$3D$ horizons in an alternative Natario warp drive using the ADM-MTW-Alcubierre formalism with constant speeds over the x-axis

Fernando Loup *†
independent researcher in warp drive spacetimes:Lisboa Portugal

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Abstract

The Natario warp drive appeared for the first time in 2001.Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.Natario defined a warp drive vector for constant speeds in Polar Coordinates using the Hodge Star but Polar Coordinates uses only two dimensions and we know that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates.In this work we present the alternative warp drive vector in $3D$ Spherical Coordinates without the Hodge Star for constant speeds.Our alternative warp drive vector also satisfies the Natario requirements for a warp drive spacetime.One the major drawbacks concerning warp drives is the problem of the Horizons(causally disconnected portions of spacetime) in which an observer in the center of the bubble cannot signal nor control the front part of the bubble.The behavior of a photon sent to the front of the warp bubble in the case of the alternative Natario warp drive with constant velocity and a lapse function is also one of the main purposes of this work.We present the behavior of a photon sent to the front of the bubble in the alternative Natario warp drive in the 3+1 spacetime with and without the lapse function using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and the ADM(Arnowitt-Dresner-Misner) formalism equations with the approach of MTW(Misner-Thorne-Wheeler) and Alcubierre.

*spacetimeshortcut@yahoo.com,spacetimeshortcut@gmail.com
†https://independent.academia.edu/FernandoLoup,https://www.researchgate.net/profile/Fernando-Loup

1 Introduction:

The Natario warp drive appeared for the first time in 2001.([1]).Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.

This propulsion vector nX uses the form $nX = X^i e_i$ where X^i are the shift vectors responsible for the spaceship propulsion or speed and e_i are the Canonical Basis of the Coordinates System where the shift vectors are based or placed.

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).(see Appendix D about Polar Coordinates in [2]).(see Appendix A for the complete mathematical demonstration of the Natario calculations for the Hodge Star in [2]).The final form of the original Natario warp drive vector is given by $nX = vs * d(r\cos\theta)$ or better:

$$nX = -2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{1}$$

or

$$nX = 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{2}$$

However Polar Coordinates are not real $3D$ coordinates since it uses only the two dimensional Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

We adopted the second expression above taken from Natario (pg 5 in [1]) to define an alternative warp drive vector that do not uses the Hodge Star but retains all the Natario requirements as will be demonstrated in this work.The final form of the alternative warp drive vector nWD is given by:

$$nWD = 2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{3}$$

Note that this alternative warp drive vector nWD is symmetrical when compared to the second original Natario warp drive vector in the shift vector and Canonical Basis $X^\theta e_\theta$.In the Natario case $X^\theta e_\theta$ is negative $[-v_s(2f + rf')\sin\theta \mathbf{e}_\theta]$ while in the new case $X^\theta e_\theta$ is positive $[+v_s(2f + rf')\sin\theta \mathbf{e}_\theta]$.The symmetry in this case lies over the shift vector X^θ where in the Natario case is $X^\theta = [-v_s(2f + rf')\sin\theta]$ and in our case is $X^\theta = [+v_s(2f + rf')\sin\theta]$

Note also that the alternative warp drive vector nWD above uses a constant speed because it was derived from the original Natario warp drive vector nX also with a constant speed.

Natario used Polar Coordinates(See pg 4 in [1]) but for a real $3D$ Spherical Coordinates another alternative warp drive vector must be calculated.Remember that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must uses all the $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.(see Appendix E about $3D$ Spherical Coordinates in [2]).

In this work we present the alternative warp drive vector nWD in $3D$ Spherical Coordinates for constant speeds.

The warp drive work that predates Natario by 7 years was written by Alcubierre in 1994.(see [14])

Alcubierre([16]) used the so-called $3+1$ original Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([15]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pg 67 in [16],see also eq 1 pg 3 in [14]) and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive spacetime.In this work concerning the ADM formalism we adopt the Alcubierre methodology.

The ADM equation with signature $(-,+,+,+)$ that obeys the original $3+1$ ADM formalism is given below:(see eq (21.40) pg 507 in [15])(see Appendix A).

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{4}$$

In the equation above α is the so-called lapse function,γ_{ij} is the $3D$ diagonalized induced metric and β^i and β^j are the so-called shift vectors.

Combining the eqs (21.40),(21.42) and (21.44) pgs $507, 508$ in [15]
with the eqs (2.2.4),(2.2.5) and (2.2.6) pg 67 in [16] using the signature $(-,+,+,+)$ we get the original matrices of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{5}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{6}$$

The alternative Natario warp drive equation with signature $(-,+,+,+)$ that obeys the original $3+1$ ADM formalism is given below:(see eq 21.40 pg 507 in [15])

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{7}$$

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ making $\alpha = 1$ and inserting the components of the alternative Natario vectors we have in Polar Coordinates:(see Appendix A).

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{8}$$

And in $3D$ Spherical Coordinates:(see also Appendix A).

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{9}$$

The equations above dont have the lapse function.The equivalent equations using the lapse function would then be:

Polar Coordinates:(see Appendix B).

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \quad (10)$$

$3D$ Spherical Coordinates:(see also Appendix B).

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (11)$$

The lapse function is equal to 1 inside and outside the Natario warp bubble while having large values in the Natario warped region.See [3]

In this work we also discuss the Horizon problem for the alternative Natario warp drive spacetime equations in the $3+1$ ADM formalisms in Polar and Spherical coordinates with and without the lapse function at constant velocities and we arrive at the conclusion that in the $3+1$ spacetime the alternative warp drive equations dont suffers from the problem of the Horizon.

