

HARMONIC DISTRIBUTION ON RUNS IN A CHAIN

THEOPHILUS AGAMA

ABSTRACT. We prove that a certain class of infinite sequences whose finite truncation is an addition chain must have a zero logarithmic density. This result is generic and can be applied to particular known infinite sequences with this property.

1. PRELIMINARIES AND SETUP

Let $l(n)$ be the length of an addition chain leading to n , denoted $E(n)$, of the form

$$E(n) : s_0 = 1, s_1 = 2, \dots, s_{l(n)} = n$$

with $2^m \leq n < 2^{m+1}$ such that $l(n) := \beta(m)$. By adapting the ideas of the paper [1], we partition the steps in an addition chain into the following classes of steps

$$\mathcal{A} := \{i : s_i = 2s_{i-1}\} \quad (\text{doubling steps})$$

$$\mathcal{B} := \{i : \gamma s_{i-1} \leq s_i < 2s_{i-1}\} \quad (\text{large steps})$$

where $\gamma := \frac{1+\sqrt{5}}{2}$ is the *golden ratio*

$$\mathcal{C} := \{i : (1 + \delta)s_{i-1} \leq s_i < \gamma s_{i-1}\} \quad (\text{medium - sized steps})$$

where $\delta := \delta(m) \rightarrow 0$ as $m \rightarrow \infty$. In particular

$$\delta := \delta(m) = \frac{1}{\log m}$$

$$\mathcal{D} := \{i : s_i < (1 + \delta)s_{i-1}\} \quad (\text{small steps}).$$

We denote the cardinality of the sets to be

$$\#\mathcal{A} = A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

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We call steps in $\mathcal{B}, \mathcal{C}, \mathcal{D}$ as *non-doubling steps*. We have therefore the relation

$$A + B + C + D = \beta(m).$$

Because each non-doubling step in an addition chain cannot grow faster than a corresponding step in a Fibonacci sequence, we have (by induction) the inequality

$$2^m \leq n \leq 2^A \gamma^{B+C+D} = 2^{\beta(m)} \left(\frac{\gamma}{2}\right)^{B+C+D}$$

and we deduce from this relation an upper control for the total number of non-doubling steps in an addition chain of length $\beta(m)$ to be

Lemma 1.1. *Put*

$$E(n) : s_0 = 1, s_1 = 2, \dots, s_{l(n)} = n$$

be an addition chain with $l(n) := \beta(m)$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be steps in an addition chain of length $\beta(m)$ with cardinality A, B, C, D , respectively. Then we have

$$B + C + D \leq \frac{\beta(m) - m}{1 - \log_2 \gamma}.$$

It turns out that the non-doubling steps in an addition chain have certain structural pattern.

Lemma 1.2. *If $j \in \mathcal{B}$, then $j - 1 \in \mathcal{C} \cup \mathcal{D}$. In particular, each large step in an addition chain must be preceded by either a small step or a medium-sized step.*

Proof. Let $j \in \mathcal{B}$ (large step) then we have by definition

$$\gamma s_{j-1} \leq s_j < 2s_{j-1}$$

where $\gamma := \frac{1+\sqrt{5}}{2}$ is the *golden ratio*. Write $s_j = s_k + s_l$ with $k \geq l$. The inequality $s_j = s_k + s_l \leq s_{j-1} + s_l$ with $s_j < 2s_{j-1}$ implies that

$$\gamma s_{j-1} \leq s_j \leq s_{j-1} + s_{j-2}$$

which further implies

$$(\gamma - 1)s_{j-1} < s_{j-2} \iff s_{j-1} < \gamma s_{j-2}$$

since $\gamma = \frac{1}{\gamma-1}$. This proves $j - 1 \in \mathcal{C} \cup \mathcal{D}$. □

2. THE RUN AND RUN LENGTH OF STEP TYPES IN AN ADDITION CHAIN

We begin this section with the following definition.

