On the calculation of both power and moment required of motors to drive a robot arm in a controllable manner

Aloys J. Sipers[*], Joh. J. Sauren[†]

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Abstract

In this letter we use complex numbers to calculate both the power and the mechanical moment required of robot motors to drive a robot arm subjected to loads on joints and on multiple parts.

1 Expressing mechanical power with complex numbers

We define a robot arm as a construction consisting of two parts, $\underline{X}_1$ and $\underline{X}_2$, see Figure 1. Part $\underline{X}_1$ is directed from joint 1 to joint 2, part $\underline{X}_2$ is directed from joint 2 to joint 3. Joint 1 is at rest, joint 2 and joint 3 are freely displaceable. The required powers of the robot motors, located at joint 1 and joint 2, respectively, depend on the forces $\underline{F}_2$ and $\underline{F}_3$ excerted at joints 2 and 3. The mechanical moments M_1 and M_2 drive and rotate the robot parts $\underline{X}_1$ and $\underline{X}_2$.

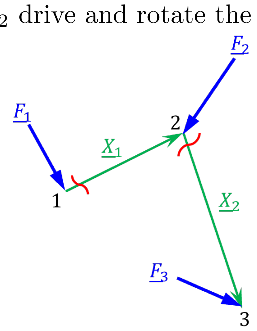

Figure 1: One motor is located at joint 1, a second motor is located at joint 2. The gripper is located at joint 3. The robot parts $\underline{X}_1$ and $\underline{X}_2$ are mechanically at equilibrium. Cuts in the vicinity of joints 1 and 2 are denoted by red tildes.

*Corresponding author, Engineering Academy, Zuyd University of Applied Sciences, NL–6419 DJ, Heerlen, The Netherlands, aloys.sipers@zuyd.nl

†Engineering Academy, Zuyd University of Applied Sciences, NL–6419 DJ, Heerlen, The Netherlands, hans.sauren@zuyd.nl

Cut at joint 2.

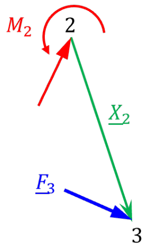

Figure 2

Equilibrium of part 2−3, see Figure 2.

$$\sum M_2 = 0$$
$$+M_2 + \mathfrak{Im}\left(\underline{F}_3 \cdot \underline{X}_2^*\right) = 0$$
$$M_2 = -\mathfrak{Im}\left(\underline{F}_3 \cdot \underline{X}_2^*\right)$$

Cut at joint 1.

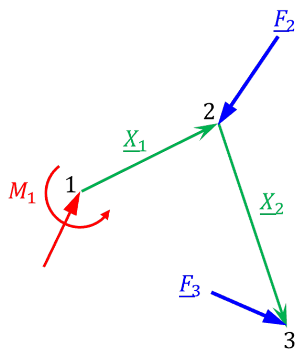

Figure 3

Equilibrium of both part 1−2 and of part 2−3, see Figure 3.

$$\sum M_1 = 0$$
$$+M_1 + \mathfrak{Im}\left(\underline{F}_2 \cdot \underline{X}_1^*\right) + \mathfrak{Im}\left(\underline{F}_3 \cdot \left(\underline{X}_1 + \underline{X}_2\right)^*\right) = 0$$
$$M_1 = -\mathfrak{Im}\left(\underline{F}_2 \cdot \underline{X}_1^* + \underline{F}_3 \cdot \left(\underline{X}_1 + \underline{X}_2\right)^*\right)$$

Lemma

$$M_2 \cdot \omega_2 = -\mathfrak{Re}\left(\underline{F}_3 \cdot \left(\underline{v}_3 - \underline{v}_2\right)^*\right) \quad (1)$$
$$\left(M_1 - M_2\right) \cdot \omega_1 = -\mathfrak{Re}\left(\left(\underline{F}_2 + \underline{F}_3\right) \cdot \underline{v}_2^*\right) \quad (2)$$

Proof

$$\underline{X}_1 \coloneqq r_1 \cdot e^{\mathrm{i}\cdot\varphi_1} \qquad \underline{X}_2 \coloneqq r_2 \cdot e^{\mathrm{i}\cdot\varphi_2}$$
$$\omega_1 \coloneqq \dot{\varphi}_1 \qquad \omega_2 \coloneqq \dot{\varphi}_2$$

