Proof that angle is not dimensionless quantity

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Abstract

In the current International System of Units (SI), it is conventionally assumed that $\mathrm{rad} = 1$, thereby treating angle as a dimensionless quantity. However, this convention presents a conceptual problem because of the relation $\mathrm{sr} = \mathrm{rad}^2$, so that the fundamental physical distinction between solid angles and plane angles is blurred. In this paper, I revisited the nature of physical equations based on the principle of dimensional homogeneity and the mathematical properties of the dimensional analysis function, demonstrated that angle is a physical quantity possessing its own fundamental dimension which means angle must be treated as one of the base quantities. Furthermore, I showed that, from the perspective of dimensional analysis, the domain of trigonometric functions and the codomain of inverse trigonometric functions—being inherently numerical-value equations—must consist of $\underline{\theta} = \theta/\mathrm{rad}$ when θ means angle. Finally, some well-known equations are reconsidered comprehensively for dimensional homogeneity.

1. Introduction

In the current International System of Units (SI), the unit of plane angle, the radian (rad), is conventionally treated as a dimensionless derived unit, with the definition $\mathrm{rad} = 1$[1]. This convention originates from formulae such as $l = r\theta$ for the arc length l of a circular sector with radius r and central angle θ, or $\nu = \omega/2\pi$ for the frequency ν of a periodic motion in terms of its angular velocity ω. At first glance, $\mathrm{rad} = 1$ seems reasonable because l and ν do not retain the unit rad although θ and ω are measured in rad and $\mathrm{rad/s}$, respectively.

However, $\mathrm{rad} = 1$ results in ambiguities. For instance, while angular velocity ω and frequency ν are clearly distinct in their physical meaning, both quantities share the same unit s^{-1} because of $\mathrm{rad} = 1$, making them indistinguishable in dimensional analysis as both are represented by T^{-1}. Similar issues arise in other physical quantities involving angular measures: luminous intensity in candela (cd) and luminous flux in lumen (lm = cd · sr), radiant flux in watts (W) and radiant intensity in $\mathrm{W/sr}$, where sr is steradian, a unit of solid angle. Each pair is dimensionally equivalent in the present system—e.g., J in the former $\mathsf{ML^2T^{-3}}$ in the latter—, despite having fundamentally different physical interpretations.

In this paper, I firstly revisit the algebraic characteristics of physical quantities and the structure of equations involving them. Secondly, the mathematical properties of dimensional analysis with dimension-extraction function Dim are clarified. Then, I demonstrate that angle is a physical quantity possessing its own intrinsic dimension, denoted as A, following the proposals by Quincey (2021)[2] and Mohr *et al.* (2022)[3]. I then apply dimensional analysis on trigonometric and inverse trigonometric function to show that angles, in certain contexts, must be treated as quantities divided by their units. Finally, I examined several equations for revision to make them hold dimensional homogeneity.

2. Algebraic Characteristics of Physical Quantities

Consider, for example, the case where the mass m of an object is 3 kilograms. This is expressed as:

$$m = 3\ \mathrm{kg}$$

which means that the physical quantity "m" is three times the base unit "kg". In other words, this expression is equivalent to

$$m = 3 \times \mathrm{kg}$$

where the multiplication symbol between the numerical value and the unit is omitted. According to ISO 80000-1 by the International Organization for Standardization (ISO), the numerical part of a quantity Q is denoted by $\{Q\}$, and the unit part by $[Q]$[4]. Thus, Q can be represented as:

$$Q = \{Q\}\,[Q]$$

From this relation, the expression $Q/[Q]$ represents the numerical value $\{Q\}$ of the quantity Q.

3. Quantity Equations and Numerical-Value Equations

Equations involving physical quantities can generally be classified into two categories: *quantity equations* and *numerical-value equations*[4]. The former do not include numerical values of quantities—that is, they are expressed only in terms of physical quantities without explicit units because the numerical value of Q is $Q/[Q]$. By contrast, the latter contain at least more than one unit.

Consider a single physical quantity Q which is just represented in two different units u_1 and u_2, denoted respectively as Q_{u_1} and Q_{u_2}. Let the corresponding numerical values be v_1 and v_2. Then, there exists a functional relationship between these values such that

$$\begin{cases} f(\mathrm{v}_1) = \mathrm{v}_2 \\ g(\mathrm{v}_2) = \mathrm{v}_1 \end{cases}$$

with $g = f^{-1}$, implying that both f and g should be bijective. Among polynomial functions, only linear functions satisfy this condition. Therefore, f may be written as:

$$f(\mathrm{v}_1) = k\mathrm{v}_1 + l$$

Since

$$\begin{cases} \mathrm{v}_2 = \dfrac{Q_{\mathrm{u}_2}}{\mathrm{u}_2} \\ \mathrm{v}_1 = \dfrac{Q_{\mathrm{u}_1}}{\mathrm{u}_1} \end{cases}$$

substituting into the previous expression yields:

$$\frac{Q_{\mathrm{u}_2}}{\mathrm{u}_2} = k\frac{Q_{\mathrm{u}_1}}{\mathrm{u}_1} + l$$

When $l \neq 0$, the equation cannot be simplified further. A well-known example of this type is the conversion between degrees Celsius and kelvin[1]:

$$\frac{T_{{}^\circ\mathrm{C}}}{{}^\circ\mathrm{C}} = \frac{T_\mathrm{K}}{\mathrm{K}} + 273.15$$

However, when $l = 0$, the equation can be rewritten as:

$$\frac{Q_{\mathrm{u}_2}}{Q_{\mathrm{u}_1}} = k\frac{\mathrm{u}_2}{\mathrm{u}_1}$$

Since Q_{u_1} and Q_{u_2} represent the same physical quantity Q, merely expressed in different units, it follows that

$$\frac{Q_{\mathrm{u}_2}}{Q_{\mathrm{u}_1}} = 1$$

and therefore, the common unit conversion formula

$$\mathrm{u}_1 = k\mathrm{u}_2$$

is obtained. The conversion factor k is uniquely determined by the units u_1 and u_2. This means that in numerical-value equation, the form changes depending on the units used, since the conversion factor k or its reciprocal must be explicitly included in the equation.

In contrast, quantity equations inherently include the units in the physical quantities themselves, so their mathematical form remains invariant regardless of the choice of units. For example, in the case of uniform linear motion, the relationship between distance s, speed v, and time t is expressed as:

$$s = vt$$

The form of this equation remains valid regardless of the units used for s, v, or t. Because their units should be substituted together with the numerical values in actual calculation, each different units with the same dimension are eventually canceled.

However, when the same relationship is expressed as a numerical-value equation by dividing each quantity by its unit, for example, $[s] = \mathrm{m}$, $[v] = \mathrm{m/s}$ and $[t] = \mathrm{s}$, we obtain:

$$\frac{s}{\mathrm{m}} = \frac{v}{\mathrm{m/s}} \cdot \frac{t}{\mathrm{s}} \implies \{s\}_{\mathrm{m}} = \{v\}_{\mathrm{m/s}} \cdot \{t\}_{\mathrm{s}}$$

If the speed v is expressed in $\mathrm{km/h}$, then using the conversion $\mathrm{km/h} = \frac{1}{3.6}\ \mathrm{m/s}$, or equivalently $\mathrm{m/s} = 3.6\ \mathrm{km/h}$, the equation becomes:

$$\frac{s}{\mathrm{m}} = \frac{1}{3.6} \cdot \frac{v}{\mathrm{km/h}} \cdot \frac{t}{\mathrm{s}} \implies \{s\}_{\mathrm{m}} = \frac{1}{3.6}\{v\}_{\mathrm{km/h}} \cdot \{t\}_{\mathrm{s}}$$

Comparing the former equation, the latter needs the conversion factor $1/3.6$ in right-hand side, the form of the equation is changed.

4. Properties of the dimension-extraction function Dim for Physical Quantities

Let us consider the function Dim that shows the dimension of a given physical quantity Q. According to the International System of Quantities (ISQ), the dimension of any physical quantity Q is expressed as a product of powers of the seven base quantities (Table 1)[5].

Table 1 Base quantities and their corresponding dimension symbols.

