No-hidden-variable theorem based on a single projection operator

Koji Nagata[1] and Tadao Nakamura[2]

[1]*Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 34141, Korea*
E-mail: ko_mi_na@yahoo.co.jp
[2]*Department of Information and Computer Science, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan*

(Dated: July 22, 2025)

Abstract

We propose no-hidden-variable theorem for the quantum measurement outcomes by measuring an observable that is obtained through a single projection operator. We are in the inconsistency within hidden-variable theories when the first predetermined hidden result is +1 by measuring a single projection operator $|\uparrow\rangle\langle\uparrow|$ in the quantum state $|\uparrow\rangle$, the second predetermined hidden result is the same +1 as by measuring the same projection operator $|\uparrow\rangle\langle\uparrow|$ in the same quantum state $|\uparrow\rangle$, and then we consider the existence of only the following proposition $[|\uparrow\rangle\langle\uparrow|, |\uparrow\rangle\langle\uparrow|] = 0$ and we assign the value "1" for the commutator. It turns out that we cannot assign the predetermined hidden result for quantum measurement outcome as +1 when measuring a single projection operator. Based on the argumentations, we propose an experimental accessible hidden-variable inconsistency in terms of the imperfect source and detector.

PACS numbers: 03.67.-a, 03.65.Ta, 42.50.-p, 03.65.Ca
Keywords: Quantum information, Foundations of quantum mechanics, Quantum optics, Formalism

I. INTRODUCTION

Quantum mechanics (cf. [1–8]) gives explanations for the microscopic behaviors of the nature. We see researches concerning the mathematical formulations of quantum mechanics. For example, the mathematical foundations of quantum mechanics are discussed by Mackey [9]. On the quantum logic approach to quantum mechanics is also discussed by Gudder [10]. Conditional probability and the axiomatic structure of quantum mechanics are also reported by Guz [11].

von Neumann's mathematical model for quantum mechanics is logically successful [5]. The axiomatic system for the mathematical model is a consistent one. Thus, we cannot say that von Neumann's mathematical model has the hidden-variable inconsistency. What is the hidden-variable inconsistency to be discussed in this paper? We cannot expand von Neumann's mathematical model more in handling real experimental data [12, 13]. Mathematically, von Neumann's model is logically consistent, which fact is true. However, von Neumann's theory is questionable in the sense that the mathematical model does not always expand to real experimental data. And there is the hidden-variable inconsistency if we apply von Neumann's model to expanding even a simple physical situation. In short, von Neumann's mathematical model might not be useful in that case.

The hidden-variable inconsistency to be discussed in this paper is significant. von Neumann's mathematical model has the qualification to be true axiomatic system for quantum mechanics. Therefore, we cannot modify the axioms based on the nature of Matrix theory. Nevertheless, we encounter the hidden-variable inconsistency, probably due to the nature of Matrix theory, within von Neumann's theory, that is, his mathematical model is perfect but his physical thought might be a little bit questionable today.

Of course our analyses rely on von Neumann's model and the possible positions of the hidden-variable inconsistencies lie outside von Neumann's model. Thus we cannot sometimes expand von Neumann's model to experimental situations. In short, we might have to limit in some sense his model for quantum measurement theories.

On the other hand, the incompleteness argument to quantum mechanics itself is discussed by Einstein, Podolsky, and Rosen [14]. A hidden-variable interpretation of quantum mechanics is a topic of research [3, 4] and the no-hidden-variable theorem is discussed by Bell, Kochen, and Specker [15, 16]. A strengthened Kochen–Specker theorem, i.e., the free will theorem is discussed by Conway and Kochen [17].

Recently, Nagata, Diep, and Nakamura discuss a novel inconsistency within hidden-variable theories, using two symmetric measurements. They are independent of the order of measurements themselves [18]. As the aim of this paper, we propose an experimental accessible hidden-variable inconsistency within quantum phenomena in terms of the imperfect source and detector based on the argumentations [18]. We hope not only theoretical physicists but also experimental physicists can easily understand our claim.

