Sequences of prime numbers and composite odd numbers generated with set theory

Muhammad Razzaq Aman Wattoo[1], Mehar Ali Malik[1], Iqra Aftab[2]

[1]Department of Mathematics, Riphah International University, Lahore, Pakistan

[2]Department of Mathematics, Government College University Faisalabad, Pakistan

Email addresses: irwattoo@gmail.com,

Abstract

In this paper, we have developed a set S named as Aman's set that is union of collection of sets generated by arithmetic sequence. Then we have used this set to generate two separate complete lists of prime numbers and composite odd numbers. Also we have run this set to find the finite list of the prime numbers and composite numbers in python successfully.

Keywords: Aman's set, prime numbers, composite number, prime generator.

1 Introduction

Prime numbers are widely used in information technology and cryptography but it is an ultimate problem to generate a complete list of prime numbers. Around 300 BC, Euclid demonstrated that there are infinitely primes [1]. It is believed that there is no known simple formula to separate prime numbers from odd composite numbers. At the end of 19th century, it was proved that the probability of randomly chosen number being prime is inversely proportional to its number of digits that is to its logarithm [2]. Many theorems and conjectures have been given about prime numbers but none of them is simple and easy to generate a complete list of prime numbers or composite odd numbers. A positive n is a prime number if and only if (n-1)! + 1 is divisible by n [3], $M = 2^n - 1$ may be a prime number [4], Many primes can be written as $(2^n + 1) / 3$ [5], there may be many prime numbers which can be expressed as $2^{2^n}+1$, there is a transcendental number $\xi > 1$ for which the number $[\xi^{n!}]$, n = 2, 3, … are all primes. Sieve of Eratosthenes technique is used to make a list of prime numbers but it becomes very complicated for the finding of huge prime numbers. Many questions regarding prime numbers as Goldbach's conjecture, Twin prime conjecture and Legendre's conjecture are still unsolved. In the following we give the definition of a set called Aman's, that will be used to derive the main results.

Definition: A set S is said to be Aman's if $S = \bigcup_{n=1}^{\infty}\{A_n\}$ and $A_n = \{2k^2 + (2k+1)n\}_{k=1}^{\infty}$ where $n, k \in \mathbb{N}$

2 Main Results

The results are presented in terms of following remarks followed by their proofs.

Remark 1. Each element of set S is composite odd generator for $2r - 1$ where $r \in S$

Proof: When $r = 2 + 3n$ then $2r - 1 = 2(2 + 3n) - 1 = 3(1 + 2n)$ this is composite odd number for all $\in \mathbb{N}$. It gives $O_1 = \{9, 15, 21, 27, \ldots\}$ => Set of composite odd numbers divisible by 3 having 1st term is 3^2
When $r = 8 + 5n$ then $2r - 1 = 2(8 + 5n) - 1 = 5(3 + 2n)$ this is composite odd number for all $n \in \mathbb{N}$. It gives $O_2 = \{25, 35, 45, 55, \ldots\}$ => Set of composite odd numbers divisible by 5 having 1st term is 5^2
When $r = 18 + 7n$ then $2r - 1 = 2(18 + 7n) - 1 = 7(5 + 2n)$ this is composite odd number for all $\in \mathbb{N}$. It gives $O_3 = \{49, 63, 77, 91, \ldots\}$ => Set of composite odd numbers divisible by 7 having 1st term is 7^2
When $r = 32 + 9n$ then $2r - 1 = 2(32 + 9n) - 1 = 9(7 + 2n)$ this is composite odd number for all $\in \mathbb{N}$. It gives $O_4 = \{81, 99, 117, 135, \ldots\}$ => Set of composite odd numbers divisible by 9 having 1st term is 9^2
When $r = 2k^2 + (2k+1)n$ then $2r - 1 = 2(2k^2 + (2k+1)n) - 1 = (2k+1)(2k - 1 + 2n)$ this is composite odd number divisible by $(2k+1)$ having 1st term is$(2k+1)^2$ for all $n \in \mathbb{N}$
.
It gives $O_c = \{\{(2k+1)^2\}, \{(2k+1)(2k+3\}, \{(2k+1)(2k+5)\}, \ldots\}$ => Set of composite Odd numbers divisible by $(2k+1)$ for all $k \in \mathbb{N}$
From the above proof we can conclude that a set O_c is said to be set of all composite numbers generated by Aman's set if
$O_c = \bigcup_{n=1}^{\infty}\{B_n\}$ and $B_n = \{(2k+1)(2k+2n-1)\}_{k=1}^{\infty}$ where $n, k \in \mathbb{N}$

