1, x_2, ..., x_n) = -\lim_{\beta \to \infty} \frac{1}{\beta} \ln \sum_{k=1}^{n} \exp \beta(-x_k) = \lim_{\beta \to \infty} \frac{1}{-\beta} \ln \sum_{k=1}^{n} \exp(-\beta x_k)$$

Now, we can plug this into equation 3 to obtain

$$\lim_{\alpha \to \infty} \left(\lim_{\beta \to \infty} \frac{1}{-\beta} \ln \left(e^{-\beta} + e^{-\beta\alpha|x|}\right) \right) = 1 - \mathfrak{d}(x) \qquad \forall x \in \mathbb{R} \qquad (4)$$

Now, we will look at some useful properties of the Kronecker Naught function. For the Kronecker delta function, we know that

$$\delta(x, y) = \delta(kx, ky) = \delta\left(x^{2k+1}, y^{2k+1}\right) = \delta\left(x^{\frac{1}{2k+1}}, y^{\frac{1}{2k+1}}\right)$$

for any nonzero real number k. If we set $y = 0$, then $\delta(x, y) = \mathfrak{d}(x)$. Therefore,

$$\mathfrak{d}(x) = \mathfrak{d}(kx) = \mathfrak{d}\left(x^{2k+1}\right) = \mathfrak{d}\left(x^{\frac{1}{2k+1}}\right) \qquad \forall k \in \mathbb{R}_{\neq 0} \tag{5}$$

Another property of the Kronecker Naught function is that for nonnegative parameters it **behaves like an exponential-type function**, which means that it obeys the exponential functional equation:

$$\mathfrak{d}(x)\,\mathfrak{d}(y) = \mathfrak{d}(x + y) \qquad \forall x, y \in \mathbb{R}_{\geq 0} \tag{6}$$

Recall the definition of the Kronecker naught function for nonnegative real arguments:

$$\mathfrak{d}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases} = 1 - \operatorname{sgn}(x) \quad \forall x \geq 0$$

where $\operatorname{sgn}(x)$ is the sign function defined as

$$\operatorname{sgn}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \quad \forall x \geq 0$$

We want to show that for all $x, y \geq 0$,

$$\mathfrak{d}(x) \cdot \mathfrak{d}(y) = \mathfrak{d}(x + y)$$

Substituting the expression in terms of sgn, the left-hand side becomes

$$\mathfrak{d}(x) \cdot \mathfrak{d}(y) = (1 - \operatorname{sgn}(x))(1 - \operatorname{sgn}(y)) = 1 - \operatorname{sgn}(x) - \operatorname{sgn}(y) + \operatorname{sgn}(x)\operatorname{sgn}(y)$$

Next, note that since $x, y \geq 0$, the sign function of the sum satisfies

$$\operatorname{sgn}(x + y) = \max(\operatorname{sgn}(x), \operatorname{sgn}(y)).$$

Thus,

$$1 - \operatorname{sgn}(x + y) = 1 - \max(\operatorname{sgn}(x), \operatorname{sgn}(y)) = \min(1 - \operatorname{sgn}(x), 1 - \operatorname{sgn}(y))$$

Given that $\operatorname{sgn}(x), \operatorname{sgn}(y) \in \{0, 1\}$, the product $(1 - \operatorname{sgn}(x))(1 - \operatorname{sgn}(y))$ equals the minimum of the two factors. Hence,

$$1 - \operatorname{sgn}(x + y) = (1 - \operatorname{sgn}(x))(1 - \operatorname{sgn}(y))$$

Therefore,

$$\mathfrak{d}(x + y) = 1 - \operatorname{sgn}(x + y) = (1 - \operatorname{sgn}(x))(1 - \operatorname{sgn}(y)) = \mathfrak{d}(x) \cdot \mathfrak{d}(y)$$

which completes the proof.

2.1.2 Using the Kronecker Naught

Consider the binary piecewise function

$$a(x) = \begin{cases} b & \text{if } x = k \\ c & \text{otherwise} \end{cases}$$

This means that if $x - k = 0$, then $a(x) = b$; if not, then $a(x) = c$. This means that we can rewrite this by using our Kronecker naught function as

$$a(x) = b \cdot \{\mathfrak{d}(x-k)\} + c \cdot \{1 - \mathfrak{d}(x-k)\} = (b-c)\ \mathfrak{d}\,(x-k) + c$$

Thus, we form the **first law of algebraic-piecewise transformation**.

$$a(x) = \begin{cases} b & \text{if } x = k \\ c & \text{otherwise} \end{cases}$$
$$\implies a(x) = (b-c)\ \mathfrak{d}\,(x-k) + c$$

Now, we introduce a function called the *conditional function*, and we define it uniquely. We define it such that $\mathfrak{C}\,(x = y) = x - y$. The conditional function is defined such that, for some mathematical statement $\mathcal{S}$, it characteristically obeys

$$\mathcal{S} \text{ is true} \qquad \iff \qquad \mathfrak{C}\,(\mathcal{S}) = 0 \tag{7}$$

If, for a statement $\mathcal{S}$, we have a conditional function $\mathfrak{C}(\mathcal{S})$, then we define the negative conditional function as

$$^{!}\mathfrak{C}(\mathcal{S}) = \mathfrak{C}(\text{not } \mathcal{S}) \tag{8}$$

Therefore, it characteristically obeys

$$\mathcal{S} \text{ is false} \qquad \iff \qquad {}^{!}\mathfrak{C}\,(\mathcal{S}) = 0 \tag{9}$$

We can use the conditional function when we have general statements that are not equations and can convert these statements into meaningful equations through some trial and tuning to obtain the desired form after which we can apply the conditional function freely. Additionally, we can say that

$$\mathfrak{d}\left[{}^{!}\mathfrak{C}(\mathcal{S})\right] = 1 - \mathfrak{d}\left[\mathfrak{C}(\mathcal{S})\right]$$

Thus, we form the ***second law of algebraic-piecewise transformation***.

For a statement $\mathcal{S}_{(x)}$,

$$a(x) = \begin{cases} b & \text{if } \mathcal{S}_{(x)} \\ c & \text{otherwise} \end{cases}$$

$$\implies a(x) = b \cdot \left\{ \mathfrak{d} \left[\mathfrak{C} \left(S \right) \right] \right\} + c \cdot \left\{ 1 - \mathfrak{d} \left[\mathfrak{C} \left(S \right) \right] \right\}$$

$$\implies a(x) = (b - c) \;\; \mathfrak{d} \left[\mathfrak{C}(\mathcal{S}) \right] + c$$

$$\implies a(x) = (c - b) \;\; \mathfrak{d} \left[{}^{!}\mathfrak{C}(\mathcal{S}) \right] + b$$

It is also essential to establish a set of governing rules to define the conditional function properly. Thus, we will form a list of *axioms* that we will follow throughout the paper.

1. **There are two main types of statements: open-form and closed-form statements. Open-form statements cannot be directly expressed as mathematical equations, while closed-form statements can be represented in this way. It is possible to *interpolate* an open-form statement into a closed-form format, although this process may result in the loss of some information from the original statement.**

2. **For two statements $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X} \iff \mathcal{Y}$ implies that $\mathfrak{C}(\mathcal{X})$ is logically equivalent, but not equal to $\mathfrak{C}(\mathcal{Y})$, which we denote as $\mathfrak{C}(\mathcal{X}) \equiv \mathfrak{C}(\mathcal{Y})$.**

3. **For two statements $\mathcal{X}$ and $\mathcal{Y}$, $\mathfrak{C}(\mathcal{X}) \equiv \mathfrak{C}(\mathcal{Y})$ does not imply that $\mathcal{X} \iff \mathcal{Y}$; only that $\mathfrak{d} \left[\mathfrak{C}(\mathcal{X}) \right] = \mathfrak{d} \left[\mathfrak{C}(\mathcal{Y}) \right]$.**

4. **If a piecewise function $a(x) = \begin{cases} b & \textbf{if } \mathcal{S}_{(x)} \\ c & \textbf{otherwise} \end{cases}$ exists such that only two possible values of $\mathfrak{C}(\mathcal{S})$ are 0 and 1, then $\mathfrak{d} \left[\mathfrak{C}(\mathcal{S}) \right] = 1 - \mathfrak{C}(\mathcal{S})$.**

5. **For a statement $\mathcal{S}$ and an expression s, if $\mathfrak{d} \left[\mathfrak{C}(\mathcal{S}) \right] = \mathfrak{d} \left(s \right)$, then $\mathfrak{C}(\mathcal{S}) \equiv s$.**

Now, we will be able to transform different variations of *binary piecewise forms.* Let us look at a few examples to better understand how the conditional function works.