In more detail, we encounter an imperfect quantum state, the dark count, and the quantum efficiency, which cannot be avoidable in a real experimental situation. If we use the quantum predictions by even number $2N$ trials, then the degree of the hidden-variable inconsistency increases by an amount that grows linearly with N. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

II. EXPLANATIONS FOR THE HIDDEN-VARIABLE INCONSISTENCY USING COMMUTING PROJECTORS

The hidden-variable contradiction in this paper is explained this: If we allow to take both of commutativeness and non-commutativeness in consideration, there is the uncertainty principle, which fact seems to be likely to be quantum theory. And that seems to be natural to have the value of "2".

On the other hand, we would discuss that the sum rule is equivalent to the product rule for commuting observables. First, we define the functional rule this:

$$f(g(O)) = g(f(O)), \tag{1}$$

where O is a Hermitian operator and f, g are appropriate functions to be used later. Second, the sum rule is defined this:

$$f(A_1 + A_2) = f(A_1) + f(A_2), \tag{2}$$

where A_1, A_2 are Hermitian operators. Finally, the product rule is defined this:

$$f(A_1 \cdot A_2) = f(A_1) \cdot f(A_2). \tag{3}$$

This fact above (the sum rule is equivalent to the product rule) is based on the property of these two Hermitian operators themselves. This leads to the propositions that they are valid even for the real numbers of the diagonal elements of the two Hermitian operators.

We may have [4, 18] the following relation between the three rules for commuting observables:

$$\begin{aligned} &\text{The functional rule} \\ &\Leftrightarrow \text{The sum rule} \\ &\Leftrightarrow \text{The product rule} \end{aligned} \tag{4}$$

For example, let us derive the sum rule and the product rule from the functional rule. Suppose now that A and B are two commuting Hermitian operators. Since A and B commute they can be diagonalized simultaneously. This means that there exists a basis $\{P_i\}$ by which we can expand $A = \sum_i a_i P_i$ and such that B can also be expanded in the form $B = \sum_i b_i P_i$. Now construct a Hermitian operator $O := \sum_i o_i P_i$ with real values o_i, which are all different. Here O is assumed to be nondegenerate by construction. Let us define respectively functions j and k by $j(o_i) := a_i$ and $k(o_i) := b_i$. Then we can see that if A and B commute, there exists a nondegenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $A \cdot B = h(O)$ where $h := j \cdot k$. Thus we have

$$\begin{aligned} &f(A \cdot B) = f(h(O)) = h(f(O)) = j(f(O)) \cdot k(f(O)) \\ &f(j(O)) \cdot f(k(O)) = f(A) \cdot f(B), \end{aligned} \tag{5}$$

where we use the functional rule. We can introduce also a function l such that $A + B = l(O)$ where $l := j + k$. Thus we have

$$\begin{aligned} &f(A + B) = f(l(O)) = l(f(O)) = j(f(O)) + k(f(O)) \\ &f(j(O)) + f(k(O)) = f(A) + f(B), \end{aligned} \tag{6}$$

where we use the functional rule. In fact, the sum rule is equivalent to the product rule for commuting observables. Additionally, we may introduce a function m, if the following conditions are satisfied, such that $m(l(O)) := h(O)$, $m(j(O)) := j(O)$, and $m(k(O)) := k(O)$. For example, we may have

$$A = B = |\uparrow\rangle\langle\uparrow| \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{7}$$

and let us take the function m as $m(2) = 1$, $m(0) = 0$, and $m(1) = 1$. Therefore we can define the function m that satisfies the requirement conditions. Then we have

$$\begin{aligned} &f(A) \cdot f(B) = f(A \cdot B) = f(h(O)) = f(m(l(O))) \\ &= f(l(m(O))) = l(f(m(O))) \\ &= j(f(m(O))) + k(f(m(O))) \\ &= f(m(j(O))) + f(m(k(O))) \\ &= f(A) + f(B) = f(A + B). \end{aligned} \tag{8}$$

Thus we may introduce a supposition that the operation Addition is equivalent to the operation Multiplication when $A = B = |\uparrow\rangle\langle\uparrow|$ and the quantum state is $|\uparrow\rangle$.