Definition 2.1. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be steps in an addition chain as in the setup. We call a **maximal** consecutive sequence of steps of a given type

$$j_1 < j_2 < \cdots < j_d$$

such that $j_1 - 1, j_d + 1$ cannot be a step of the given type a **run** of the step type. We call the number of terms in the *run* the **run length**.

In the sequel, we develop bounds for the distribution function $\sum \frac{1}{s_i}$ on a run of various types in an addition chain leading to n of the form

$$E(n) : s_o = 1, s_1 = 2, \dots, s_{l(n)} = n$$

with $2^m \leq n < 2^{m+1}$ such that $l(n) := \beta(m)$.

Lemma 2.2. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type \mathcal{A} . Then*

$$\frac{1}{2^{\alpha_o}} \left(1 - \frac{1}{2^d} \right) \leq \sum_{k=1}^d \frac{1}{s_{j_k}} \leq 1 - \frac{1}{2^d}$$

for some $\alpha_o := \alpha_o(d) > 0$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

be a **run** of a step of type \mathcal{A} . By induction, we can write

$$2^{d+\beta_o} \geq s_{j_d} = 2s_{j_{d-1}} \cdots = 2^{d-1}s_{j_1} \geq 2^d$$

for some $\beta_o := \beta_o(d) > 0$. Similarly, we can write

$$2^{d-1+\beta_1} \geq s_{j_{d-1}} = 2s_{j_{d-2}} \cdots = 2^{d-2}s_{j_1} \geq 2^{d-1}$$

for some $\beta_1 := \beta_1(d) > 0$. By induction, we write

$$2^{d-r+\beta_r} \geq s_{j_{d-r}} = 2s_{j_{d-r-1}} \cdots = 2^{d-r-1}s_{j_1} \geq 2^{d-r}$$

for any $0 \leq r \leq d-1$ and $\beta_r := \beta_r(d) > 0$. Setting $\max\{\beta_r\}_{r=0}^{d-1}$, we get

$$\sum_{s=1}^d \frac{1}{2^{s+\alpha_o}} \leq \sum_{k=1}^d \frac{1}{s_{j_k}} \leq \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^d} = 1 - \frac{1}{2^d}.$$

□

Lemma 2.3. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

be a **run** of a step of type \mathcal{C} . Then

$$\frac{1}{\gamma^{\alpha_o}(\gamma-1)} \left(1 - \frac{1}{\gamma^d}\right) \leq \sum_{k=1}^d \frac{1}{s_{j_k}} \leq \frac{1}{\delta} \left(1 - \frac{1}{(1+\delta)^d}\right)$$

for some $\alpha_o := \alpha_o(d) > 0$, where $\delta := \delta(m) = \frac{1}{\log m}$ and $\gamma := \frac{1+\sqrt{5}}{2}$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

of a step of type \mathcal{C} . We can write

$$(1+\delta)s_{j_{d-1}} \leq s_{j_d} < \gamma s_{j_{d-1}}$$

so that by induction, we deduce further

$$(1+\delta)^d \leq s_{j_d} < \gamma^{d+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$. Similarly, we obtained (by induction) the inequality

$$(1+\delta)^{d-1} \leq s_{j_{d-1}} < \gamma^{d-1+\beta_1}$$

for $\beta_1 := \beta_1(d) > 0$. We get

$$(1+\delta)^{d-r} \leq s_{j_{d-r}} < \gamma^{d-r+\beta_r}$$

for $0 \leq r \leq d-1$ and $\beta_r := \beta_r(d) > 0$. We obtain for the upper bound

$$\sum_{k=1}^d \frac{1}{s_{j_k}} \leq \sum_{s=1}^d \frac{1}{(1+\delta)^s} = \frac{1}{\delta} \left(1 - \frac{1}{(1+\delta)^d}\right).$$

