The Conflict Between Non-definable Reals and the Axiom of Choice in ZFC

On the Incompatibility of Non-computable Reals with the Axiom of Choice

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Abstract

This paper analyzes the consequences of applying the Axiom of Choice (AC) within ZFC to the set $D := \mathbb{R} \setminus C$, where C is the set of computable real numbers. We demonstrate that any choice function defined over D implicitly renders its elements nameable or indexable, contradicting the very definition of D as the class of non-definable reals. This contradiction reveals an internal inconsistency in ZFC when AC is applied without restriction.

Importantly, the entire argument is developed strictly within standard ZFC logic, without any use of forcing, generic filters, metamathematical hierarchies, or external models. To resolve the conflict, we propose a revised system, ZFC D/AC, which explicitly prohibits applying AC to sets composed of non-definable elements. The result calls into question the universal applicability of AC and highlights the fragile boundary between definability and existence in modern set theory.

This result is not philosophical speculation, but a formal consequence of the current axiomatic foundations.

1 Introduction

The Axiom of Choice (AC) is among the most powerful and controversial axioms in the Zermelo-Fraenkel set theory (ZFC). It permits the selection of elements from arbitrary collections of nonempty sets, and supports numerous fundamental results across modern mathematics. Yet, AC also gives rise to counterintuitive outcomes, such as the Banach–Tarski paradox, and its relationship to non-constructive or non-definable entities remains insufficiently explored.

Historically, debates surrounding AC have focused on its non-constructive nature and its independence from ZF. Gödel showed that ZF + AC is consistent if ZF is, while Cohen later proved that AC is not provable within ZF. However, both results operate through external model constructions. In contrast, this paper investigates a contradiction that arises *internally* within ZFC, without invoking forcing, large cardinals, or alternate set-theoretic universes.

We focus on the implications of applying AC to a particular subset of real numbers: let $C \subset \mathbb{R}$ denote the set of computable real numbers, and define $D := \mathbb{R} \setminus C$, the set of non-computable reals. Elements of D are not only uncomputable, but also undefinable and unindexable — they cannot be described by any algorithm, function, or formal expression within ZFC.

We argue that applying a choice function $f : D \to D$ inevitably gives rise to a structure in which elements of D become distinguishable or referable, violating the very definition of D. This leads to a logical contradiction within the system ZFC + AC. To maintain consistency, we propose a revised framework, ZFC D/AC, in which AC is explicitly barred from acting on non-definable sets.

This contradiction between the act of choice and the principle of unnameability raises deeper questions about the nature of mathematical existence. We contend that the very operation of choosing presupposes a minimal degree of identifiability — a property that cannot coherently be attributed to entities that are, by construction, unidentifiable.

2 Definitions

Definition 1 (Computable Reals)**.** *Let $C \subset \mathbb{R}$ denote the set of* computable *real numbers. A real number $r \in \mathbb{R}$ is computable if there exists a Turing machine M such that, for every $n \in \mathbb{N}$, the machine M, on input n, halts and outputs the n-th digit of the decimal expansion of r.* [1]

Definition 2 (Non-computable Reals)**.** *Let $C \subset \mathbb{R}$ denote the set of* computable *real numbers, i.e.,*

$$C := \{ r \in \mathbb{R} \mid \exists M \ \forall n \in \mathbb{N},\ M(n) \downarrow = r_n \},$$

where M is a Turing machine that outputs the n-th digit r_n of r's decimal expansion. Define the set of non-computable *reals as the complement:*

$$D := \mathbb{R} \setminus C.$$

Since C is countable and $\mathbb{R}$ is uncountable, it follows that D is uncountable. [2]