Remark 2. Each element of set $\mathbb{N}\backslash S - \{1\}$ is prime generator for $2r - 1$ where $r \in \mathbb{N}\backslash S - \{1\}$

Proof: As each prime number greater than 2 is odd number but every odd is not prime because set of odd number greater than 1 is union of set of prime numbers greater than 2 and set of composite numbers. From the given set $\mathbb{N}\backslash S - \{1\}$ it is clear that it does not contain Aman's set S the generator of set of all composite odd numbers and also $2r - 1 \neq 1$ because $r \neq 1$. Hence for $r \in \mathbb{N}\backslash S - \{1\}$ the odd number $2r - 1$ is always prime number.
From the above proof we can conclude that the set of all prime number is as following

$$P = \{2r - 1 : r \in \mathbb{N}\backslash S - \{1\}\} \cup \{2\}$$

Theorem 1. A finite set S_o whose largest element is less or equal to $\left\lfloor \frac{\aleph_0}{2} \right\rfloor$ is said to be finite Aman's set if $S_o = \bigcup_{n=1}^{\lambda}\{A_n\}$ such that $A_n = \{2k^2 + (2k+1)n\}_{k=1}^{\rho}$ for $n, k \in \mathbb{N}$ where $\lambda = \left\lfloor \frac{\aleph_0 - 4}{6} \right\rfloor$ and $\rho = \left\lfloor \frac{-1+\sqrt{\aleph_0 - 1}}{2} \right\rfloor$ where $\aleph_0 = |\mathbb{N}|$

E.g. For $\aleph_0 = 100000$ set S_o whose largest element is less or equal to $\left\lfloor \frac{100000}{2} \right\rfloor = 50000$ $S_o = \bigcup_{n=1}^{16666}\{A_n\}$ such that $A_n = \{2k^2 + (2k+1)n\}_{k=1}^{157}$ where $\lambda = \left\lfloor \frac{100000-4}{6} \right\rfloor = 16666$ and $\rho = \left\lfloor \frac{-1+\sqrt{100000-1}}{2} \right\rfloor = 157$.

Theorem 2. The set of all prime numbers in first consecutive $\aleph_0$ natural numbers can be listed as

$P = \{2r - 1 : r \in \mathbb{N} \backslash S_0 - \{1\}\} \cup \{2\}$ Where $|\mathbb{N}| = \left\lfloor \frac{\aleph_0}{2} \right\rfloor$ or

$$P = \{\{2r-1\}_{r=1}^{u}\} - \bigcup_{n=1}^{v}\{B_n\}\} \cup \{2\}$$

where $u = \left\lfloor \frac{\aleph_0}{2} \right\rfloor$ *and* $v = \left\lfloor \frac{\aleph_0 - 3}{6} \right\rfloor$ and $B_n = \{(2k+1)(2k+2n-1)\}_{k=1}^{w}$ and $w = \left\lfloor \frac{-1+\sqrt{\aleph_0}}{2} \right\rfloor$.

Theorem 3. For the set of Mersenne prime numbers $P = \{\{2^r - 1\}_{r=1}^{u}\} - \bigcup_{n=1}^{v}\{B_n\}\}$, where $u = \left\lfloor \frac{Log(\aleph_0 + 1)}{Log2} \right\rfloor$ *and* $v = \left\lfloor \frac{\aleph_0 - 3}{6} \right\rfloor$ and $B_n = \{(2k+1)(2k+2n-1)\}_{k=1}^{w}$ and $w = \left\lfloor \frac{-1+\sqrt{\aleph_0}}{2} \right\rfloor$.

Theorem 4. For the Set of Fermat Primes $P = \left\{\{2^{2^r} - 1\}_{r=1}^{u}\right\} - \bigcup_{n=1}^{v}\{B_n\}\} \cup \{2\}$ where $u = \left\lfloor \frac{\left\lfloor Log\left(\frac{Log(\aleph_0 - 1)}{Log2}\right)\right\rfloor}{Log2} \right\rfloor$ *and* $v = \left\lfloor \frac{\aleph_0 - 3}{6} \right\rfloor$ and $B_n = \{(2k+1)(2k+2n-1)\}_{k=1}^{w}$ where $w = \left\lfloor \frac{-1+\sqrt{\aleph_0}}{2} \right\rfloor$.

The proofs of Theorems 1-4 can be observed by the following algorithm.

