Deriving the Restitution Coefficient from Energy and Momentum Conservation

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Abstract

The coefficient of restitution is a dimensionless parameter that quantifies the elasticity of collisions. In this paper, we present a derivation that expresses the coefficient directly in terms of the change in kinetic energy for a two-body system undergoing a one-dimensional collision. This approach clarifies the relationship between energy dissipation, mass, and velocity, offering a transparent analytic framework useful for both physical insight and computational modeling.

1 Introduction

The restitution coefficient ε is commonly introduced as a ratio of relative velocities before and after collision. While effective descriptively, this definition often obscures its underlying connection to energy conservation. In this paper, we re-express ε in terms of the kinetic energy loss T of a two-body system, offering a physically intuitive formulation that aligns with theoretical and computational modeling needs.

2 Definitions and Setup

Consider two point masses m_a and m_b undergoing a head-on collision. Let v_a and v_b be their initial velocities, and v'_a and v'_b their velocities after the collision. Define:

- **Restitution coefficient:**

$$\varepsilon = \frac{v'_b - v'_a}{v_a - v_b}$$

- **Kinetic energy change:**

$$\Delta T = \frac{1}{2}m_a {v'_a}^2 + \frac{1}{2}m_b {v'_b}^2 - \left(\frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2\right)$$

3 Main Result

Lemma 1 (Restitution Coefficient from Energy Loss)**.** *Let m_a, m_b be the masses of two colliding bodies, with initial velocities v_a, v_b and final velocities v'_a, v'_b. If momentum is conserved, then the restitution coefficient ε can be written in terms of the kinetic energy loss ΔT as:*

$$\varepsilon = \sqrt{1 - \frac{2\Delta T}{\mu(v_a - v_b)^2}}\ ,\ \ \mu = \frac{m_a m_b}{m_a + m_b}$$

Proof. We define:

$$\varepsilon = \frac{v_b' - v_a'}{v_a - v_b}, \quad \Delta T = \frac{1}{2}m_a v_a'^2 + \frac{1}{2}m_b v_b'^2 - \left(\frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2\right)$$

Rewriting v_b' from ε:

$$v_b' = v_a' + \varepsilon(v_a - v_b)$$

From conservation of momentum:

$$m_a v_a + m_b v_b = m_a v_a' + m_b v_b'$$

Substitute v_b' in:

$$\begin{aligned} m_a v_a + m_b v_b &= m_a v_a' + m_b(v_a' + \varepsilon(v_a - v_b)) \\ &= (m_a + m_b)v_a' + m_b\varepsilon(v_a - v_b) \end{aligned}$$

Solving for v_a':

$$v_a' = \frac{m_a v_a + m_b v_b - m_b\varepsilon(v_a - v_b)}{m_a + m_b}$$

Then:

$$v_b' = v_a' + \varepsilon(v_a - v_b) = \frac{m_a v_a + m_b v_b + m_a\varepsilon(v_a - v_b)}{m_a + m_b}$$

Now compute ΔT:

$$\Delta T = \frac{1}{2}m_a v_a'^2 + \frac{1}{2}m_b v_b'^2 - \left(\frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2\right)$$

Let $A = m_a v_a + m_b v_b$, $D = v_a - v_b$, $M = m_a + m_b$. Then:

$$v_a' = \frac{A - m_b\varepsilon D}{M}, \quad v_b' = \frac{A + m_a\varepsilon D}{M}$$

Now plug into ΔT:

$$\Delta T = \frac{1}{2}m_a\left(\frac{A - m_b\varepsilon D}{M}\right)^2 + \frac{1}{2}m_b\left(\frac{A + m_a\varepsilon D}{M}\right)^2 - \left(\frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2\right)$$

Factor:

$$= \frac{1}{2M^2}\left[m_a(A - m_b\varepsilon D)^2 + m_b(A + m_a\varepsilon D)^2\right] - T_{\text{initial}}$$

Expand both squares:

$$\begin{aligned} (A - m_b\varepsilon D)^2 &= A^2 - 2Am_b\varepsilon D + m_b^2\varepsilon^2 D^2 \\ (A + m_a\varepsilon D)^2 &= A^2 + 2Am_a\varepsilon D + m_a^2\varepsilon^2 D^2 \end{aligned}$$

Now plug back in:

$$\Delta T = \frac{1}{2M^2}\left[(m_a + m_b)A^2 + \varepsilon^2 D^2(m_a m_b^2 + m_b m_a^2)\right] - T_{\text{initial}}$$

Note:

$$m_a m_b^2 + m_b m_a^2 = m_a m_b(m_a + m_b)$$

So:

$$\Delta T = \frac{1}{2M^2}\left[MA^2 + \varepsilon^2 D^2 m_a m_b M\right] - T_{\text{initial}}$$

