Do quantified flavor neutrino masses exist?

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ABSTRACT

Mass-squared differences and neutrino mixing angles can be determined from neutrino oscillations. Three neutrino masses m_i (i = 1, 2, 3) corresponding to three mass eigenstates can all be calculated, when one additional relation is available. The so-called geometric mean mass relation may serve as such.

Alternatively, when the neutrino mass m_1 is known, the other two masses can be calculated from measured mass-squared differences. As an example, a mass m_1 of 1.530 meV/c^2 has previously been obtained by combination of the magnetic moment of a massive Dirac neutrino deduced in the context of the electroweak interaction at the one-loop level and the so-called Wilson-Blackett law. Curiously, about the same result for m_1 is found from the geometric mean mass relation.

Furthermore, the existence of neutrino masses m_α ($\alpha = e, \mu, \tau$) corresponding to the flavor eigenstates will be conjectured. They are defined by the product of the unitary PMNS mixing matrix and the column vector with masses m_i as components. For a zero Dirac CP phase δ, the masses m_α appear to be approximately quantized, where the ratio of the three masses is equal to: $m_e : m_\mu : m_\tau$ = 1.00 : 3.02 : 2.03. The mass set m_α appears to possess a higher total energy than the mass set m_i.

Finally, the toroidal model of neutrinos is applied to all neutrino masses. It appears that neutrinos with masses m_i (i = 2, 3) and m_α ($\alpha = e, \mu, \tau$) can all be described by the same *spindle torus model*. In addition, expressions for the radii and angular momenta of all neutrinos are deduced. The obtained parameters all depend on simple functions of the toroidal factor N.

1. INTRODUCTION

The standard model of particle physics contains three generations or flavors of neutrinos the electron neutrino ν_e, the muon neutrino ν_μ and the tau neutrino ν_τ. The subscript α (= e, μ, τ) denotes the charged lepton that takes part in the weak interaction. In addition, it follows from neutrino oscillations that the freely propagating neutrinos possess a definite mass, usually labelled as m_1, m_2 and m_3.

In the case of normal hierarchy, a simple relation between the three neutrinos masses have been proposed by a number of authors, e.g., by He and Zee [1], and Sazdović [2]. In particular, they investigated the validity of the so-called geometric mean mass relation

$$m_2 = \sqrt{m_1 m_3}, \quad \text{or} \quad \frac{m_2^2}{m_1 m_3} = 1. \tag{1.1}$$

It is assumed that mass m_1 in (1.1) possesses a non-zero value. In addition, two other relations between the masses m_1, m_2 and m_3 follow from neutrino oscillation experiments

$$m_{21}^2 \equiv m_2^2 - m_1^2 \quad \text{and} \quad m_{31}^2 \equiv m_3^2 - m_1^2. \tag{1.2}$$

In total, eqs. (1.1) and (1.2) contain three relations and three unknowns, so that in general all masses m_1, m_2 and m_3 can be calculated. For example, combination of (1.1) and (1.2) gave the following values for mass m_1: 1.58 meV/c^2 [1] and 1.55 meV/c^2 [2], respectively.

Since these authors used slightly different experimental values for the mass-squared differences of (1.2), the obtained values for mass m_1 also differ.

Alternatively, a known value of mass m_1 of neutrino 1 can also serve as the missing third relation. Such a mass m_1 can be obtained by combination of two magnetic moments from different origin. A first expression for the magnetic moment for a Dirac neutrino, containing the neutrino mass m_i (i = 1, 2, 3), has previously been deduced by Lee and Shrock [3] and Fujikawa and Shrock [4]. A second magnetic moment from gravitational origin, the so-called Wilson-Blackett law, has been proposed by Biemond [5–7]. Combination of both magnetic moments then provides a value of mass m_1.

In the context of electroweak interactions at the one-loop level a magnetic moment $\boldsymbol{\mu}_i$ of a left-handed neutrino arises from a minimal extension of the standard model with right-handed neutrinos. In that case the following expression of $\boldsymbol{\mu}_i$ for a Dirac neutrino with a positive mass m_i has been deduced [3, 4]

$$\boldsymbol{\mu}_i = \frac{3|e|G_\mathrm{F} m_i c^4 \hbar}{8\pi^2\sqrt{2}}\boldsymbol{\sigma} = \frac{3G_\mathrm{F} m_i m_e c^4 \mu_B}{4\pi^2\sqrt{2}}\boldsymbol{\sigma} = 3.2026\times10^{-22}\left(\frac{m_i}{\mathrm{meV}}\right)\mu_B\boldsymbol{\sigma}, \quad (1.3)$$

where $G_\mathrm{F} = 1.16638\times10^{-5}$ GeV^{-2} is the Fermi coupling constant, $\boldsymbol{\sigma}$ is the Pauli matrix and $\mu_B = |e|\hbar/2m_e$ is the Bohr magneton. Note that $\boldsymbol{\mu}_i$ is proportional to mass m_i.

At present, no magnetic dipole moment of any neutrino has been measured. Applying a new calibration of the brightness of the tip of the red-giant branch in the global cluster ω Centauri, Capozzi and Raffelt [8] found an improved lower bound on a possible neutrino magnetic moment of $\mu_i < 1.2\times10^{-12}\mu_B$ at 95% C. L. This result, however, is many orders larger than the value $\mu_i = 3.2\times10^{-22}(m_i/\mathrm{meV})\mu_B$ for $m_i = 1$ meV/c^2. So, conformation of the existence of a neutrino magnetic moment by observation seems still far away.

