AN OPTIMIZATION PROBLEM FOR ADDITION CHAINS

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Abstract. In this paper, we formulate an optimization problem for addition chains.

1. Introduction

An addition chain of length h leading to n is a sequence of positive integers $s_o = 1, s_1 = 2, \ldots, s_h = n$ where $s_i = s_k + s_s$ for $i > k \geq s \geq 0$. The number of terms (excluding the first term) in an addition chain leading to n is the length of the chain. We call an addition chain leading to n with a minimal length an *optimal* addition chain leading to n. In standard practice, we denote by $\ell(n)$ the length of an optimal addition chain that leads to n.

In our previous investigations, we proved the following identities for the partial sums (mass) and the reciprocal sum of terms in an addition chain.

Proposition 1.1. *Let $n \geq 3$ be fixed positive integer and let $s_o = 1, s_1 = 2, \ldots, s_h = n$ be an addition chain leading to n with*

$$\mathbb{G}_n := \{(a_i, r_i) \ : \ a_{i+1} = a_i + r_i, \ s_i = a_{i+1}, \ 1 \leq i \leq h\}$$

then

$$\sum_{i=1}^{h} s_i = 2(n-1) + h - r_h + \int_{1}^{h-1} \left(\sum_{1 \leq i \leq t} r_i \right) dt.$$

Proposition 1.2. *Let $s_o = 1, s_1 = 2, \ldots, s_h = n$ be an addition chain leading to n with*

$$\mathbb{G}_n := \{(a_i, r_i) \ : \ a_{i+1} = a_i + r_i, \ s_i = a_{i+1}, \ 1 \leq i \leq h\}.$$

then

$$\sum_{l=0}^{h} \frac{1}{s_l} = \frac{3}{2} + \frac{h-1}{n} + \sum_{l=3}^{h} \sum_{v=1}^{\infty} \frac{1}{n^{v+1}} \left(\sum_{j=l}^{h} r_j \right)^{v}$$

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where $\sum_{j=l}^{h} r_j < n-1$ *for each* $3 \leq l \leq h$.

We can verify Proposition 1.2 as follows:
Consider an addition chain $s_o = 1, s_1 = 2, \ldots, s_h = n$ leading to n with

$$\mathbb{G}_n := \{(a_i, r_i) \ : \ a_{i+1} = a_i + r_i, \ s_i = a_{i+1}, \ 1 \leq i \leq h\}.$$

We make the following observations: $s_{h-1} = a_h = a_{h-1} + r_{h-1} = s_{h-2} + r_{h-1} = a_{h-2} + r_{h-2} + r_{h-1} = \cdots = 1 + \sum_{j=1}^{n-1} r_j = n - r_h$. Similarly, we can write $a_{h-1} = 1 + \sum_{j=1}^{h-2} = n - r_h - r_{h-1}$. Thus, by induction, we can write $a_l = n - \sum_{j=l}^{h} r_j$ for each $3 \leq l \leq h$. We observe

$$\sum_{l=1}^{h} \frac{1}{s_l} = \frac{3}{2} + \sum_{l=3}^{h} \frac{1}{a_l} + \frac{1}{n}.$$

We now analyze the latter sum of the right side. We can write

$$\sum_{l=3}^{h} \frac{1}{a_l} = \sum_{l=3}^{h} \frac{1}{n - \sum_{i=l}^{h} r_i}$$

which can be recast as

$$\sum_{l=3}^{h} \frac{1}{a_l} = \sum_{l=3}^{h} \frac{1}{n} + \sum_{l=3}^{h} \sum_{v=1}^{\infty} \frac{1}{n^{v+1}} \left(\sum_{i=l}^{h} r_i\right)^{v}$$

with $\sum_{i=l}^{h} r_i < n-1$ for each $3 \leq l \leq h$. It follows that

$$\sum_{l=0}^{h} \frac{1}{s_l} = \frac{3}{2} + \frac{h-2}{n} + \sum_{l=3}^{h} \sum_{v=1}^{\infty} \frac{1}{n^{v+1}} \left(\sum_{i=l}^{h} r_i\right)^{v} + \frac{1}{n}$$

where $\sum_{i=l}^{h} r_i < n-1$ for each $3 \leq l \leq h$.

It is also possible to track the worst growth rate of each term in an addition chain

$$\mathcal{C}_n : s_o = 1 < s_1 = 2 < \cdots < s_h = n$$

in a way that observes

Lemma 1.3 (Local linear control)**.** *For each term s_j in an addition chain leading to n of length h, we have*

$$s_j \leq \frac{n-1}{h}(j+1)$$

for all $0 \leq j \leq h-1$.