Base quantity	Dimension Symbol
Mass	M
Length	L
Time	T
Electric current	I
Thermodynamic temperature	Θ
Amount of substance	N
Luminous intensity	J

Thus, the dimension of a physical quantity Q can be written as:

$$\mathrm{Dim}(Q) = \mathsf{M}^{\alpha}\mathsf{L}^{\beta}\mathsf{T}^{\gamma}\mathsf{I}^{\delta}\Theta^{\epsilon}\mathsf{N}^{\zeta}\mathsf{J}^{\eta}$$

If all the exponents are zero as a result of calculation, then Q is said to be a dimensionless quantity, and $\mathrm{Dim}(Q) = 1$. Since pure numbers do not possess any physical units, the numerical value $\{Q\}$ is inherently dimensionless:

$$\mathrm{Dim}(\{Q\}) = 1$$

Therefore, the dimension of Q is determined solely by its unit $[Q]$, such that:

$$\mathrm{Dim}(Q) = \mathrm{Dim}([Q])$$

Now, suppose Q is expressed as a combination of products and quotients of other physical quantities p_i, q_j:

$$Q = \prod_{i,j} \frac{p_i}{q_j} = \frac{p_1 p_2 p_3 \cdots}{q_1 q_2 q_3 \cdots}$$

When the Dim is applied to both sides for dimensional analysis, since $p_i = \{p_i\}\,[p_i]$, $q_j = \{q_j\}\,[q_j]$, it follows that:

$$\mathrm{Dim}(Q) = \mathrm{Dim}\left(\frac{p_1 p_2 p_3 \cdots}{q_1 q_2 q_3 \cdots}\right)$$
$$= \frac{\mathrm{Dim}([p_1])\mathrm{Dim}([p_2])\mathrm{Dim}([p_3])\cdots}{\mathrm{Dim}([q_1])\mathrm{Dim}([q_2])\mathrm{Dim}([q_3])\cdots}$$

Indeed, dimensional analysis by Dim is an Abelian group under multiplication[6]; that is, there exists an identity and inverse element, it is closed and satisfies both the associative and commutative laws.

On the other hand, for addition and subtraction operations, all terms in equation or inequation should share the same dimension by the principle of dimensional homogeneity. This is clearly seen in cases such as a force **F** which can be expressed as the sum of its components:

$$\mathbf{F} = \sum_i \mathbf{F}_i = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots$$

Since each term $\mathbf{F}_i$ has the same dimension with $\mathbf{F}$, applying the Dim to the sum yields the same result as applying it to any single term. In this sense, additive or subtractive operations in dimensional analysis are ignored. In practice, applying Dim on the equation above, we have:

$$\mathrm{Dim}(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots) = \mathrm{Dim}([\mathbf{F}_1]) + \mathrm{Dim}([\mathbf{F}_2]) + \mathrm{Dim}([\mathbf{F}_3]) + \cdots = \mathrm{Dim}([\mathbf{F}])$$
$$= \mathsf{MLT}^{-2} + \mathsf{MLT}^{-2} + \mathsf{MLT}^{-2} + \cdots = \mathsf{MLT}^{-2}$$

Conversely, if the dimensions on both sides of a physical equation are not identical, or if any term within the equation possesses a different dimension from the others, then the equation is dimensionally incomplete. Therefore, Dim can be the discriminator for the completeness of dimension in the equations.

Besides pure numbers, unit vectors of physical vector quantities are remarkable example of dimensionless quantities. For a vector quantity $\mathbf{Q}$ that is not itself dimensionless, its unit vector $\widehat{\mathbf{Q}}$ is defined as the normalized vector:

$$\widehat{\mathbf{Q}} = \frac{\mathbf{Q}}{\|\mathbf{Q}\|}$$

Since the magnitude $\|\mathbf{Q}\|$ contains the same unit as that of $\mathbf{Q}$, their ratio has the same dimension as $\{\mathbf{Q}\}$, which is dimensionless. Consequently, $\widehat{\mathbf{Q}}$ is a dimensionless quantity, with a magnitude equal to the pure number 1.

5. Inconsistency Arising from the Relation $\mathbf{sr} = \mathbf{rad}^2$ under $\mathbf{rad} = \mathbf{1}$

A solid angle means the extent of expansion of something having linearity—such as a beam of light—from a point into three-dimensional space. Therefore, the solid angle is independent of the distance from the origin. Suppose a sphere of radius r, and consider light emitted from the center of the sphere penetrates an area A on the spherical surface. In this case, the area A is proportional to the solid angle Ω.

Under the steradian (sr) system, the total solid angle corresponding to the entire space from a point is 4π sr and the whole surface of a sphere is $4\pi r^2$. As the same as the case of plane angle, where an arc length l in a circular sector with radius r and central angle θ satisfies the proportion:

$$l : 2\pi r = \theta : 2\pi \text{ rad} \iff \theta = \frac{l}{r} \text{ rad}$$

the following proportion holds:

$$A : 4\pi r^2 = \Omega : 4\pi \text{ sr} \iff \Omega = \frac{A}{r^2} \text{ sr}$$

In spherical coordinates (r, θ, ϕ), this surface area can be calculated as:

$$A = \iint_S \mathrm{d}A$$
$$= \iint_S r^2 \sin\theta \ \mathrm{d}\theta \, \mathrm{d}\phi$$

Since Ω characterizes the spatial spread independently of r, and the geometry is determined solely by the zenith angle θ $(0 \text{ rad} \le \theta \le \pi \text{ rad})$ and the azimuthal angle ϕ $(0 \text{ rad} \le \phi \le 2\pi \text{ rad})$, Ω can be calculated as:

$$\Omega = \iint_V \sin\theta \ \mathrm{d}\theta \, \mathrm{d}\phi \ \text{ sr}$$

From this formulation, it is clear that while 1 rad is defined as a fixed and unique geometric measure, 1 sr is determined by two independent angular variables θ and ϕ, and therefore cannot correspond to a single specific geometric instance. Fig. 1 illustrates a representative example of a region corresponding to 1 sr.

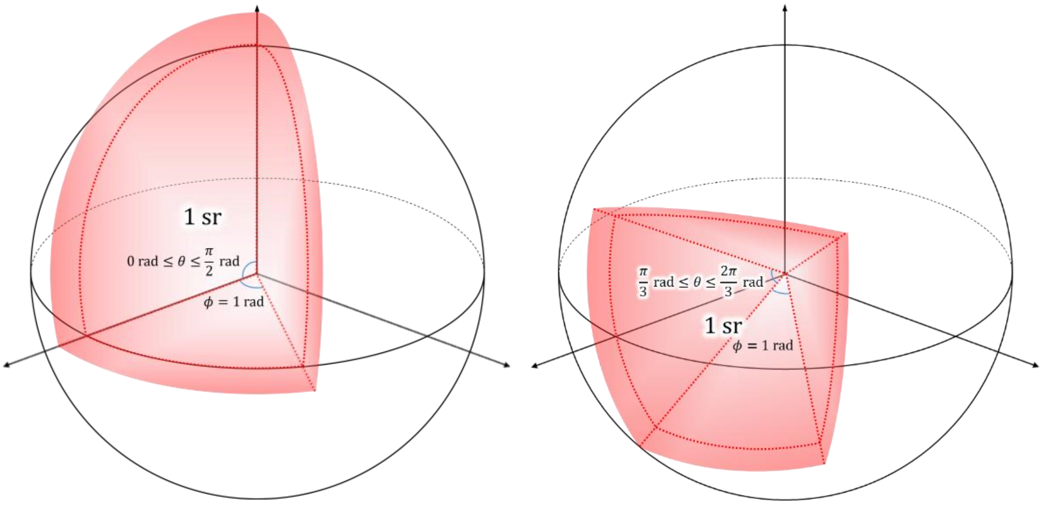

Fig. 1 Examples of 1 sr.

Let us now consider another unit of solid angle, *square degrees* (deg^2). In the system using degrees (°) as angular units, the solid angle is defined as the product of the two angular quantities $\theta_\circ$ and $\phi_\circ$; hence, $^{\circ 2} = \mathrm{deg}^2$. As $\theta_\circ$, $\phi_\circ$ is written as:

$$\begin{cases} \theta = \dfrac{\pi\theta_\circ}{180^\circ} \\ \phi = \dfrac{\pi\phi_\circ}{180^\circ} \end{cases}$$

The integral over the full solid angle becomes:

$$\Omega = \iint_V \sin\left(\frac{\pi\theta_\circ}{180^\circ}\right)\frac{\pi\, \mathrm{d}\theta_\circ}{180^\circ}\frac{\pi\, \mathrm{d}\phi_\circ}{180^\circ}\ \mathrm{sr}$$
$$= \left(\frac{\pi}{180^\circ}\right)^2 \int_{0^\circ}^{180^\circ} \sin\frac{\pi\theta_\circ}{180^\circ}\,\mathrm{d}\theta_\circ \int_{0^\circ}^{360^\circ} \mathrm{d}\phi_\circ\ \mathrm{sr}$$

Using the relation $180^\circ = \pi\ \mathrm{rad}$, we obtain:

$$\Omega = \mathrm{rad}^{-2} \cdot \frac{180^\circ}{\pi} \cdot 2 \cdot 360^\circ\ \mathrm{sr}$$
$$= \frac{360^2}{\pi}\ \mathrm{deg}^2\ \mathrm{sr}/\mathrm{rad}^2$$

As this result corresponds to the total solid angle, $4\pi\ \mathrm{sr}$, it follows that:

$$\frac{360^2}{\pi}\ \mathrm{deg}^2 = 4\pi\ \mathrm{rad}^2$$

Here, deg^2 is a unit of solid angle, so the right-hand side, $4\pi\ \mathrm{rad}^2$, must also represent a solid angle. As numerical value 4π corresponds to the total solid angle, therefore:

$$4\pi\ \mathrm{sr} = 4\pi\ \mathrm{rad}^2 \Longrightarrow \mathrm{sr} = \mathrm{rad}^2$$

Now, if we apply the SI convention that $\mathrm{rad} = 1$, it leads to a contradiction:

$$\mathrm{sr} = \mathrm{rad}^2 = \mathrm{rad} \cdot \mathrm{rad} = \mathrm{rad} \cdot 1 = \mathrm{rad}$$

despite the fact that $1\ \mathrm{sr}$ and $1\ \mathrm{rad}$ are fundamentally different geometrical quantities. Therefore, this contradiction suggests that the definition $\mathrm{rad} = 1$ is conceptually flawed and must be reconsidered.