By setting $\max\{\beta_r\}_{r=0}^{d-1} = \alpha_o$, we get for the lower bound

$$\sum_{k=1}^d \frac{1}{s_{j_k}} > \sum_{s=1}^d \frac{1}{\gamma^{s+\alpha_o}} = \frac{1}{\gamma^{\alpha_o}(\gamma-1)} \left(1 - \frac{1}{\gamma^d}\right).$$

□

Lemma 2.4. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

be a **run** of a step of type $\mathcal{B} \cup \mathcal{C}$. Then

$$\frac{1}{2^{\alpha_o}} \left(1 - \frac{1}{2^d}\right) \leq \sum_{k=1}^d \frac{1}{s_{j_k}} \leq \frac{1}{\delta} \left(1 - \frac{1}{(1+\delta)^d}\right)$$

for some $\alpha_o := \alpha_o(d) > 0$ and $\gamma := \frac{1+\sqrt{5}}{2}$ and $\delta := \delta(m) = \frac{1}{\log m}$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

of a step of type $\mathcal{B} \cup \mathcal{C}$. Then we can write

$$(1+\delta)s_{j_{d-1}} \leq s_{j_d} < 2s_{j_{d-1}}$$

so that by induction, we get

$$(1+\delta)^d \leq s_{j_d} < 2^{d+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$. Similarly, by induction, we can write

$$(1+\delta)^{d-1} \leq s_{j_{d-1}} < 2^{d-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. Thus

$$(1+\delta)^{d-r} \leq s_{j_{d-r}} < 2^{d-r+\beta_r}$$

for $0 \leq r \leq d-1$ and $\beta_r := \beta_r(d) > 0$. Now, we get for the reciprocal

$$\sum_{k=1}^d \frac{1}{s_{j_k}} \leq \sum_{s=1}^d \frac{1}{(1+\delta)^s} = \frac{1}{\delta} \left(1 - \frac{1}{(1+\delta)^d}\right).$$

Similarly for the lower bound of the reciprocal sum, we set $\max\{\beta_r\}_{r=0}^{d-1} = \alpha_o$ and get

$$\sum_{k=1}^d \frac{1}{s_{j_k}} > \sum_{s=1}^d \frac{1}{2^{s+\alpha_o}} = \frac{1}{2^{\alpha_o}} \left(1 - \frac{1}{2^d}\right)$$

□

Lemma 2.5. *Let*

$$E(n) : s_0 = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type $\mathcal{D} \cup \mathcal{C}$. Then*

$$\frac{1}{\gamma^\alpha(\gamma-1)} \left(1 - \frac{1}{\gamma^d}\right) \leq \sum_{k=1}^d \frac{1}{s_{j_k}} \leq \frac{1}{\delta} \left(1 - \frac{1}{(1+\delta)^d}\right)$$

for some $\alpha_o := \alpha_o(d) > 0$ and $\gamma := \frac{1+\sqrt{5}}{2}$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

of a step of type $\mathcal{D} \cup \mathcal{C}$. Then we can write

$$(1+\delta)s_{j_d-1} \leq s_{j_d} < \gamma s_{j_d-1}$$

if $j_d \in \mathcal{C}$. On the other hand, we get

$$s_{j_d} < (1+\delta)s_{j_d-1}$$

if $j_d \in \mathcal{D}$. By induction, we can write

$$(1+\delta)^{d'} \leq s_{j_d} < \gamma^{d+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$ and where d' is the number of steps of type \mathcal{C} in the run. Similarly, we obtain (by induction) the inequality

$$(1+\delta)^{d'-1} \leq s_{j_d-1} < \gamma^{d-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. Thus

$$(1+\delta)^{d'-r} \leq s_{j_d-r} < \gamma^{d-r+\beta_r}$$

for $0 \leq r \leq d' - 1$ and $\beta_r := \beta_r(d) > 0$. Now, we get for the reciprocal sum

$$\sum_{k=1}^d \frac{1}{s_{j_k}} \leq \sum_{s=1}^{d'} \frac{1}{(1+\delta)^s} = \frac{1}{\delta} \left(1 - \frac{1}{(1+\delta)^{d'}}\right) \leq \frac{1}{\delta} \left(1 - \frac{1}{(1+\delta)^d}\right).$$