Using this observation (Lemma 1.3), we obtain an upper bound for the difference:

$$\sum_{i=0}^{h} s_i - \sum_{i=0}^{h} \frac{1}{s_i} \leq \left(\frac{n-1}{h}\right) \sum_{i=0}^{h-1}(i+1) + n - \left(\frac{h}{n-1}\right) \sum_{i=0}^{h-1} \frac{1}{i+1} - \frac{1}{n}$$
$$= \frac{(n-1)(h+1)}{2} + n - \left(\frac{h}{n-1}\right) \sum_{i=1}^{h} \frac{1}{i} - \frac{1}{n}.$$

2. An optimization problem

Problem 2.1. *For each positive integer $n \geq 3$, consider the set of all addition chains*

$$\mathcal{C}_n : s_o = 1 < s_1 < \cdots < s_h = n$$

of arbitrary length h. Define the cost of the chain $\mathcal{C}_n$ by

$$A(\mathcal{C}_n) := \sum_{i=0}^{h} s_i - \sum_{i=0}^{h} \frac{1}{s_i}.$$

Which addition chain(s) minimizes the cost $A(\mathcal{C}_n)$? Equivalently, determine

$$\min_{\mathcal{C}_n}\{A(\mathcal{C}_n)\} = \min \left\{ \sum_{i=0}^{h} s_i - \sum_{i=0}^{h} \frac{1}{s_i} \right\}$$

and characterize the chain(s) that achieve this minimum.

Now, consider the set of all addition chains

$$\mathcal{C}_n : s_o = 1 < s_1 < \cdots < s_h = n$$

of arbitrary length h. The cost $A(\mathcal{C}_n)$ of an addition chain - as defined - is determined by two competing forces, namely the partial sum (mass) and the reciprocal sum (penalty)

$$A(\mathcal{C}_n) := \sum_{i=0}^{h} s_i - \sum_{i=0}^{h} \frac{1}{s_i}.$$

Since

$$\inf_{1\leq i<h}(s_i) \ll i\frac{n}{h}$$

and

$$\sup_{1\leq i<h}(s_i) \gg i\frac{n}{h}$$

we have on average

$$s_i \approx l\frac{n}{h}.$$

Thus for optimal or near-optimal addition chains $\{s_o, s_1, \ldots, s_h\}$, one expects the worst *penalty*

$$\sum_{i=0}^{h}\frac{1}{s_i}$$

on average. On the other hand, for addition chains that are neither optimal nor near optimal, the *penalty* is relatively small on average. Therefore, addition chain

$$C_n : s_o = 1 < s_1 < \cdots < s_h = n$$

which minimizes the cost

$$A(\mathcal{C}_n) := \sum_{i=0}^{h} s_i - \sum_{i=0}^{h}\frac{1}{s_i}$$

is the one that balances the trade-off between the partial sum (mass) $\sum\limits_{i=0}^{h} s_i$ and the reciprocal sum (penalty) $\sum\limits_{i=0}^{h}\frac{1}{s_i}$. This, however, seems to be a difficult and non-trivial problem. However, we expect that the minimal cost will be

$$\min_{\mathcal{C}_n}\{A(\mathcal{C}_n)\} \sim Cn\log n$$

for some constant $C > 0$ and the length h of the minimizer to satisfy

$$h = \kappa \log n + O(1)$$

for some $\kappa > 0$. Determining C and κ also seems to be a non-trivial problem. Even if this problem is eventually solved, the question of the existence of a unique minimizer will still remain another challenge. For a collection of all the addition chains that leads to a fixed target n without a unique minimizer, one could also ask how many of these addition chains $\mathcal{C}_n$ in the collection minimize the cost $A(\mathcal{C}_n)$ over all chains $\mathcal{C}_n$. The following table provides a detailed cost measurement for all possible addition chains leading to $n = 7$.

Chain	**Length**	**Cost**
$1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7$	6	25.407143
$1 \to 2 \to 3 \to 4 \to 5 \to 7$	5	19.573810
$1 \to 2 \to 3 \to 4 \to 6 \to 7$	5	20.607143
$1 \to 2 \to 3 \to 4 \to 7$	4	14.773810
$1 \to 2 \to 3 \to 5 \to 6 \to 7$	5	21.657143
$1 \to 2 \to 3 \to 5 \to 7$	4	15.823810
$1 \to 2 \to 3 \to 6 \to 7$	4	16.857143
$1 \to 2 \to 4 \to 5 \to 6 \to 7$	5	22.740476
$1 \to 2 \to 4 \to 5 \to 7$	4	16.907143
$1 \to 2 \to 4 \to 6 \to 7$	4	17.940476

TABLE 1. All valid addition chains to 7 (up to length 6), their lengths, and the cost $\sum s_i - \sum 1/s_i$.

In this example, the minimal-length addition chain has length four and the cost minimizing chain is

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 7$$

with the cost $\inf_{\mathcal{C}_7}\{A(\mathcal{C}_7)\} \approx 14.773810$ of length four coincides with the minimum possible length. The table confirms that the chain

$$\mathcal{C}_7 : s_o = 1, s_1 = 2, s_2 = 3, s_3 = 4, s_4 = 7$$

uniquely minimizes the cost

$$A(\mathcal{C}_7) := \sum s_i - \sum \frac{1}{s_i}.$$

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