Existence and Nature of D in ZFC. Both $\mathbb{R}$ and C are sets formally definable within ZFC: $\mathbb{R}$ is constructed via Dedekind cuts or Cauchy sequences over $\mathbb{Q}$, and C is the set of real numbers computable by Turing machines, which form a countable set. Therefore, by applying the Separation and Power Set axioms, the complement $D := \mathbb{R} \setminus C$ is also a valid set in ZFC. We thus affirm:

$$\mathrm{ZFC} \vdash \exists D := \mathbb{R} \setminus C.$$

More significantly, we observe that for any $d \in D$, the absence of a computing machine implies the absence of a definitional formula in ZFC:

$$\nexists \phi(x) \in \mathrm{Lang(ZFC)} \text{ such that } \begin{cases} \mathrm{ZFC} \vdash \exists! x\ \phi(x), \\ \mathrm{ZFC} \vdash \phi(d) \end{cases} \quad \Rightarrow \quad d \text{ is non-definable in ZFC.}$$

Consequently, there exists no definable injective function

$$f : \mathbb{N} \to D$$

[1] See A. M. Turing, "On Computable Numbers, with an Application to the Entscheidungsproblem," *Proc. London Math. Soc.*, Series 2, Vol. 42 (1937), pp. 230–265.

[2] See Jech, *Set Theory* (2002), p. 38.

with $f(n) = d_n \in D$ such that f is definable in ZFC. Thus, D is non-indexable.

Furthermore, since $|C| = \aleph_0$ and $|\mathbb{R}| = 2^{\aleph_0}$, the standard model of ZFC satisfies $|D| = 2^{\aleph_0}$, and hence $D \neq \emptyset$. Therefore, in any standard model of ZFC, D is not only a formally definable set, but one that concretely contains non-definable, non-computable, and non-indexable real numbers.

Definition 3 (Non-definability and Non-indexability)**.** *Let $D := \mathbb{R} \setminus C$ be the set of non-computable reals.*

An element $d \in D$ is said to be non-definable *within ZFC if there does not exist any formula $\varphi(x)$ in the language of ZFC such that*

$$\mathit{ZFC} \vdash \exists! x\, \varphi(x) \quad \text{and} \quad \mathit{ZFC} \vdash \varphi(d).$$

That is, no uniquely specifying definition of d exists within the system.

Consequently, the set D is non-indexable*: there exists no definable injective function*

$$f : \mathbb{N} \to D$$

such that $f(n) = d_n \in D$ for all $n \in \mathbb{N}$, and f is definable in ZFC.

This aligns with the canonical understanding of uncountability*, which entails the impossibility of indexing the elements of D via any definable mapping from $\mathbb{N}$.* [3]

Definition 4 (Axiom of Choice and Indexing Implication)**.** *The Axiom of Choice (AC) asserts that for any nonempty set S, there exists a function*

$$f : \mathcal{P}(S) \setminus \{\emptyset\} \to S \quad \text{such that} \quad f(A) \in A$$

for all nonempty subsets $A \subseteq S$.

When applied to the set D, AC guarantees the existence of a choice function that selects a single element from each nonempty subset of D. This selection process imposes an implicit indexing structure on D, making the elements of D distinguishable, which contradicts the non-indexability of D as previously defined.

3 Main Argument

Let $C \subset \mathbb{R}$ denote the set of all computable real numbers. Define the set

$$D := \mathbb{R} \setminus C.$$

Then D is uncountable, and by construction, every $d \in D$ is non-computable and unnameable within ZFC.

1. **Assumption of the Choice Function.**
   Assume the Axiom of Choice (AC). Then there exists a choice function

   $$f : \mathcal{P}(D) \setminus \{\emptyset\} \to D \quad \text{such that} \quad f(A) \in A \quad \forall A \subseteq D,\ A \neq \emptyset.$$

   This function selects one element from each non-empty subset of D.

[3] See Jech, *Set Theory* (2002), Theorem 4.3, p. 38.

2. **Indexing via Single Element.**
   Let $d_1 := f(D)$. Then we have
   $$D \setminus \{d_1\} \subseteq D, \quad d_1 \notin D \setminus \{d_1\}.$$
   Now, define a labeling function
   $$\text{Index}_1 : D \to \{0,1\}, \quad \text{Index}_1(d) = \begin{cases} 1 & \text{if } d = d_1, \\ 0 & \text{otherwise.} \end{cases}$$
   This function identifies a unique element d_1 in D, contradicting the assumption that D is unindexable.