Horizons were deeply covered in the warp drive literature but always for constant velocities and without lapse functions in the $1+1$ spacetime.(see pg 6 in [1],pg 34 in [23],pgs 268 in [24]).The behavior of a photon sent to the front of the warp bubble in the case of an alternative warp drive with constant velocity and with a lapse function(or not) is one of the main purposes of this work.We present the behavior of a photon sent to the front of the bubble in the alternative Natario warp drive in the $3+1$ spacetimes in Polar and Spherical coordinates with or without the lapse function at constant velocities using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provide here the step by step mathematical calculations in order to outline(or underline or reinforce) the final results found in our work which are the following ones:

In the $3+1$ spacetimes wether in Polar or Spherical Coordinates the Horizon do not exists at all!!.

In the solutions with $3+1$ dimensions wether the lapse function exists or not the whole spacetime geometries are affected by presence of the $3+1$ dimensions and have different results when compared to the solutions with only $1+1$ dimensions.These $3+1$ dimensions affects the behavior of the Horizon.For results in the $1+1$ dimensions see sections 4 and 5 in [13].The behavior of a photon moving in a $3+1$ dimensions is completely different than the behavior the same photon moving in a $1+1$ dimensions affecting the Horizon.

Remember that we are presenting our results using step by step mathematics in order to better illustrate our point of view. For the solutions of the quadratic forms in $3+1$ spacetimes see Appendices C and D.These solutions are different than the ones obtained only in $1+1$ spacetimes.

We adopt here the Geometrized system of units in which $c = G = 1$ for geometric purposes.

This presentation about Horizons is a complement to our work in [13].The alternative Natario warp drives have geometrical advantages in terms of Horizons when compared with the original Natario warp drives.

In order to fully understand the main idea presented in this work(an alternative warp drive vector nWD in $3D$ Spherical Coordinates obtained independently from the Natario Hodge Star but retaining all the Natario physical features and properties) acquaintance or familiarity with the Natario original warp drive paper in [1] or familiarity with our works in [2] and [13] are a previous reading requirement.We provide useful mathematical demonstrations QED(Quod Erad Demonstratum) of the process Natario used to obtain the original warp drives using the Hodge Star in the Appendices of [2] and [13].

Also we analyze the behavior of a photon sent to the front of the bubble in the alternative Natario warp drive in $3+1$ spacetimes in Polar and Spherical coordinates with or without the lapse function at constant velocities from the point of view of the Horizon problem.

Remember that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates so a photon sent to the front of the bubble fundamentally moves in a $3D$ spacetime.

The Arnowitt-Dresner-Misner(ADM) formalism using the approaches of Misner-Thorne-Wheeler(MTW)([15]) and Alcubierre([16] are a fundamental requirement and we provide in this work all the mathematical demonstrations QED(Quod Erad Demonstratum) in the Appendices A and B.

This work is organized as follows:

- A)-Section 2 introduces the alternative Natario warp drive vector nWD in $2D$ Polar Coordinates for constant speeds.
- B)-Section 3 introduces the alternative Natario warp drive vector nWD in $3D$ Spherical Coordinates for constant speeds..
- C)-Section 4 introduces the Horizon in the alternative Natario warp drive vector nWD in $2D$ Polar Coordinates for constant speeds without the lapse function.
- D)-Section 5 introduces the Horizon in the alternative Natario warp drive vector nWD in $2D$ Polar Coordinates for constant speeds with the lapse function.
- E)-Section 6 introduces the Horizon in the alternative Natario warp drive vector nWD in $3D$ Spherical Coordinates for constant speeds without the lapse function.
- F)-Section 7 introduces the Horizon in the alternative Natario warp drive vector nWD in $3D$ Spherical Coordinates for constant speeds with the lapse function.
- G)-Section 8 the most important section and the main reason for this work:advantages of the alternative Natario warp drives when compared to the original Natario warp drives due to a different shift vector X^θ in the behavior of the Horizons.

We adopted in this work a pedagogical language and a presentation style that perhaps will be considered as tedious,monotonous, exhaustive or extensive by experienced or seasoned readers and we designated this work for novices,newcomers,beginners or intermediate students providing in our work all the mathematical references and Appendices required for the background needed to understand the process we used to generate these alternative warp drive vectors independently from the Natario Hodge Star but retaining all the Natario physical features and properties and also the behavior of a photon sent to the front of the bubble in the alternative Natario warp drive in $3 + 1$ spacetimes with or without the lapse function at constant velocities.

We hope our paper is suitable to fill this proposed task.

This work was designed as a companion work to our works in [2],[12] and [13].

2 The equation of the alternative Natario warp drive vector nWD in $2D$ polar coordinates with a constant speed vs

The equation of the alternative Natario warp drive vector nWD is given by:

$$nWD = X^r e_r + X^\theta e_\theta \tag{12}$$

With the contravariant shift vector components X^r and X^θ given by:

$$X^r = 2v_s f(r) \cos\theta \tag{13}$$

$$X^\theta = +v_s(2f(r) + (r)f'(r)) \sin\theta \tag{14}$$

Compare the expressions above with the original Natario warp drive vector in $2D$ polar coordinates with a constant speed vs presented in section 2 in [13].The symmetry in this case lies over the shift vector X^θ where in the Natario case is $X^\theta = -v_s(2f(r) + (r)f'(r)) \sin\theta$ and in our case is $X^\theta = +v_s(2f(r) + (r)f'(r)) \sin\theta$

Considering a valid $f(r)$ as a shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small r(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the warped region(see pg 5 in [1]):

We must demonstrate that the alternative Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any new vector nWD generates a warp drive spacetime if $nWD = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nWD = vs(t)$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(see pg 4 in [1])(see Appendix G in [2] for an explanation about this statement)

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x-axis$ or the equatorial plane r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).