Calculating the velocities:

$$\underline{v}_2 = \frac{d}{dt}\underline{X}_1 = \mathrm{i} \cdot \dot{\varphi}_1 \cdot r_1 \cdot e^{\mathrm{i}\cdot\varphi_1} = \mathrm{i} \cdot \omega_1 \cdot \underline{X}_1 \Leftrightarrow \underline{v}_2^* = -\mathrm{i} \cdot \omega_1 \cdot \underline{X}_1^* \Leftrightarrow$$
$$\omega_1 \cdot \underline{X}_1^* = \mathrm{i} \cdot \underline{v}_2^*$$
$$\underline{v}_3 = \frac{d}{dt}\left(\underline{X}_1 + \underline{X}_2\right) = \underline{v}_2 + \mathrm{i} \cdot \dot{\varphi}_2 \cdot r_2 \cdot e^{\mathrm{i}\cdot\varphi_2} = \underline{v}_2 + \mathrm{i} \cdot \ \omega_2 \cdot \underline{X}_2$$
$$\underline{v}_3 - \underline{v}_2 = +\mathrm{i} \cdot \omega_2 \cdot \underline{X}_2 \Leftrightarrow \left(\underline{v}_3 - \underline{v}_2\right)^* = -\mathrm{i} \cdot \omega_2 \cdot \underline{X}_2^* \Leftrightarrow \omega_2 \cdot \underline{X}_2^* = \mathrm{i} \cdot \left(\underline{v}_3 - \underline{v}_2\right)^*$$

Calculating the moments and the powers:

$$M_2 = -\mathfrak{Im}\left(\underline{F}_3 \cdot \underline{X}_2^*\right) \tag{1}$$
$$M_2 \cdot \omega_2 = -\mathfrak{Im}\left(\underline{F}_3 \cdot \underline{X}_2^* \cdot \omega_2\right) = -\mathfrak{Im}\left(\underline{F}_3 \cdot \mathrm{i} \cdot \left(\underline{v}_3 - \underline{v}_2\right)^*\right)$$
$$M_2 \cdot \omega_2 = -\mathfrak{Re}\left(\underline{F}_3 \cdot \left(\underline{v}_3 - \underline{v}_2\right)^*\right)$$
$$M_1 = -\mathfrak{Im}\left(\underline{F}_2 \cdot \underline{X}_1^* + \underline{F}_3 \cdot \left(\underline{X}_1 + \underline{X}_2\right)^*\right) \tag{2}$$
$$M_1 = -\mathfrak{Im}\left(\left(\underline{F}_2 + \underline{F}_3\right) \cdot \underline{X}_1^* + \underline{F}_3 \cdot \underline{X}_2^*\right)$$
$$M_1 = -\mathfrak{Im}\left(\left(\underline{F}_2 + \underline{F}_3\right) \cdot \underline{X}_1^*\right) - \mathfrak{Im}\left(\underline{F}_3 \cdot \underline{X}_2^*\right)$$
$$M_1 = -\mathfrak{Im}\left(\left(\underline{F}_2 + \underline{F}_3\right) \cdot \underline{X}_1^*\right) + M_2$$

$$M_1 - M_2 = -\mathfrak{Im}\left(\left(\underline{F}_2 + \underline{F}_3\right) \cdot \underline{X}_1^*\right)$$
$$\left(M_1 - M_2\right) \cdot \omega_1 = -\mathfrak{Im}\left(\left(\underline{F}_2 + \underline{F}_3\right) \cdot \underline{X}_1^* \cdot \omega_1\right) = -\mathfrak{Im}\left(\left(\underline{F}_2 + \underline{F}_3\right) \cdot \mathrm{i} \cdot \underline{v}_2^*\right)$$
$$\left(M_1 - M_2\right) \cdot \omega_1 = -\mathfrak{Re}\left(\left(\underline{F}_2 + \underline{F}_3\right) \cdot \underline{v}_2^*\right)$$

□

Theorem

$$M_1 \cdot \omega_1 + M_2 \cdot \left(\omega_2 - \omega_1\right) + \mathfrak{Re}\left(\underline{F}_2 \cdot \underline{v}_2^*\right) + \mathfrak{Re}\left(\underline{F}_3 \cdot \underline{v}_3^*\right) = 0$$

Remark: note that

- $M_1 \cdot \omega_1$ is the power of the moment M_1 acting at joint 1.
- $M_2 \cdot (\omega_2 - \omega_1)$ is the power of the moment M_2 acting at joint 2. In case $\omega_1 = \omega_2$, the motor at joint 2 supplies no power, i.e. the robot arm rotates as a rigid body.
- $\mathfrak{Re}\left(\underline{F}_2 \cdot \underline{v}_2^*\right)$ is the power of the force $\underline{F}_2$ acting at joint 2.
- $\mathfrak{Re}\left(\underline{F}_3 \cdot \underline{v}_3^*\right)$ is the power of the force $\underline{F}_3$ acting at joint 3.