3. Algorithm

Following are two simple python algorithms to find complete list of prime number by using the results

<u>Step 1</u>.

```
N = set(i for i in range(1,                    ⌊ℵ_0/2⌋ + 1               ))
j =                                                                       ρ
n =                                                                       λ
M = set((((2 * k + 1) * i) + (2 * (k ** 2))) for i in range
(1, n + 1) for k in range(1, j + 1))

S = set(i for i in M if i <=                    ⌊ℵ_0/2⌋                   )
```

```
A = N - S
B = A.copy()
B.remove(1)
print(sorted(B))
C = set(2 * b - 1 for b in B)
print(sorted(C))

P = C.copy()
P.add(2)
print(P)

print(f'Length of the set={len(P)}')
```

Step 2.

$$J = \left\lfloor \frac{\aleph_0}{2} \right\rfloor$$

$$K = \left\lfloor \frac{-1+\sqrt{\aleph_0}}{2} \right\rfloor$$

$$N = \left\lfloor \frac{\aleph_0 - 3}{6} \right\rfloor$$

```
A = {2 * j - 1 for j in range(2, J + 1)}
B = {4 * k * k + 4 * k * n + 2 * n - 1 for n in range(1, N
+ 1) for k in range(1, K + 1)}
P = A - B
P.add(2)
print(P)
print(f'Length of the set={len(P)}')
```

Following is the python algorithm for the finding of complete list of composite odd numbers number by using the results

```
N = set(i for i in range(1,  $\left\lfloor \frac{\aleph_0}{2} \right\rfloor + 1$  ))
j =  $\rho$
n =  $\lambda$
M = set((((2 * k + 1) * i) + (2 * (k ** 2))) for i in range
(1, n + 1) for k in range(1, j + 1))

S = set(i for i in M if i <=  $\left\lfloor \frac{\aleph_0}{2} \right\rfloor$  )


print(sorted(S))
C = set(2 * b - 1 for b in S)
print(sorted(C))

O = C.copy()
print(O)

print(f'Length of the set={len(O)}')
```

4. Use of Aman's Set to find the Mersenne prime numbers

In this section we use the above algorithm to find the Mersenne primes.

$J = \left\lfloor \frac{Log(\aleph_0+1)}{Log2} \right\rfloor$

$K = \left\lfloor \frac{-1+\sqrt{\aleph_0}}{2} \right\rfloor$

$N = \left\lfloor \frac{\aleph_0-3}{6} \right\rfloor$

```
A = {2**j - 1 for j in range(2, J + 1)}
B = {4 * k * k + 4 * k * n + 2 * n - 1 for n in range(1, N
+ 1) for k in range(1, K + 1)}
P = A - B
print(P)
print(f'Length of the set={len(P)}')
```

5. Use of Aman's Set to find the Fermat prime numbers

In this section we use the algorithm given in Section 3 to find the Fermat primes.

$J = \left\lfloor \frac{\left\lfloor Log\left(\frac{Log(\aleph_0-1)}{Log2}\right)\right\rfloor}{Log2} \right\rfloor$

$K = \left\lfloor \frac{-1+\sqrt{\aleph_0}}{2} \right\rfloor$

$N = \left\lfloor \frac{\aleph_0-3}{6} \right\rfloor$

```
A = {2**2**j - 1 for j in range(0, J + 1)}
B = {4 * k * k + 4 * k * n + 2 * n - 1 for n in range(1, N
+ 1) for k in range(1, K + 1)}
P = A - B
print(P)
print(f'Length of the set={len(P)}')
```

6. Conclusion

In this paper, we give a sequence to generate all the prime numbers. We give an algorithm in python to run the sequence that generates all the prime numbers. Further, we use this sequence to obtain the Meresenne and Fermat's primer numbers.

References

[1] J. Williamson (translator and commentator), The Elements of Euclid, With Dissertations,
Clarendon Press, Oxford, 1782, page 63.

[2] D. J. Newman, Simple Analytic Proof of the Prime Number Theorem, The American Mathematical Monthly, Vol. 87, No. 9 (Nov., 1980), Taylor & Francis, Ltd, pp. 693-696.

[3] C.S. Dalawat, Wilson’s theorem. J. Th´eor. Nombres Bordeaux 21 (2009), 517–521.

[4] G. Woltman, “The Great Internet Mersenne Prime Search,” March 2001,http://www.mersenne.org

[5] J. Williamson (translator and commentator), The Elements of Euclid, With Dissertations,
Clarendon Press, Oxford, 1782, page 63