And then, we can create a novel algebra that might be called commuting observable algebra. The space for commuting observable algebra is different from our common space in which Newton's mechanics is created. In the space holding commuting observable algebra, the operation Addition and the operation Multiplication are the same as each other. In the normal space where Newton's mechanics is held, the operation Addition and the operation Multiplication are different from each other.

The space holding commuting observable algebra in considering only commutativeness is opposite to quantum theory. The reason of the hidden-variable inconsistency is because there is not the uncertainty principle because of not being noncommutative. Besides, holding commuting observable algebra allows any unnatural value to be "1". Therefore, we have the hidden-variable contradiction. The point of the hidden-variable contradiction is based on our consideration that we have noncommutative property in quantum operations. This fact is based on the uncertainty principle. Here we must notice theoretically that the space holding commuting observable algebra thinking of only commutativeness is different from the normal space holding Newton's mechanics.

The scenario of this paper tells us that Newton's mechanics is not held when thinking of only commutativeness. The other case is in the space holding commuting observable algebra. On the other hand, in case of holding both of commutativeness and non-commutativeness, the space not permitting commuting observable algebra is allowed, and quantum theory is permitted, and then the uncertainty principle exists.

III. NO-HIDDEN-VARIABLE THEOREM FOR QUANTUM MEASUREMENT OUTCOMES ON A SINGLE PROJECTION OPERATOR

Let $P \equiv |\uparrow\rangle\langle\uparrow|$ be a single projection operator. Let $|\uparrow\rangle$ be an eigenstate of P such that $P|\uparrow\rangle = +1|\uparrow\rangle$. The predetermined hidden results of quantum measurement outcomes are only $+1$ in the ideal case.

We might be in the inconsistency within hidden-variable theories when the first predetermined hidden result is $+1$ by measuring the projection operator P in the quantum state $|\uparrow\rangle$, the second predetermined hidden result is the same $+1$ as by measuring the same projection operator P in the same quantum state $|\uparrow\rangle$, and then we consider the existence of only the following proposition $[P, P] = 0$ and we assign the value "1" for the commutator. This means the uncertainty principle does not exist because we think only commutativeness.

In general, the physical situation is either $[P, P] \neq 0$ or $[P, P] = 0$. This means the uncertainty principle exists because we think both of commutativeness and non-commutativeness.

We consider a value V which is the sum of the two predetermined hidden results in a thought experiment. The predetermined hidden results of outcomes are only $+1$. If the number of outcomes is two, then we have

$$V = (+1) + (+1) = +2. \tag{9}$$

We derive a general and natural necessary condition of the value V. In the general and natural case, the uncertainty principle exists because we think both of commutativeness and non-commutativeness, and we have

$$V = +2. \tag{10}$$

This is the general and natural necessary condition when we consider both the propositions $[P, P] \neq 0$ and $[P, P] = 0$ and we assign simultaneously the different two values ("0" and "1") for the propositions $[P, P] \neq 0$ and $[P, P] = 0$. We assign the value "0" for the proposition $[P, P] \neq 0$.

On the other hand, we can depict the predetermined hidden results using v_1, v_2 this: $v_1 = +1$ and $v_2 = +1$. Let us write V this:

$$V = v_1 + v_2. \tag{11}$$

In the following, we evaluate a value V and derive a specific and unnatural necessary condition under the supposition that the measured projection operators are commuting (that is, $[P, P] = 0$ and we assign the value "1" for the commutator) and we do not consider the existence of the following proposition $[P, P] \neq 0$. In the specific and unnatural case, the uncertainty principle does not exist because we think only commutativeness.

We may introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have using commuting observable algebra,

$$V = (v_1 + v_2) = (v_1 \times v_2) = +1. \tag{12}$$

This is possible for the specific and unnatural case that we consider the existence of only the following proposition $[P, P] = 0$ and we assign the value "1" for the commutator. In the specific and unnatural case, the uncertainty principle does not exist because we think only commutativeness.