6. Can the Unit of Angle Be Expressed in Terms of Other SI Base Units?

Consider the formula for the central angle θ of a circular sector:

$$\theta = \frac{l}{r}\ \mathrm{rad}$$

Since this equation includes the unit rad, it is evidently a numerical-value equation, not a quantity equation. That is, the expression l/r represents the numerical value $\{\theta\}$, while the unit part rad is not determined by l or r. Thus, interpreting rad is equal to $\mathrm{m/m}$ in the SI framework is a misunderstanding of the meaning of $\{\theta\}$. Consequently, the area S of a circular sector should be

$$S = \frac{1}{2} r^2 \theta/\mathrm{rad}$$

which is also numerical-value equation.

According to the ISO 80000-1, a base quantity is defined as "a quantity in a conventionally chosen subset of a given system of quantities, where no quantity in the subset can be expressed in terms of the other quantities within that subset[5]."

Let me now examine whether the unit of angle can be expressed as a combination of the other SI base units: kilogram (kg), meter (m), second (s), kelvin (K), ampere (A), mole (mol), and candela (cd). Consider the plane angle θ_{tr} expressed in the unit of one turn (tr). If angle can be rewritten with other quantities, then tr is also replaced by a combination of other units, eventually only a number is remained. For a plane angle θ expressed in radians, the following proportion holds:

$$\theta : 2\pi\ \mathrm{rad} = \theta_{\mathrm{tr}} : 1\ \ \mathrm{tr} \iff \frac{\theta_{\mathrm{tr}}}{\theta} = \frac{1}{2\pi\ \mathrm{rad}}\ \ \mathrm{tr}$$

Here, θ_{tr} and θ represent the same physical quantity—an angle—merely expressed in different units. Therefore, it follows that:

$$\frac{\theta_{\mathrm{tr}}}{\theta} = 1$$

Thus, we obtain the relationship:

$$1\ \mathrm{rad} = \frac{1}{2\pi}\ \mathrm{tr}$$

Since $\mathrm{rad} = r\theta/l$, we get:

$$\frac{r\theta}{l} = \frac{1}{2\pi}\ \mathrm{tr} \iff \theta = \frac{1}{2\pi} \cdot \frac{l}{r}\ \mathrm{tr}$$

Now θ is expressed in tr. This exemplifies the nature of a numerical-value equation: when using the unit tr, the conversion factor $1/2\pi$ must be included explicitly. Furthermore, since tr is not rewritten with a combination of kg, m, s, or other SI base units, implying that angular quantities possess an **intrinsic dimension**.

Following the proposals of Quincey P. (2021)[2] and Mohr P. (2022)[3], I denote the dimension of angle as A. Given the equivalences:

$$2\pi\ \mathrm{rad} = 360° = 1\ \mathrm{tr}$$

then, dimensional analysis for each unit is:

$$\mathrm{Dim}(\mathrm{rad}) = \mathrm{Dim}(°) = \mathrm{Dim}(\mathrm{tr}) = \mathsf{A}$$

Similarly, for the gradian ($^{\mathrm{g}}$), since $400^{\mathrm{g}} = 2\pi\ \mathrm{rad}$, it also follows that $\mathrm{Dim}(^{\mathrm{g}}) = \mathsf{A}$.

From a perspective of dimensional analysis, the conventional form $\theta = l/r$ is dimensionally inconsistent. That is:

$$\mathrm{Dim}(\theta) = \mathsf{A}$$

$$\mathrm{Dim}\left(\frac{l}{r}\right) = \frac{\mathrm{Dim}(l)}{\mathrm{Dim}(r)} = \frac{\mathsf{L}}{\mathsf{L}} = 1$$

where the left-hand and right-hand sides of the equation have different dimensions. A dimensionally consistent formula would be with the numerical value $\{\theta\} = \theta/\mathrm{rad}$ as:

$$\{\theta\} = \theta/\mathrm{rad} = \frac{l}{r}$$

In the same way, the equation for the solid angle:

$$\{\Omega\} = \Omega/\mathrm{sr} = \frac{A}{r^2}$$

is also a numerical-value equation. Since $\mathrm{sr} = \mathrm{rad}^2$, any unit of solid angle cannot be expressed as a combination of other SI base units either. On the dimension of units for solid angle,

$$\mathrm{Dim}(\mathrm{sr}) = \mathrm{Dim}(\mathrm{rad}^2) = \mathrm{Dim}(\mathrm{deg}^2) = \mathsf{A}^2$$

Therefore, the solid angle is a quantity of the dimension A^2

As a side note, the dimensionless units, such as the percent (%), can be expressed as pure number in the same manner above that the dimension is 1. Given two physical quantities Q and Q_0, let r be a ratio of Q to Q_0, and $r_{\%}$ is a ratio expressed in %. Then,

$$\begin{aligned} r_{\%} &= \frac{Q}{Q_0} \times 100\ \% \\ &= 100r\ \% \end{aligned}$$

and further,

$$\frac{r_{\%}}{r} = 100\ \%$$

Here, as r and $r_{\%}$ refer to the same physical quantity, $r_{\%}/r = 1$, and rearranging this expression yields

$$\% = \frac{1}{100}$$

Therefore, $\mathrm{Dim}(\%) = 1$.

7. Reconsideration of Several Equations Involving Angular Quantities

7-1. Dimensional Analysis on Trigonometric and Inverse Trigonometric Functions

At the elementary level, trigonometric functions are defined as ratios of two sides of a right triangle, making them pure numbers without any physical units—that is, they are dimensionless quantities. This fact can be rigorously confirmed by applying dimensional analysis to their infinite series representations. For instance, the sine function is expressed as:

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned}$$

Applying Dim to both sides, we obtain:

$$\mathrm{Dim}(\sin x) = 1$$

and

$$\begin{aligned} &\mathrm{Dim}\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= \mathrm{Dim}(x) + \{\mathrm{Dim}(x)\}^3 + \{\mathrm{Dim}(x)\}^5 + \{\mathrm{Dim}(x)\}^7 + \cdots \end{aligned}$$

This implies that $\mathrm{Dim}(x) = 1$ by the dimensional homogeneity. Therefore, x must be a dimensionless quantity. However, since angle θ inherently possess its own dimension A, does not satisfy this condition unless it is normalized. The simplest consistent form of such normalization is:

$$x = \{\theta\}$$

Similarly, inverse trigonometric functions can be treated via infinite series. For example, the arcsine function is given by:

$$\begin{aligned} \arcsin z &= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!! \cdot (2n+1)} z^{2n+1} \\ &= z + \frac{1}{6}z^3 + \frac{3}{40}z^5 + \frac{5}{112}z^7 + \cdots \end{aligned}$$

For clarifying the meaning of z, consider a circle of radius r centered at the origin in a Cartesian coordinate. For a point (x, y) on the circumference satisfies:

$$\sin x = \frac{y}{r} \iff x = \arcsin\frac{y}{r}$$

Since $z = y/r$, which is a pure number, $\mathrm{Dim}(z) = 1$.

It is important to note that these series expansions above are valid under a unit system in which the following limit holds:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\,(\cos h - 1) + \cos x \sin h}{h}$$
$$= \lim_{h\to 0}\left(\frac{\sin h}{h}\cos x - \frac{\sin h}{h}\cdot\frac{\sin h}{\cos h + 1}\sin x\right)$$
$$= \cos x$$

which requires

$$\lim_{h\to 0}\frac{\sin h}{h} = 1$$

The value of limit above is only valid when h is a numerical value of angles in radians. This identity can be derived from geometric inequalities for areas of isosceles triangle, sector, and right triangle (Fig. 2):

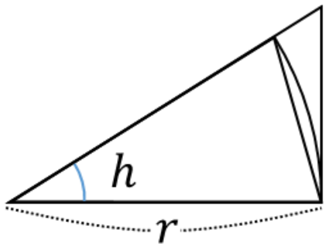

Fig. 2 Geometric inequalities for areas isosceles triangle, sector, right triangle.

$$\sin\{h\} < \{h\} < \tan\{h\}$$

where $\{h\}$ is a numerical value of angles in radians.

