ERRATUM TO EXERCISE A4.2 IN "AN INTRODUCTION TO THE THEORY OF THE RIEMANN ZETA FUNCTION" (1988) BY S. J. PATTERSON

RICHARD J. MATHAR

ABSTRACT. The evaluation of coefficients of the Laurent series of $\Gamma(x)$ on page 135 of Patterson's book "An introduction to the theory of the Riemann zeta function" has sign and other errors which are corrected here.

1. LAURENT SERIES OF $\Gamma(z)$

The Laurent series of Euler's Γ-function near the pole at $z = 0$ (with residuum 1) is

$$\Gamma(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots . \tag{1}$$

With the functional equation $z\Gamma(z) = \Gamma(z+1)$ [1, 6.1.15] the coefficients resurface as coefficients of the Taylor series at $z = 1$:

$$\Gamma(z+1) = 1 + a_0 z + a_1 z^2 + a_2 z^3 + a_3 z^4 + \cdots = \sum_{k\ge 0} a_{k-1} z^k \tag{2}$$

and its derivative:

$$\Gamma'(z+1) = \sum_{k\ge 1} k a_{k-1} z^{k-1} \tag{3}$$

with supporting definition

$$a_{-1} \equiv 1. \tag{4}$$

We may start the evaluation of the a_i with the series of the digamma function [1, 6.3.14]

$$\psi(1+z) = -\gamma + \sum_{n\ge 2} (-)^n \zeta(n) z^{n-1} \tag{5}$$

where ζ is Riemann's zeta function and [3, A001620]

$$\gamma \approx 0.577215664\ldots \tag{6}$$

the Euler-Mascheroni constant.

Remark 1. *In numerical practise this is replaced by Chebyshev series* [9, 2, 8, 7].

Date: July 13, 2025.

2020 *Mathematics Subject Classification.* Primary 33B15 Secondary 26A09.

The digamma function is the derivative of the logarithm of the Γ-function [1, 6.3.1]

$$\psi(z)\Gamma(z) = \Gamma'(z). \tag{7}$$

Transition $z \to z+1$

$$\psi(z+1)\Gamma(z+1) = \Gamma'(z+1) \tag{8}$$

and insertion of the three power series shows that a convolution of the coefficients of ψ and Γ gives Γ', essentially again the coefficients of Γ:

$$\left[-\gamma - \sum_{m\ge 1} (-)^m \zeta(m+1) z^m\right] \sum_{k\ge 0} a_{k-1} z^k = \sum_{k\ge 0} (k+1) a_k z^k, \tag{9}$$

which is Jensen's Equation (49') [4]. Equating coefficients of z^0, z^1, z^2 etc. on both sides of the equation yields

$$-\gamma a_{-1} = 1a_0; \tag{10}$$

$$-\gamma a_0 + \zeta(2) a_{-1} = 2a_1; \tag{11}$$

$$-\gamma a_1 + \zeta(2) a_0 - \zeta(3) a_{-1} = 3a_2; \tag{12}$$

$$-\gamma a_2 + \zeta(2) a_1 - \zeta(3) a_0 + \zeta(4) a_{-1} = 4a_3, \tag{13}$$

so the a_i on the right hand sides are recursively computed starting from (4):

Algorithm 1.

$$a_i = \frac{1}{i+1}\left[-\gamma a_{i-1} + \sum_{k=2}^{i+1} (-)^k \zeta(k) a_{i-k}\right]. \tag{14}$$

Example 1. *The first examples are*

$$a_0 = -\gamma \approx -0.5772156649015329; \tag{15}$$

$$a_1 = \frac{1}{2}[\gamma^2 + \zeta(2)] \approx 0.9890559953279726; \tag{16}$$

$$a_2 = \frac{1}{3!}[-\gamma^3 - 3\gamma\zeta(2) - 2\zeta(3)] \approx -0.9074790760808863; \tag{17}$$

$$a_3 = \frac{1}{4!}[\gamma^4 + 6\gamma^2\zeta(2) + 8\gamma\zeta(3) + 3\zeta(2)^2 + 6\zeta(4)] \approx 0.9817280868344002; \tag{18}$$

$$a_4 = \frac{1}{5!}[-\gamma^5 - 10\gamma^3\zeta(2) - 20\gamma^2\zeta(3) - (15\zeta(2)^2 + 30\zeta(4))\gamma - 20\zeta(2)\zeta(3) - 24\zeta(5)] \approx -0.9819950689031452; \tag{19}$$

The main result: On page 135 of Patterson's book [6], there is

- a sign error in front of $\zeta(2)$ in (16),
- a sign error in front of $3\gamma\zeta(2)$ and a missing factor in front of $\zeta(3)$ in (17),
- two sign errors and a wrong factor for $\gamma^2\zeta(2)$ in (18),
- two sign errors and a missing mixed term $\zeta(2)\zeta(3)$ in (19).