- *Example 1:*

  Consider the binary piecewise function

$$a(x) = \begin{cases} b & \text{if } n \mid x \\ c & \text{otherwise} \end{cases} \qquad \forall n, x \in \mathbb{Z}$$

where $n \mid x$ means that n divides x.

We can represent the conditional function here as $\mathfrak{C}(n \mid x)$. If $n \mid x$, $^{x}/_{n}$ is an integer. Therefore, we can say that

$$\mathfrak{C}(n \mid x) \equiv \mathfrak{C}\left(\underbrace{\frac{x}{n} \in \mathbb{Z}}_{\text{open-form statement}}\right) \overset{\text{interpolate}}{\equiv} \mathfrak{C}\left(\frac{x}{n} = \left\lfloor \frac{x}{n} \right\rfloor\right)$$

Now, we have interpolated this statement into a closed-form statement, which means this is of the form $\mathfrak{C}(x = y)$. Therefore,

$$\mathfrak{C}\left(\frac{x}{n} = \left\lfloor \frac{x}{n} \right\rfloor\right) = \frac{x}{n} - \left\lfloor \frac{x}{n} \right\rfloor$$

By the second law of algebraic piecewise transformation,

$$a(x) = \begin{cases} b & \text{if } n \mid x \\ c & \text{otherwise} \end{cases}$$

$$\implies a(x) = (b - c)\ \mathfrak{d}\left[\mathfrak{C}(n \mid x)\right] + c$$

$$\implies a(x) = (b - c)\ \mathfrak{d}\left[\mathfrak{C}(n \mid x)\right] + c$$

$$\implies a(x) = (b - c)\ \mathfrak{d}\left(\frac{x}{n} - \left\lfloor \frac{x}{n} \right\rfloor\right) + c$$

- *Example 2:*

Consider the binary piecewise function

$$a(x) = \begin{cases} b & \text{if } x > k \\ c & \text{otherwise} \end{cases}$$

We can represent the conditional function here as $\mathfrak{C}(x > k)$. If $x > k$, then $x - k$ is its own absolute value. However, this also includes the case where $x = k$; thus, we need to use the negative conditional version of this.

Since $x > k$ implies that $x \not\leq k$, $\mathfrak{C}(x > k) = {}^{!}\mathfrak{C}(x \leq k)$. Now we can use the logic that $k - x$ is its own absolute value.

Therefore, we can say that

$$\mathfrak{C}(x > k) = {}^{!}\mathfrak{C}(x \leq k) \overset{\text{interpolate}}{\equiv} {}^{!}\mathfrak{C}\left(|k - x| = k - x\right) = {}^{!}\mathfrak{C}\left(|x - k| = k - x\right)$$

Now, this is of the form $\mathfrak{C}(x = y)$, so we can say that

$$\mathfrak{C}\left(|x - k| = k - x\right) = \Big(|x - k| + x - k\Big)$$

By the second law of algebraic piecewise transformation,

$$a(x) = \begin{cases} b & \text{if } x > k \\ c & \text{otherwise} \end{cases}$$
$$\implies a(x) = (c - b)\ \mathfrak{d}\Big[|x - k| + x - k\Big] + b$$

2.1.3 Multiple Conditionals

Suppose we have a binary piecewise function that has multiple conditionals such as:

$$a(x) = \begin{cases} b & \text{if } x = \alpha \vee \beta \vee \gamma \vee \cdots \\ c & \text{otherwise} \end{cases}$$

This is a unique form of binary piecewise representation, but we can still utilize the same method. The conditional function here is $\mathfrak{C}(x = \alpha \vee \beta \vee \cdots)$ or $\mathfrak{C}\left(\bigvee_{\alpha,\beta,\cdots}(x = \alpha)\right)$. We will stylize it to write it as $\mathfrak{C}\begin{pmatrix} & x = \alpha \\ \bigvee & x = \beta \\ \bigvee & \vdots \end{pmatrix}$. By the second law of algebraic piecewise transformation,

$$a(x) = \begin{cases} b & \text{if } x = \alpha \vee \beta \vee \gamma \vee \cdots \\ c & \text{otherwise} \end{cases}$$

$$\implies a(x) = (b-c)\,\mathfrak{d}\left[\mathfrak{e}\left(\begin{matrix} x=\alpha \\ \bigvee x=\beta \\ \bigvee \quad \vdots \end{matrix}\right)\right] + c$$

Here, we introduce our first few *lemmas*.

Lemma 1.1 If $\mathcal{S}_1 \nLeftrightarrow \mathcal{S}_2 \nLeftrightarrow \mathcal{S}_3 \nLeftrightarrow \cdots$ strictly, then

$$\mathfrak{d}\left[\mathfrak{e}\left(\begin{matrix} \mathcal{S}_1 \\ \bigvee \mathcal{S}_2 \\ \bigvee \vdots \end{matrix}\right)\right] = \sum \mathfrak{d}\Big[\mathfrak{e}\Big(\mathcal{S}_i\Big)\Big]$$

where $\mathcal{S}_n(x)$ denotes the n^{th} statement involving x.

Proof.

Suppose there is a piecewise function

$$a(x) = \begin{cases} b & \text{if } \mathcal{S}_1 \bigvee \mathcal{S}_2 \bigvee \mathcal{S}_3 \bigvee \cdots \\ c & \text{otherwise} \end{cases}$$

This can be rewritten as:

$$a(x) = \begin{cases} b & \text{if } \mathcal{S}_1 \\ b & \text{if } \mathcal{S}_2 \\ b & \text{if } \mathcal{S}_3 \\ & \vdots \\ c & \text{otherwise} \end{cases}$$

By the Second Law of Algebraic-Piecewise Transformation:

$$a(x) = \left(b \cdot \Big\{\mathfrak{d}\Big[\mathfrak{e}\Big(\mathcal{S}_1\Big)\Big]\Big\} + b \cdot \Big\{\mathfrak{d}\Big[\mathfrak{e}\Big(\mathcal{S}_2\Big)\Big]\Big\} + \cdots\right) + c \cdot \{\cdots\}$$

$$\implies a(x) = b \cdot \left\{\mathfrak{d}\Big[\mathfrak{e}\Big(\mathcal{S}_1\Big)\Big] + \mathfrak{d}\Big[\mathfrak{e}\Big(\mathcal{S}_2\Big)\Big] + \cdots\right\} + c \cdot \cdots$$

$$\Rightarrow a(x) = b \cdot \left\{ \sum \mathfrak{d}\left[\mathfrak{C}\left(\mathcal{S}_i\right)\right]\right\} + c \cdot \{\cdots\}$$

However, we also know that:

$$a(x) = \begin{cases} b & \text{if } \mathcal{S}_1 \bigvee \mathcal{S}_2 \bigvee \mathcal{S}_3 \bigvee \cdots \\ c & \text{otherwise} \end{cases}$$

$$\Rightarrow a(x) = b \cdot \left\{ \mathfrak{d}\left[\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigvee \mathcal{S}_2 \\ \bigvee \vdots \end{pmatrix}\right]\right\} + c \cdot \{\cdots\}$$

Since the coefficient of b in both equations must be the same (as b is independent of c), the following equation is used:

$$\mathfrak{d}\left[\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigvee \mathcal{S}_2 \\ \bigvee \vdots \end{pmatrix}\right] = \sum \mathfrak{d}\left[\mathfrak{C}\left(\mathcal{S}_i\right)\right]$$

Thus, we proved.

Lemma 1.2

$$\mathfrak{d}\left[\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigvee \mathcal{S}_2 \\ \bigvee \\ \vdots \end{pmatrix}\right] = \mathfrak{d}\left[\prod \mathfrak{C}\left(\mathcal{S}_i\right)\right]$$

where $\mathcal{S}_n(x)$ denotes the n^{th} statement involving x.

Proof.