However, if we allow the domain of trigonometric functions can be a numerical value of angles in other units than radians, for instance, in degrees denoted as $\{h_\circ\}$, this inequality becomes:

$$\sin\{h_\circ\} < \frac{\pi}{180}\{h_\circ\} < \tan\{h_\circ\}$$

Therefore, the value of limit become different as:

$$\lim_{\{h_\circ\}\to 0}\frac{\sin\{h_\circ\}}{\{h_\circ\}} = \frac{\pi}{180}$$

And by the definition of Taylor series, $\sin\{h_\circ\}$ and $\{h_\circ\} = \arcsin x$ would be:

$$\sin\{h_\circ\} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{180}\{h_\circ\}\right)^{2n+1}$$

$$\arcsin x = \{h_\circ\} = \frac{180}{\pi}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!\cdot(2n+1)}x^{2n+1}$$

These show that trigonometric functions are defined only for the numerical value of angles expressed in radians. Similarly, the values of inverse trigonometric functions necessarily converge to the numerical values of angles in radians. This is essential for the mathematical requirement of functions that one element in the domain must correspond to one element in the codomain. For example, the value of $\arcsin 1$ should always be $\pi/2$, not 90 (numerical value in degrees), 100 (numerical value in gradians) and so on. Therefore, if the output x of $\arcsin z$ is interpreted as angle, then x must be expressed as:

$$\arcsin z = x = \{\theta\} = \frac{\theta}{\text{rad}}$$

To convert to units other than radians (e.g., degrees), using $\text{rad} = (180°)/\pi$,

$$\theta = (\arcsin z)\,\text{rad} = \frac{180°}{\pi}\arcsin z$$
$$= \frac{180°}{\pi}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!\cdot(2n+1)}z^{2n+1}$$

In summary, the domain of trigonometric functions should be expressed not as:

$$\sin\theta \quad \cos\theta \quad \tan\theta$$

but rather:

$$\sin\{\theta\} \quad \cos\{\theta\} \quad \tan\{\theta\}$$

The notation above is the clear answer for a question that the unit "rad" is not explicitly used in trigonometric functions despite being introduced as a unit for angle system.

Since curly brackets may be easily confused with ordinary parentheses and fraction expression θ/rad is also cumbersome, so that I propose an alternative simple notation for numerical value of angle as:

$$\begin{cases} \dfrac{\theta}{\mathrm{rad}} = \underline{\theta} \\ \dfrac{\Omega}{\mathrm{sr}} = \dfrac{\Omega}{\mathrm{rad}^2} = \underline{\underline{\Omega}} \end{cases}$$

Accordingly, the integral for the total solid angle subtended by a full sphere, discussed earlier, should be rewritten as:

$$\Omega = \iint_V \frac{\mathrm{d}A}{r^2}\ \mathrm{sr}$$
$$= \iint_V \sin\underline{\theta}\ \mathrm{d}\underline{\theta}\,\mathrm{d}\underline{\phi}\ \mathrm{sr}$$
$$= 4\pi\ \mathrm{sr}$$

When angular units other than rad, such as degrees (°) or gradians ($^{\mathrm{g}}$), are used, the trigonometric expressions must reflect the change in unit. For example:

$$\sin\frac{\pi\theta_{\circ}}{180°} \quad \cos\frac{\pi\theta_{\mathrm{g}}}{200^{\mathrm{g}}} \quad \tan\frac{2\pi\theta_{\mathrm{tr}}}{\mathrm{tr}}$$

where $\theta_{\circ}$, θ_{g} and θ_{tr} are angles in degree, gradian and turn, respectively. If we introduce the underline notation such as $\theta_{\circ}/° = \underline{\theta_{\circ}}$, then they can be expressed more simply as:

$$\sin\frac{\pi\underline{\underline{\theta_{\circ}}}}{180} \quad \cos\frac{\pi\underline{\underline{\theta_{\mathrm{g}}}}}{200} \quad \tan 2\pi\underline{\underline{\theta_{\mathrm{tr}}}}$$

These forms show a key characteristic of numerical-value equations: the form of the equation varies depending on the units used. For example, when differentiating the sine function with respect to an angle measured in degrees, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\underline{\theta_{\circ}}}\sin\frac{\pi\underline{\underline{\theta_{\circ}}}}{180} = \frac{\pi}{180}\cos\frac{\pi\underline{\underline{\theta_{\circ}}}}{180}$$

which differs in form from the commonly known derivative:

$$\frac{\mathrm{d}}{\mathrm{d}\underline{\theta}}\sin\underline{\theta} = \cos\underline{\theta}$$

Additionally, the complex exponential representations of trigonometric functions are:

$$\sin\underline{\theta} = \frac{e^{i\underline{\theta}} - e^{-i\underline{\theta}}}{2i} \qquad \cos\underline{\theta} = \frac{e^{i\underline{\theta}} + e^{-i\underline{\theta}}}{2}$$

From these, the relations $\sin(i\underline{\theta}) = i\sinh\underline{\theta}$, $\cos i\underline{\theta} = \cosh\underline{\theta}$ are derived. Mohr *et al.* (2022) has suggested that the hyperbolic functions might be similarly required normalized arguments[3]. However, the geometrical meaning of the imaginary angle $i\theta$ in trigonometric or just θ in hyperbolic geometry remains unclear. In practice, this consideration does not go beyond the requirement that both the domain and the codomain of hyperbolic functions be pure numbers. For example, consider the hyperbola $C: x^2 - y^2 = r^2$, and the line $y = (\tanh a)x$. The area between this line, the curve, and the y-axis is given by $r^2 a$. By dimensional analysis:

$$\mathrm{Dim}(r^2 a) = \{\mathrm{Dim}(r)\}^2 \mathrm{Dim}(a) = \mathsf{L}^2 \mathrm{Dim}(a) = \mathsf{L}^2$$

Thus, $\mathrm{Dim}(a) = 1$, which means the domain of hyperbolic functions must be dimensionless, but not such a numerical value of angle. Lastly, note that the hyperbolic functions are ultimately derived from exponential functions as follows:

$$e^x = \sinh x + \cosh x$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The variable x can be any of dimensionless quantity, not limited to angles. Consequently, there is no evident physical reason to adopt the normalized angle $\underline{\theta} = \theta/\mathrm{rad}$ for hyperbolic or inverse hyperbolic functions.

7-2. Infinitesimal Rotations in Three-Dimensional Space

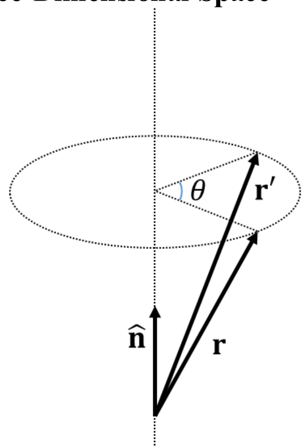

Fig. 3 Rotation of $\mathbf{r}$ to $\mathbf{r}'$ around the axis $\hat{\mathbf{n}}$.

Let us consider a vector $\mathbf{r}$ that undergoes a rotation by an angle θ around an axis defined by a unit vector $\hat{\mathbf{n}}$, resulting in a new vector $\mathbf{r}'$. According to Rodrigues' rotation formula[7–10], $\mathbf{r}'$ is expressed as:

$$\mathbf{r}' = (\cos\underline{\theta})\mathbf{r} + (1 - \cos\underline{\theta})(\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\sin\underline{\theta})\hat{\mathbf{n}}\times\mathbf{r}$$
$$= \mathbf{r} + (1 - \cos\underline{\theta})\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{r}) + (\sin\underline{\theta})\hat{\mathbf{n}}\times\mathbf{r}$$

To investigate the infinitesimal arc vector $\mathbf{dl}$ corresponding to an infinitesimal rotation $\mathrm{d}\theta$, let $\Delta\mathbf{r}$ is the displacement vector $\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$, and take the total derivative of $\Delta\mathbf{r}(\underline{\theta})$ with respect to $\underline{\theta}$, then consider the limit of $\underline{\theta} \to 0$:

$$\mathbf{dl} = \lim_{\underline{\theta}\to 0} \mathrm{d}\{\Delta\mathbf{r}(\underline{\theta})\}$$
$$= \lim_{\underline{\theta}\to 0}\{(\sin\underline{\theta}\ \mathrm{d}\underline{\theta})\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{r}) + (\cos\underline{\theta}\ \mathrm{d}\underline{\theta})\hat{\mathbf{n}}\times\mathbf{r}\}$$
$$= \mathrm{d}\underline{\theta}\ \hat{\mathbf{n}}\times\mathbf{r}$$
$$= \mathrm{d}\theta\ \hat{\mathbf{n}}\times\mathbf{r}/\mathrm{rad}$$

This result can also be derived using the infinite series for small $\underline{\theta}$:

$$1 - \cos\mathrm{d}\underline{\theta} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\mathrm{d}\underline{\theta})^{2n}}{(2n)!} = \frac{(\mathrm{d}\underline{\theta})^2}{2!} - \frac{(\mathrm{d}\underline{\theta})^4}{4!} + \frac{(\mathrm{d}\underline{\theta})^6}{6!} - \frac{(\mathrm{d}\underline{\theta})^8}{8!} + \cdots \approx 0$$
$$\sin\mathrm{d}\underline{\theta} = \sum_{n=0}^{\infty}\frac{(-1)^{n}(\mathrm{d}\underline{\theta})^{2n+1}}{(2n+1)!} = \frac{(\mathrm{d}\underline{\theta})}{1!} - \frac{(\mathrm{d}\underline{\theta})^3}{3!} + \frac{(\mathrm{d}\underline{\theta})^5}{5!} - \frac{(\mathrm{d}\underline{\theta})^7}{7!} + \cdots \approx \mathrm{d}\underline{\theta}$$