In a $1+1$ spacetime the equatorial plane we get¿:

$$nWD = X^r e_r \tag{15}$$

The contravariant shift vector component X^{rs} is then:

$$X^r = 2v_s f(r) \tag{16}$$

Remember that Natario(see pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f(r) = 0$ resulting in a $X^r = 0$ and outside the bubble $f(r) = \frac{1}{2}$ resulting in a $X^r = vs$ and this illustrates the Natario definition for a warp drive spacetime.See Appendix D in [2].

The difference between the alternative Natario warp drive vector and the Natario original warp drive vector occurs if the motion occurs also with the shift vector X^θ

Placing again the original Natario warp drive vector nX(see pg 5 in [1]) and the alternative Natario warp drive vector nWD side by side and together we have:

$$nX = -2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf') \sin\theta \mathbf{e}_\theta \tag{17}$$

or

$$nX = 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf') \sin\theta \mathbf{e}_\theta \tag{18}$$

With:

$$nWD = 2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf') \sin\theta \mathbf{e}_\theta \tag{19}$$

We use the second expression for the original Natario warp drive vector nX.We consider motion in the $x - axis$ or the equatorial plane r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).Now all the warp drive vectors in the $1+1$ spacetime are equal.

$$nX = 2v_s f \cos\theta \mathbf{e}_r \longrightarrow 2v_s f \mathbf{e}_r \tag{20}$$

$$nWD = 2v_s f \cos\theta \mathbf{e}_r \longrightarrow 2v_s f \mathbf{e}_r \tag{21}$$

The energy density in the original Natario warp drive $3+1$ spacetime is being given by the following expressions(pg 5 in [1]):

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(f'(r))^2 \cos^2\theta + \left(f'(r) + \frac{r}{2} f''(r) \right)^2 \sin^2\theta \right]. \tag{22}$$

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(\frac{df(r)}{dr})^2 \cos^2\theta + \left(\frac{df(r)}{dr} + \frac{r}{2} \frac{d^2 f(r)}{dr^2} \right)^2 \sin^2\theta \right]. \tag{23}$$

Is being distributed around all the space involving the ship(above the ship $\sin\theta = 1$ and $\cos\theta = 0$ while in front of the ship $\sin\theta = 0$ and $\cos\theta = 1$).The negative energy in front of the ship "deflect" photons or other particles so these will not reach the ship inside the bubble.

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg 116 in [17])

The Appendices E and F shows the composition of the Interstellar Medium IM and since the Interstellar Medium is not empty a collision between a spaceship with hazardous objects would potentially be catastrophic.

-)-Energy directly above the ship($y - axis$)

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[\left(\frac{df(r)}{dr} + \frac{r}{2} \frac{d^2 f(r)}{dr^2} \right)^2 \sin^2 \theta \right]. \tag{24}$$

-)-Energy directly in front of the ship($x - axis$)

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3 (\frac{df(r)}{dr})^2 \cos^2 \theta \right]. \tag{25}$$

The distribution of energy presented here is valid only for the original Natario warp drive vector in Polar Coordinates without the lapse function.For the case of the lapse function see Section 3 and Appendix I in [3].

But for the case of the alternative warp drive vector we can say nothing about the negative energy density at first sight and we need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors.

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pg 342 in [15], pg 276 in [19],pgs $454, 457, 560$ in [20] pg 98 in [21],pg 178 in [22]).

Appendix C pgs $551 - 555$ in [20] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3 + 1$ spacetime metric using $Mathematica$.

But we know that in the $1 + 1$ spacetime in the equatorial plane over the x-axis the original Natario vector is equal to the alternative Natario vector.Since in this case $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$ the negative energy density reduces to:

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3 (\frac{df(r)}{dr})^2 \right]. \tag{26}$$

Note that even in the $1 + 1$ spacetime the alternative Natario vector possesses negative energy density in front of the ship with repulsive gravitational behavior(see pg 116 in [17]) and this is very important concerning collisions against the Interstellar Medium.See Appendices E and F.

Unfortunately we dont know the negative energy distribution for the alternative Natario vector in a $3 + 1$ spacetime because we dont have access to none of these programs so we have our hands "tied up" but we believe that this distribution of energy would possess negative energy density also in the front of the ship.

Since in the $1+1$ spacetime in the equatorial plane the original Natario vector in Polar Coordinates is equal to the alternative Natario vector in Polar Coordinates then in the question of the Horizons both have the same results.(see sections 4 and 5 in [13]).

The difference between the alternative Natario warp drive vector and the Natario original warp drive vector occurs if the motion occurs also with the shift vector X^{θ} in the $3+1$ spacetime.

For the original Natario shift vector is:

$$X^{\theta} = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{27}$$

And for the alternative Natario shift vector is:

$$X^{\theta} = +v_s(2f(r) + (r)f'(r))\sin\theta \tag{28}$$

This is the reason why in this work we cover Horizons only in the $3+1$ spacetime.