Furthermore, a magnetic moment for a massive neutrino arising from gravitational origin follows from the so-called Wilson-Blackett law. As has been discussed previously [5–7], the magnetic moment $\boldsymbol{\mu}_i$ for an elementary particle like a neutrino with mass m_i and angular momentum $\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$ may be given by

$$\boldsymbol{\mu}_i = \frac{1}{2}\left(\frac{G}{k}\right)^{1/2}\hbar\boldsymbol{\sigma}, \quad (1.4)$$

where G is the gravitational constant and $k = (4\pi\varepsilon_0)^{-1}$ is the constant in Coulomb's law. Both formulas for the magnetic moment can be combined, yielding a value for mass m_i. For normal hierarchy only the lightest neutrino with mass $m_i = m_1$ is compatible with the found values of $\Delta m_{21}^2 \equiv m_2^2 - m_1^2$ and $\Delta m_{31}^2 \equiv m_3^2 - m_1^2$, so that

$$m_1 = \frac{4\pi^2\sqrt{2}}{3|e|G_F c^4}\left(\frac{G}{k}\right)^{1/2}. \quad (1.5)$$

When all constants are inserted into (1.5), a value of 1.530 meV/c^2 = 2.727×10^{-39} kg is obtained for the neutrino with mass m_1.

2. DEFINITION OF FLAVOR MASSES

The concept of neutrino mixing in its simplest form can be expressed as a unitary transformation relating flavor and mass eigenstates. The transformations are generally written as

$$\nu_\alpha = \sum_i U_{\alpha i}^* \nu_i \quad \text{and} \quad \nu_i = \sum_\alpha U_{\alpha i}\nu_\alpha, \quad (2.1)$$

where ν_α represents the flavor eigenstate α ($\alpha = e, \mu, \tau$) and ν_i denotes the mass eigenstate i (i = 1, 2, 3). It is supposed that each neutrino is created by weak interactions in one of the flavor states ν_α. In addition, it is assumed that each eigenstate ν_i corresponds to one of the masses m_i (i = 1, 2, 3). The elements $U^*_{\alpha i}$ are given by the 3×3 unitary Pontecorvo-Maki-Nakagawa-Sakata mixing matrix [9–11] (element $U^*_{\alpha i}$ is the complex conjugate of element $U_{\alpha i}$). This matrix, usually denoted as the mixing matrix U_{PMNS}, connects each flavor eigenstate ν_α to three mass eigenstates ν_i. In matrix representation one obtains

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U^*_{e1} & U^*_{\mu 1} & U^*_{\tau 1} \\ U^*_{e2} & U^*_{\mu 2} & U^*_{\tau 2} \\ U^*_{e3} & U^*_{\mu 3} & U^*_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \tag{2.2}$$

In order to conserve probability, the U_{PMNS} matrix has to be unitary, so that each mass eigenstate ν_i is also connected to three flavor eigenstates ν_α

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}. \tag{2.3}$$

The usual parameterization U_{PDG}, used by the Particle Data Group for the mixing matrix U_{PMNS} for Dirac particles [9–11], is given by

$$U_{\mathrm{PMNS}} = U_{\mathrm{PDG}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\mathrm{CP}}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{\mathrm{CP}}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{\mathrm{CP}}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{\mathrm{CP}}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{\mathrm{CP}}} & c_{13}c_{23} \end{pmatrix}, \tag{2.4}$$

where $c_{12} = \cos\theta_{12}$, $s_{12} = \sin\theta_{12}$, and so on. So, the mixing matrix U_{PDG} is characterized by four parameters: three mixing angles (θ_{12}, θ_{13}, θ_{23}) and the Dirac CP phase δ_{CP}.

According to the standard model of particle physics neutrinos are supposed to be massless, but neutrino oscillation experiments show that at least two of them are massive. From oscillation experiments two mass-squared differences $\Delta m^2_{21} \equiv m_2^2 - m_1^2$ and $\Delta m^2_{31} \equiv m_3^2 - m_1^2$ and three mixing angles θ_{12}, θ_{13} and θ_{23} can be calculated. Although current neutrino oscillation data shed some light on the range of the Dirac phase δ_{CP}, its value is uncertain (see, e.g., discussion in sections 2 and 3 of ref. [11]).

As assumed above, neutrinos i (i = 1, 2, 3) with mass m_i correspond to the mass eigen states ν_i (i = 1, 2, 3). It will be proposed that neutrinos α ($\alpha = e, \mu, \tau$) corresponding to the flavor eigenstates ν_α ($\alpha = e, \mu, \tau$) and possess three masses m_α. More in detail, it will be conjectured that the vector components m_α are defined by the product of the unitary PMNS mixing matrix and the vector components m_i, analogous to the transformation of (2.2)

$$\begin{pmatrix} m_e \\ m_\mu \\ m_\tau \end{pmatrix} = \begin{pmatrix} U^*_{e1} & U^*_{\mu 1} & U^*_{\tau 1} \\ U^*_{e2} & U^*_{\mu 2} & U^*_{\tau 2} \\ U^*_{e3} & U^*_{\mu 3} & U^*_{\tau 3} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}. \tag{2.5}$$

The masses m_α form a linear combination of the masses m_i.

The complete set of masses m_1, m_2 and m_3 are deduced from the three relations of (1.1) and (1.2) and are given in table 1 of section 3. In addition, the masses m_α ($\alpha = e, \mu, \tau$) are calculated from (2.5) and are shown in table 2 of section 3. In section 4 the toroidal model of leptons is applied to all neutrinos. For neutrino 1 with mass m_1 the *ring torus model* appears to be a suitable choice. In addition, quantitative expressions for the radius r_1 of the torus and the radius r_2 of the tube of the torus, as well as for the ratio r_2/r_1, for all neutrinos are calculated. Moreover, the angular momenta coupled to radii r_1 and r_2 are deduced in this section. In section 5 the *spindle torus model* is considered more in detail. The latter model appears to apply to all other neutrinos, i.e., to neutrinos $i = 2, 3$ and $\alpha = e, \mu, \tau$. In section 6 a summary is given of the remarkable magnitudes of m_1, m_2 and m_3 and m_e, m_μ and m_τ. In addition, consequences of the toroidal model are discussed.

