Vector kinematic analysis of Keplerian velocity from Hamilton’s hodograph

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Abstract: Many authors in the literature focused on Hamilton's hodograph of the Keplerian motion but none of them developed a vector kinematic analysis of the peculiar velocity shown by the hodograph, so we perform it here. As expected, the analysis predicts Kepler's laws, the mathematical structure of Newton's acceleration, and it also leads to new expressions for the classical mechanical energy and the eccentricity. We then discuss the relationship between the geometric acceleration given by the kinematics and the physical one stated by Newton.

Introduction

In 1845 W.R. Hamilton proposed a new way of representing the motion of a body submitted to Newton's gravitation law, the hodograph[1]. Many authors have since discussed this approach[2-8], some of them giving figures of this geometric problem that lacked in Hamilton's communication. Among them is E.I. Butikov who pictured a most interesting property of the Keplerian motion, in the figure 4 of one of his article[8]: the velocity of a Keplerian orbiter is always the simple addition of a constant rotation velocity, that he called **u** while we will call it $\mathbf{v_R}$, and a constant translation velocity, that he called **w** while we will call it $\mathbf{v_T}$. The overall velocity is then given by :

$$\mathbf{v} = \mathbf{v_R} + \mathbf{v_T} \tag{1}$$

Because the authors were focused on the structure of the hodograph rather than the kinematics of this peculiar velocity, we could not find in the literature any vector kinematic analysis of this very simple vector expression, therefore we decided to develop one.

We will show that, as expected, we are able to confirm that such a velocity is in full agreement with the three laws of Kepler and with the mathematical expression of Newton's gravitational acceleration. Nonetheless, Newton's constant physical factor GM (G being the constant of gravitation and M the mass at the focus of the orbit) is replaced by another constant factor purely kinematic, which is trivial as far as the above equation (1) embeds no physical parameter but only geometric ones.

We are then driven to discuss the equality between the kinematic acceleration and Newton's physical one.

Kinematic analysis

As explained in the introduction, the velocity **v** of any Keplerian orbiter is simply the vector addition of two uniform velocities, one of rotation, $\mathbf{v_R}$, plus one of translation, $\mathbf{v_T}$ (see equation 1). Its simplest mathematical expression is then as follows:

$$\mathbf{v} = \mathbf{v_R} + \mathbf{v_T}$$

with

$$\begin{aligned} &\mathbf{v_R} = \boldsymbol{\omega} \times \mathbf{r} \\ &v_R = \|\mathbf{v_R}\| = \omega\, r = \text{constant} \\ &\mathbf{v_T} = \mathbf{constant} \end{aligned} \tag{2}$$

In this expression $\boldsymbol{\omega}$ is the vector frequency of rotation, perpendicular to the plane of the orbit, and **r** is the vector radius, from the focus of the orbit to the orbiter. Note that $\mathbf{v_T}$ and $\mathbf{v_R}$ are coplanar all along the orbit. Take care, in this expression the index R means “rotation” but not “radial”, while the index T stands for “translation” but not “tangential”. The figure 1 shows these two velocities on a typical Keplerian orbit.

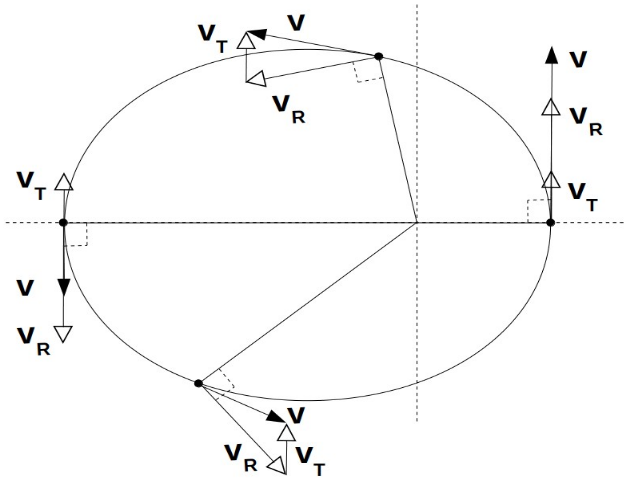

***Figure 1** : the Keplerian velocity* **v** *decomposed into its rotation velocity* $\mathbf{v_R}$ *and its translation velocity* $\mathbf{v_T}$.