Suppose there is a piecewise function

$$a(x) = \begin{cases} b & \text{if } \mathcal{S}_1 \bigvee \mathcal{S}_2 \bigvee \mathcal{S}_3 \bigvee \cdots \\ c & \text{otherwise} \end{cases}$$

The conditional function here is

$$\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigvee \mathcal{S}_2 \\ \bigvee \\ \vdots \end{pmatrix}$$

Therefore, the coefficient of b is

$$\mathfrak{d}\left[\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigvee \mathcal{S}_2 \\ \bigvee \vdots \end{pmatrix}\right]$$

The logical reasoning behind this coefficient is that if any one relation mentioned holds, then $a(x) = b$. This can only happen if any one of $\mathfrak{d}\left[\mathfrak{C}\left(\mathcal{S}_i\right)\right]$ is equal to 1, i.e., if any one $\mathfrak{C}\left(\mathcal{S}_i\right)$ is equal to 0. We can implement this by taking the product of each conditional function $\mathfrak{C}\left(\mathcal{S}_i\right)$, and applying the Kronecker naught operator, so that if any one conditional function is 0, the entire product becomes 0, and the Kronecker naught of 0 is 1. Therefore,

$$\mathfrak{d}\left[\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigvee \mathcal{S}_2 \\ \bigvee \vdots \end{pmatrix}\right] = \mathfrak{d}\left[\prod \mathfrak{C}\left(\mathcal{S}_i\right)\right]$$

Thus, we proved.

Lemma 2

$$\mathfrak{d}\left[\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigwedge \mathcal{S}_2 \\ \bigwedge \vdots \end{pmatrix}\right] = \prod \mathfrak{d}\left[\mathfrak{C}\left(\mathcal{S}_i\right)\right]$$

where $\mathcal{S}_n(x)$ denotes the n^{th} statement involving x.

Proof.

Suppose there is a piecewise function

$$a(x) = \begin{cases} b & \text{if } \mathcal{S}_1 \bigwedge \mathcal{S}_2 \bigwedge \mathcal{S}_3 \bigwedge \cdots \\ c & \text{otherwise} \end{cases}$$

The conditional function here is:

$$\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigwedge \mathcal{S}_2 \\ \bigwedge \vdots \end{pmatrix}$$

Therefore, the coefficient of b is:

$$\mathfrak{d}\left[\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigwedge \mathcal{S}_2 \\ \bigwedge \vdots \end{pmatrix}\right]$$

The logical reasoning behind this coefficient is that if every single relation mentioned holds, then only $a(x) = b$. This can only happen if

$$\mathfrak{d}\left[\mathfrak{C}\Big(\mathcal{S}_1\Big)\right] = 1, \mathfrak{d}\left[\mathfrak{C}\Big(\mathcal{S}_2\Big)\right] = 1, \cdots$$

i.e., only if every conditional function yields 0. If any one of them is nonzero, then the entire coefficient will be 0, so we logically multiply all such $\mathfrak{d}\left[\mathfrak{C}\Big(\mathcal{S}_i\Big)\right]$, so that if any one of them is 0, the entire coefficient becomes 0; if all of them are 1, the coefficient becomes 1. Therefore,

$$\mathfrak{d}\left[\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigwedge \mathcal{S}_2 \\ \bigwedge \vdots \end{pmatrix}\right] = \prod \mathfrak{d}\left[\mathfrak{C}\Big(\mathcal{S}_i\Big)\right]$$

Thus, we proved.

Lemma 3

$$\mathfrak{d}\left[{}^{!}\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigvee \mathcal{S}_2 \\ \bigvee \vdots \end{pmatrix}\right] = \prod 1 - \mathfrak{d}\left[\mathfrak{C}\Big(\mathcal{S}_i\Big)\right] = \prod \mathfrak{d}\left[{}^{!}\mathfrak{C}\Big(\mathcal{S}_i\Big)\right]$$

where $\mathcal{S}_n(x)$ denotes the n^{th} statement involving x.

Proof.

Suppose there is a piecewise function

$$a(x) = \begin{cases} b & \text{if } \mathcal{S}_1 \bigvee \mathcal{S}_2 \bigvee \mathcal{S}_3 \bigvee \cdots \\ c & \text{otherwise} \end{cases}$$

The conditional function here is:

$$\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigwedge \mathcal{S}_2 \\ \bigwedge \vdots \end{pmatrix}$$

Therefore, the coefficient of c is

$$\mathfrak{d}\left[{}^{!}\mathfrak{C}\begin{pmatrix} & \mathcal{S}_1 \\ \bigwedge & \mathcal{S}_2 \\ \bigwedge & \vdots \end{pmatrix}\right]$$

The logical reasoning behind this coefficient is that if not a single relation mentioned holds, then only $a(x) = c$.

[1]This essentially means that

$${}^{!}\mathfrak{C}\begin{pmatrix} & \mathcal{S}_1 \\ \bigvee & \mathcal{S}_2 \\ \bigvee & \vdots \end{pmatrix} = \mathfrak{C}\begin{pmatrix} & \neg\mathcal{S}_1 \\ \bigwedge & \neg\mathcal{S}_2 \\ \bigwedge & \vdots \end{pmatrix}$$

By Lemma 2,

$$\mathfrak{d}\left[\mathfrak{C}\begin{pmatrix} & \neg\mathcal{S}_1 \\ \bigwedge & \neg\mathcal{S}_2 \\ \bigwedge & \vdots \end{pmatrix}\right] = \prod_{\alpha,\beta,\cdots}\mathfrak{d}\left[\mathfrak{C}\Big(\neg\mathcal{S}_i\Big)\right] = \prod\mathfrak{d}\left[{}^{!}\mathfrak{C}\Big(\mathcal{S}_i\Big)\right]$$

Therefore,

$$\mathfrak{d}\left[{}^{!}\mathfrak{C}\begin{pmatrix} & \mathcal{S}_1 \\ \bigvee & \mathcal{S}_2 \\ \bigvee & \vdots \end{pmatrix}\right] = \prod 1 - \mathfrak{d}\left[\mathfrak{C}\Big(\mathcal{S}_i\Big)\right] = \prod\mathfrak{d}\left[{}^{!}\mathfrak{C}\Big(\mathcal{S}_i\Big)\right]$$

Thus, we proved.

Lemma 4

$$\mathfrak{d}\left[{}^{!}\mathfrak{C}\begin{pmatrix} & \mathcal{S}_1 \\ \bigwedge & \mathcal{S}_2 \\ \bigwedge & \vdots \end{pmatrix}\right] = 1 - \prod\mathfrak{d}\left[\mathfrak{C}\Big(\mathcal{S}_i\Big)\right] = 1 - \mathfrak{d}\left[\mathfrak{C}\begin{pmatrix} & \mathcal{S}_1 \\ \bigwedge & \mathcal{S}_2 \\ \bigwedge & \vdots \end{pmatrix}\right]$$

where $\mathcal{S}_n(x)$ denotes the n^{th} statement involving x.

Proof.

Suppose there is a piecewise function

[1]The symbol "¬" refers to *negation*, also known as *logical NOT*, and is an operation that takes a proposition and returns the opposite truth value. Specifically, it considers all scenarios where the proposition does not hold true or is false. This means we can write ${}^{!}\mathfrak{C}\,(\mathcal{X})$ as $\mathfrak{C}\,(\neg\mathcal{X})$ for any mathematical statement $\mathcal{X}$.

$$a(x) = \begin{cases} b & \text{if } \mathcal{S}_1 \bigwedge \mathcal{S}_2 \bigwedge \mathcal{S}_3 \bigwedge \cdots \\ c & \text{otherwise} \end{cases}$$

The conditional function here is $\mathfrak{C}\begin{pmatrix} x?_1\alpha \\ \bigwedge x?_2\beta \\ \bigwedge \quad \vdots \end{pmatrix}$, so the coefficient of c is

$$\mathfrak{d}\left[{}^{!}\mathfrak{C}\begin{pmatrix} x?_1\alpha \\ \bigwedge x?_2\beta \\ \bigwedge \quad \vdots \end{pmatrix}\right]$$

The logical reasoning behind this coefficient is that if any one relation does not hold, then $a(x) = c$. This happens when $\mathfrak{d}\left[\mathfrak{C}\left(\mathcal{S}_1\right)\right] = 0$ or $\mathfrak{d}\left[\mathfrak{C}\left(\mathcal{S}_2\right)\right] = 0$, etc. Since we need to check if any one of them is zero, we multiply all such $\mathfrak{d}\left[\mathfrak{C}\left(\mathcal{S}_i\right)\right]$ and subtract it from 1, so that if any one of them is 0, the entire coefficient becomes 0, and the coefficient becomes 1. Therefore,

$$\mathfrak{d}\left[{}^{!}\mathfrak{C}\begin{pmatrix} \mathcal{S}_1 \\ \bigwedge \mathcal{S}_2 \\ \bigwedge \quad \vdots \end{pmatrix}\right] = 1 - \prod \mathfrak{d}\left[\mathfrak{C}\left(\mathcal{S}_i\right)\right]$$

Thus, we proved.