We cannot assign simultaneously the same two values ("1" and "1") or ("0" and "0") for the two propositions (10) and (12) when we consider only the following proposition $[P, P] = 0$ and we assign the value "1" for the commutator. We derive the inconsistency within hidden-variable theories when we consider the existence of only the following proposition $[P, P] = 0$ and we assign the value "1" for the commutator.

In summary, we have been in the inconsistency within hidden-variable theories when the first predetermined hidden result is $+1$ by measuring the projection operator P in the quantum state $|\uparrow\rangle$, the second predetermined hidden result is the same $+1$ as by measuring the same projection operator P in the same quantum state $|\uparrow\rangle$, and then we consider the existence of only the following proposition $[P, P] = 0$ and we assign the value "1" for the commutator. This has meant the uncertainty principle does not exist because we think only commutativeness and then we derive the specific and unnatural value to be "1".

IV. EXPERIMENTALLY FEASIBLE NO-HIDDEN-VARIABLE THEOREM BASED ON A SINGLE PROJECTION OPERATOR

In a real experiment, there are no perfect detectors, but the good ones with some errors. There is an unforeseen effect that an imperfect detector does not count even though the particle indeed passes through the detector (the quantum efficiency). There is also an unforeseen effect that an imperfect detector counts even though the particle does not pass through the detector (the dark count). In this case, we increase measurement outcomes to even number $2N(\gg 1)$ and then we change such errors into trivial things. If we use the quantum predictions by even number $2N$ trials, then the degree of the hidden-variable inconsistency increases by an amount that grows linearly with N. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

When we consider a quantum optical experiment, we have the following relation with the photon polarization state:

$$|\uparrow\rangle \leftrightarrow |H\rangle, \tag{13}$$

where $|H\rangle$ is a quantum state interpreted by a horizontally polarized photon.

Let us introduce the random noise admixture $\rho_{\text{noise}}(= \frac{1}{2}I)$ into the quantum state $|\uparrow\rangle$, where I is the two-dimensional identity operator. We consider the noisy quantum state emerged from an imperfect source this:

$$\rho = (1-\epsilon)|\uparrow\rangle\langle\uparrow| + \epsilon \times \rho_{\text{noise}}. \tag{14}$$

The value of $\epsilon(< 1)$ is interpreted as the reduction factor of the contrast observed in the single-particle experiment. Then we have $\mathrm{tr}[\rho P] = +1-\epsilon$.

We suppose the number of outcomes of obtaining the predetermined hidden results $+1-\epsilon$ is even number $2N$. We consider the following value $V_i, (i = 1, 2, ..., N)$ which is the sum of the two predetermined hidden results in a thought experiment:

$$V_i = (+1-\epsilon) + (+1-\epsilon). \tag{15}$$

We introduce the function $S(N)$ which is the sum over i of V_i in the thought experiment:

$$S(N) = V_1 + V_2 + ... + V_N. \tag{16}$$

If the number of outcomes is even number $2N$, then we have

$$S(N) = V_1 + V_2 + ... + V_N = 2N(+1-\epsilon), \tag{17}$$

where we use $V_i = (+1-\epsilon) + (+1-\epsilon)$. We derive a general and natural necessary condition of the function value $S(N)$. In the general and natural case, the uncertainty principle exists because we think both of commutativeness and non-commutativeness, and we have

$$S(N) = 2N(+1-\epsilon). \tag{18}$$

This is the general and natural necessary condition when we consider both of the propositions $[P, P] \neq 0$ and $[P, P] = 0$ and we assign simultaneously the different two values ("0" and "1") for the two propositions $[P, P] \neq 0$ and $[P, P] = 0$. We assign the value "0" for the proposition $[P, P] \neq 0$.