3 CALCLATION OF THE NEUTRINO MASSES

The complete set of masses m_1, m_2 and m_3 can be deduced from the relations of (1.1) and (1.2). Their values have been calculated from recent neutrino oscillation data [11] (Nu-Fit 6.0 data, without SK atm data) for $\Delta m^2_{21} \equiv m_2^2 - m_1^2$ and $\Delta m^2_{31} \equiv m_3^2 - m_1^2$ in the case of normal hierarchy. The results are summarized in table 1.

Table 1. Neutrino masses m_1, m_2, m_3 and their sum $\Sigma_i m_i$ are obtained by combining the best fit values ($\pm 1\sigma$) for Δm^2_{21} and Δm^2_{31} from ref. [11] and relation (1.1). All masses are given in units of meV/c^2, or in meV in short. Mass ratios m_2/m_1, m_3/m_2 and m_3/m_1 are also shown.

m_2^2/m_1m_3	Δm^2_{21} [11] (meV^2)	Δm^2_{31} [11] (meV^2)	m_1 (meV)	m_2 (meV)	m_3 (meV)	$\Sigma_i m_i$ (meV)	m_2/m_1	m_3/m_2	m_3/m_1
1	$74.9^{+1.9}_{-1.9}$	2534^{+25}_{-23}	1.53	8.79	50.4	60.7	5.73	5.73	32.8

It appears that within experimental accuracy the value of m_1 in table 1 is equal to the value 1.530 meV/c^2 from (1.5). According to (1.1), the ratios m_2/m_1 and m_3/m_2 must be equal. The result of m_3/m_1 may be important in view of a possible link to the weak coupling constant at low energy (see below). Furthermore, the sum $\Sigma_i m_i = 60.7$ meV approaches to the lower bound $\Sigma_i m_i = 59$ meV, when mass $m_1 = 0$ in the expressions for Δm^2_{21} and Δm^2_{31}. It is noticed that no value for Dirac phase δ_{CP} is used in the calculations of table 1.

In order to calculate the constituent neutrino masses m_e, m_μ, m_τ and their sum $\Sigma_\alpha m_\alpha$ by combination of (1.1) and (2.5), knowledge of a total of six parameters is necessary: the mass differences of Δm^2_{21} and Δm^2_{31}, the mixing angles θ_{12}, θ_{13}, θ_{23} and the value of δ_{CP}. The values for Δm^2_{21} and Δm^2_{31} and θ_{12}, θ_{13}, θ_{23} are again taken from ref. [11] (Nu-Fit 6.0 data, w/o SK atm, in case of normal hierarchy). Since the value of δ_{CP} is uncertain, a value of $\delta_{CP} = 0^\circ$ will first be considered as an example. The results are summarized in table 2.

Table 2. The constituent masses m_e, m_μ, m_τ and their sum $\Sigma_\alpha m_\alpha$ are obtained by substituting the best fit values ($\pm 1\sigma$) for the mixing angles θ_{12}, θ_{13}, θ_{23} from ref. [11] into (2.5). In addition, the masses m_1, m_2 and m_3 from table 1 are inserted into (2.5). Subsequently, the mass sum $\Sigma_\alpha m_\alpha$ and the mass ratios m_e/m_1, m_μ/m_1 and m_τ/m_1 are calculated, together with mass ratios m_μ/m_e, m_τ/m_e and m_μ/m_τ. For comment, see text.

θ_{12} (º)	θ_{13} (º)	θ_{23} (º)	δ_{CP} (º)	m_e (meV)	m_μ (meV)	m_τ (meV)	$\Sigma_\alpha m_\alpha$ (meV)	m_μ/m_e	m_τ/m_e	m_μ/m_τ
33.68	8.52	48.5	0	13.5	40.9	27.6	82.0	3.02	2.03	1.48
				m_e/m_1	m_μ/m_1	m_τ/m_1				
				8.83	26.7	18.0				

These results are remarkable in many respects. For example, the ratio of the three masses m_e, m_μ and m_τ is equal to: $m_e : m_\mu : m_\tau = 1.00 : 3.02 : 2.03$, close to the simple integer sequence 1 : 3 : 2. Moreover, the ratios m_e/m_1, m_μ/m_1 and m_τ/m_1 show that the masses m_e, m_μ and m_τ can approximately be written as: $m_e = 9\ m_1$, $m_\mu = 27\ m_1$ and $m_\tau = 18\ m_1$, consistent with the sequence $m_e : m_\mu : m_\tau = 1 : 3 : 2$. These results suggest a type of quantization of the m_α masses. Furthermore, the mass ratio m_μ/m_τ is equal to the value 1.48, close to the remarkable factor 3/2. In table 2 a sum $\Sigma_i m_i = 82$ meV is found, higher than the upper bound of $\Sigma_i m_i \leq 72$ meV at 95% C. L. deduced from baryon acoustic oscillations (BAO) and cosmic microwave background (CMB) data, given by the 2024 DESI collaboration [12].

An alternative value for $\delta_{\rm CP}$ can be deduced from relation (2.5) for mass m_e

$$m_e = c_{12}c_{13}m_1 + s_{12}c_{13}m_2 + s_{13}\cos\delta_{\rm CP}m_3. \tag{3.1}$$

When a value $m_e = 0$ is assumed, one obtains a value of $\delta_{\rm CP} = 215^{\rm o}$. The latter value deviates from the Nu-Fit data for normal hierarchy in ref. [11]: $\delta_{\rm CP} = 177^{\rm o}$, w/o SK atm and $\delta_{\rm CP} = 212^{\rm o}$, with SK atm. In table 3 the values of m_e, m_μ, m_τ, and so on, are summarized. It is noted that mass m_e becomes negative in the interval $145^{\rm o} < \delta_{\rm CP} < 215^{\rm o}$, whereas a maximal value of $m_e = 13.5$ meV is obtained for $\delta_{\rm CP} = 0^{\rm o}$ (see table 2).