3 The equation of the alternative Natario warp drive vector nWD in $3D$ spherical coordinates with a constant speed vs

The equation of the alternative Natario warp drive vector in $3D$ spherical coordinates with a constant speed vs nWD is given by:

$$nWD = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{29}$$

With the contravariant shift vector components X^{rs}, X^θ and X^ϕ given by:

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{30}$$

$$X^\theta = +vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{31}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{32}$$

Compare the expressions above with the warp drive vector in $3D$ spherical coordinates with a constant speed vs presented in section 3 in [13].The symmetry in this case lies over the shift vector X^θ where in the section 3 in [13] case is $X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta]$ and in our case is $X^\theta = +vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta]$

Considering a valid $f(r)$ as a shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small rs(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the warped region(see pg 5 in [1]):

We must demonstrate that the alternative Natario warp drive vector satisfies the Natario criteria for a warp drive defined by:

any warp drive vector nWD generates a warp drive spacetime if $nWD = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nWD = vs(t)$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(see pg 4 in [1])(see Appendix G in [2] for an explanation about this statement)

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x - axis$ or $1+1$ spacetime (like Natario did) or the equatorial plane $x - y$ in r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).Also the equatorial plane $x - y$ makes an angle of 90 degrees with the $z - axis$ so $\phi = 90$,$\sin\phi = 1$ and $cos\phi = 0$.

Then the contravariant components reduces to:

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \rightarrow X^r = vs(t)[2f(r)] \rightarrow \sin\phi = 1 \rightarrow\ cos\theta = 1 \tag{33}$$

$$X^\theta = +vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] = 0 \rightarrow \sin\phi = 1 \rightarrow \sin\theta = 0 \tag{34}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] = 0 \rightarrow cos\phi = 0 \tag{35}$$

Remember that Natario(see pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f(r) = 0$ resulting in a $X^r = 0$ and outside the bubble $f(r) = \frac{1}{2}$ resulting in a $X^r = vs$ and this illustrates the Natario definition for a warp drive spacetime.See Appendix E in [2].

Only in $3D$ motion the results becomes different.

The difference between the alternative Natario warp drive vector and the original Natario warp drive vector presented in section 4 in [2] occurs only if the motion occurs also with the shift vector X^θ.

Examining again the equation of the alternative warp drive vector in $3D$ spherical coordinates with a constant speed vs nWD:

$$nWD = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{36}$$

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{37}$$

$$X^\theta = +vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{38}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{39}$$

We know that the equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\phi = 90$,$\sin\phi = 1$ and $cos\phi = 0$.In this case the alternative warp drive vector in $3D$ spherical coordinates reduces to the alternative warp drive vector in polar coordinates.

$$nWD = X^r e_r + X^\theta e_\theta \tag{40}$$

$$X^r = vs(t)[2f(r)\ cos\theta] \tag{41}$$

$$X^\theta = +vs(t)[[2f(r) + rf'(r)]\sin\theta] \tag{42}$$

Then in a $1+1$ spacetime reduction with motion over the x-axis $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$ the alternative warp drive vector in $3D$ spherical coordinates gives the same results of the alternative warp drive vector in polar coordinates and these results are equal to the ones obtained with the Natario original warp drive vector in polar coordinates in the same $1+1$ spacetime reduction.

Like in the previous section we dont know the negative energy density distribution of the alternative warp drive vector in $3D$ spherical coordinates because we dont have the $Maple$ or $Mathematica$ programs needed to handle all the tensor algebra and we will not do all the calculations from the Christoffel Symbols Riemann,Ricci and Einstein tensors "by hand" but we know that in a 1+1 spacetime reduction with motion over the x-axis the alternative warp drive vector in $3D$ spherical coordinates would also possess negative energy density in front of the ship since is mathematically equivalent to the Natario original warp drive vector in polar coordinates in the same $1+1$ spacetime reduction.This is very important see Appendices E and F

For the $Maple$ or $Mathematica$ see pg 342 in [15], pg 276 in [19],pgs $454, 457, 560$ in [20] pg 98 in [21],pg 178 in [22],see also Appendix C pgs $551-555$ in [20].

Since in the $1+1$ spacetime in the equatorial plane the original Natario warp drive vector in spherical(or polar) coordinates is equal to the alternative $3D$ spherical coordinates Natario warp drive vector when reduced to a $1+1$ spacetime then in the question of the Horizons both have the same results.(see sections 4 and 5 in [13]).

The difference between the alternative $3D$ spherical coordinates Natario warp drive vector and the original $3D$ spherical coordinates Natario warp drive vector occurs if the motion occurs also with the shift vector X^{θ} in the $3+1$ spacetime.

For the original Natario shift vector is:(see section 3 in [13])

$$X^{\theta} = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{43}$$

And for the alternative Natario shift vector is:

$$X^{\theta} = +vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{44}$$

This is the reason why in this work we cover Horizons only in the $3+1$ spacetime.

4 Horizons in the alternative Natario warp drive with a constant speed vs in the original $3+1$ ADM formalism without a lapse function α in Polar Coordinates

In the sections 4 and 5 in [13] we used photons sent to the front of the bubble in the original Natario warp drive with a constant speed in the $1+1$ spacetime.($1+1$ dimensions).

Now we use photons sent to the front of the bubble in the alternative Natario warp drive with a constant speed vs in the $2+1$ spacetime.($2+1$ dimensions).