On the other hand, we can depict the predetermined hidden results using $r_1, r_2, r_3, ..., r_{2N}$ this: $r_1 = +1-\epsilon$, $r_2 = +1-\epsilon$, $r_3 = +1-\epsilon$,..., $r_{2N} = +1-\epsilon$. We can write V_i this:

$$V_i = r_{2i-1} + r_{2i}, \tag{19}$$

where

$$r_{2i-1} = +1-\epsilon, \ r_{2i} = +1-\epsilon. \tag{20}$$

Thus, we have

$$V_i = (+1-\epsilon) + (+1-\epsilon). \tag{21}$$

In the following, using commuting observable algebra, we evaluate another value of V_i and derive a specific and unnatural necessary condition for the function value $S(N)$ under the supposition that the two measured observables are commuting (that is, $[P, P] = 0$ and we assign the value "1" for the commutator) and we do not consider the existence of the following proposition $[P, P] \neq 0$. In the specific and unnatural case, the uncertainty principle does not exist because we think only commutativeness.

We may introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have, after commuting observable algebra,

$$\begin{aligned} V_i &= (+1-\epsilon) + (+1-\epsilon) \\ &= (+1-\epsilon) \times (+1-\epsilon) = (+1-\epsilon)^2. \end{aligned} \tag{22}$$

Thus, we have

$$S(N) = V_1 + V_2 + ... + V_N = N(+1-\epsilon)^2. \tag{23}$$

This is possible for the specific and unnatural case that we consider the existence of only the following proposition $[P, P] = 0$ and we assign the value "1" for the commutator. In the specific and unnatural case, the uncertainty principle does not exist because we think only commutativeness.

We cannot assign simultaneously the same two values ("1" and "1") or ("0" and "0") for the two propositions (18) and (23) when we consider the existence of only the following proposition $[P, P] = 0$ and we assign the value "1" for the commutator. We derive the inconsistency within hidden-variable theories when we consider the existence of only the following proposition $[P, P] = 0$ and we assign the value "1" for the commutator.

If we use the quantum predictions by even number $2N$ trials, then the degree of the hidden-variable inconsistency increases by an amount that grows linearly with N. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

We note that our argumentations here agree with the discussions in Section III when $N = 1$ and $\epsilon = 0$. From the relation (18), we have the following general and natural value:

$$S(1) = V_1 = +2. \tag{24}$$

From the relation (23), using commuting observable algebra, we have the following specific and unnatural value:

$$S(1) = V_1 = +1. \tag{25}$$

Thus, our discussions here are a natural expansion of Section III.

In summary, we have been in the inconsistency within hidden-variable theories when measurement outcomes are even number $2N(\gg 1)$, the odd number predetermined hidden results are $+1-\epsilon$ by measuring the projection operator P in the quantum state ρ, the even number predetermined hidden results are the same $+1-\epsilon$ as by measuring the same projection operator P in the same quantum state ρ, and then we consider the existence of only the following proposition $[P, P] = 0$ and we assign the value "1" for the commutator. This has meant the uncertainty principle does not exist because we think only commutativeness and then we derive the specific and unnatural value to be "1".

V. CONCLUSIONS

In conclusions, we have proposed no-hidden-variable theorem for the quantum measurement outcomes by measuring an observable that is obtained through a single projection operator. We have been in the inconsistency within hidden-variable theories when the first predetermined hidden result is $+1$ by measuring a single projection operator $|\uparrow\rangle\langle\uparrow|$ in the quantum state $|\uparrow\rangle$, the second predetermined hidden result is the same $+1$ as by measuring the same projection operator $|\uparrow\rangle\langle\uparrow|$ in the same quantum state $|\uparrow\rangle$, and then we consider the existence of only the following proposition $[|\uparrow\rangle\langle\uparrow|, |\uparrow\rangle\langle\uparrow|] = 0$ and we assign the value "1" for the commutator. It has turned out that we cannot assign the predetermined hidden result for quantum measurement outcome as $+1$ when measuring a single projection operator.

Based on the argumentations, we have proposed an experimental accessible hidden-variable inconsistency in terms of the imperfect source and detector. In more detail, we have encountered an imperfect quantum state, the dark count, and the quantum efficiency, which cannot be avoidable from a real experimental situation. However, such an error of the number of particles has become less and less important as we increase trials more and more by using the strong law of large numbers.