Now let us demonstrate that coming from this definition of the orbital velocity we can predict the existence of Kepler’s laws as well as Newton’s acceleration, or at least its mathematical structure.

The first consequence of the above expression (2) is the validity of the following one by derivation of v_R with respect to time ($\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ being colinear) :

$$\dot{\boldsymbol{\omega}}\, r = -\dot{r}\, \boldsymbol{\omega} \tag{3}$$

From the relations (2) and (3) we can then calculate the acceleration which is the derivative of the velocity with respect to time :

$$\mathbf{a} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v} = -\frac{\boldsymbol{\omega}}{r^2} \times \left(\mathbf{r} \times \left(\mathbf{r} \times \mathbf{v}\right)\right) \tag{4}$$

Defining the massless angular momentum as

$$\mathbf{L} = \mathbf{r} \times \mathbf{v} \tag{5}$$

the final expression of the acceleration is given by :

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$$\mathbf{a} = -\frac{k}{r^3}\,\mathbf{r} \quad \text{with} \quad k = L\,v_R = \text{constant} \tag{6}$$

Therefore, the acceleration and the vector radius are colinear and this forces the angular momentum to be a constant, because $\dot{\mathbf{L}} = \mathbf{r} \times \mathbf{a} = \mathbf{0}$, as awaited for a central field motion :

$$\mathbf{L} = \mathbf{constant} \tag{7}$$

Note that the expression (6) of the acceleration is mathematically identical to the one of Newton, at the condition that Kepler's constant k verifies:

$$GM = L\,v_R = k \tag{8}$$

Now from this we observe that the vector product of the rotation velocity with the angular momentum leads trivially to :

$$\mathbf{v_R} \times \mathbf{L} = v_R^2 \left(1 + \frac{\mathbf{v_R} \cdot \mathbf{v_T}}{v_R^2}\right)\mathbf{r} \tag{9}$$

The scalar version of this equation is therefore :

$$p = (1 + e\cos\theta)\,r \qquad \text{with} \quad p = \frac{L}{v_R} = \frac{k}{v_R^2} \quad \text{and} \quad e = \frac{v_T}{v_R} \tag{10}$$

This is the equation of a conic where p is the semi-latus rectum, e is the eccentricity and θ is the true anomaly, i.e. the angle between $\mathbf{v_T}$ and $\mathbf{v_R}$ which is also the angle between the direction of the perigee and the vector radius. This is the expression of Kepler's first law[9].

Now the vector multiplication of $\mathbf{v}$ and $\mathbf{L}$ leads to $\mathbf{v} \times \mathbf{L} = \mathbf{v_T} \times \mathbf{L} + \mathbf{v_R} \times \mathbf{L} = \mathbf{v_T} \times \mathbf{L} + k\,\mathbf{r}/r$, and thus to

$$\frac{\mathbf{v_T} \times \mathbf{L}}{k} = \frac{\mathbf{v} \times L}{k} - \frac{\mathbf{r}}{r} \tag{11}$$

The right hand side of this equation is known as the eccentricity vector[10], that can be thus given by a new expression :

$$\mathbf{e} = \frac{\mathbf{v_T} \times \mathbf{L}}{L\,v_R} \tag{12}$$

Therefore the translation velocity is always perpendicular to the main axis of the conic, which direction is the one of the vector eccentricity. The figure 1 exhibits both the rotation and the translation velocities at different positions on a conic.

Let us now notice that the scalar multiplication of the total velocity and the vector radius leads to :

$$\mathbf{r} \cdot \mathbf{v} = \mathbf{r} \cdot \mathbf{v_T} = r\,\dot{r} \quad \text{thus} \quad \dot{r} = v_T \sin\theta \tag{13}$$

Using this last expression associated to the time derivation of the conic equation (10), it is trivial to show that the angular momentum can be presented as the multiplication of the square of the vector radius and the derivative of the true anomaly with respect to time :

$$L = r^2\,\dot{\theta} \tag{14}$$

This last expression is very well known, being described for instance by L. Landau and E. Lifchitz in their course "Mechanics"[11]. It shows that the areal velocity, defined as $\dot{f} = r^2\,\dot{\theta}/2 = L/2$, must be a constant as far as the angular momentum also is. Therefore the expression (14) is nothing else but the second law of Kepler[9].