Table 3. The masses m_e, m_μ, m_τ and their sum $\Sigma_\alpha m_\alpha$ are obtained by inserting the best fit values ($\pm 1\sigma$) for the mixing angles θ_{12}, θ_{13}, θ_{23} from ref. [11] into (2.5), together with the masses m_1, m_2 and m_3 from table 1. Subsequently, the mass ratios m_e/m_1, m_μ/m_1, m_τ/m_1 and m_μ/m_τ are calculated.

θ_{12} (º)	θ_{13} (º)	θ_{23} (º)	$\delta_{\rm CP}$ (º)	m_e (meV)	m_μ (meV)	m_τ (meV)	$\Sigma_\alpha m_\alpha$ (meV)	m_μ/m_τ
33.68	8.52	48.5	215	0	42.1	28.7	70.8	1.47
				m_e/m_1	m_μ/m_1	m_τ/m_1		
				0	27.5	18.7		

More definite conclusions about the values of the masses m_e, m_μ and m_τ and the mass ratios m_μ/m_τ can be drawn, when more accurate input parameters become available.

In addition, it is found that the mass sum $\Sigma_i m_i = 60.7$ meV in table 1 is smaller than the flavor mass sums $\Sigma_\alpha m_\alpha$ in tables 2 and 3, both for $\delta_{\rm CP} = 0^{\rm o}$ and $\delta_{\rm CP} = 215^{\rm o}$. These results indicate that the mass set m_i as a whole is more stable than the two flavor mass sets of m_α. In case of $\delta_{\rm CP} = 215^{\rm o}$ the energy difference between the mass sum $\Sigma_\alpha m_\alpha$ and the mass sum $\Sigma_i m_i$ is 70.8 – 60.7 = 10.1 meV. When the flavor mass sum $\Sigma_\alpha m_\alpha$ transforms into the mass sum $\Sigma_i m_i$, the extra energy may be released as kinetic energy of the neutrinos with mass m_i.

Furthermore, it is noted that combination of values of Δm^2_{21} and Δm^2_{31} for normal hierarchy from ref. [11] (Nu-Fit 6.0 data, w/o SK atm) and the value $m_1 = 1.530$ meV from (1.5) leads to almost the same set of masses m_1, m_2 and m_3, as is shown in table 4.

Table 4. Neutrino masses m_2 and m_3 are calculated by combining the best fit values ($\pm 1\sigma$) of Δm^2_{21} and Δm^2_{31} from ref. [11] and the value $m_1 = 1.530$ meV from (1.5). Mass ratios m_2/m_1, m_3/m_2 and m_3/m_1 are also added. For comment, see text.

Δm^2_{21} [11] (meV^2)	Δm^2_{31} [11] (meV^2)	m_1 (meV)	m_2 (meV)	m_3 (meV)	$\Sigma_i m_i$ (meV)	m_2/m_1	m_3/m_2	m_3/m_1	m_2^2/m_1m_3
$74.9^{+1.9}_{-1.9}$	2534^{+25}_{-23}	1.530	8.79	50.4	60.7	5.74	5.73	32.9	1.00

In this case the values of the mass ratios m_2/m_1 and m_3/m_2 are also nearly equal. Moreover, the quantity $m_2^2/(m_1m_3)$ obtains unity value, in agreement with the unity value of relation (1.1). Since in tables 1 and 4 the same values for Δm^2_{21} and Δm^2_{31} are combined with slightly different values for m_1, the sums of masses $\Sigma_i m_i$ also slightly differ.

It is noticed that the value $m_3/m_1 = 32.9$ deviates from the value $m_3/m_1 = 33$ in ref. [6]. The quantity $(m_3/m_1 - 1) = 31.9$ is important, for it may be identified as the reciprocal value of the weak coupling constant α_W at low energy. The value of α_W^{-1} is still uncertain, as has been discussed in [5, 6].

Instead of the values Δm_{21}^2 and Δm_{31}^2 in table 3, the values Δm_{21}^2 = 74.9 meV^2 and $\Delta m_{31}^2 = \Delta m_{32}^2 - \Delta m_{21}^2$ = 2545.9 meV^2 are used in ref. [6]. Compared to the results in table 1, the values m_2, m_3 and the sum $\Sigma_i m_i$ only slightly differ, whereas the quantity $m_2^2/(m_1 m_3)$ is again equal to 1.00, in agreement with (1.1). When more accurate data for masses m_i become available, a more definite conclusion can be drawn about the equivalence of the mean geometric mass relation (1.1) and the value m_1 = 1.530 meV from (1.5).

4. TOROIDAL MODEL APPLIED TO NEUTRINOS

Recently. the toroidal model of leptons as developed by Biemond [13, 14] has been applied to the neutrinos with masses m_1, m_2 and m_3. In this section this approach will be extended to the masses m_e, m_μ and m_τ. It appears that this model provides a clear picture of the geometric structure of all neutrinos. The following basic equations are assumed

$$\begin{aligned} x(t) &= (r_1 + r_2 \cos N\omega t)\cos \omega t, \\ y(t) &= (r_1 + r_2 \cos N\omega t)\sin \omega t, \\ z(t) &= -r_2 \sin N\omega t, \end{aligned} \tag{4.1}$$

where r_1 is the radius of the torus and r_2 is the radius of the tube of the torus. The dimensionless factor N influences the magnitude of the so-called toroidal moment and will be denoted as the toroidal factor. Therefore, in order to characterize some lepton in (4.1) four quantities: r_1, r_2, ω and N must be known.