Here, since $\hat{\mathbf{n}}$ is a unit vector—a dimensionless quantity—in the direction of the rotation axis, it can be treated as defining the direction of the infinitesimal rotation vector. Denoting this infinitesimal rotation as $\mathrm{d}\theta\ \hat{\mathbf{n}} = \mathrm{d}\boldsymbol{\theta}$, we obtain:

$$\mathbf{dl} = \mathrm{d}\underline{\boldsymbol{\theta}}\times\mathbf{r} = \mathrm{d}\boldsymbol{\theta}\times\mathbf{r}/\mathrm{rad}$$

Applying dimensional analysis to this expression:

$$\mathrm{Dim}(\mathbf{dl}) = \mathrm{Dim}(\mathrm{d}\boldsymbol{\theta}\times\mathbf{r}/\mathrm{rad})$$
$$= \frac{\mathrm{Dim}(\mathrm{d}\boldsymbol{\theta})\cdot\mathrm{Dim}(\mathbf{r})}{\mathrm{Dim}(\mathrm{rad})}$$
$$\mathsf{L} = \frac{\mathsf{A}\cdot\mathsf{L}}{\mathsf{A}} = \mathsf{L}$$

confirming that both sides have the same dimension. Note that this is a numerical-value equation, as it explicitly contains the unit rad.

Consequently, the tangential velocity $\mathbf{v}$ of a rotational motion with constant radius is given by:

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t} = \frac{\mathrm{d}\underline{\boldsymbol{\theta}}}{\mathrm{d}t} \times \mathbf{r} = \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} \times \mathbf{r}/\mathrm{rad}$$
$$= \boldsymbol{\omega} \times \mathbf{r}/\mathrm{rad}$$

Letting $\boldsymbol{\omega}/\mathrm{rad} = \underline{\boldsymbol{\omega}}$, we obtain the simplified form:

$$\mathbf{v} = \underline{\boldsymbol{\omega}} \times \mathbf{r}$$

Similarly, the acceleration $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$ becomes:

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega} \times \mathbf{r})/\mathrm{rad}$$
$$= \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)/\mathrm{rad}$$
$$= (\boldsymbol{\alpha} \times \mathbf{r})/\mathrm{rad} + \underline{\boldsymbol{\omega}} \times \left(\underline{\boldsymbol{\omega}} \times \mathbf{r}\right)$$

Defining $\underline{\boldsymbol{\alpha}} = \boldsymbol{\alpha}/\mathrm{rad}$, we obtain:

$$\mathbf{a} = \underline{\boldsymbol{\alpha}} \times \mathbf{r} + \underline{\boldsymbol{\omega}} \times \left(\underline{\boldsymbol{\omega}} \times \mathbf{r}\right)$$

Using the vector triple product identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$ in the second term, then:

$$\mathbf{a} = \underline{\boldsymbol{\alpha}} \times \mathbf{r} + \left\|\underline{\boldsymbol{\omega}}\right\|^2 \mathbf{r}$$
$$= (\boldsymbol{\alpha} \times \mathbf{r}/\mathrm{rad}) + \|\boldsymbol{\omega}\|^2 \mathbf{r}/\mathrm{rad}^2$$

where the first term represents tangential acceleration $\mathbf{a_t}$ and the second term represents centripetal acceleration $\mathbf{a_c}$. All of these equations explicitly include the unit rad and therefore represent numerical-value equations rather than quantity equations.

7-3. Rotational Motion of a Rigid Body

7-3-1. On the Torque

Consider a situation in which an external force $\mathbf{F}$ is applied to a rigid body, causing it to rotate by an infinitesimal displacement $\mathrm{d}\mathbf{l}$ around a certain axis. The resulting infinitesimal work $\mathrm{d}W$ is expressed as:

$$\mathrm{d}W = \mathbf{F} \cdot \mathrm{d}\mathbf{l}$$

Since $\mathrm{d}\mathbf{l} = \mathrm{d}\underline{\boldsymbol{\theta}} \times \mathbf{r}$, this becomes:

$$\mathrm{d}W = \mathbf{F} \cdot \left(\mathrm{d}\underline{\boldsymbol{\theta}} \times \mathbf{r}\right)$$

Using the identity for mixed scalar and vector products, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$, this expression can be rewritten as:

$$\mathrm{d}W = \mathrm{d}\underline{\boldsymbol{\theta}} \cdot (\mathbf{r} \times \mathbf{F}) = (\mathbf{r} \times \mathbf{F}) \cdot \mathrm{d}\underline{\boldsymbol{\theta}}$$
$$= (\mathbf{r} \times \mathbf{F}) \cdot \mathrm{d}\boldsymbol{\theta}/\mathrm{rad}$$

This formula clearly shows that rotational work involving the angle $\boldsymbol{\theta}$ should be treated as a numerical-value equation, due to the explicit presence of the unit rad.

Torque $\boldsymbol{\tau}$ is defined as the quantity obtained by differentiating rotational work W with respect to angular displacement $\boldsymbol{\theta}$. Thus, the torque is given by:

$$\boldsymbol{\tau} = \frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{\theta}} = \mathbf{r} \times \mathbf{F}/\mathrm{rad}$$

This shows that torque, as conventionally written as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, actually omits the unit and should correctly be divided by rad in the right-hand side. This derivation confirms that torque is also a numerical-value equation.

A dimensional analysis yields:

$$\mathrm{Dim}(\boldsymbol{\tau}) = \mathrm{Dim}(\mathbf{r} \times \mathbf{F}/\mathrm{rad})$$
$$= \frac{\mathrm{Dim}(\mathbf{r}) \cdot \mathrm{Dim}(\mathbf{F})}{\mathrm{Dim}(\mathrm{rad})}$$
$$= \frac{\mathsf{L} \cdot \mathsf{MLT}^{-2}}{\mathsf{A}}$$

$$= \mathsf{ML^2T^{-2}A^{-1}}$$

Thus, torque has dimensions distinct from that of energy ($\mathsf{ML^2T^{-2}}$), and it is more accurate to express the unit of torque as $\mathrm{J/rad} = \mathrm{N\cdot m/rad}$ rather than $\mathrm{N\cdot m}$.

7-3-2. On the Angular Momentum

Since force is given by $\mathbf{F} = \mathrm{d}\mathbf{p}/\mathrm{d}t$, and using the property of the cross product that $(\mathrm{d}\mathbf{r}/\mathrm{d}t)\times\mathbf{p} = \mathbf{v}\times\mathbf{p} = \mathbf{v}\times m\mathbf{v} = \mathbf{0}\ \mathrm{kg\cdot m^2/s^2}$, the torque becomes:

$$\begin{aligned}\boldsymbol{\tau} &= \mathbf{r}\times\mathbf{F}/\mathrm{rad}\\ &= \mathbf{r}\times\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}/\mathrm{rad}\\ &= \left(\mathbf{r}\times\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}+\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\mathbf{p}\right)/\mathrm{rad}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\times\mathbf{p})/\mathrm{rad}\end{aligned}$$

This shows that torque is the time derivative of angular momentum. Integrating both sides with respect to time yields the angular momentum

$$\mathbf{L} = \int \boldsymbol{\tau}\,\mathrm{d}t = \mathbf{r}\times\mathbf{p}/\mathrm{rad}$$

Therefore, the conventional expression $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ must also be divided by rad on the right-hand side. Angular momentum is also a numerical-value equation, given its explicit dependence on rad.

By dimensional analysis:

$$\begin{aligned}\mathrm{Dim}(\mathbf{L}) &= \mathrm{Dim}(\mathbf{r}\times\mathbf{p}/\mathrm{rad})\\ &= \frac{\mathrm{Dim}(\mathbf{r})\cdot\mathrm{Dim}(\mathbf{p})}{\mathrm{Dim}(\mathrm{rad})}\\ &= \frac{\mathsf{L\cdot MLT^{-1}}}{\mathsf{A}}\\ &= \mathsf{ML^2T^{-1}A^{-1}}\end{aligned}$$

This confirms that angular momentum possesses a dimension of angle. Since $\mathbf{p} = m\mathbf{v} = m\underline{\boldsymbol{\omega}}\times\mathbf{r}$ and if $\underline{\boldsymbol{\omega}}$ is perpendicular to $\mathbf{r}$, i.e., $(\mathbf{r}\cdot\underline{\boldsymbol{\omega}}) = 0\ \mathrm{m/s}$, substituting this into the formula yields

$$\begin{aligned}\mathbf{L} &= \mathbf{r}\times\mathbf{p}/\mathrm{rad}\\ &= \mathbf{r}\times(m\underline{\boldsymbol{\omega}}\times\mathbf{r})/\mathrm{rad}\\ &= m\|\mathbf{r}\|^2\underline{\boldsymbol{\omega}}/\mathrm{rad}\\ &= m\|\mathbf{r}\|^2\boldsymbol{\omega}/\mathrm{rad}^2\end{aligned}$$

indicating that the commonly known formula $\mathbf{L} = m\|\mathbf{r}\|^2\boldsymbol{\omega}$ must, in fact, be divided by rad^2 on the right-hand side.