Initial energy is:

$$T_{\text{initial}} = \frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2$$

Subtracting leaves:

$$\Delta T = \frac{1}{2}\varepsilon^2 D^2 \cdot \frac{m_a m_b}{M} = \frac{1}{2}\mu\varepsilon^2(v_a - v_b)^2$$

Solve for ε:

$$\varepsilon^2 = \frac{2\Delta T}{\mu(v_a - v_b)^2} \Rightarrow \varepsilon = \sqrt{\frac{2\Delta T}{\mu(v_a - v_b)^2}}$$

If ΔT is defined as final minus initial (negative for loss):

$$\Delta T = -\frac{1}{2}\mu(1 - \varepsilon^2)(v_a - v_b)^2 \Rightarrow \varepsilon = \sqrt{1 - \frac{2\Delta T}{\mu(v_a - v_b)^2}}$$

□

4 Discussion

The derived expression bridges a scalar energy-based description of collisions with the vectorial framework defined by the coefficient of restitution. This connection reinforces physical intuition by clarifying that:

$\varepsilon = 1$ corresponds to $\Delta T = 0$, indicating a perfectly elastic collision.

$\varepsilon = 0$ corresponds to maximal kinetic energy loss, representing a perfectly inelastic collision.

Additionally, the appearance of the reduced mass $\mu = \frac{m_a m_b}{m_a + m_b}$ emphasizes the role of system symmetry and mass distribution in the dynamics of energy loss.

Notably, the formula can be inverted to extract ε from empirical measurements of energy dissipation, enabling its application in experimental and data-driven modeling contexts.

4.1 Corollaries: Collision Types and Their Kinetic Signatures

We explore the specific outcomes of collisions by evaluating the restitution coefficient and the change in kinetic energy ΔT for key cases. Each case provides a distinct form of the underlying energy equation.

4.2 Perfectly Elastic Collision

Corollary 1 (**Elastic Case** ($\varepsilon = 1$))**.** *The kinetic energy is fully conserved. Thus,*

$$\Delta T = 0$$

Velocities after collision:

$$v_a' = \frac{(m_a - m_b)v_a + 2m_b v_b}{m_a + m_b}$$
$$v_b' = \frac{(m_b - m_a)v_b + 2m_a v_a}{m_a + m_b}$$

These expressions are derived under conservation of both momentum and kinetic energy.

4.3 Perfectly Inelastic Collision

Corollary 2 (**Inelastic Case** ($\varepsilon = 0$))**.** *The objects stick together post-collision:*

$$v_f = \frac{m_a v_a + m_b v_b}{m_a + m_b}$$

The kinetic energy loss is maximized:

$$\Delta T = -\frac{1}{2}\frac{m_a m_b}{m_a + m_b}(v_a - v_b)^2$$

4.4 Partially Elastic Collision

Corollary 3 (**General Case** $(0 < \varepsilon < 1)$). *A fraction of the kinetic energy is lost:*

$$\Delta T = -\frac{1}{2}\frac{m_a m_b}{m_a + m_b}(1 - \varepsilon^2)(v_a - v_b)^2$$

Final velocities are:

$$v_a' = \frac{m_a v_a + m_b v_b - m_b \varepsilon (v_a - v_b)}{m_a + m_b}$$
$$v_b' = \frac{m_a v_a + m_b v_b + m_a \varepsilon (v_a - v_b)}{m_a + m_b}$$

These forms reduce to the elastic or inelastic cases under $\varepsilon = 1$ *or* $\varepsilon = 0$*, respectively.*

Graphical Summary

4.5 Limitations and Scope

While the derived expression for the restitution coefficient is both analytically elegant and physically consistent, it is essential to outline the constraints under which this formulation holds. The following limitations define the scope of its validity:

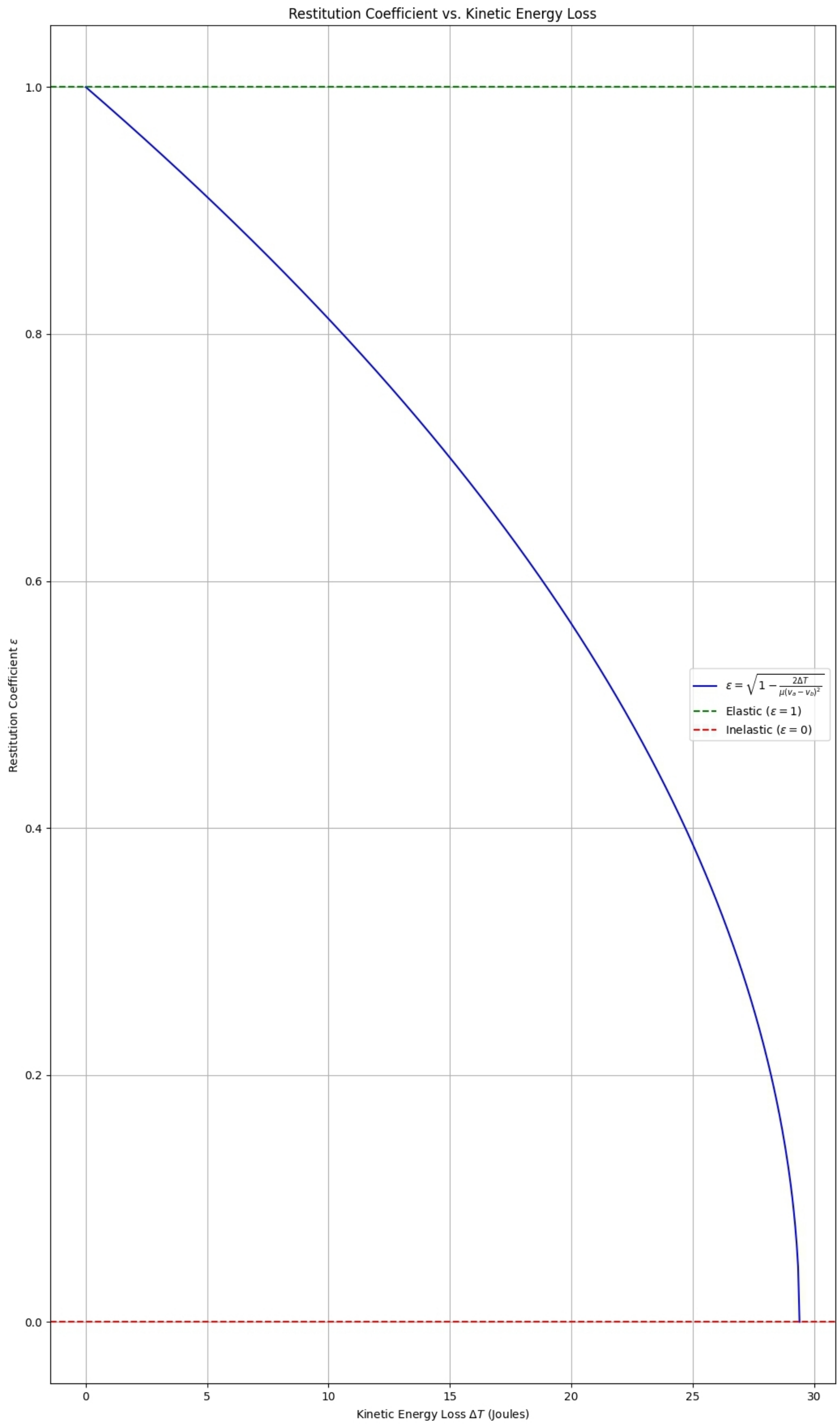
1. **One-Dimensional Collisions Only:** The derivation assumes purely linear motion along a single axis. Extension to two or three dimensions would require tensorial treatment of momentum and energy, accounting for angular degrees of freedom and vector projections [3, 2].

2. **Rigid Body Assumption:** The objects involved are considered point masses or rigid bodies. Internal deformations, flexing, and vibrational modes during impact are neglected, which are critical in real-world materials [4, **?**].

3. **Negligible External Forces:** The system is assumed isolated during collision, with no net external forces acting during the interaction time. This excludes gravitational gradients, electromagnetic effects, and friction from the environment [1].

4. **No Rotational Dynamics:** The derivation does not account for angular momentum or torque. For spinning or off-center impacts, the formulation becomes insufficient and would require moment of inertia and angular velocity analysis.

5. **Perfect Contact Model:** The collision is modeled as an instantaneous event with no surface compliance. Time-dependent contact forces or energy losses due to sound, heat, or microscopic friction are not included [4, **?**].



Restitution vs Energy Loss: Graphical relationship between kinetic energy loss and restitution coefficient. The curve illustrates nonlinear dissipation behavior under varying ΔT.

6. **Scalar Energy Representation:** The kinetic energy used is scalar and does not distinguish between directional motion. Thus, this model does not capture vectorial

asymmetries or directional energy redistributions [3].

7. **Symmetric Frame Reference:** The derivation implicitly assumes an inertial frame where the total system momentum is defined consistently pre- and post-collision. Frame-dependent interpretations may distort results [1].

This model could be effective for idealized, symmetrical, and controlled collisions in elementary mechanics. For applications in materials science, robotics, or astrophysical dynamics, higher-order generalizations are necessary.

5 Conclusion

The restitution coefficient is often regarded as a purely kinematic descriptor. By deriving its dependence on kinetic energy loss, we establish a bridge between dynamics and energy methods in classical mechanics. This relationship adds interpretive power to both simulation and theoretical frameworks. This paper is open for generalizations to other concepts, such as in rotational mechanics, application to real world systems, and higher dimension generalizations.

References

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