Now, we can return to the problem. By lemmas 1 and 3, if $\alpha \neq \beta \neq \gamma \neq \cdots$, then

$$a(x) = \begin{cases} b & \text{if } x = \alpha \vee \beta \vee \gamma \vee \cdots \\ c & \text{otherwise} \end{cases}$$

$$\implies a(x) = (b - c)\,\mathfrak{d}\left[\mathfrak{C}\begin{pmatrix} x = \alpha \\ \bigwedge x = \beta \\ \bigwedge \quad \vdots \end{pmatrix}\right] + c$$

$$\implies a(x) = (b - c) \sum_{\alpha,\beta,\cdots} \mathfrak{d}\left[\mathfrak{C}\left(x = \alpha\right)\right] + c$$

$$\implies a(x) = (b - c)\left\{\sum_{\alpha,\beta,\cdots} \mathfrak{d}\left(x - \alpha\right)\right\} + c$$

However, this has a slight issue, because we cannot, in the general scenario, guarantee that $\alpha \neq \beta \neq \gamma \neq \cdots$. In this case, we usually use Lemma 1.2; alternatively, we can apply Lemma 1.1 with a slight modification by introducing a *tuning term* σ such that

$$a(x) = \begin{cases} b & \text{if } x = \alpha \vee \beta \vee \gamma \vee \cdots \\ c & \text{otherwise} \end{cases}$$
$$\implies a(x) = (b-c) \cdot \sigma \left\{ \sum_{\alpha,\beta,\cdots} \mathfrak{d}\left(x - \alpha\right) \right\} + c$$

In our scenario, the tuning term is defined such that

$$\sigma = \lim_{\varepsilon \to \sum_{\alpha,\beta,\cdots} \mathfrak{d}(x-\alpha)} \left\{ \frac{1}{\varepsilon} \right\}$$

We can similarly define the tuning term for general scenarios. This can be better understood by examining several examples.

- *Example 1:*

  Consider the binary piecewise function

$$a(x) = \begin{cases} b & \text{if } x = \pm y \\ c & \text{otherwise} \end{cases}$$

  for $y > 0$.

  We can represent the conditional function here as $\mathfrak{C}(x = \pm y)$ or $\mathfrak{C}\left(\bigvee \begin{matrix} x = y \\ x = -y \end{matrix} \right)$. By the second law of algebraic piecewise transformation,

$$a(x) = \begin{cases} b & \text{if } x = \pm y \\ c & \text{otherwise} \end{cases}$$
$$\implies a(x) = (b-c) \ \left\{ \mathfrak{d}\left[\mathfrak{C}\left(\bigvee \begin{matrix} x = y \\ x = -y \end{matrix} \right) \right] \right\} + c$$

By lemmas 1 and 3,

$$a(x) = \begin{cases} b & \text{if } x = \pm y \\ c & \text{otherwise} \end{cases}$$

$$\implies a(x) = (b-c) \cdot \sigma \left\{ \mathfrak{d}\Big[\mathfrak{C}(x=y)\Big] + \mathfrak{d}\Big[\mathfrak{C}(x=-y)\Big] \right\} + c$$

$$\implies a(x) = (b-c) \cdot \sigma \left\{ \mathfrak{d}\Big(x-y\Big) + \mathfrak{d}\Big(x+y\Big) \right\} + c$$

Here, the tuning term σ will be equal to 1, because the only case when tuning is required is when $y = -y = 0$; however, since $y > 0$ is given, there is no such case,

$$a(x) = (b-c) \cdot \sigma \left\{ \mathfrak{d}\Big(x+y\Big) + \mathfrak{d}\Big(x-y\Big) \right\} + c$$

- *Example 2:*

Consider the binary piecewise function

$$a(x) = \begin{cases} b & \text{if } x \in (\alpha, \beta] \\ c & \text{otherwise} \end{cases}$$

We can represent the conditional function here as $\mathfrak{C}\left(x \in (\alpha, \beta]\right)$, or in other words, $\mathfrak{C}(\alpha < x \le \beta)$, which we can rewrite as $\mathfrak{C}\left(\bigwedge \begin{matrix} \alpha < x \\ x \le \beta \end{matrix}\right)$.

By Lemma 2,

$$\mathfrak{d}\left[\mathfrak{C}\left(\bigwedge \begin{matrix} \alpha < x \\ x \le \beta \end{matrix}\right)\right] = \mathfrak{d}\left[\mathfrak{C}\Big(\alpha < x\Big)\right] \mathfrak{d}\left[\mathfrak{C}\Big(x \le \beta\Big)\right] = \mathfrak{d}\left[{}^{!}\mathfrak{C}\Big(x \le \alpha\Big)\right] \mathfrak{d}\left[\mathfrak{C}\Big(x \le \beta\Big)\right]$$

$$= \mathfrak{d}\left[{}^{!}\mathfrak{C}(\alpha - x \ge 0)\right] \mathfrak{d}\left[\mathfrak{C}(\beta - x \ge 0)\right]$$

From here, we can solve for ${}^{!}\mathfrak{C}(\alpha - x \ge 0)$ and $\mathfrak{C}(\beta - x \ge 0)$.

$${}^{!}\mathfrak{C}(\alpha - x \ge 0) \equiv {}^{!}\mathfrak{C}\left(|\alpha - x| = \alpha - x\right) = {}^{!}\left(|x-\alpha| + x - \alpha\right)$$

$$\mathfrak{C}(\beta - x \geq 0) \equiv \mathfrak{C}\left(|\beta - x| = \beta - x\right) = |x - \beta| + x - \beta$$

Therefore,

$$\begin{aligned}\mathfrak{d}\left[\mathfrak{C}\left(\bigwedge \begin{matrix}\alpha < x \\ x \leq \beta\end{matrix}\right)\right] &= \mathfrak{d}\left[^{!}\left(|x-\alpha| + x - \alpha\right)\right] \mathfrak{d}\left[|x-\beta| + x - \beta\right] \\ &= \left(1 - \mathfrak{d}\Big\{|x-\alpha| + x - \alpha\Big\}\right)\left(\mathfrak{d}\Big\{|x-\beta| + x - \beta\Big\}\right)\end{aligned}$$

By the second law of algebraic piecewise transformation,

$$a(x) = \begin{cases} b & \text{if } x \in (\alpha, \beta] \\ c & \text{otherwise} \end{cases}$$

$$\implies a(x) = (b-c)\ \left(1 - \mathfrak{d}\Big\{|x-\alpha| + x - \alpha\Big\}\right)\left(\mathfrak{d}\Big\{|x-\beta| + x - \beta\Big\}\right) + c$$

$$\implies a(x) = (b-c)\ \left\{\mathfrak{d}\left[\mathfrak{C}\left(\bigwedge \begin{matrix}\alpha < x \\ x \leq \beta\end{matrix}\right)\right]\right\} + c$$

2.1.4 N-ary piecewise forms

Now that we have covered binary piecewise forms and laid the foundation for algebraic-piecewise transformation, we can talk a bit about general n-ary piecewise forms. Suppose that a general n-ary piecewise function of the form

$$a(x) = \begin{cases} b & \text{if} \quad x = \alpha \\ c & \text{if} \quad x = \beta \\ d & \text{if} \quad x = \gamma \\ & \quad \vdots \end{cases}$$

When addressing cases involving multiple conditions, the structure closely resembles that of n-ary piecewise-defined functions. As we discussed earlier, an example of a piecewise function with multiple conditions is

$$a(x) = \begin{cases} b & \text{if } x = \alpha \vee \beta \vee \gamma \vee \cdots \\ c & \text{otherwise} \end{cases}$$

This can be restated as

$$a(x) = \begin{cases} b & \text{if} \quad x = \alpha \\ b & \text{if} \quad x = \beta \\ b & \text{if} \quad x = \gamma \\ & \vdots \\ c & \text{otherwise} \end{cases}$$

This means that we can use laws 1 and 2 in these types of scenarios because piecewise functions do not depend on what result you get for some condition, but rather on condition itself. Therefore, we can apply the first and second laws of algebraic piecewise transformation for the n-ary piecewise function as follows:

$$a(x) = \begin{cases} b & \text{if} \quad x = \alpha \\ c & \text{if} \quad x = \beta \\ d & \text{if} \quad x = \gamma \\ & \vdots \end{cases}$$

$$\implies a(x) = b \cdot \{\mathfrak{d}\,[\mathfrak{C}(x = \alpha)]\} + c \cdot \{\mathfrak{d}\,[\mathfrak{C}(x = \beta)]\} + d \cdot \{\mathfrak{d}\,[\mathfrak{C}(x = \gamma)]\} + \cdots$$

$$\implies a(x) = b \cdot \Big\{\mathfrak{d}\Big(x - \alpha\Big)\Big\} + c \cdot \Big\{\mathfrak{d}\Big(x - \beta\Big)\Big\} + d \cdot \Big\{\mathfrak{d}\Big(x - \gamma\Big)\Big\} + \cdots$$

Similarly, we generalize this for all the relations.