Dimensional analysis supports this revision, as

$$\begin{aligned}\mathrm{Dim}(m\|\mathbf{r}\|^2\boldsymbol{\omega}/\mathrm{rad}^2) &= \frac{\mathrm{Dim}(m)\mathrm{Dim}(\|\mathbf{r}\|^2)\mathrm{Dim}(\boldsymbol{\omega})}{\mathrm{Dim}(\mathrm{rad}^2)}\\ &= \frac{\mathsf{M\cdot L^2\cdot AT^{-1}}}{\mathsf{A^2}}\\ &= \mathsf{ML^2T^{-1}A^{-1}}\end{aligned}$$

thus confirming consistency with the expected dimensions.

7-3-3. On the Moment of Inertia

As discussed in the earlier section *Infinitesimal Rotation in Three-Dimensional Space*, the acceleration of a point undergoing rotational motion is given by:

$$\mathbf{a} = \underline{\boldsymbol{\alpha}}\times\mathbf{r} + \|\underline{\boldsymbol{\omega}}\|^2\mathbf{r}$$

Substituting this into the expression for torque yields:

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}/\mathrm{rad}$$

$$= \mathbf{r} \times m\mathbf{a}/\text{rad}$$
$$= \mathbf{r} \times m\left(\underline{\boldsymbol{\alpha}} \times \mathbf{r} + \|\underline{\boldsymbol{\omega}}\|^2 \mathbf{r}\right)/\text{rad}$$
$$= m\mathbf{r} \times \left(\underline{\boldsymbol{\alpha}} \times \mathbf{r}\right)/\text{rad} + m\mathbf{r} \times \left(\|\underline{\boldsymbol{\omega}}\|^2 \mathbf{r}\right)/\text{rad}$$

Using the vector identity $\mathbf{r} \times (\underline{\boldsymbol{\alpha}} \times \mathbf{r}) = \|\mathbf{r}\|^2 \underline{\boldsymbol{\alpha}} - (\mathbf{r} \cdot \underline{\boldsymbol{\alpha}})\mathbf{r}$, and noting that $\mathbf{r} \times \mathbf{r} = \mathbf{0}\ \text{m}^2$, the second term vanishes. If the rotation axis is perpendicular to $\mathbf{r}$, i.e., $(\mathbf{r} \cdot \underline{\boldsymbol{\alpha}}) = 0\ \text{m/s}^2$, then the expression simplifies to:

$$\boldsymbol{\tau} = m\|\mathbf{r}\|^2 \underline{\boldsymbol{\alpha}}/\text{rad}$$
$$= m\|\mathbf{r}\|^2 \boldsymbol{\alpha}/\text{rad}^2$$
$$= (m\|\mathbf{r}\|^2/\text{rad}^2)\boldsymbol{\alpha}$$

This leads to the expression:

$$\boldsymbol{\tau} = I\boldsymbol{\alpha}$$

where the moment of inertia I is given by:

$$I = m\|\mathbf{r}\|^2/\text{rad}^2$$

This result indicates that the moment of inertia, as well as torque and angular momentum, must also be treated as a numerical-value equation; that is, the commonly used formula $I = m\|\mathbf{r}\|^2$ must be divided by rad^2 in the right-hand side.

7-4. Periodic Quantities and Dimensional Consistency of Angular Units

Consider a system undergoing periodic motion with angular velocity ω. Since one full rotation corresponds to an angle of 2π rad, the period T of this motion is given by:

$$T = \frac{2\pi\ \text{rad}}{\omega}$$

Accordingly, the frequency ν, defined as the reciprocal of T, is:

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi\ \text{rad}} = \frac{\underline{\omega}}{2\pi}$$

This indicates that in the commonly cited formula $\nu = \omega/2\pi$, the right-hand side must be divided by rad to yield a dimensionally consistent expression.

Likewise, the wavenumber $\tilde{\nu}$ is defined as the reciprocal of the wavelength λ, and the angular wavenumber k, commonly used in physics, is defined via the identity:

$$k = 2\pi\tilde{\nu}\ \text{rad} = \frac{2\pi\ \text{rad}}{\lambda}$$

From this, we can define the angular wavelength $\bar{\lambda}$, as the reciprocal of k:

$$\bar{\lambda} = \frac{1}{k} = \frac{\lambda}{2\pi\ \text{rad}}$$

In a similar manner, the energy E of a quantum can be expressed as:

$$E = h\nu = \frac{h}{2\pi\ \text{rad}} \cdot 2\pi\nu\ \text{rad}$$
$$= \hbar\omega$$

Since $\omega = 2\pi\nu$ rad, this leads to a natural definition of the reduced Planck constant $\hbar$ as:

$$\hbar = \frac{h}{2\pi\ \text{rad}}$$

This definition ensures consistency with quantum mechanical expressions, such as total angular momentum $\mathbf{J}$, which is expressed using the dimensionless azimuthal quantum number l as[11]:

$$\|\mathbf{J}\| = \sqrt{l(l+1)}\hbar$$

From the perspective of dimensional analysis, this also maintains internal consistency; $\text{Dim}(\mathbf{J}) = \mathsf{ML^2T^{-1}A^{-1}}$, which is the same as that of $h/(2\pi\ \text{rad})$:

$$\text{Dim}(\hbar) = \frac{\text{Dim}(h)}{\text{Dim}(2\pi\ \text{rad})}$$
$$= \frac{\mathsf{ML^2T^{-1}}}{\mathsf{A}} = \mathsf{ML^2T^{-1}A^{-1}}$$

Therefore, we can distinguish $\hbar$ from h dimensionally. This suggests that the commonly cited definition $\hbar = h/2\pi$ should be corrected to include the angular unit. The notation $\check{h} = \hbar\,\mathrm{rad} = h/2\pi$, as suggested by Quincey (2021)[2], may be introduced to distinguish $\hbar$ when necessary. This correction is essential, for example, for the uncertainty principle:

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$$

Dimensionally, the left-hand side is:

$$\mathrm{Dim}(\hat{x}\hat{p} - \hat{p}\hat{x}) = \mathrm{Dim}(\hat{x}\hat{p})$$
$$= \mathsf{L} \cdot \mathsf{MLT^{-1}} = \mathsf{ML^{2}T^{-1}}$$

Thus, the right-hand side must also have this dimension. Under the definition of $\hbar = h/(2\pi\,\mathrm{rad})$ the left-hand side lacks the angular unit and hence does not match the dimension. Therefore, it is more appropriate to express the relation as:

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\check{h}$$

where $\check{h} = \hbar\,\mathrm{rad} = h/2\pi$, or

$$(\hat{x}\hat{p} - \hat{p}\hat{x})/\mathrm{rad} = i\hbar$$

7-5. On the Planck's Law

Planck's law describes the energy density $u_\nu(\nu, T)\,\mathrm{d}\nu$ of blackbody radiation at frequency ν and absolute temperature T as[12,13]:

$$u_\nu(\nu, T)\,\mathrm{d}\nu = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}} - 1}\,\mathrm{d}\nu$$

Performing dimensional analysis:

$$\begin{cases} \mathrm{Dim}(h) = \mathsf{ML^{2}T^{-1}} \\ \mathrm{Dim}\left(\frac{\nu}{c}\right) = \mathsf{L^{-1}} \\ \mathrm{Dim}(h\nu) = \mathrm{Dim}(k_\mathrm{B}T) = \mathsf{ML^{2}T^{-2}} \\ \mathrm{Dim}(\mathrm{d}\nu) = \mathsf{T^{-1}} \end{cases}$$

we obtain:

$$\mathrm{Dim}(u_\nu(\nu, T)\,\mathrm{d}\nu) = \mathsf{ML^{2}T^{-1}} \cdot \mathsf{L^{-3}} \cdot 1 \cdot \mathsf{T^{-1}} = \frac{\mathsf{ML^{2}T^{-2}}}{\mathsf{L^{3}}}$$

which is indeed the dimension of energy density.