We set $\max\{\beta_r\}_{r=0}^{d-1} = \alpha_o$ and deduce the lower bound

$$\sum_{k=1}^d \frac{1}{s_{j_k}} > \sum_{s=1}^d \frac{1}{\gamma^{s+\alpha_o}} = \frac{1}{\gamma^\alpha(\gamma-1)} \left(1 - \frac{1}{\gamma^d}\right).$$

□

Lemma 2.6. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type \mathcal{D} . Then*

$$\frac{1}{(1+\delta)^{\alpha_o} \delta} \left(1 - \frac{1}{(1+\delta)^d} \right) \leq \sum_{k=1}^d \frac{1}{s_{j_k}} \leq d$$

for some $\alpha_o := \alpha_o(d) > 0$ and where $\delta := \delta(m) = \frac{1}{\log m}$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

of a step of type \mathcal{D} . Then we write

$$s_{j_d} < (1+\delta)s_{j_{d-1}}$$

so that (by induction), we can write

$$s_{j_d} < (1+\delta)^{d+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$. Similarly, we obtain (by induction) the inequality

$$s_{j_{d-1}} < (1+\delta)^{d-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. Thus

$$s_{j_{d-r}} < (1+\delta)^{d-r+\beta_r}$$

for $0 \leq r \leq d-1$ and $\beta_r := \beta_r(d) > 0$. We deduce for the reciprocal sum

$$\sum_{s=1}^d \frac{1}{(1+\delta)^{s+\alpha_o}} < \sum_{k=1}^d \frac{1}{s_{j_k}} < d$$

where $\alpha_o = \max\{\beta_r\}_{r=0}^{d-1}$. Thus, we deduce

$$\frac{1}{(1+\delta)^{\alpha_o} \delta} \left(1 - \frac{1}{(1+\delta)^d} \right) < \sum_{k=1}^d \frac{1}{s_{j_k}} < d.$$

□

Theorem 2.7. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type \mathcal{D} . Then*

$$d \gg \log \log n.$$

Proof. This follows from Lemma 2.6 and using $\delta := \delta(m) = \frac{1}{\log m}$ and the fact that $m \gg \log n$. \square

Lemma 2.8. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type $\mathcal{A} \cup \mathcal{B}$. Then*

$$\frac{1}{2^{\alpha_o}} \left(1 - \frac{1}{2^d}\right) \leq \sum_{k=1}^d \frac{1}{s_{j_k}} \leq \frac{1}{(\gamma-1)} \left(1 - \frac{1}{\gamma^d}\right)$$

for some $\alpha_o := \alpha_o(d) > 0$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

of a step of type $\mathcal{A} \cup \mathcal{B}$. If $j_d \in \mathcal{A}$, then we can write

$$s_{j_d} = 2s_{j_d-1}$$

so that (by induction) we obtain the inequality

$$2^{d'} \leq s_{j_d} \leq 2^{d'+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$, where d' is the number of steps of type \mathcal{A} in the run. Again, if $j_d - 1 \in \mathcal{A}$, then we get (by induction) the inequality

$$2^{d'-1} \leq s_{j_d-1} \leq 2^{d'-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. On the other hand, if $j_d \in \mathcal{B}$, then we write

$$\gamma s_{j_d-1} \leq s_{j_d} < 2s_{j_d-1}$$

so that by induction, we get

$$\gamma^{d''} \leq s_{j_d} < 2^{d''+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$, and where d'' is the number of steps of type \mathcal{B} in the run. Similarly, if $j_d - 1 \in \mathcal{B}$, then we write (by induction)

$$\gamma^{d''-1} \leq s_{j_d-1} < 2^{d''-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. Combining both cases, we can write

$$\gamma^{d-r} \leq s_{j_d-r} < 2^{d-r+\beta_r}$$

for $0 \leq r \leq d-1$ and $\beta_r := \beta_r(d) > 0$. We deduce

$$\sum_{k=1}^d \frac{1}{s_{j_k}} \leq \sum_{s=1}^d \frac{1}{\gamma^s} = \frac{1}{(\gamma-1)} \left(1 - \frac{1}{\gamma^d}\right).$$