2.2 Using Trigonometric Functions

Many piecewise functions exhibit periodic behavior, which allows for their representation in terms of trigonometric functions by principle, in accordance with Fourier series expansions.

2.2.1 Modulos and divisibility

A good example of this is having a piecewise function of a variable x with a condition involving a modulo by some number n. In these scenarios, the conditional function is $\mathfrak{C}\,(n \mid x)$. In this situation, we can use a trigonometric function to represent the conditional function.

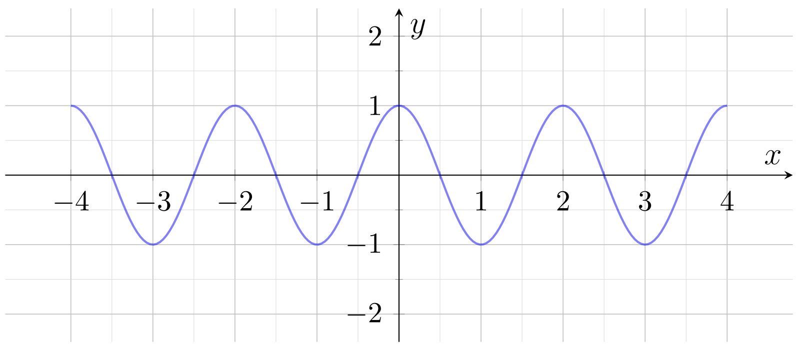
Let us take a simpler example where $n = 2$, so that the conditional function becomes $\mathfrak{C}\,(2 \mid x)$. This is much easier to address since we need only find a function that can detect odd or even signals. We ideally want such a function f to make $f(x) = 0$ when x is even and 1 when x is odd (i.e. $f(x) = x \mod 2$) such that

$$\mathfrak{d}\left[\mathfrak{C}\,(2 \mid x)\right] = 1 - f(x)$$

and

$$\mathfrak{d}\left[\mathfrak{C}(2 \nmid x)\right] = f(x).$$

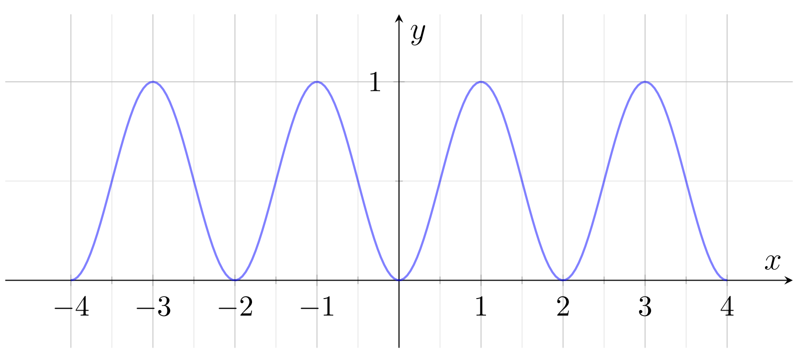
First, consider the function $\cos \pi x$. Its graph looks like this:



We can see that 1 whenever x is even and -1 when it is odd. This means that when x is even, $\cos(\pi x) = 1$, which we can write as $1 - \cos(\pi x) = 0$. This form is excellent because 0 when x is even and 2 (which is greater than 0) when x is odd. Therefore, it is a great candidate for our desired function f. It would be better, however, if it yielded 1 when x is odd. This can be done by simply halving the function so that it becomes

$$\frac{1 - \cos(\pi x)}{2}.$$

This makes the graph for this function look like this:



This approach is the perfect candidate for our desired function; however, let us try writing this function in terms of other trigonometric functions to determine which one has the clearest form.

We can write it in terms of a sine function using the half-angle identity $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$. If we substitute πx for θ and square both sides, we obtain

$$\sin^2\left(\frac{\pi x}{2}\right) = \frac{1-\cos(\pi x)}{2}$$

This is a much cleaner form and will be the most useful when we talk about modulos involving numbers other than 2.

Writing it in terms of tangent will not yield a clean form if we use only trigonometric identities, so we do not need one.

Of these three, the most useful is the one involving sine. Thus, the most ideal f will be $\sin^2\left(\frac{\pi x}{2}\right)$, so that

$$\mathfrak{C}\left(2 \mid x\right) = x \mod 2 = \sin^2\left(\frac{\pi x}{2}\right)$$

Additionally, according to [2]Axiom 5 for conditional functions,

$$\mathfrak{d}\left[\mathfrak{C}\left(2 \mid x\right)\right] = 1 - \sin^2\left(\frac{\pi x}{2}\right) = \cos^2\left(\frac{\pi x}{2}\right) \tag{10}$$

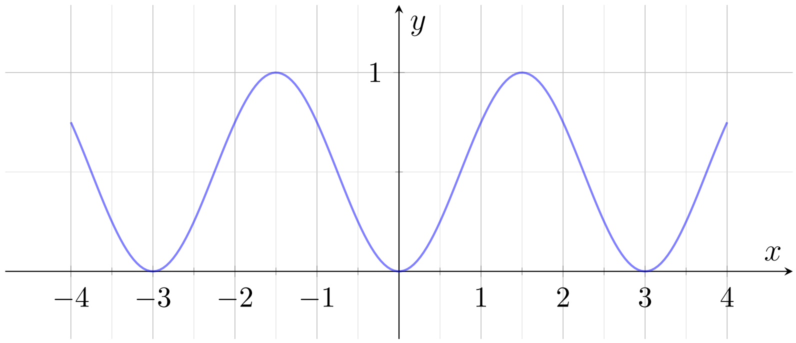
and

$$\mathfrak{d}\left[\mathfrak{C}\left(2 \nmid x\right)\right] = \sin^2\left(\frac{\pi x}{2}\right) \tag{11}$$

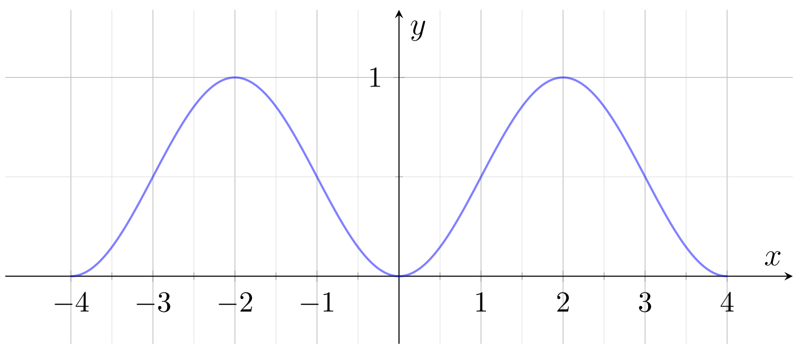
for all integers x.

However, our job here has not yet been done, as we still have to find a general form for $\mathfrak{d}\left[\mathfrak{C}\left(n \mid x\right)\right]$. For this purpose, the sine result we derived earlier will be useful. Right now, we have only dealt with $\sin^2\left(\frac{\pi x}{2}\right)$. If, instead of 2, we divide by 3 in the parameter to obtain $\sin^2\left(\frac{\pi x}{3}\right)$, the graph will look like this:

[2]If the only two possible values of $\mathfrak{C}\left(f(x)\right)$ are 0 and 1, then $\mathfrak{d}\left[\mathfrak{C}\Big(f(x)\Big)\right] = 1 - \mathfrak{C}\Big(f(x)\Big)$.



As you can see, the function has roots at all values divisible by 3. This implies that $\mathfrak{C}\,(3 \mid x) = \sin^2\left(\frac{\pi x}{3}\right)$. Similarly, if we look at the graph of $\sin^2\left(\frac{\pi x}{4}\right)$, it looks like this:



This, once again, has roots at all values divisible by 4 such that $\mathfrak{C}\,(4 \mid x) = \sin^2\left(\frac{\pi x}{4}\right)$. This leads us to propose the general case, for any $n \in \mathbb{Z}$, $\mathfrak{C}\,(n \mid x) = \sin^2\left(\frac{\pi x}{n}\right)$.