2. Reciprocal Γ-function

The expansion coefficients c_k of [1, 6.1.34]

(20)
$$\frac{1}{\Gamma(z)} = \sum_{k\ge 1} c_k z^k = \frac{1}{\frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \cdots} = \frac{z}{1 + a_0 z + a_1 z^2 z + a_2 z^3 + a_3 z^4 + \cdots}$$

are obtained by series reversion [1, 3.6.25], which means comparison of like powers of z of both sides of

(21)
$$\left[\sum_{k\ge 1} c_k z^k\right]\left[\sum_{l\ge 0} a_{l-1} z^l\right] = z,$$

which yields the recurrence

Algorithm 2.

(22)
$$c_m = -\sum_{k=1}^{m-1} c_k a_{m-k-1}, \quad m > 1.$$

Algorithm 3. *This can be simplified to* [5, 5.7.2]

(23)
$$c_m = \frac{1}{m-1}\left[\gamma c_{m-1} - \sum_{i=2}^{m-1} (-)^i \zeta(i) c_{m-i}\right], \quad m > 3.$$

Example 2.

(24) $c_1 = 1;$

(25) $c_2 = \gamma \approx 0.577215664901533;$

(26) $c_3 = \frac{1}{12}[-\pi^2 + 6\gamma^2] \approx -0.655878071520254;$

(27) $c_4 = \frac{1}{12}[4\zeta(3) - \pi^2\gamma + 2\gamma^3] \approx -0.042002635034095;$

(28) $c_5 = \frac{1}{1440}[\pi^4 - 60\pi^2\gamma^2 + 60\gamma^4 + 480\gamma\zeta(3)] \approx 0.166538611382291;$

(29) $c_6 = \frac{1}{1440}[288\zeta(5) - 20\pi^2\gamma^3 + \pi^4\gamma + 12\gamma^5$

(30) $-40\pi^2\zeta(3) + 240\zeta(3)\gamma^2] \approx -0.042197734555544;$

Numerical values have been tabulated up to $k = 40$ by Wrench [10, Table 5]. Splitting off a factor $z + 1$ on the right hand side of (20),

(31)
$$\frac{1}{z\Gamma(z)} = (1+z)\sum_{k\ge 0} b_k z^k,$$

defines coefficients b_k that have been tabulated up to $k = 38$ by Wrench [10, Table 4].

References

1. Milton Abramowitz and Irene A. Stegun (eds.), *Handbook of mathematical functions*, 9th ed., Dover Publications, New York, 1972. MR 0167642
2. W. J. Cody, Anthony J. Strecok, and Henry C. Thacher, Jr., *Chebyshev approximations for the psi function*, Math. Comp. **27** (1973), no. 121, 123–127. MR 0326986

3. O. E. I. S. Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2025), https://oeis.org/. MR 3822822
4. J. L. W. V. Jensen, *An elementary exposition of the theory of the gamma function*, Ann. Math. **17** (1916), no. 3, 124–166. MR 1503576
5. Natl. Inst. Stand. Technol., *Digital library of mathematical functions*, NIST, 2025. MR 1990416
6. S. J. Patterson, *An introduction to the theory of the Riemann zeta function*, Cambridge studies in advanced Mathematics, Cambridge University Press, 1988. MR 0933558
7. Karl Dieter Reinartz, *Chebyshev interpolations of the gamma and polygamma functions and their analytical properties*, arXiv:1605.02776 [math.CA] (2016).
8. Helmut Werner and Robert Collinge, *Chebyshev approximations to the gamma function*, Math. Comp. **15** (1961), no. 74, 195–197.
9. Jet Wimp, *Polynomial approximation to integral transforms*, Math. Comp. **15** (1961), no. 74, 174–178.
10. J. W. Wrench Jr., *Concerning two series for the gamma function*, Math. Comput. **22** (1968), no. 103, 617–626, E: [11]. MR 0237078
11. ———, *Erratum: "concerning two series for the gamma function"*, Math. Comput. **27** (1973), 681–682. MR 0319344

Max-Planck Institute for Astronomy, Königstuhl 17, 69117 Heidelberg, Germany