The equation of the alternative Natario warp drive spacetime in Polar Coordinates with a constant speed vs in the original $3+1$ ADM formalism without the lapse function is given by:(see Appendix A)

$$ds^2 = (1 - X_rX^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{45}$$

Actually Polar Coordinates are given in the $2+1$ spacetime.The generic quadratic form and its solutions for the $2+1$ spacetime are given by:(see Appendix C)

$$ds^2 = (1 - X_1X^1 - X_2X^2)dt^2 + 2(X_1dx^1 + X_2dx^2)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 \tag{46}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 + \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{47}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 - \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{48}$$

Remember that $dl^2 = \gamma_{ii}dx^idx^i = dr^2 + r^2d\theta^2$ with $\gamma_{11} = \gamma_{rr} = 1$ and $\gamma_{22} = \gamma_{\theta\theta} = r^2$. Then the covariant shift vector components $X_1 = X_r$ and $X_2 = X_\theta$ are given by:

$$X_r = \gamma_{rr}X^r = X_r = \gamma_{rr}X^r = 2v_sf(r)\cos\theta = X^r \tag{49}$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = r^2X^\theta = +r^2v_s(f(r) + (r)f'(r))\sin\theta \tag{50}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \sqrt{1+r^2}}{1+r^2} \tag{51}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta - \sqrt{1+r^2}}{1+r^2} \tag{52}$$

Note that now the photon moves in a $2+1$ spacetime and this means motion in r and θ.

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \sqrt{1+r^2}}{1+r^2} = \frac{[2v_sf(r)\cos\theta] + [+r^2v_s(f(r) + (r)f'(r))\sin\theta] + \sqrt{1+r^2}}{1+r^2} \tag{53}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta - \sqrt{1+r^2}}{1+r^2} = \frac{[2v_sf(r)\cos\theta] + [+r^2v_s(f(r) + (r)f'(r))\sin\theta] - \sqrt{1+r^2}}{1+r^2} \tag{54}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}=\frac{X_r+X_\theta+\sqrt{1+r^2}}{1+r^2}=\frac{[2v_s f(r)\cos\theta]+[r^2 v_s(f(r)+(r)f'(r))\sin\theta]+\sqrt{1+r^2}}{1+r^2} \quad (55)$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}=\frac{X_r+X_\theta-\sqrt{1+r^2}}{1+r^2}=\frac{[2v_s f(r)\cos\theta]+[r^2 v_s(f(r)+(r)f'(r))\sin\theta]-\sqrt{1+r^2}}{1+r^2} \quad (56)$$

In two dimensions the photon moves in a $2+1$ spacetime and this means motion in r and θ the Horizon do not occurs unless $\theta=0$,$\cos\theta=1$,$\sin\theta=0$ and $r^2 d\theta^2=0$ and we recover in this case the problem of the Horizon without the lapse function in the $1+1$ spacetime.See sections 4 and 5 in [13]

Of course this point of view about the Horizons reflects only the geometrical point of view of the alternative Natario warp drive equation for constant speed vs in a $3+1$ spacetime.But in the alternative Natario warp drive we expect that the negative energy density may covers the entire bubble.Since the negative energy density have repulsive gravitational behavior(see pg 116 in [17]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.

The solution that allows contact with the bubble walls was presented in pg 83 in [18].Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg 83 in [18] to be sent to the large warp bubble keeping it in causal contact.Remember that one source of negative energy repels a source of positive energy but attracts another source of negative energy.This idea seems to be endorsed by pg 34 in [23],pg 268 in [24] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

5 Horizons in the alternative Natario warp drive with a constant speed vs in the original $3+1$ ADM formalism with a lapse function α in Polar Coordinates

In the sections 4 and 5 in [13] we used photons sent to the front of the bubble in the original Natario warp drive with a constant speed in the $1+1$ spacetime.($1+1$ dimensions).

Now we use photons sent to the front of the bubble in the alternative Natario warp drive with a constant speed in the $2+1$ spacetime.($2+1$ dimensions).

The equation of the alternative Natario warp drive spacetime in Polar Coordinates with a constant speed vs in the original $3+1$ ADM formalism with the lapse function is given by:(see Appendix B)

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{57}$$

Actually Polar Coordinates are given in the $2+1$ spacetime.The generic quadratic form and its solutions for the $2+1$ spacetime are given by:(see Appendix D)

$$ds^2 = (\alpha^2 - X_1 X^1 - X_2 X^2)dt^2 + 2(X_1 dx^1 + X_2 dx^2)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 \tag{58}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 + \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{59}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 - \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{60}$$

Remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{11} = \gamma_{rr} = 1$ and $\gamma_{22} = \gamma_{\theta\theta} = r^2$. Then the covariant shift vector components $X_1 = X_r$ and $X_2 = X_\theta$ are given by:

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s f(r)\cos\theta = X^r \tag{61}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = +r^2 v_s (f(r) + (r)f'(r))\sin\theta \tag{62}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \alpha\sqrt{1+r^2}}{1+r^2} \tag{63}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta - \alpha\sqrt{1+r^2}}{1+r^2} \tag{64}$$

Note that now the photon moves in a $2+1$ spacetime and this means motion in r and θ.