ACKNOWLEDGMENTS

We thank Soliman Abdalla, Jaewook Ahn, Josep Batle, Do Ngoc Diep, Mark Behzad Doost, Ahmed Farouk, Han Geurdes, Preston Guynn, Shahrokh Heidari, Wenliang Jin, Hamed Daei Kasmaei, Janusz Milek, Mosayeb Naseri, Santanu Kumar Patro, Germano Resconi, and Renata Wong for their valuable support.

DECLARATIONS

Ethical approval

The authors are in an applicable thought to ethical approval.

Competing interests

The authors state that there is no conflict of interest.

Author contributions

Koji Nagata and Tadao Nakamura wrote and read the manuscript.

Funding

Not applicable.

Data availability

No data associated in the manuscript.

REFERENCES

[1] R. P. Feynman, R. B. Leighton, and M. Sands, "Lectures on Physics, Volume III, Quantum Mechanics," Addison-Wesley Publishing Company (1965).

[2] J. J. Sakurai, "Modern Quantum Mechanics," Addison-Wesley Publishing Company, Revised ed. (1995).

[3] A. Peres, "Quantum Theory: Concepts and Methods," Kluwer Academic, Dordrecht, The Netherlands (1993).

[4] M. Redhead, "Incompleteness, Nonlocality, and Realism," Clarendon Press, Oxford, 2nd ed. (1989).

[5] J. von Neumann, "Mathematical Foundations of Quantum Mechanics," Princeton University Press, Princeton, New Jersey (1955).

[6] M. A. Nielsen and I. L. Chuang, "Quantum Computation and Quantum Information," Cambridge University Press (2000).

[7] A. S. Holevo, "Quantum Systems, Channels, Information, A Mathematical Introduction," De Gruyter (2012). https://doi.org/10.1515/9783110273403

[8] K. Nagata, D. N. Diep, A. Farouk, and T. Nakamura, "Simplified Quantum Computing with Applications," IOP Publishing, Bristol, UK (2022). https://iopscience.iop.org/book/mono/978-0-7503-4700-6

[9] G. W. Mackey, "The Mathematical Foundations of Quantum Mechanics," Dover Books on Physics, Illustrated Edition (1963).

[10] S. Gudder, "On the Quantum Logic Approach to Quantum Mechanics," Commun. math. Phys. 12, 1–15 (1969). https://doi.org/10.1007/BF01646431

[11] W. Guz, "Conditional Probability and the Axiomatic Structure of Quantum Mechanics," Fortschritte der Physik, Volume 29, Issue 8, Pages 345-379 (1981). https://doi.org/10.1002/prop.19810290802

[12] K. Nagata, "There is No Axiomatic System for the Quantum Theory," Int. J. Theor. Phys. **48**, 3532 (2009). https://doi.org/10.1007/s10773-009-0158-z

[13] K. Nagata and T. Nakamura, "Can von Neumann's Theory Meet the Deutsch-Jozsa Algorithm?," Int. J. Theor. Phys. **49**, 162 (2010). https://doi.org/10.1007/s10773-009-0189-5

[14] A. Einstein, B. Podolsky, and N. Rosen, "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?," Phys. Rev. **47**, 777 (1935). https://doi.org/10.1103/PhysRev.47.777

[15] J. S. Bell, “On the Einstein Podolsky Rosen Paradox,” Physics (Long Island City, N.Y.) **1**, 195 (1964). https://doi.org/10.1103/PhysicsPhysiqueFizika.1.195
[16] S. Kochen and E. P. Specker, “The Problem of Hidden Variables in Quantum Mechanics,” J. Math. Mech. **17**, 59 (1967). https://www.jstor.org/stable/24902153
[17] J. Conway and S. Kochen, “The Free Will Theorem,” Foundations of Physics, Volume 36, Issue 10, 1441 (2006). https://doi.org/10.1007/s10701-006-9068-6
[18] K. Nagata, D. N. Diep, and T. Nakamura, “Two Symmetric Measurements May Cause an Unforeseen Effect,” Quantum Information Processing, Volume 22, Issue 2, Article number: 94 (2023). https://doi.org/10.1007/s11128-023-03841-5.