We can prove this trivially. If $n \mid x$, then there exists an integer k such that $x = kn$. This means that $\sin^2\left(\frac{\pi x}{n}\right) = \sin^2\left(\frac{\pi k n}{n}\right) = \sin^2(\pi k)$. We know that, for any integer a, $\sin(\pi a) = 0$, and since k is an integer, $\sin^2\left(\frac{\pi x}{n}\right) = \sin^2(\pi k) = 0$, is proven. Therefore,

$$\mathfrak{C}\,(n \mid x) = \sin^2\left(\frac{\pi x}{n}\right)$$

2.2.2 Membership criteria for number sets

Trigonometric functions can also be used to address conditions where a variable x belongs to a number set $\mathbb{A}$, which could be the set of all integers ($\mathbb{Z}$), natural numbers ($\mathbb{N}$), or something else. In these scenarios, the conditional function is $\mathfrak{C}\,(x \in \mathbb{A})$. This is another situation where we can represent the conditional function as a trigonometric function.

Let us start with the conditional function involving the set of all integers so that the conditional function becomes $\mathfrak{C}\,(x \in \mathbb{Z})$. Interestingly, saying that a number is an integer is mathematically equivalent to saying that it is congruent to 0 modulo one. This means that $x \in \mathbb{Z}$ is the same as $1 \mid x$. This means that the conditional function is equivalent to $\mathfrak{C}\,(1 \mid x)$. In the previous section, we established that $\mathfrak{C}\,(n \mid x) = \sin^2\left(\frac{\pi x}{n}\right)$. This means that our case is a special case of this scenario where $n = 1$. Therefore, $\mathfrak{C}\,(x \in \mathbb{Z}) \equiv \mathfrak{C}\,(1 \mid x) = \sin^2\left(\frac{\pi x}{1}\right) = \sin^2 \pi x$. Therefore, to conclude, we have

$$\mathfrak{C}\,(x \in \mathbb{Z}) = \sin^2(\pi x) \tag{12}$$

This works because for any integer a, $\sin(\pi a) = 0$.

Now that we have established the conditional function for a variable belonging to the set of integers, we can address the one involving the set of natural numbers so that the conditional function becomes $\mathfrak{C}\,(x \in \mathbb{N})$. This task becomes trivial if we use the fact that the set of natural numbers is simply the positive set of integers, i.e. $\mathbb{N} = \mathbb{Z}^+$.

This means that we can write the conditional function as

$$\mathfrak{C}(x \in \mathbb{N}) \equiv \mathfrak{C}(x \in \mathbb{Z}^+) = \mathfrak{C}\left(\bigwedge \begin{matrix} x \in \mathbb{Z} \\ x > 0 \end{matrix}\right)$$

By Lemma 2,

$$\mathfrak{d}\left[\mathfrak{C}\left(\bigwedge \begin{matrix} x \in \mathbb{Z} \\ x > 0 \end{matrix}\right)\right] = \mathfrak{d}\left[\mathfrak{C}(x \in \mathbb{Z})\right]\mathfrak{d}\left[\mathfrak{C}(x > 0)\right]$$

We already know that $\mathfrak{d}\left[\mathfrak{C}(x \in \mathbb{Z})\right] = \mathfrak{d}\left[\sin^2 \pi x\right]$, so we need to simplify only $\mathfrak{d}\left[\mathfrak{C}(x > 0)\right]$.

We can write $x > 0$ as $\neg(x \leq 0)$, which becomes $\neg(-x \geq 0)$. This means that

$$\mathfrak{d}\left[\mathfrak{C}(x > 0)\right] = \mathfrak{d}\left[\mathfrak{C}(\neg(x \leq 0))\right] = \mathfrak{d}\left[{}^{!}\mathfrak{C}(x \leq 0)\right] = 1 - \mathfrak{d}\left[\mathfrak{C}(x \leq 0)\right]$$

$$= 1 - \mathfrak{d}\left[\mathfrak{C}(-x \geq 0)\right] = 1 - \mathfrak{d}\left[|-x| - (-x)\right] = 1 - \mathfrak{d}\left[|x| + x\right]$$

In total, we have

$$\mathfrak{d}\left[\mathfrak{C}(x \in \mathbb{N})\right] = \mathfrak{d}\left[\mathfrak{C}(x \in \mathbb{Z})\right]\mathfrak{d}\left[\mathfrak{C}(x > 0)\right] = \mathfrak{d}\left[\sin^2 \pi x\right] \cdot \left(1 - \mathfrak{d}\left[|x| + x\right]\right)$$

Therefore, to conclude, we have

$$\mathfrak{d}\left[\mathfrak{C}(x \in \mathbb{N})\right] = \mathfrak{d}\Big(\sin^2 \pi x\Big) \cdot \Big\{1 - \mathfrak{d}\Big(|x| + x\Big)\Big\} \tag{13}$$

If we are talking about whole numbers, which we will denote with $\mathbb{W}$ so that the conditional function becomes $\mathfrak{C}(x \in \mathbb{W})$, we can use the fact that

$$\mathbb{W} = \mathbb{N} \cup \{0\} = \mathbb{Z}^+ \cup \{0\}.$$

This means that we can write the conditional function as

$$\mathfrak{C}(x \in \mathbb{W}) = \mathfrak{C}(x \in \mathbb{Z}^+ \cup \{0\}) = \mathfrak{C}\left(\bigwedge \begin{matrix} x \in \mathbb{Z} \cup \{0\} \\ x > 0 \end{matrix}\right) = \mathfrak{C}\left(\bigwedge \begin{matrix} x \in \mathbb{Z} \\ x \geq 0 \end{matrix}\right)$$

By Lemma 2,

$$\mathfrak{d}\left[\mathfrak{C}\left(\bigwedge \begin{matrix} x \in \mathbb{Z} \\ x \geq 0 \end{matrix}\right)\right] = \mathfrak{d}\left[\mathfrak{C}(x \in \mathbb{Z})\right] \mathfrak{d}\left[\mathfrak{C}(x \geq 0)\right]$$

We know that

$$\mathfrak{d}\left[\mathfrak{C}(x \in \mathbb{Z})\right] = \mathfrak{d}\left[\sin^2 \pi x\right]$$

and since $\mathfrak{C}(x \geq 0) = |x| - x$, therefore

$$\mathfrak{d}\left[\mathfrak{C}(x \geq 0)\right] = \mathfrak{d}\left[|x| - x\right]$$

In total, we have

$$\mathfrak{d}\left[\mathfrak{C}(x \in \mathbb{W})\right] = \mathfrak{d}\left[\mathfrak{C}(x \in \mathbb{Z})\right] \mathfrak{d}\left[\mathfrak{C}(x \geq 0)\right] = \mathfrak{d}\left[\sin^2 \pi x\right] \cdot \mathfrak{d}\left[|x| - x\right]$$

Using the exponential property of the Kronecker naught,

$$\mathfrak{d}\left[\sin^2 \pi x\right] \cdot \mathfrak{d}\left[|x| - x\right] = \mathfrak{d}\left[\sin^2 \pi x + |x| - x\right]$$

Therefore, we can rewrite this as

$$\mathfrak{d}\left[\mathfrak{C}\Big(x \in \mathbb{W}\Big)\right] = \mathfrak{d}\left[\sin^2 \pi x + |x| - x\right] \tag{14}$$

Additionally, according to [3]Axiom 6 for conditional functions, we can say that

[3]For a statement $\mathcal{S}$ and an expression s, if $\mathfrak{d}\left[\mathfrak{C}(\mathcal{S})\right] = \mathfrak{d}\left(s\right)$, then $\mathfrak{C}(\mathcal{S}) \equiv s$

$$\mathfrak{C}\Big(x \in \mathbb{W}\Big) \equiv \sin^2(\pi x) + |x| - x \tag{15}$$

Now, let us explore how to approach the membership condition for a variable x belonging to the set of prime numbers, which we denote with $\wp$ so that the conditional function becomes $\mathfrak{C}(x \in \wp)$. Although this approach seems challenging because prime numbers have limited properties, it is quite trivial.

For this purpose, we use **Wilson's theorem**, which states that if a natural number n satisfies

$$(n-1)! \equiv -1 \pmod{n},$$

then it is a prime number.