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \alpha\sqrt{1+r^2}}{1+r^2} = \frac{[2v_s f(r)\cos\theta] + [+r^2 v_s (f(r) + (r)f'(r))\sin\theta] + \alpha\sqrt{1+r^2}}{1+r^2} \tag{65}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta - \alpha\sqrt{1+r^2}}{1+r^2} = \frac{[2v_s f(r)\cos\theta] + [+r^2 v_s (f(r) + (r)f'(r))\sin\theta] - \alpha\sqrt{1+r^2}}{1+r^2} \tag{66}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \alpha\sqrt{1+r^2}}{1+r^2} = \frac{[2v_s f(r)\cos\theta] + [r^2 v_s (f(r) + (r)f'(r))\sin\theta] + \alpha\sqrt{1+r^2}}{1+r^2} \tag{67}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta - \alpha\sqrt{1+r^2}}{1+r^2} = \frac{[2v_s f(r)\cos\theta] + [r^2 v_s (f(r) + (r)f'(r))\sin\theta] - \alpha\sqrt{1+r^2}}{1+r^2} \tag{68}$$

In two dimensions the photon moves in a $2+1$ spacetime and this means motion in r and θ the Horizon do not occurs even if $\theta = 0$,$\cos\theta = 1$,$\sin\theta = 0$ and $r^2 d\theta^2 = 0$ and we recover in this case the problem of the Horizon with the lapse function in the $1+1$ spacetime.See sections 4 and 5 in [13]

A large lapse function may perhaps keeps the alternative Natario warped region causally connected.Like in the previous case this is the geometrical point of view of the alternative Natario warp drive equation for constant speed vs in a $3+1$ spacetime with a lapse function.But in the alternative Natario warp drive we expect that the negative energy density may covers the entire bubble.Since the negative energy density have repulsive gravitational behavior(see pg 116 in [17]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.

The lapse function is 1 inside and outside the bubble but with a large value in the Natario warped region.(see [3]).

6 Horizons in the alternative Natario warp drive with a constant speed vs in the original $3+1$ ADM formalism without a lapse function α in $3D$ Spherical Coordinates

In the sections 4 and 5 in [13] we used photons sent to the front of the bubble in the original Natario warp drive with a constant speed in the $1+1$ spacetime.($1+1$ dimensions).

Now we use photons sent to the front of the bubble in the $3+1$ alternative Natario warp drive spacetime in $3D$ Spherical Coordinates.($3+1$ dimensions).

The equation of the Natario warp drive spacetime in $3D$ Spherical Coordinates with a constant speed vs in the original $3+1$ ADM formalism without the lapse function is given by:(see Appendix A)

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (69)$$

The generic quadratic form and its solutions for the $3+1$ spacetime are given by:(see Appendix C)

$$ds^2 = (1-X_1X^1-X_2X^2-X_3X^3)dt^2+2(X_1dx^1+X_2dx^2+X_3dx^3)dt-\gamma_{11}(dx^1)^2-\gamma_{22}(dx^2)^2-\gamma_{33}(dx^3)^2 \quad (70)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 + \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \quad (71)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 - \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \quad (72)$$

Remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ with $\gamma_{11} = \gamma_{rr} = 1$, $\gamma_{22} = \gamma_{\theta\theta} = r^2$ and $\gamma_{33} = \gamma_{\phi\phi} = r^2 \sin^2\theta$. Then the covariant shift vector components $X_1 = X_r$,$X_2 = X_\theta$ and $X_3 = X_\phi$ are given by:

$$X_r = \gamma_{rr}X^r = X_r = \gamma_{rr}X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] = X^r \quad (73)$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = r^2 X^\theta = +r^2 vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \quad (74)$$

$$X_\phi = \gamma_{\phi\phi}X^\phi = r^2 sin^2\theta X^\phi = r^2 sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \quad (75)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi + \sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \quad (76)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi - \sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \quad (77)$$

Note that now the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ.

Note that now the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ.

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{X_r+X_\theta+X_\phi+\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{78}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{X_r+X_\theta+X_\phi-\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{79}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{U+\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{80}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{U-\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{81}$$

$$U=X_r+X_\theta+X_\phi \tag{82}$$

$$U=vs(t)[\sin\phi][2f(r)\ cos\theta]+(+r^2vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta])+r^2sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r))+(rf'(r))] \tag{83}$$

$$U=vs(t)[\sin\phi][2f(r)\ cos\theta]+r^2vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta]+r^2sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r))+(rf'(r))] \tag{84}$$

In three dimensions the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ the Horizon do not occurs unless $\theta=0$,$\cos\theta=1$,$\sin\theta=0$,$\phi=90$,$\sin\phi=1$,$\cos\phi=0$,$r^2d\theta^2=0$ and $r^2\sin^2 d\phi^2=0$ and we recover in this case the problem of the Horizon without the lapse function in the $1+1$ spacetime.See sections 4 and 5 in [13]

Of course this point of view about the Horizons reflects only the geometrical point of view of the alternative Natario warp drive equation for constant speed vs in Polar Coordinates in a $3+1$ spacetime. But in the alternative Natario warp drive we expect that the negative energy density may covers the entire bubble.Since the negative energy density have repulsive gravitational behavior(see pg 116 in [17]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.

But now we are in $3D$ Spherical Coordinates and we know that the alternative Natario warp drive vector in this case can be reduced to a alternative Natario warp drive vector in Polar Coordinates. So it is reasonable to suppose that the negative energy in this case may perhaps cover the entire bubble although we dont have the distribution of energy in the $3D$ Spherical Coordinates.