3. **Explicit Indexing via Iteration.**
   Define recursively:
   $$D_1 := D, \quad d_n := f(D_n), \quad D_{n+1} := D_n \setminus \{d_n\}.$$
   Let
   $$D' := \{d_n \mid n \in \mathbb{N}\} \subseteq D,$$
   and define the map
   $$\varphi : \mathbb{N} \to D', \quad \varphi(n) := d_n.$$
   Then φ is a bijection, and hence D' is explicitly indexable. Again, this violates the assumption that D is unindexable.

4. **Logical Incompatibility.**
   Both the single-element case and the iterative construction yield identifiable elements in D. Therefore, we conclude:
   $$\text{AC} \Rightarrow \text{“Indexing of } D\text{”} \Rightarrow \text{Contradiction.}$$
   Hence, we deduce:
   $$\text{AC} \Rightarrow \neg\text{“D exists in ZFC”}, \quad \text{or} \quad \text{D exists} \Rightarrow \neg\text{AC}.$$

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4 Conclusion

1. Let
   $$C := \{r \in \mathbb{R} \mid r \text{ is computable}\}, \quad D := \mathbb{R} \setminus C.$$
   Then
   $$\forall d \in D, \quad \neg\text{Computable}(d), \quad \neg\text{Definable}_{\text{ZFC}}(d),$$
   and D is uncountable and non-indexable.

2. Assume the Axiom of Choice (AC):

$$\text{AC} \Rightarrow \exists f : \mathcal{P}(D) \setminus \{\emptyset\} \to D, \quad f(A) \in A.$$

Then we obtain:

$$d_1 := f(D) \Rightarrow D \setminus \{d_1\} \text{ is definable in terms of } d_1.$$

Define

$$\text{Index}_1 : D \to \{0,1\}, \quad \text{Index}_1(d) = \begin{cases} 1 & \text{if } d = d_1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, d_1 becomes indexable, violating the assumption that D is non-indexable.

3. More generally, define:

$$d_1 := f(D), \quad d_n := f(D \setminus \{d_1, \dots, d_{n-1}\}) \quad (n \geq 2),$$

and

$$D' := \{d_n \mid n \in \mathbb{N}\}, \quad \varphi : \mathbb{N} \to D', \quad \varphi(n) := d_n.$$

Then φ is a definable bijection from $\mathbb{N}$ to a subset of D, which contradicts the assumption that D is unindexable.

4. Therefore, we conclude:

$$\text{AC} \Rightarrow \text{Indexable}(D) \Rightarrow \bot,$$

and so

$$\neg(\text{AC} \wedge \text{Exists}(D)).$$

5. To resolve this contradiction, we propose a modified system, denoted **ZFC D/AC**, which restricts the domain of AC. We add the following axiom:

   Restricted Choice Axiom (RCA):

$$\forall S \neq \emptyset, \quad [\forall x \in S, \ \neg\text{Definable}_{\text{ZFC}}(x)] \Rightarrow \neg\exists f : \mathcal{P}(S)\setminus\{\emptyset\} \to S, \quad f(A) \in A.$$

6. This axiom preserves the standard utility of AC for definable sets, while preventing contradictions arising from its application to purely non-definable sets such as D.

5 Anticipated Objections and Rebuttals

Objection 1: Gödel's constructible universe L proves that ZF + AC is consistent.

Rebuttal: We examine this claim in light of the structure of L, Gödel's inner model of constructible sets.

1. Construction of L. The universe L is defined by transfinite recursion on the ordinals as follows:

$$\begin{aligned}
&L_0 := \emptyset \\
&L_{\alpha+1} := \mathrm{Def}(L_\alpha) := \left\{ x \subseteq L_\alpha \;\middle|\; \begin{array}{l} \exists \phi(v_1, \ldots, v_n) \in \mathrm{Lang(ZF)},\ \exists a_1, \ldots, a_n \in L_\alpha, \\ x = \{ y \in L_\alpha \mid \phi(y, a_1, \ldots, a_n) \} \end{array} \right\} \\
&L_\lambda := \bigcup_{\beta<\lambda} L_\beta \quad \text{for limit ordinal } \lambda \\
&L := \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha
\end{aligned}$$