Proof

Adding the results of part (1) and part (2) of the previous lemma:

$$\begin{aligned} M_2 \cdot \omega_2 + (M_1 - M_2) \cdot \omega_1 &= - \mathfrak{Re}\left(\underline{F}_3 \cdot (\underline{v}_3 - \underline{v}_2)^*\right) - \mathfrak{Re}\left((\underline{F}_2 + \underline{F}_3) \cdot \underline{v}_2^*\right) \\ &= - \mathfrak{Re}\left(\underline{F}_3 \cdot (\underline{v}_3 - \underline{v}_2)^* + (\underline{F}_2 + \underline{F}_3) \cdot \underline{v}_2^*\right) \\ &= - \mathfrak{Re}\left(\underline{F}_2 \cdot \underline{v}_2^* + \underline{F}_3 \cdot \underline{v}_3^*\right) \end{aligned}$$

$$(M_1 - M_2) \cdot \omega_1 + M_2 \cdot \omega_2 + \mathfrak{Re}\left(\underline{F}_2 \cdot \underline{v}_2^* + \underline{F}_3 \cdot \underline{v}_3^*\right) = 0$$
$$M_1 \cdot \omega_1 + M_2 \cdot (\omega_2 - \omega_1) + \mathfrak{Re}\left(\underline{F}_2 \cdot \underline{v}_2^*\right) + \mathfrak{Re}\left(\underline{F}_3 \cdot \underline{v}_3^*\right) = 0$$

□

2 Numerical example

$$\begin{aligned} \underline{X}_1 &= 7 + 14\mathfrak{i}\ m, & \underline{X}_2 &= 6 - 2\mathfrak{i}\ m \\ \omega_1 &= 7\ s^{-1}, & \omega_2 &= -5\ s^{-1} \\ \underline{F}_2 &= 5 - 3\mathfrak{i}\ N, & \underline{F}_3 &= -7 - 4\mathfrak{i}\ N \end{aligned}$$

Calculating the moments

$$\begin{aligned} &M_2 = -\mathfrak{Im}\left(\underline{F}_3 \cdot \underline{X}_2^*\right) = -\mathfrak{Im}\left((-7 - 4\mathfrak{i}) \cdot (6 - 2\mathfrak{i})^*\right) = 38\ N \cdot m \\ &M_1 - M_2 = -\mathfrak{Im}\left((\underline{F}_2 + \underline{F}_3) \cdot \underline{X}_1^*\right) = -\mathfrak{Im}\left((-2 - 7\mathfrak{i}) \cdot (7 + 14\mathfrak{i})^*\right) = 21\ N \cdot m \\ &M_1 - M_2 = M_1 - 38\ N \cdot m \Leftrightarrow M_1 = 21 + 38 = 59\ N \cdot m \end{aligned}$$

Calculating the velocities

$$\begin{aligned} &\underline{v}_2 = \mathfrak{i} \cdot \omega_1 \cdot \underline{X}_1 = \mathfrak{i} \cdot 7 \cdot (7 + 15\mathfrak{i}) = -98 + 49\mathfrak{i}\ m \cdot s^{-1} \\ &\underline{v}_3 = \underline{v}_2 + \mathfrak{i} \cdot \omega_2 \cdot \underline{X}_2 = (-98 + 49\mathfrak{i}) + \mathfrak{i} \cdot -5 \cdot (6 - 2\mathfrak{i}) = -108 + 19\mathfrak{i}\ m \cdot s^{-1} \end{aligned}$$

Calculating the power of the forces required at the joints 2 and 3

$$\begin{aligned} &\mathfrak{Re}\left(\underline{F}_2 \cdot \underline{v}_2^*\right) = \mathfrak{Re}\left((5 - 3\mathfrak{i}) \cdot (-98 + 49\mathfrak{i})^*\right) = -637\ N \cdot m \\ &\mathfrak{Re}\left(\underline{F}_3 \cdot \underline{v}_3^*\right) = \mathfrak{Re}\left((-7 - 4\mathfrak{i}) \cdot (-108 + 19\mathfrak{i})^*\right) = 680\ N \cdot m \end{aligned}$$

Verifying the theorem numerically:

$$\begin{aligned} &M_1 \cdot \omega_1 + M_2 \cdot (\omega_2 - \omega_1) + \mathfrak{Re}\left(\underline{F}_2 \cdot \underline{v}_2^*\right) + \mathfrak{Re}\left(\underline{F}_3 \cdot \underline{v}_3^*\right) = \\ &59 \cdot 7 + 38 \cdot (-5 - 7) + (-637) + (680) = 413 - 456 - 637 + 680 = 0 \end{aligned}$$

□

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