This means that for all $p \in \wp$,

$$(p-1)! + 1 \equiv 0 \pmod{p}$$

Therefore, we can rewrite the conditional function $\mathfrak{C}(x \in \wp)$ as $\mathfrak{C}\left(x \mid (x-1)! + 1\right)$, which is of the form $\mathfrak{C}\left(n \mid x\right)$ and is equal to

$$\sin^2\left(\frac{\pi x}{n}\right).$$

In our case, we have

$$\mathfrak{C}(x \in \wp) = \sin^2\left(\frac{\pi\left[(x-1)! + 1\right]}{x}\right) = \sin^2\left(\frac{\pi(x-1)! + \pi}{x}\right)$$

We can use the [4]**gamma function ($\boldsymbol{\Gamma}$)** to rewrite this as

$$\mathfrak{C}\left(x \in \wp\right) = \sin^2\left(\frac{\pi\Gamma\left(x\right) + \pi}{x}\right) \tag{16}$$

This is, for now, the best we can do regarding the conditional function involving the set of primes. Although working with factorials and gamma functions is time-consuming, most lemmas related to prime numbers incorporate them due to the strong connection between the two concepts.

[4]The Gamma Function is an extension of the factorial to all complex values and is defined as $\Gamma(x) = (x-1)!$.

2.2.3 Additional Approximations

The maximum of two variables can be written as a piecewise function as follows:

$$\max(x,y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{otherwise} \end{cases}$$

By the second law of algebraic piecewise transformation,

$$\max(x,y) = (x-y)\ \mathfrak{d}\left[\mathfrak{C}\Big(x \geq y\Big)\right] + y$$

As we discussed before, we can apply the **softmax approximation** again, giving us

$$\frac{1}{\alpha}\ln\left[e^{\alpha x} + e^{\alpha y}\right] = (x-y)\ \mathfrak{d}\left[\mathfrak{C}\Big(x \geq y\Big)\right] + y$$

For now, we will avoid writing a limit for convenience. Simplifying,

$$e^{\alpha x} + e^{\alpha y} = \exp\left\{\alpha\,(x-y)\ \mathfrak{d}\left[\mathfrak{C}\Big(x \geq y\Big)\right]\right\} e^{\alpha y}$$

$$\implies (x-y)\,\mathfrak{d}\left[\mathfrak{C}\Big(x \geq y\Big)\right] = \frac{1}{\alpha}\ln\left[e^{\alpha(x-y)} + 1\right]$$

Therefore, we finally have

$$(x-y)\,\mathfrak{d}\left[\mathfrak{C}\Big(x \geq y\Big)\right] = \lim_{\alpha\to\infty}\frac{1}{\alpha}\ln\left[e^{\alpha(x-y)} + 1\right] \tag{17}$$

Similarly,

$$(x-y)\,\mathfrak{d}\left[\mathfrak{C}\Big(x \leq y\Big)\right] = \lim_{\alpha\to\infty}\frac{1}{-\alpha}\ln\left[e^{-\alpha(x-y)} + 1\right] \tag{18}$$

3 Applications

In this section, we will look at a few practical applications where this theory can be used. This helps us obtain a better understanding of the nature of the form and how it simplifies problems.

3.1 The Collatz conjecture

A well-known unsolved mathematical problem is the Collatz conjecture, which is set up as follows: For a natural number $n \in \mathbb{N}$, a function $\mathcal{K}(n)$ is defined as half of n if it is even and thrice of n plus one if it is odd. The conjecture states that if we apply $\mathcal{K}$ indefinitely to any natural n, it will eventually reach the cycle of

$$4 \to 2 \to 1 \to 4 \to \cdots$$

We can write the function in piecewise form as

$$\mathcal{K}(n) = \begin{cases} n/2 & \text{if } 2 \mid n \\ 3n+1 & \text{otherwise} \end{cases}$$

We can represent the conditional function here as $\mathfrak{C}\,(2 \mid n)$. If $2 \mid n$, $n/2$ is an integer. Therefore, we can say that

$$\mathfrak{C}\,(2 \mid n) = \mathfrak{C}\left(\frac{n}{2} \in \mathbb{Z}\right) \overset{\text{interpolate}}{\equiv} \mathfrak{C}\left(\frac{n}{2} = \left\lfloor\frac{n}{2}\right\rfloor\right)$$

Now, this is of the form $\mathfrak{C}(x = y)$, so we can say that

$$\mathfrak{C}\,(2 \mid n) = \frac{n}{2} - \left\lfloor\frac{n}{2}\right\rfloor$$

By the second law of algebraic piecewise transformation,

$$\mathcal{K}(n) = \begin{cases} n/2 & \text{if } 2 \mid n, \\ 3n+1 & \text{otherwise.} \end{cases}$$

$$\implies \mathcal{K}(n) = \frac{n}{2} \cdot \left\{\mathfrak{d}\left(\frac{n}{2} - \left\lfloor\frac{n}{2}\right\rfloor\right)\right\} + (3n+1) \cdot \left\{1 - \mathfrak{d}\left(\frac{n}{2} - \left\lfloor\frac{n}{2}\right\rfloor\right)\right\}$$

We know that $\mathfrak{d}\,(n) = \mathfrak{d}\,(2n)$, so we can write this as

$$\mathcal{K}(n) = \frac{n}{2} \cdot \left\{\mathfrak{d}\left(n - 2\left\lfloor\frac{n}{2}\right\rfloor\right)\right\} + (3n+1) \cdot \left\{1 - \mathfrak{d}\left(n - 2\left\lfloor\frac{n}{2}\right\rfloor\right)\right\}$$

Let us now look at the nature of $\lfloor n/2 \rfloor$. Interestingly, it is also a piecewise function, and it is defined as

$$\left\lfloor\frac{n}{2}\right\rfloor = \begin{cases} \frac{n}{2} & \text{if } 2 \mid n \\ \frac{n-1}{2} & \text{otherwise} \end{cases}$$

By Multiplying -2 on both sides,

$$-2\left\lfloor\frac{n}{2}\right\rfloor = \begin{cases} -n & \text{if } 2 \mid n \\ -n+1 & \text{otherwise} \end{cases}$$

Adding n on both sides,

$$n-2\left\lfloor\frac{n}{2}\right\rfloor = \begin{cases} 0 & \text{if } 2 \mid n \\ 1 & \text{otherwise} \end{cases}$$

Again, we can represent the conditional function here as $\mathfrak{C}(2 \mid n)$, which we know is $\frac{n}{2} - \left\lfloor\frac{n}{2}\right\rfloor$; therefore,

$$n-2\left\lfloor\frac{n}{2}\right\rfloor = (0-1)\cdot\mathfrak{d}\left(\frac{n}{2}-\left\lfloor\frac{n}{2}\right\rfloor\right)+1 = 1-\mathfrak{d}\left(\frac{n}{2}-\left\lfloor\frac{n}{2}\right\rfloor\right) = 1-\mathfrak{d}\left(n-2\left\lfloor\frac{n}{2}\right\rfloor\right)$$

$$\implies \mathfrak{d}\left(n-2\left\lfloor\frac{n}{2}\right\rfloor\right) = 2\left\lfloor\frac{n}{2}\right\rfloor - n + 1$$

We can substitute this into the equation for $\mathcal{K}$ so that it becomes

$$\mathcal{K}(n) = \frac{n}{2}\cdot\left\{2\left\lfloor\frac{n}{2}\right\rfloor - n + 1\right\} + (3n+1)\cdot\left\{n-2\left\lfloor\frac{n}{2}\right\rfloor\right\}$$

This simplifies to

$$\boxed{\mathcal{K}(n) = \frac{n}{2}(3+5n) - \left\lfloor\frac{n}{2}\right\rfloor(2+5n)}$$

We name this the **Keshet-Collatz function**.

Therefore, this equation for $\mathcal{K}$ should also work in the same way. We can test this out for ourselves, by checking for when n is even (i.e. when $\left\lfloor\frac{n}{2}\right\rfloor = \frac{n}{2}$), and when n is odd (i.e. when $\left\lfloor\frac{n}{2}\right\rfloor = \frac{n-1}{2}$). If we plug them in, we obtain

$$\mathcal{K}(n) = \frac{n}{2}(3+5n) - \frac{n}{2}(2+5n) = \frac{n}{2}$$

and

$$\mathcal{K}(n) = \frac{n}{2}(3+5n) - \frac{n-1}{2}(2+5n) = 3n+1$$

We can improve the Keshet-Collatz function and make it differentiable by using trigonometric functions as we discussed earlier. This approach is much easier. We know that the conditional function is $\mathfrak{C}(2 \mid n)$, which means we are dealing with

$\mathfrak{d}\left[\mathfrak{C}\left(2 \mid n\right)\right]$ and $\mathfrak{d}\left[\mathfrak{C}(2 \nmid n)\right]$, and we know they are equal to $\cos^2\left(\frac{\pi n}{2}\right)$ and $\sin^2\left(\frac{\pi n}{2}\right)$ respectively.