Note that the time derivative of the true anomaly $\dot{\theta}$ and the frequency of rotation ω are then related by the following formula :

$$\dot{\theta} = \omega(1 + e\cos\theta) \quad \text{or} \quad r\,\dot{\theta} = p\,\omega \tag{15}$$

Now integrating the expression (14) over a complete period T of revolution for an ellipse (see L. Landau and E. Lifchitz[9]), knowing that the surface of an ellipse is $f = \pi\,a\,b$, a being the semi major axis and $b = \sqrt{a\,p}$ the minor one, we can integrate as so :

$$\int_0^T L\,dt = L\,T = \int_0^T 2\,\dot{f}\,dt = 2\,f = 2\,\pi\,a\sqrt{a\,p} \tag{16}$$

Then using the equations (10), we are trivially led to the following formula :

$$L\,v_R = 4\,\pi^2\,\frac{a^3}{T^2} = k \tag{17}$$

This is the expression of Kepler's third law[9].

In addition, we can see a deep connection between this Keplerian kinematics and the classical mechanics. Indeed by calculating the square of the velocity (2) it is trivial to get the following relationship :

$$\frac{1}{2}\,v^2 - \frac{k}{r} = \frac{1}{2}\,v_R^2\,(e^2 - 1) = \text{constant} \tag{18}$$

If we multiply this expression by the mass m of the orbiter, we get its classical mechanical energy[9], which is therefore :

$$E_M = \frac{1}{2}\,m\,v^2 - \frac{mk}{r} = \frac{1}{2}\,m\,v_R^2\,(e^2 - 1) \tag{19}$$

This expression of the classical energy by the means of the rotation velocity and the eccentricity has never been reported at our knowledge.

Discussion

As expected, because the authors have already shown it, the kinematic analysis of the velocity derived from Hamilton's hodograph of the Keplerian motion predicts Kepler's laws as well as the mathematical structure of Newton's acceleration.

In addition it gives two results that have not been described yet in the literature. The first one is the expression (12) of the eccentricity vector, so also of the Runge-Lenz vector[10]. The second one is the value (19) of the mechanical energy depending on the rotation velocity and the eccentricity.

A first obvious notice has to be made about the velocity (2) : mathematically it cannot describe a linear accelerated trajectory. This geometric fact is consistent with the radial equation of the conics (10), also demonstrated by Lev Landau in his physics class "Mechanics"[9] as the equation to consider when explaining the mechanics of the Keplerian motion, and that is mathematically also unable to describe a linear accelerated trajectory.

A second obvious notice concerns the gravitational circular motion around a central body of mass M. Indeed, for Newton the velocity must then be:

$$v = \sqrt{\frac{GM}{r}} \tag{20}$$

Consequently, GM being postulated as a constant independent of r, the velocity can only decrease if the radius increases. The situation is different for the kinematics that states:

$$v = \sqrt{\frac{L\,v_R}{r}} = \sqrt{L\,\omega} \tag{21}$$

Therefore the velocity can be constant, whatever r, if L ω also is.

The whole connection between the kinematics's interpretation of the acceleration and Newton's one stands in the relation (8): $GM = L\,v_R$. On the left side of this equality is a constant physical factor, where G is postulated by Newton to be the same universal constant, applicable to all masses, at all scales. On the contrary, on the right side is the kinematic factor $L\,v_R$, where L and v_R are defined for a given orbit but are both mathematically free to vary at any scale and for any phenomenon.

With no doubt at all the equation (8) is a reality because we measured it for so many phenomenons at the solar system scale. We may wonder however if this is also true at other scales, like the galactic one or the atomic one, because nothing limits mathematically L and v_R. It would be so, G would not be a universal constant any more but would vary according to the scale of the phenomenon, nonetheless the mathematical structure (6) of Newton's acceleration would still be universal.

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