For radius r_1 the following expression has already been postulated by [13]

$$r_1 = g\frac{\hbar}{mc}, \tag{4.2}$$

where the quantity g is given by

$$g \equiv \sqrt{1 + N^2\frac{r_2^2}{r_1^2} + \tfrac{1}{2}\frac{r_2^2}{r_1^2}}. \tag{4.3}$$

More information about the parameters N, r_1 and r_2 can be obtained from the z-component of the magnetic moment $\mu_z(i)$. From the analysis in refs. [13, 14] follows

$$\mu_z(i) = \left(\frac{G}{k}\right)^{1/2}\frac{\hbar}{2}\left(1 + \tfrac{1}{2}\frac{r_2^2}{r_1^2}\right) = \left(\frac{G}{k}\right)^{1/2}\frac{\hbar}{2}g_i'. \tag{4.4}$$

It is noted that the magnetic moment $\mu_z(i)$ in (4.4) of all neutrinos i from gravitomagnetic origin is analogous to the magnetic moment $\mu_z(l)$, given in eq. (2.11) of ref. [13], of all charged leptons l from electromagnetic origin.

When radius r_2 of neutrino 1, or $r_2(1)$, is zero, the corresponding magnetic moment $\mu_z(1)$ reduces to

$$\mu_z(1) = \left(\frac{G}{k}\right)^{1/2}\frac{\hbar}{2}. \tag{4.5}$$

In this case unity values are obtained for $g(1)$ and g_1' from (4.3) and (4.4), respectively. Moreover, for neutrino 1 the equations of motion (4.1) become simpler, whereas radius $r_1(1)$ from (4.2) simplifies to the following reduced Compton wavelength

$$r_1(1) = \frac{\hbar}{m_1 c} \equiv r_0. \tag{4.6}$$

For mass m_1 = 1.530 meV/c^2 radius r_0 is equal to 1.29×10^{-4} m. The resulting model is denoted as Parson's *ring model* (see, e.g., ref. [14]).

As has been discussed in refs. [13, 14], the correction term r_2^2/r_1^2 in (4.4) for neutrino 1 might be equal to

$$\frac{r_2^2}{r_1^2} = \frac{\alpha_W}{\pi}, \tag{4.7}$$

where α_W is the weak coupling constant at low energy. Choosing the value α_W = 1/32, one finds from combination of (4.4) and (4.7) for neutrino 1

$$\mu_z(1) = \left(\frac{G}{k}\right)^{1/2} \frac{\hbar}{2}\left(1 + \frac{\alpha_W}{2\pi}\right) \quad \text{and} \quad g_1' = 1 + \frac{\alpha_W}{2\pi} = 1.005. \tag{4.8}$$

It is noticed that magnetic moment $\mu_z(1)$ is analogous to the z-component of the magnetic moment $\mu_z(e)$ of the electron (see ref. [13]). Furthermore, combination of (4.2), (4.3) and (4.7) yields for neutrino 1

$$\frac{r_2(1)}{r_1(1)} = \sqrt{\frac{\alpha_W}{\pi}} = 0.10, \;\; g(1) = \sqrt{1 + \tfrac{3}{2}\frac{\alpha_W}{\pi}} = 1.007 \;\; \text{and} \;\; r_1(1) = g(1)\frac{\hbar}{m_1 c} = 1.007\, r_0. \tag{4.9}$$

It is shown in ref. [13] that a value N = 1 applies to the electron, muon and tauon and may apply to neutrino 1. Then, the latter neutrino also belongs to the limiting case $r_1 >> Nr_2$. Owing to the used coordinates of (4.1) in the case of neutrino 1, this model can be characterized as a *ring torus model*. An illustration of this model is given in figure 1.

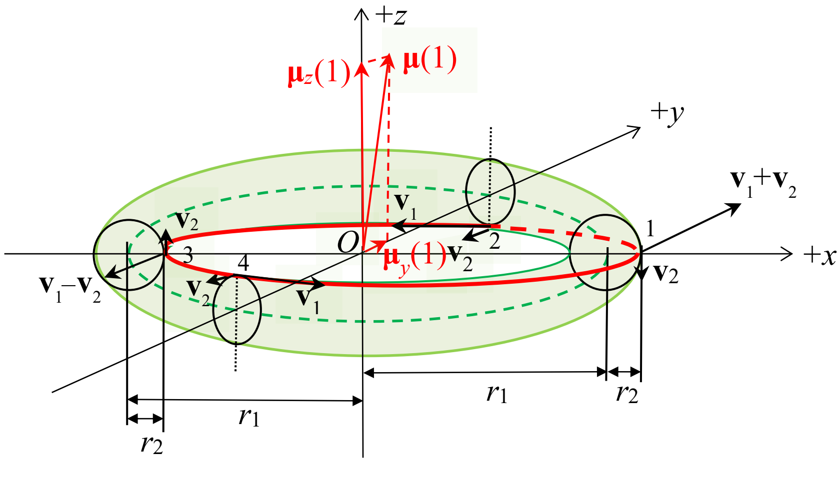

Figure 1. *Ring torus model* of neutrino 1 with mass m_1 following from (4.1) for N = 1 and $r_1 > r_2$. When O is the origin of the coordinate system, the location of mass m_1 is fixed by the Cartesian coordinates $x(t) = x$, $y(t) = y$ and $z(t) = z$. The mass m_1 moves with an average speed v_1 in a ring of radius r_1 and a speed v_2 ($v_1 > v_2$) in a circle of radius r_2. The green blocked line is a circle with radius r_1 in the x-y plane and the orbit of neutrino 1 is drawn in red. For clarity reasons the values of r_1, r_2, v_1 and v_2 are not drawn to scale. The vectors of the y- and z-component of the magnetic dipole moment $\mathbf{\mu}(1)$ of neutrino 1 are also shown (note that the x-component of $\mathbf{\mu}(1)$ is zero). See refs. [13, 14] for further details.