By the second law of algebraic piecewise transformation,

$$\mathcal{K}(n) = \begin{cases} n/2 & \text{if } 2 \mid n, \\ 3n+1 & \text{otherwise.} \end{cases}$$

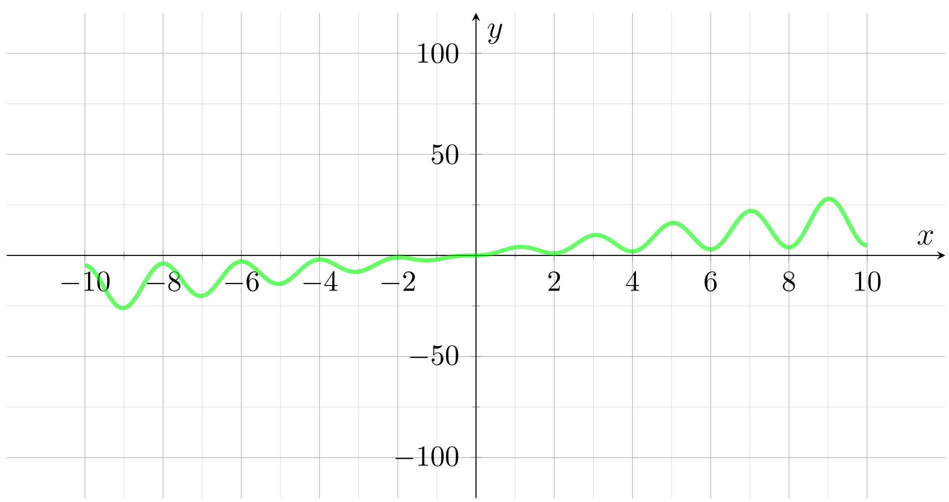
$$\Rightarrow \mathcal{K}(n) = \frac{n}{2} \cdot \cos^2\left(\frac{\pi n}{2}\right) + (3n+1) \cdot \sin^2\left(\frac{\pi n}{2}\right)$$

By simplifying, we obtain

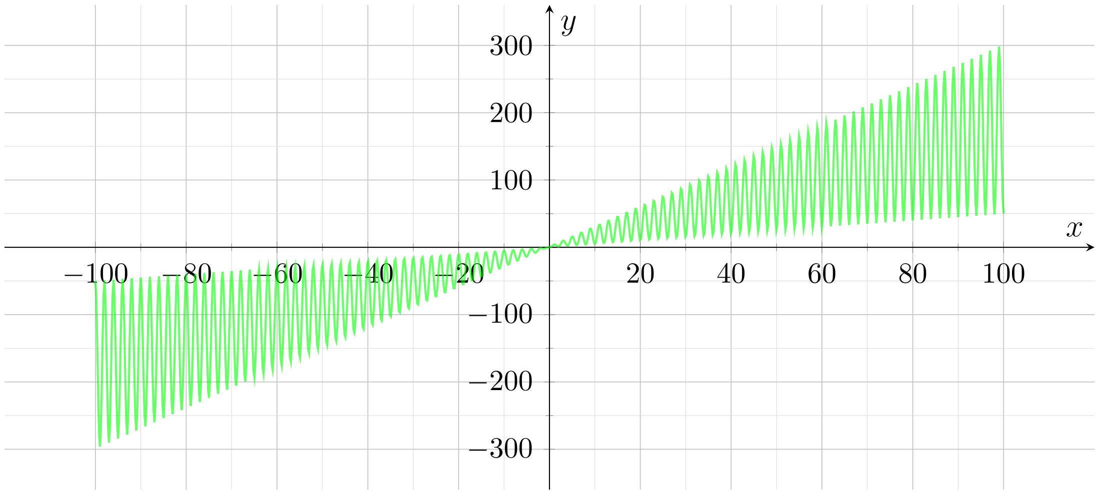
$$\boxed{\mathcal{K}(n) = \frac{n}{2} + \left(\frac{5n+2}{2}\right) \sin^2\left(\frac{\pi n}{2}\right)}$$

This is another form of the **Keshet-Collatz function**. Keshet-Collatz function can also be thought of as an extension of the Collatz function to integers.

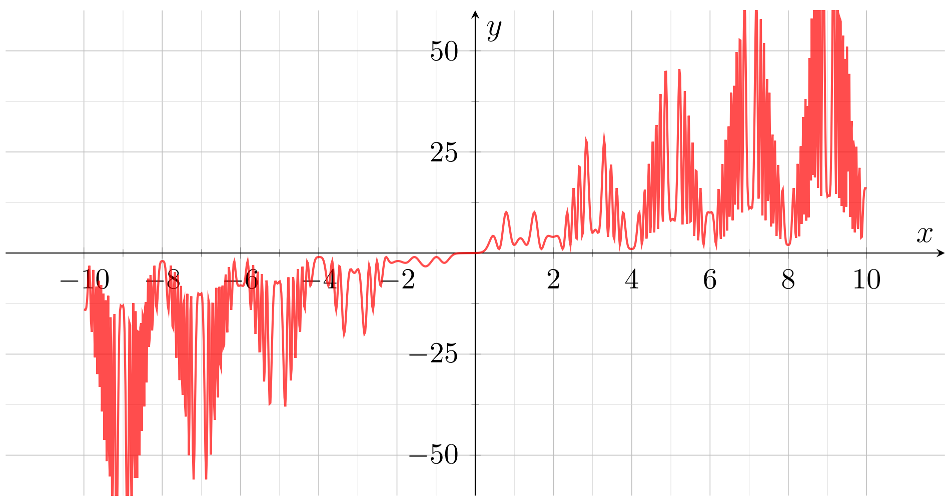
Graphing $\mathcal{K}(x)$ over small values of x (including negative values) gives:



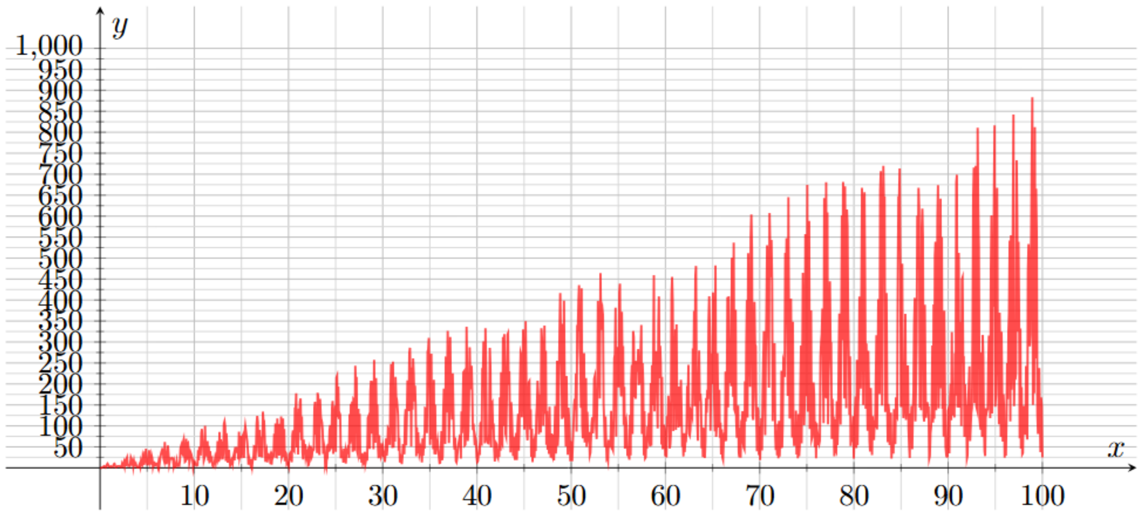
For large values,



Graphing $\mathcal{K}\left(\mathcal{K}(x)\right)$ over small values gives:



For large values (ignoring negative values for a less compile-heavy graph),



Author's Note

All the core definitions and transformations (e.g., the Kronecker naught, conditional function, and laws of algebraic-piecewise transformation) presented in this paper are original contributions by the author. The listed references are provided for general background and context but are not directly cited within the body of the article.

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