For the lower bound, we set $\max\{\beta_r\}_{r=0}^{d-1} = \alpha_o$ and obtain

$$\sum_{k=1}^d \frac{1}{s_{j_k}} \geq \sum_{s=1}^d \frac{1}{2^{s+\alpha_o}} = \frac{1}{2^{\alpha_o}} \left(1 - \frac{1}{2^d}\right).$$

□

Lemma 2.9. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type $\mathcal{A} \cup \mathcal{C}$. Then*

$$\frac{1}{2^{\alpha_o}} \left(1 - \frac{1}{2^d}\right) \leq \sum_{k=1}^d \frac{1}{s_{j_k}} \leq \frac{1}{\delta} \left(1 - \frac{1}{(1+\delta)^d}\right)$$

for some $\alpha_o := \alpha_o(d) > 0$, where $\delta := \delta(m) = \frac{1}{\log m}$ and $\gamma := \frac{1+\sqrt{5}}{2}$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

of a step of type $\mathcal{A} \cup \mathcal{C}$. If $j_d \in \mathcal{A}$, then we can write

$$s_{j_d} = 2s_{j_d-1}$$

so that (by induction) we obtain the inequality

$$2^{d'} \leq s_{j_d} \leq 2^{d'+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$, where d' is the number of steps of type \mathcal{A} in the run. Again, if $j_d - 1 \in \mathcal{A}$, then we get (by induction) the inequality

$$2^{d'-1} \leq s_{j_d-1} \leq 2^{d'-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. On the other hand, if $j_d \in \mathcal{C}$ then

$$(1 + \delta)s_{j_d-1} \leq s_{j_d} < \gamma s_{j_d-1}$$

so that (by induction)

$$(1 + \delta)^{d''} \leq s_{j_d} < \gamma^{d''+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$ and where d'' is the number of steps of type \mathcal{C} in the run. Similarly, if $j_d - 1 \in \mathcal{C}$ then

$$(1 + \delta)^{d''-1} \leq s_{j_d-1} < \gamma^{d''-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. Combining both cases, we can write

$$(1 + \delta)^{d-r} \leq s_{j_d-r} < 2^{d-r+\beta_r}$$

for $0 \leq r \leq d-1$ and $\beta_r := \beta_r(d) > 0$. We deduce for the upper bound

$$\sum_{k=1}^d \frac{1}{s_{j_k}} < \sum_{s=1}^d \frac{1}{(1 + \delta)^s} = \frac{1}{\delta} \left(1 - \frac{1}{(1 + \delta)^d} \right).$$

By setting $\max\{\beta_r\}_{r=0}^{d-1} = \alpha_o$, we deduce for the lower bound

$$\sum_{k=1}^d \frac{1}{s_{j_k}} > \sum_{s=1}^d \frac{1}{2^{s+\alpha_o}} = \frac{1}{2^{\alpha_o}} \left(1 - \frac{1}{2^d} \right).$$

□

We are now ready to state the main results of the current studies. It is a general observation that applies to a certain class of infinite sequences of positive integers $L := \{T_m\}_{m=0}^\infty$ with $T_o = 1$, $T_1 = 2$ and that each finite truncation of the sequence

$$\{T_m\}_{m=0}^k$$

is an addition chain leading to T_k with the property that for each $k \geq 1$ the finite truncation $\{T_n\}_{n=0}^k$ as an addition chain leading to T_k has length of runs of steps of type \mathcal{D} satisfying the upper bound

$$\ll (\log T_k)^{1-\epsilon}$$

for a small $\epsilon > 0$. We define the *logarithmic* density of $L := \{T_m\}_{m=0}^\infty$ as the limit

$$\mathcal{D}_{\log}(L) := \lim_{n \rightarrow \infty} \frac{1}{\log x} \sum_{\substack{n \leq x \\ n \in L}} \frac{1}{n}$$

if the limit exists.