According to formula (1.3) the magnetic moment $\mu_z(i)$ is proportional to mass m_i. Utilizing (1.4), the magnetic moment then may be written as

$$\mu_z(i) = \mu_z(1)\frac{m_i}{m_1} = \left(\frac{G}{k}\right)^{1/2}\frac{\hbar}{2}\frac{m_i}{m_1}. \tag{4.10}$$

Relation (4.10) has been deduced for masses m_i, but it is assumed that this equation also applies to masses m_α ($\alpha = e, \mu, \tau$). For mass m_i combination of (4.10) and (4.4) yields for g_i'

$$\left(1+\tfrac{1}{2}\frac{r_2^2(i)}{r_1^2(i)}\right) = g_i' = \frac{m_i}{m_1}. \tag{4.11}$$

Note that for $i = 1$ the value of g_1' in (4.8) is more precise than the value $g_1' = 1$ following from (4.11). All other masses m_i and m_α are much larger than m_1, so that the ratios r_2/r_1 of all values of g_i' and g_α' are considerably higher than unity value. Furthermore, in the limiting case $Nr_2 >> r_1$, the quantities g_i' (and g_α') may satisfy the following condition [13]

$$g_i' = N\left(1+\tfrac{1}{2}\frac{1}{N^2}\frac{r_1^2}{r_2^2}+\tfrac{1}{4}\frac{1}{N^2}-\tfrac{3}{8}\frac{1}{N^4}\frac{r_1^2}{r_2^2}-\tfrac{1}{8}\frac{1}{N^4}\frac{r_1^4}{r_2^4}-\tfrac{3}{64}\frac{1}{N^4}+\ldots\right). \tag{4.12}$$

As an example, the value of N for mass m_e will be calculated. Substitution of $m_e/m_1 = 8.83$ from table 2 into (4.11) yields a value of 8.83 for g_e' and a value of 3.96 for $r_2(e)/r_1(e)$. Substitution of the *reciprocal* values of $r_2(e)/r_1(e)$ into (4.12), followed by simulation, then leads to a value of $N = 8.80$, a value only slightly smaller than the input value $g_e' = 8.83$. Subsequently, the separate values of the radii $r_1(e)$ and $r_2(e)$ can be found by combining the obtained results for $r_2(e)/r_1(e)$ and N with (4.2) and (4.3).

For high values of N, combination of (4.11) and (4.12) approximately yields for masses m_2, m_3 (and all masses m_α)

$$\frac{r_2^2(i)}{r_1^2(i)} \approx 2\left(\frac{m_i}{m_1}-1\right) \approx 2(N-1) \quad \text{or} \quad \frac{r_2(i)}{r_1(i)} \approx \sqrt{2(N-1)}. \tag{4.13}$$

As an example, for $N = 8.80$ the approximate value $r_2(e)/r_1(e) = 3.95$ follows from (4.13). In addition, combination of (4.2), (4.3) and (4.13) yields

$$r_1(i) \approx N\frac{r_2(i)}{r_1(i)}\frac{\hbar}{m_i c} = N\sqrt{2(N-1)}\frac{\hbar}{m_i c} \quad \text{and} \quad r_2(i) \approx 2N(N-1)\frac{\hbar}{m_i c}. \tag{4.14}$$

Both radius $r_1(i)$ and $r_2(i)$ appear to depend on the factor N, a factor that reminds us of quantum behaviour. Moreover, from (4.14) the following angular momenta coupled to radius $r_1(i)$ and $r_2(i)$ can be calculated, respectively

$$m_i c r_1(i) \approx N\sqrt{2(N-1)}\,\hbar, \quad m_i c r_2(i) \approx 2N(N-1)\,\hbar. \tag{4.15}$$

Both angular momenta also depend on the factor N and are reminiscent of quantization, too. It is assumed that eqs. (4.14) and (4.15) deduced for neutrinos $i = 2, 3$ also apply to neutrinos α ($\alpha = e, \mu, \tau$).

In table 5 all calculated values of the neutrino masses m_1, m_2, m_3 and m_e, m_μ, m_τ from tables 1 and 2 are summarized, in sequence of increasing magnitude. Only the limiting case of $\delta_{CP} = 0^\circ$ is considered. In addition, the ratios of r_2/r_1 are given, together with the radii r_1

and r_2 in units r_0 for all masses m_i (i = 1, 2, 3) ànd m_α ($\alpha = e, \mu, \tau$). Apart from the ratio r_2/r_1 of mass m_1, all other masses belong to the limiting case $Nr_2 >> r_1$. In addition, the obtained values for N are all higher than unity value and are lying between 5.68 and 32.8. It is noticed, however, that the treatment on the toroidal model in refs. [13, 14] starting with (4.1) was mainly restricted to integer values of N. The obtained results, however, may retain their approximate validity. Parallel to the values of N, the values of the ratios r_2/r_1 increase from 3.08 up to 7.98. Likewise, the values of radius r_1 increase from 3.08 r_0 up to 7.98 r_0, whereas the values of r_2 increase from 9.46 r_0 up to 63.7 r_0.