To derive radiance, we first consider energy $\mathrm{d}E_\nu(\nu, T)\,\mathrm{d}\nu = u_\nu(\nu, T)\,\mathrm{d}\nu\,\mathrm{d}V$ on a infinitesimal volume $\mathrm{d}V = \mathrm{d}A\,\mathrm{d}r$. Using $\mathrm{d}r = c\,\mathrm{d}t$, we have:

$$\mathrm{d}V = c\,\mathrm{d}A\,\mathrm{d}t$$

Substituting this into the energy expression yields:

$$\mathrm{d}E_\nu(\nu, T)\,\mathrm{d}\nu = u_\nu(\nu, T)\,\mathrm{d}\nu\,\mathrm{d}V = \frac{8\pi h\nu^3}{c^2} \cdot \frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}} - 1}\,\mathrm{d}\nu\,\mathrm{d}A\,\mathrm{d}t$$

Dividing both sides by $\mathrm{d}t$, we obtain the infinitesimal radiant flux $\mathrm{d}\Phi_{\mathrm{e}_\nu}(\nu, T)\,\mathrm{d}\nu$:

$$\mathrm{d}\Phi_{\mathrm{e}_\nu}(\nu, T)\,\mathrm{d}\nu = \frac{\mathrm{d}E_\nu(\nu, T)}{\mathrm{d}t}\,\mathrm{d}\nu = \frac{8\pi h\nu^3}{c^2} \cdot \frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}} - 1}\,\mathrm{d}\nu\,\mathrm{d}A$$

Further dividing both side by $\mathrm{d}A$ yields radiant intensity $I_{\mathrm{e}_\nu}(\nu, T)\,\mathrm{d}\nu$:

$$I_{\mathrm{e}_\nu}(\nu, T)\,\mathrm{d}\nu = \frac{\mathrm{d}\Phi_{\mathrm{e}_\nu}(\nu, T)}{\mathrm{d}A}\,\mathrm{d}\nu = \frac{8\pi h\nu^3}{c^2} \cdot \frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}} - 1}\,\mathrm{d}\nu$$

Finally, since radiation from a blackbody is emitted over the entire solid angle of 4π sr, we obtain the radiance (radiant intensity per unit solid angle) $B_\nu(\nu, T)\,\mathrm{d}\nu$ by dividing $I_{\mathrm{e}_\nu}(\nu, T)\,\mathrm{d}\nu$ by 4π sr:

$$B_\nu(\nu, T)\,\mathrm{d}\nu = \frac{I_{\mathrm{e}_\nu}(\nu, T)}{4\pi\,\mathrm{sr}}\,\mathrm{d}\nu = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}} - 1}\,\mathrm{sr}^{-1}\,\mathrm{d}\nu$$

This suggests that the conventional equations of radiance must include sr^{-1} in the right-hand side.

By dimensional analysis:

$$\mathrm{Dim}(B_\nu(\nu,T)\,\mathrm{d}\nu) = \mathsf{ML^2T^{-1}} \cdot \mathsf{L^{-2}T^{-1}} \cdot 1 \cdot \mathsf{A^{-2}} \cdot \mathsf{T^{-1}} = \frac{\mathsf{ML^2T^{-2}}}{\mathsf{T}} \cdot \frac{1}{\mathsf{L^2}} \cdot \frac{1}{\mathsf{A^2}}$$

which matches the SI unit $\mathrm{W}/(\mathrm{m}^2 \cdot \mathrm{sr})$.

Integrating $B_\nu(\nu,T)\,\mathrm{d}\nu$ over all frequencies yields the total radiance $L(T)$:

$$L(T) = \int_{0\ \mathrm{Hz}}^{\infty\ \mathrm{Hz}} B_\nu(\nu,T)\,d\nu$$
$$= \int_{0\ \mathrm{Hz}}^{\infty\ \mathrm{Hz}} \frac{2h\nu^3}{c^2} \cdot \frac{1}{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right) - 1}\ \mathrm{sr}^{-1}\,d\nu$$
$$= \frac{2\pi^4 {k_\mathrm{B}}^4}{15c^2h^3} T^4\ \mathrm{sr}^{-1}$$
$$= \frac{\sigma}{\pi} T^4/\mathrm{sr}$$

This shows that the conventional Stefan–Boltzmann expression for radiance

$$L(T) = \frac{\sigma}{\pi} T^4$$

must include sr^{-1} in the right-hand side, treated as a numerical-value equation.

7-6. In optics

An infinitesimal luminous flux $\mathrm{d}\Phi_\mathrm{V}$ is defined as the product of luminous intensity I_V and an infinitesimal solid angle $\mathrm{d}\Omega$[14]:

$$\mathrm{d}\Phi_\mathrm{V} = I_\mathrm{V}\,\mathrm{d}\Omega$$

Given that $\mathrm{Dim}(I_\mathrm{V}) = \mathsf{J}$ and $\mathrm{Dim}(\mathrm{d}\Omega) = \mathsf{A}^2$, dimensional analysis on above yields:

$$\mathrm{Dim}(\mathrm{d}\Phi_\mathrm{V}) = \mathrm{Dim}(I_\mathrm{V}\,\mathrm{d}\Omega) = \mathrm{Dim}(I_\mathrm{V}) \cdot \mathrm{Dim}(\mathrm{d}\Omega)$$
$$= \mathsf{J} \cdot \mathsf{A}^2$$

This result aligns with the dimension of the unit of luminous flux, where the lumen (lm) is defined as $\mathrm{lm} = \mathrm{cd} \cdot \mathrm{sr}$ considering $\mathrm{Dim}(\mathrm{cd}) = \mathsf{J}$ and $\mathrm{Dim}(\mathrm{sr}) = \mathsf{A}^2$. Hence, the distinction between $\mathrm{d}\Phi_\mathrm{V}$ and I_V is maintained both conceptually and dimensionally.

8. On SI Defining Constant When Angle Is Treated as a Base Quantity

In agreement with the view of Mohr (2022)[3], it is reasonable to adopt a quantity denoted by Θ, representing one full revolution, as this notation is simple and intuitive. Since physical quantities are conventionally expressed in italic, the italicized $\mathit{\Theta}$ is considered more appropriate. Specifically, if the radian is treated as an SI base unit, the value of $\mathit{\Theta}$ is defined as $\mathit{\Theta} = 2\pi\ \mathrm{rad}$.

9. Summary of Formulae Requiring Correction

Table 2 summarizes the formulae that require correction in this paper. In some equations, the underline notation $\underline{\theta} = \theta/\mathrm{rad}$, $\underline{\omega} = \omega/\mathrm{rad}$, $\underline{\alpha} = \alpha/\mathrm{rad}$, and $\underline{\Omega}_2 = \Omega/\mathrm{sr}$, $\underline{\mathbf{k}} = \mathbf{k}/\mathrm{rad}$, $\underline{\lambda} = \lambda/\mathrm{rad}$ are introduced for simplicity. In the dimensional analysis, the dimension of plane angle is denoted as A.

Table 2 Formulae revisited in this paper for dimensional correction

Equations	Conventional equation (Dimensional analysis)	Dimensionally corrected equation (Dimensional analysis)
Arc length l	$l = r\theta$ $(\mathsf{L} \neq \mathsf{L} \cdot \mathsf{A})$	$l = r\theta/\mathrm{rad} = r\underline{\theta}$ $(\mathsf{L} = \mathsf{L} \cdot \mathsf{A} \cdot \mathsf{A}^{-1} = \mathsf{L} \cdot 1)$