Theorem 2.10. *[The run length-density theorem] Let $L := \{T_m\}_{m=0}^\infty$ be an infinite sequence of positive integers with $T_0 := 1$, $T_1 := 2$ and $T_i = T_j + T_k$ for each $i \geq 2$. If for any $r \geq 2$ the length of the runs of steps of type \mathcal{D} in the addition chain $\{T_m\}_{m=0}^r$ satisfies the upper bound*

$$\ll (\log T_r)^{1-\epsilon}$$

for a small $\epsilon > 0$, then

$$\lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{\substack{n \leq x \\ n \in L}} \frac{1}{n} = 0.$$

Proof. Let $L := \{T_m\}_{m=0}^\infty$ be an infinite sequence of positive integers with $T_0 := 1$, $T_1 := 2$ and $T_i = T_j + T_k$ for each $i \geq 2$. For a fixed real number $x > 2$, we develop upper bounds for the reciprocal sum

$$\sum_{\substack{n \leq x \\ n \in L}} \frac{1}{n}$$

where T_r is the largest term in L no more than x . The finite truncation $\{T_m\}_{m=0}^r$ is an addition chain leading to T_r . It suffices to analyze the reciprocal sum

$$\sum_{\substack{n \leq x \\ n \in L}} \frac{1}{n}$$

on runs of all types in the addition chain $\{T_m\}_{m=0}^r$. For runs of steps of type \mathcal{A} , Lemma 2.2 gives

$$\sum_{m \in \mathcal{A}} \frac{1}{T_m} \ll 1.$$

For runs of steps of type \mathcal{C} , Lemma 2.3 gives

$$\sum_{m \in \mathcal{C}} \frac{1}{T_m} \ll \log \log T_r.$$

For runs of steps of type $\mathcal{B} \cup \mathcal{C}$, Lemma 2.4 gives

$$\sum_{m \in \mathcal{B} \cup \mathcal{C}} \frac{1}{T_m} \ll \log \log T_r.$$

Similarly, for runs of steps of type $\mathcal{C} \cup \mathcal{D}$, Lemma 2.5 gives

$$\sum_{m \in \mathcal{C} \cup \mathcal{D}} \frac{1}{T_m} \ll \log \log T_r.$$

For those runs of steps of type $\mathcal{A} \cup \mathcal{B}$, Lemma 2.8 gives

$$\sum_{m \in \mathcal{A} \cup \mathcal{B}} \frac{1}{T_m} \ll 1.$$

For runs of steps of type $\mathcal{A} \cup \mathcal{C}$, Lemma 2.9 gives

$$\sum_{m \in \mathcal{A} \cup \mathcal{C}} \frac{1}{T_m} \ll \log \log T_r.$$

Finally, for those runs of steps of type \mathcal{D} , Lemma 2.6 and the requirement of the statement gives

$$\sum_{m \in \mathcal{D}} \frac{1}{T_m} \ll (\log T_r)^{1-\epsilon}$$

for a small $\epsilon > 0$. Assembling the contributions of the reciprocal sum on runs of steps of various types gives

$$\sum_{\substack{n \leq x \\ n \in L}} \frac{1}{n} \ll (\log x)^{1-\epsilon}.$$

□

REFERENCES

1. J.M. De Koninck, N. Doyon and W. Verreault *On the minimal length of addition chains*, arXiv preprint arXiv:2504.07332, 2025.