Table 5. Values of the neutrino masses m_1, m_2, m_3 and m_e, m_μ, m_τ are given in units meV and in units m_1, in sequence of increasing magnitude. In addition, the ratios of r_2/r_1 are shown, together with the radii r_1 and r_2 in units r_0 for all masses m_i (i = 1, 2, 3) and m_α ($\alpha = e, \mu, \tau$). See text for further comment.

	m_1 (meV)	m_2 (meV)	m_e (meV)	m_τ (meV)	m_μ (meV)	m_3 (meV)
m_i [a] or m_α [b]	1.53 [a]	8.79 [a]	13.5 [b]	27.6 [b]	40.9 [b]	50.4 [a]
m_i/m_1 [a] or m_α/m_1 [b]	1 [a]	5.73 [a]	8.83 [b]	18.0 [b]	26.7 [b]	32.8 [a]
N [c]	1	5.68 [c]	8.80 [c]	18.0 [c]	26.7 [c]	32.8 [c]
r_2/r_1	0, or $(\alpha_W/\pi)^{1/2} = 0.10$ [d]	3.08 [c]	3.96 [c]	5.83 [c]	7.16 [c]	7.98 [c]
	$r_1(1)$ $r_2(1)$ (r_0)	$r_1(2)$ $r_2(2)$ (r_0)	$r_1(e)$ $r_2(e)$ (r_0)	$r_1(\tau)$ $r_2(\tau)$ (r_0)	$r_1(\mu)$ $r_2(\mu)$ (r_0)	$r_1(3)$ $r_2(3)$ (r_0)
$r_1(i)$ [e] or $r_1(\alpha)$ [e] $r_2(i)$ [e] or $r_2(\alpha)$ [e]	1 $r_2(1) = 0$, or $r_2(1) \approx 0.10$ [d]	3.08 [e] 9.46 [e]	3.96 [e] 15.7 [e]	5.83 [e] 33.9 [e]	7.16 [e] 51.3 [e]	7.98 [e] 63.7 [e]

[a] Table 1. [b] Table 2. [c] Combination of (4.11) and (4.12). [d] See (4.9). [e] Combination of (4.2) and (4.3).

It has previously been discussed [14], that the geometric shape of the neutrinos with mass m_2 and m_3 can be described by the so-called *spindle torus model*. Especially, a more detailed description of the spindle torus model of neutrino 3 with mass m_3 has been given in ref. [14]. This model is extended to the masses m_e, m_μ and m_τ. As an example, the structure of mass m_e is considered in section 5.

5. SPINDLE TORUS MODEL OF NEUTRINOS

As an example of the limiting case with $Nr_2 >> r_1$, the *spindle torus* of neutrino e with mass m_e, factor $N = 8.80$ and ratio $r_2/r_1 = 3.96$ is illustrated in figure 2. The values for N, r_2/r_1, and $r_1 = 3.96\ r_0$ and $r_2 = 15.7\ r_0$ can all be found in table 5. For clarity reasons, only one open orbit of mass m_e is sketched from point P until point T passing through the points Q, R and S. The coordinates of all these points can be calculated from (4.1). For point P the coordinates are: $x = (r_1 + r_2) = +\ 4.96\ r_1$, $y = 0$, $z = 0$, for Q: $x = r_1 \cos(90°/8.80) = +\ 0.984\ r_1$, $y = r_1 \sin(90°/8.80) = +\ 0.178\ r_1$, $z = -\ r_2$, for R: $x = (r_1 - r_2) \cos(180°/8.80) = -\ 2.77\ r_1$, $y = (r_1 - r_2) \sin(180°/8.80) = -\ 1.03\ r_1$, $z = 0$, for S: $x = r_1 \cos(270°/8.80) = 0.860\ r_1$, $y = r_1 \sin(270°/8.80) = 0.510\ r_1$, $z = +\ r_2$ and for T: $x = (r_1 + r_2) \cos(360°/8.80) = +\ 3.75\ r_1$, $y = (r_1 + r_2) \sin(360°/8.80) = +\ 3.25\ r_1$, $z = 0$.

The points P, T and the auxiliary point U (with coordinates $x = -\ r_1 - r_2 = -\ 4.96\ r_1$, $y = 0$, $z = 0$) are all lying on a circle with radius $r_1 + r_2 = 4.96\ r_1$, so that $OP = OT = OU = 4.96\ r_1$, where the origin O is the centre of the circle. In addition, the point R is lying in the same plane, but the distance $OR = |r_1 - r_2| = 2.96\ r_1$ is smaller than radius OP. Note that the angle between OR and OU is 180°/8.80 = 20.5°, whereas the angle between OP and OT is 360°/8.80 = 40.9°.

From (4.1) it follows that the orbit from P to T passes through the z-axis when $r_1 + r_2 \cos N\omega t = 0$, or $\cos 8.80\omega t = -\ r_1/r_2 = -\ 0.253$. In that case the z-coordinate can be calculated from $z = -r_2 \sin 8.80\omega t = \pm 0.968 r_2$. These z-values can be compared with the value $z = +r_2$ of point S and $z = -r_2$ of point Q. The depression $d = r_2 - 0.968 r_2 = 0.032 r_2$ of the orbit at the poles resembles the shape of an apple. Therefore, the surface of the spindle torus is sometimes denoted as the *apple torus*. It is noted that for the value of $r_2/r_1 = 3.96$ for neutrino e the depression $d = 0.032 r_2$ is small, so that the surface of the *spindle torus* approaches to the surface of a *sphere*. It is found that the neutrinos 2, e, μ, τ and 3 can all adequately be described by the *spindle torus model* (compare the increasing values of r_2/r_1 in table 5), or approximately by a *sphere*. Neutrino 1 is an exception: it can be described by the *ring model* of Parson, or more accurately by the *ring torus model* of figure 1.