Thus, each level L_α contains only those sets that are first-order definable over L_α with parameters from L_α. This implies:

$$\forall x \in L,\ \exists \alpha \in \mathrm{Ord},\ \exists \phi(x_1, \ldots, x_n, y) \in \mathrm{Lang(ZF)},\ \exists a_1, \ldots, a_n \in L_\alpha, \\ \text{such that } x = \{ y \in L_\alpha \mid \phi(a_1, \ldots, a_n, y) \}$$

2. Real numbers in L. Let $\mathbb{R}_L := \mathbb{R} \cap L$ denote the reals in L. Since L only contains sets that are explicitly definable within its levels, it follows that:

$$\forall r \in \mathbb{R}_L,\ \exists \phi(x) \in \mathrm{Lang(ZF)} \text{ such that } L \models \exists! x\ \phi(x) \wedge \phi(r)$$

Hence,

$$\mathbb{R}_L \subseteq \{ r \in \mathbb{R} \mid r \text{ is definable in ZFC} \}$$

In particular, any real number that is not first-order definable cannot appear in L. Therefore, the set

$$D := \mathbb{R} \setminus C$$

which consists of reals that are not computable and not definable in ZFC, satisfies:

$$D \cap L = \emptyset \quad \Rightarrow \quad D \not\subseteq L$$

3. Consequence. Gödel's model L satisfies ZF + AC, i.e.,

$$L \models \mathrm{ZF} + \mathrm{AC}$$

But it does not satisfy:

$$L \models \exists D := \mathbb{R} \setminus C$$

since $D \not\subseteq L$. Hence,

$$L \models \mathrm{Consistent}(\mathrm{ZF} + \mathrm{AC}) \Rightarrow \mathrm{Consistent}(\mathrm{ZF} + \mathrm{AC} + \exists D)$$

Conclusion. Gödel's construction shows only that AC is consistent *in the absence of* non-definable reals. Our contradiction arises precisely when D exists — as it does in any standard model of $\mathbb{R}$, where

$$|C| = \aleph_0, \quad |\mathbb{R}| = 2^{\aleph_0}, \quad \Rightarrow \quad |D| = 2^{\aleph_0}$$

and $D \neq \emptyset$. Therefore, our argument targets the inconsistency of AC in the presence of D, not in models where D is absent by construction.

Objection 2: Perhaps D is a proper class, not a set.
Rebuttal: $D \subset \mathbb{R}$ and $\mathbb{R}$ is a set in ZFC. Therefore, D is also a set.

Objection 3: AC is foundational to modern mathematics.
Rebuttal: Foundational axioms must preserve consistency. If

$$\mathrm{AC} \Rightarrow \bot$$

then the scope of AC must be restricted to avoid contradictions in the theory.

Objection 4: Selecting $d_i \in D$ does not entail indexing all of D.
Rebuttal: If $f(D) = d_i$, then

$$\mathrm{ZFC} \vdash \mathrm{Identifiable}(d_i) \Rightarrow \mathrm{Indexable}(\{d_i\}).$$

Repeated applications yield:

$$\{d_1, d_2, \dots\} \Rightarrow \text{Indexed subset } D' \subset D \Rightarrow \bot.$$

Since D is unindexable, this leads to a contradiction.

Objection 5: A countable subset $D' \subset D$ does not contradict the uncountability of D.
Rebuttal: True, but

$$\exists \varphi : \mathbb{N} \to D' \text{ definable in ZFC} \Rightarrow \exists d \in D \text{ with Definable}(d) \Rightarrow \bot$$

Objection 6: An element of D may exist without being constructible or selectable.
Rebuttal: That is consistent. However,

$$\mathrm{AC} \Rightarrow \mathrm{Select}(d_i \in D) \Rightarrow \mathrm{Definable}(d_i) \Rightarrow \bot$$

Objection 7: AC and the existence of D can coexist.
Rebuttal: We have shown:

$$\mathrm{AC} + \mathrm{Exists}(D) \Rightarrow \mathrm{Index}(D) \Rightarrow \bot$$

$$\Rightarrow \neg(\mathrm{AC} \wedge \mathrm{Exists}(D))$$

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