Sector area S	$S=\frac{1}{2}r^2\theta$ $(\mathrm{L}^2 \neq \mathrm{L}^2\cdot\mathrm{A})$	$S=\frac{1}{2}r^2\theta/\mathrm{rad}=\frac{1}{2}r^2\underline{\theta}$ $(\mathrm{L}^2 = \mathrm{L}^2\cdot\mathrm{A}\cdot\mathrm{A}^{-1} = \mathrm{L}^2\cdot 1)$
Area on spherical surface A	$A=\Omega r^2$ $(\mathrm{L}^2 \neq \mathrm{A}^2\cdot\mathrm{L}^2)$	$A=\Omega r^2/\mathrm{sr}=\underline{\Omega}_2 r^2$ $(\mathrm{L}^2 = \mathrm{A}^2\cdot\mathrm{L}^2\cdot\mathrm{A}^{-2} = 1\cdot\mathrm{L}^2)$
Trigonometric and inverse trigonometric functions	$a=\sin\theta \Leftrightarrow \theta=\arcsin a$ (On the equation of θ, $\mathrm{A}\neq 1$)	$a=\sin\underline{\theta} \Leftrightarrow \underline{\theta}=\arcsin a$ (On the equation of θ, $1=1$)
Period T	$T=\frac{2\pi}{\omega}$ $(\mathrm{T} \neq \mathrm{TA}^{-1})$	$T=\frac{2\pi\ \mathrm{rad}}{\omega}=\frac{2\pi}{\underline{\omega}}$ $(\mathrm{T} = \mathrm{A}\cdot\mathrm{TA}^{-1} = \mathrm{T})$
Angular frequency ω	$\omega=2\pi\nu$ $(\mathrm{T}^{-1}\mathrm{A} \neq \mathrm{T}^{-1})$	$\omega/\mathrm{rad}=\underline{\omega}=2\pi\nu$ $(\mathrm{T}^{-1}\mathrm{A}\cdot\mathrm{A}^{-1} = \mathrm{T}^{-1} = \mathrm{T}^{-1})$
Angular wave number k	$k=2\pi\tilde{\nu}=\frac{2\pi}{\lambda}$ $(\mathrm{L}^{-1}\mathrm{A} \neq \mathrm{L}^{-1})$	$k=2\pi\tilde{\nu}\ \mathrm{rad}=\frac{2\pi\ \mathrm{rad}}{\lambda}$ $(\mathrm{L}^{-1}\mathrm{A} = \mathrm{L}^{-1}\cdot\mathrm{A} = \mathrm{A}\cdot\mathrm{L}^{-1})$
Angular wavelength ƛ	ƛ $=\frac{\lambda}{2\pi}$ $(\mathrm{LA}^{-1} \neq \mathrm{L})$	ƛ $=\frac{\lambda}{2\pi\ \mathrm{rad}}=\frac{\underline{\lambda}}{2\pi}$ $(\mathrm{LA}^{-1} = \mathrm{L}\cdot\mathrm{A}^{-1})$
Wave equation $u(\mathbf{x},t)$ of amplitude A	$u(\mathbf{x},t)=\frac{A}{r}e^{i(\omega t\pm\mathbf{k}\cdot\mathbf{x})}$ (The exponent part is $\mathrm{A}\neq 1$)	$u(\mathbf{x},t)=\frac{A}{r}e^{i\{\underline{\omega}t\pm\underline{\mathbf{k}}\cdot\mathbf{x}\}}$ (The exponent part is 1)
Infinitesimal displacement $\mathrm{d}\mathbf{l}$ by infinitesimal rotation	$\mathrm{d}\mathbf{l}=\mathrm{d}\boldsymbol{\theta}\times\mathbf{r}$ $(\mathrm{L} \neq \mathrm{A}\cdot\mathrm{L})$	$\mathrm{d}\mathbf{l}=\mathrm{d}\boldsymbol{\theta}\times\mathbf{r}/\mathrm{rad}=\mathrm{d}\underline{\boldsymbol{\theta}}\times\mathbf{r}$ $(\mathrm{L} = \mathrm{A}\cdot\mathrm{L}\cdot\mathrm{A}^{-1} = 1\cdot\mathrm{L})$
Tangential velocity $\mathbf{v}$	$\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$ $(\mathrm{LT}^{-1} \neq \mathrm{T}^{-1}\mathrm{A}\cdot\mathrm{L})$	$\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}/\mathrm{rad}=\underline{\boldsymbol{\omega}}\times\mathbf{r}$ $(\mathrm{LT}^{-1} = \mathrm{T}^{-1}\mathrm{A}\cdot\mathrm{L}\cdot\mathrm{A}^{-1} = \mathrm{T}^{-1}\cdot\mathrm{L})$
Tangential acceleration $\mathbf{a_t}$	$\mathbf{a_t}=\boldsymbol{\alpha}\times\mathbf{r}$ $(\mathrm{LT}^{-2} \neq \mathrm{T}^{-2}\mathrm{A}\cdot\mathrm{L})$	$\mathbf{a_t}=\boldsymbol{\alpha}\times\mathbf{r}/\mathrm{rad}=\underline{\boldsymbol{\alpha}}\times\mathbf{r}$ $(\mathrm{LT}^{-2} = \mathrm{T}^{-2}\mathrm{A}\cdot\mathrm{L}\cdot\mathrm{A}^{-1} = \mathrm{T}^{-2}\cdot\mathrm{L})$
Centripetal acceleration $\mathbf{a_c}$	$\mathbf{a_c}=\lVert\boldsymbol{\omega}\rVert^2\mathbf{r}$ $(\mathrm{LT}^{-2} \neq \mathrm{T}^{-2}\mathrm{A}^2\cdot\mathrm{L})$	$\mathbf{a_c}=\lVert\boldsymbol{\omega}\rVert^2\mathbf{r}/\mathrm{rad}^2=\lVert\underline{\boldsymbol{\omega}}\rVert^2\mathbf{r}$ $(\mathrm{LT}^{-2} = \mathrm{T}^{-2}\mathrm{A}^2\cdot\mathrm{L}\cdot\mathrm{A}^{-2} = \mathrm{T}^{-2}\cdot\mathrm{L})$
Torque $\boldsymbol{\tau}$	$\boldsymbol{\tau}=\frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{\theta}}=\mathbf{r}\times\mathbf{F}$ $(\mathrm{ML}^2\mathrm{T}^{-2}\mathrm{A}^{-1} \neq \mathrm{L}\cdot\mathrm{MLT}^{-2})$	$\boldsymbol{\tau}=\frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{\theta}}=\mathbf{r}\times\mathbf{F}/\mathrm{rad}$ $(\mathrm{ML}^2\mathrm{T}^{-2}\mathrm{A}^{-1} = \mathrm{L}\cdot\mathrm{MLT}^{-2}\cdot\mathrm{A}^{-1})$
Angular momentum $\mathbf{L}$	$\mathbf{L}=\int\boldsymbol{\tau}\,\mathrm{d}t=\mathbf{r}\times\mathbf{p}$ $(\mathrm{ML}^2\mathrm{T}^{-1}\mathrm{A}^{-1} \neq \mathrm{L}\cdot\mathrm{MLT}^{-1})$	$\mathbf{L}=\int\boldsymbol{\tau}\,\mathrm{d}t=\mathbf{r}\times\mathbf{p}/\mathrm{rad}$ $(\mathrm{ML}^2\mathrm{T}^{-1}\mathrm{A}^{-1} = \mathrm{L}\cdot\mathrm{MLT}^{-1}\cdot\mathrm{A}^{-1})$
	$\mathbf{L}=\mathbf{r}\times(m\boldsymbol{\omega}\times\mathbf{r})=m\lVert\mathbf{r}\rVert^2\boldsymbol{\omega}$ $(\mathrm{ML}^2\mathrm{T}^{-1}\mathrm{A}^{-1} \neq \mathrm{M}\cdot\mathrm{L}^2\cdot\mathrm{AT}^{-1})$	$\mathbf{L}=\mathbf{r}\times(m\underline{\boldsymbol{\omega}}\times\mathbf{r})/\mathrm{rad}=m\lVert\mathbf{r}\rVert^2\boldsymbol{\omega}/\mathrm{rad}^2$ $(\mathrm{ML}^2\mathrm{T}^{-1}\mathrm{A}^{-1} = \mathrm{M}\cdot\mathrm{L}^2\cdot\mathrm{AT}^{-1}\cdot\mathrm{A}^{-2})$
Moment of inertia I	$I=\frac{\lVert\boldsymbol{\tau}\rVert}{\lVert\boldsymbol{\alpha}\rVert}=\sum_i m_i\lVert\mathbf{r}_i\rVert^2$ $(\mathrm{ML}^2\mathrm{A}^{-2} \neq \mathrm{M}\cdot\mathrm{L}^2)$	$I=\frac{\lVert\boldsymbol{\tau}\rVert}{\lVert\boldsymbol{\alpha}\rVert}=\sum_i m_i\lVert\mathbf{r}_i\rVert^2/\mathrm{rad}^2$ $(\mathrm{ML}^2\mathrm{A}^{-2} = \mathrm{M}\cdot\mathrm{L}^2\cdot\mathrm{A}^{-2})$
Stefan–Boltzmann law for radiance of black body	$L(T)=\frac{\sigma}{\pi}T^4$ $(\mathrm{MT}^{-3}\mathrm{A}^{-2} \neq \mathrm{MT}^{-3}\Theta^{-4}\cdot\Theta^4)$	$L(T)=\frac{\sigma}{\pi}T^4/\mathrm{sr}$ $(\mathrm{MT}^{-3}\mathrm{A}^{-2} = \mathrm{MT}^{-3}\Theta^{-4}\cdot\Theta^4\cdot\mathrm{A}^{-2})$
Reduced Planck constant $\hbar$	$\hbar=\frac{h\nu}{\omega}=\frac{h}{2\pi}$ $(\mathrm{ML}^2\mathrm{T}^{-1}\mathrm{A}^{-1} \neq \mathrm{ML}^2\mathrm{T}^{-1})$	$\hbar=\frac{h\nu}{\omega}=\frac{h}{2\pi\ \mathrm{rad}}$ $(\mathrm{ML}^2\mathrm{T}^{-1}\mathrm{A}^{-1} = \mathrm{ML}^2\mathrm{T}^{-1}\cdot\mathrm{A})$

Conclusion

This study has focused on the characteristics of equations involving physical quantities, and has demonstrated that some equations involving angles must be regarded as a numerical-value equation that includes units, since angle cannot be expressed in terms of other SI base units. Furthermore, by applying dimensional analysis with dimension-extraction function Dim to mathematical analysis, it is shown that transcendental functions such as trigonometric and inverse trigonometric functions are likewise numerical-value equations and must be dealt with in terms of the numerical component of angle in radians. This approach was further extended to vector analysis, where it is shown that formulae related to rotational motion should also be treated as numerical-value equations. By treating angle as a fundamental quantity, this framework provides a clearer understanding of various physical quantities that have traditionally lacked distinction in scientific disciplines, thereby offering a promising perspective for advancing our understanding of angular quantities.

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