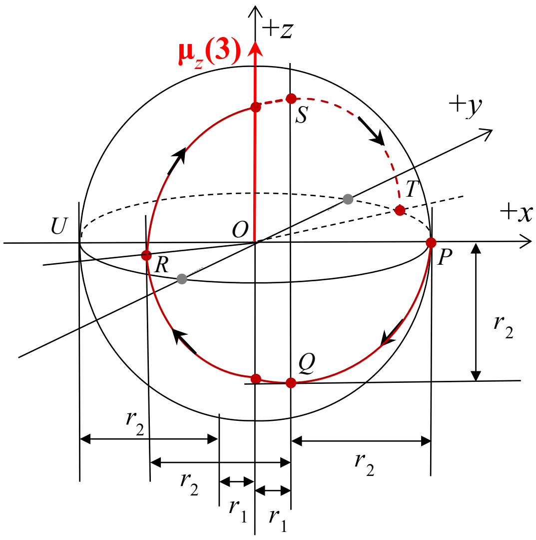

Figure 2. *Spindle torus model* of neutrino e with mass m_e, $N = 8.80$ and $r_2/r_1 = 3.96$. For clarity reasons, only one open orbit of neutrino e is sketched from point P until point T passing through the points Q, R and S. After insertion of a total of 8.80 such partial orbits, the last orbit ends in the starting point P, completing one full closed orbit. The direction of the magnetic dipole moment $\boldsymbol{\mu}_z(e)$ is also shown. See text for further comment.

The next open orbit of neutrino e starts at point T and ends on the equatorial circle of radius $4.96 r_1$ in the direction of U. After covering 8.80 such open orbits, looking like the first orbit $PQRST$, the neutrino e will return to point P, completing one full orbit. It is noticed that the *spindle torus* of neutrino e can approximately be considered as an association of $N = 8.80$ deformed *ring tori* of neutrino 1. Likewise, the neutrinos 2, μ, τ and 3 may be built up by associations of a varying number N of deformed *ring tori* of neutrino 1.

6. SUMMARY OF THE RESULTS

Two mass-squared differences $\Delta m^2_{21} \equiv m_2{}^2 - m_1{}^2$ and $\Delta m^2_{31} \equiv m_3{}^2 - m_1{}^2$ can be determined from neutrino oscillation experiments. In order to obtain the three neutrino masses m_i ($i = 1, 2, 3$) a third relation is necessary. The geometric mean mass relation of (1.1) may provide this third relation. He and Zee [1] already proposed this option in 2007 and found a value of 1.58 meV/c^2 for mass m_1, followed by Sazdović [2], who found a value of 1.55 meV/c^2 for mass m_1.

Alternatively, a value of the neutrino mass m_1 of 1.530 meV/c^2, calculated from (1.5), can serve as the third relation. This result has been obtained by combination of the magnetic moment of a massive Dirac neutrino deduced in the context of the electroweak interaction at the one-loop level (see refs. [3, 4]) and the magnetic moment from gravitational origin, the so-called Wilson-Blackett law, discussed by Biemond [5–7]. At present, the values m^2_{21} and Δm^2_{31} are not accurate enough to decide whether the values for the masses m_1 from refs. [1, 2] and [5, 6], respectively, are similar.

Furthermore, the existence of neutrino masses m_α ($\alpha = e, \mu, \tau$) corresponding to the flavor eigenstates ν_α is conjectured in this work. They are defined by the product of the unitary PMNS mixing matrix and the column vector with masses m_i (i = 1, 2, 3) as components. When the Dirac phase $\delta_{CP} = 0^o$ (see table 2), the masses m_α appear to be approximately quantized, where the ratio between the three masses is equal to: $m_e : m_\mu : m_\tau$ = 1.00 : 3.02 : 2.03. Moreover, the mass ratio m_μ/m_τ is equal to 1.48, close to the remarkable factor 3/2. For the value $\delta_{CP} = 215^o$ (see table 3) the mass m_e reduces to zero value, whereas the mass ratio m_μ/m_τ = 1.47 does not change much. Further observations are necessary to decide whether the masses m_α and ratio m_μ/m_τ possess these particular values.

It is generally supposed that neutrino α is created in the flavor eigenstate ν_α by weak interactions. In this work, it is assumed that each mass eigenstate ν_i corresponds to one of the masses m_i (i = 1, 2, 3). It is found that the sums of the masses $\Sigma_\alpha m_\alpha$ ($\alpha = e, \mu, \tau$) for δ_{CP} = 0º and δ_{CP} = 215º in table 2 and 3, respectively, are greater than the sum $\Sigma_i m_i$ (i = 1, 2, 3) in table 1. For δ_{CP} = 215º the energy difference is equal to 70.8 – 60.7 = 10.1 meV. This excess energy may be released as kinetic energy of the neutrinos of mass m_i.

The toroidal model [13, 14] based on the equations of motion of (4.1) provides a clear picture of the geometrical structure of all neutrinos. It is striking that the neutrinos with masses m_i (i = 2, 3) and m_α ($\alpha = e, \mu, \tau$), can all be described by the same *spindle torus model*, that predicts an almost spherical shape for all these neutrinos. Only neutrino 1 can be described by the *ring model* of Parson, or more accurately by the *ring torus model*. It is noticed that the spindle tori of masses m_i (i = 2, 3) and masses m_α ($\alpha = e, \mu, \tau$) can be considered as associations of N deformed *ring tori* of neutrino 1. Therefore, a transition between the mass set m_i (i = 1, 2, 3) and flavor mass set m_α ($\alpha = e, \mu, \tau$) seems conceivable.

Furthermore, quantitative expressions for the radius r_1 of the torus and the radius r_2 of the tube of the torus, as well as for the ratio r_2/r_1, are obtained. In addition, the angular momenta coupled to radii r_1 and r_2 are calculated. All radii and angular momenta appear to depend on simple functions of the toroidal factor N, too. Therefore, all radii and angular momenta may display a type of quantification.

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