The solution that allows contact with the bubble walls was presented in pg 83 in [18].Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg 83 in [18] to be sent to the large warp bubble keeping it in causal contact.Remember that one source of negative energy repels a source of positive energy but attracts another source of negative energy.This idea seems to be endorsed by pg 34 in [23],pg 268 in [24] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

7 Horizons in the alternative Natario warp drive with a constant speed vs in the original $3+1$ ADM formalism with a lapse function α in $3D$ Spherical Coordinates

In the sections 4 and 5 in [13] we used photons sent to the front of the bubble in the original Natario warp drive with a constant speed in the $1+1$ spacetime.($1+1$ dimensions).

Now we use photons sent to the front of the bubble in the alternative Natario warp drive in $3D$ Spherical Coordinates in the $3+1$ spacetime.($3+1$ dimensions).

The equation of the alternative Natario warp drive spacetime in $3D$ Spherical Coordinates with a constant speed vs in the original $3+1$ ADM formalism with a lapse function is given by:(see Appendix B)

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (85)$$

The generic quadratic form and its solutions for the $3+1$ spacetime are given by:(see Appendix D)

$$ds^2 = (\alpha^2 - X_1 X^1 - X_2 X^2 - X_3 X^3)dt^2 + 2(X_1 dx^1 + X_2 dx^2 + X_3 dx^3)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 - \gamma_{33}(dx^3)^2 \quad (86)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 + \alpha\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \quad (87)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 - \alpha\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \quad (88)$$

Remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ with $\gamma_{11} = \gamma_{rr} = 1$, $\gamma_{22} = \gamma_{\theta\theta} = r^2$ and $\gamma_{33} = \gamma_{\phi\phi} = r^2 \sin^2\theta$. Then the covariant shift vector components $X_1 = X_r$,$X_2 = X_\theta$ and $X_3 = X_\phi$ are given by:

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] = X^r \quad (89)$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = +r^2 vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \quad (90)$$

$$X_\phi = \gamma_{\phi\phi} X^\phi = r^2 sin^2\theta X^\phi = r^2 sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \quad (91)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi + \alpha\sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \quad (92)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi - \alpha\sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \quad (93)$$

Note that now the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ.

Note that now the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ.

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{X_r+X_\theta+X_\phi+\alpha\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{94}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{X_r+X_\theta+X_\phi-\alpha\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{95}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{U+\alpha\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{96}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{U-\alpha\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{97}$$

$$U=X_r+X_\theta+X_\phi \tag{98}$$

$$U=vs(t)[\sin\phi][2f(r)\ cos\theta]+(+r^2vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta])+r^2sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r))+(rf'(r))] \tag{99}$$

$$U=vs(t)[\sin\phi][2f(r)\ cos\theta]+r^2vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta]+r^2sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r))+(rf'(r))] \tag{100}$$

In three dimensions the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ the Horizon do not occurs unless $\theta=0$,$\cos\theta=1$,$\sin\theta=0$,$\phi=90$,$\sin\phi=1$,$\cos\phi=0$,$r^2d\theta^2=0$ and $r^2\sin^2 d\phi^2=0$ and we recover in this case the problem of the Horizon with the lapse function in the $1+1$ spacetime.See sections 4 and 5 in [13]

Of course this point of view about the Horizons reflects only the geometrical point of view of the alternative Natario warp drive equation for constant speed vs in a $3+1$ spacetime.But we expect that in this case of the alternative Natario warp drive in Polar Coordinates the negative energy density covers the entire bubble.Since the negative energy density have repulsive gravitational behavior(see pg 116 in [17]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.

But now we are in $3D$ Spherical Coordinates and we know that the alternative Natario warp drive vector in this case can be reduced to a alternative Natario warp drive vector in Polar Coordinates. So it is reasonable to suppose that the negative energy in this case may perhaps cover the entire bubble although we dont have the distribution of energy in the $3D$ Spherical Coordinates.

The lapse function is 1 inside and outside the bubble but with a large value in the Natario warped region.See [3].

8 Advantages of the alternative Natario warp drives when compared to the original Natario warp drives due to a different shift vector X^θ in the behavior of the Horizons.

This section do not uses the lapse function because our purpose here is to demonstrate the advantage of the alternative Natario warp drives when compared to the original Natario warp drives due to a different shift vector X^θ in the alternative geometry that affects the whole Horizon.The lapse function is not necessary here(avoiding an even more lengthly and tedious work).

In $3+1$ polar coordinates the difference between the alternative Natario warp drive vector and the original Natario warp drive vector occurs if the motion occurs with the shift vector X^θ in the $3+1$ spacetime.

For the original Natario shift vector is:

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{101}$$

And for the alternative Natario shift vector is:

$$X^\theta = +v_s(2f(r) + (r)f'(r))\sin\theta \tag{102}$$

The equations of the Horizon in $3+1$ polar coordinates where the photon moves in r and θ for the alternative Natario warp drive are the following ones:(see section 4)

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \sqrt{1+r^2}}{1+r^2} \dashrightarrow X_\theta > 0 \tag{103}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta - \sqrt{1+r^2}}{1+r^2} \dashrightarrow X_\theta > 0 \tag{104}$$

Note that $X_r + X_\theta$ in the case of the alternative Natario warp drive in polar coordinates is always positive because X_θ is also always positive and this affects the whole Horizon equations in a "good" way.

But in the case of the original Natario warp drive in polar coordinates $X_r + X_\theta$ is affected by a negative X_θ and this affects the whole Horizon equations in a way that is not so "good":(see eqs 72 and 73 section 6 in [13] but we reproduce the equations below)

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \sqrt{1+r^2}}{1+r^2} \dashrightarrow X_\theta < 0 \tag{105}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta - \sqrt{1+r^2}}{1+r^2} \dashrightarrow X_\theta < 0 \tag{106}$$

In spherical coordinates the difference between the alternative Natario $3D$ warp drive vector and the original Natario $3D$ warp drive vector occurs if the motion occurs with the shift vector X^{θ} in the $3+1$ spacetime.

For the original Natario shift vector is:(see section 3 in [13])

$$X^{\theta} = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{107}$$

And for the alternative Natario shift vector is:

$$X^{\theta} = +vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{108}$$

The Horizon equations for the alternative Natario warp drive in spherical coordinates in three dimensions where the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ are the following ones:(see section 6)

$$\frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi + \sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \dashrightarrow X_\theta > 0 \tag{109}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi - \sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \dashrightarrow X_\theta > 0 \tag{110}$$

Like in the previous case in polar coordinates we have the fact that $X_r + X_\theta + X_\phi$ in the case of the alternative Natario warp drive in $3D$ spherical coordinates is always positive because X_θ is also always positive and this affects the whole Horizon equations in a "good" way.

But in the case of the original Natario warp drive in $3D$ spherical coordinates $X_r + X_\theta + X_\phi$ is affected by a negative X_θ and this affects the whole Horizon equations in a way that is not so "good":(see eqs 99 and 100 section 8 in [13] but we reproduce the equations below)

$$\frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi + \sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \dashrightarrow X_\theta < 0 \tag{111}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi - \sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \dashrightarrow X_\theta < 0 \tag{112}$$

9 Conclusion

In this work we introduced alternative warp drive vectors using independent mathematical techniques.We focused ourselves in the $3D$ spherical coordinates for constant speeds.

Remember that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must uses all the $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.

Our focus were concentrated in the alternative methods to obtain warp drive vectors with the same physical properties of the Natario warp drive.As a matter of fact our alternative warp drives are symmetrical when compared with the Natario warp drive and also when compared with the warp drives presented in [2].

One the major drawbacks concerning warp drives is the problem of the Horizons(causally disconnected portions of spacetime) in which an observer in the center of the bubble cannot signal nor control the front part of the bubble. The behavior of a photon sent to the front of the warp bubble in the case of the alternative Natario warp drive with constant velocity and a lapse function was also one of the main purposes of this work.We presented the behavior of a photon sent to the front of the bubble in the alternative Natario warp drive $3+1$ spacetimes with and without the lapse function using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provided here the step by step mathematical calculations in order to outline the final results found in our work which are the following ones:

For the case of the lapse function or the $3+1$ spacetime the Horizon do not exists at all.Due to the extra terms in the lapse function or the presence of the $3D$ dimensions that affects the whole spacetime geometry these solutions allows to circumvent the problem of the Horizon.

In this work we developed Horizons but for constant velocities and the alternative Natario warp drive have geometrical advantages when compared with the original one.

The Natario warp drive is probably the best candidate(known until now) for an interstellar space travel considering the fact that a spaceship in a real superluminal spaceflight will encounter(or collide against) hazardous objects(asteroids,comets,interstellar dust and debris etc) and the Natario spacetime offers an excellent protection to the crew members as depicted in the works [7],[8],[9] and [10].However since these works were based in the original Natario 2001 paper this line of reason must be extended to encompass the alternative $3D$ spherical coordinates symmetrical warp drive vector

In the original Natario warp drive in polar coordinates the negative energy density with repulsive gravitational behavior lies in front of the ship protecting the ship from hazardous interstellar collisions even in a $1+1$ spacetime.(see Appendices E and F).

Our proposed alternative Natario warp drive vectors in polar or $3D$ spherical coordinates when reduced to a $1+1$ spacetime gives the same results of the original Natario warp drive in polar coordinates in a $1+1$ spacetime so it is reasonable to suppose that the negative energy density with repulsive gravitational behavior lies in front of the ship also in these $3+1$ spacetimes.

We dont know the negative energy density distribution of the alternative warp drive vector in $3D$ spherical coordinates because we dont have the *Maple* or *Mathematica* programs needed to handle all the tensor algebra

For the *Maple* or *Mathematica* see pg 342 in [15], pg 276 in [19],pgs $454, 457, 560$ in [20] pg 98 in [21],pg 178 in [22],see also Appendix C pgs $551 - 555$ in [20].

The application of the alternative $3D$ spherical coordinates symmetric warp drive vector with variable speeds to the ADM(Arnowitt-Dresner-Misner) formalism equations in General Relativity using the approach of MTW(Misner-Thorne-Wheeler)-Alcubierre resembling the works [3],[4],[5] and [6] will appear in a future work.

The warp drive as an artificial superluminal geometric tool that allows to travel faster than light may well have an equivalent in the Nature.According to the modern Astronomy the Universe is expanding and as farther a galaxy is from us as faster the same galaxy recedes from us.The expansion of the Universe is accelerating and if the distance between us and a galaxy far and far away is extremely large the speed of the recession may well exceed the light speed limit.(see pg 98 in [25] and pg 377 in [26]).

For the experimental verification of the acceleration of the Universe see for example the bottom of pg 355 and top of pg 356 eq 8.155 in [28].

10 Appendix A:mathematical demonstration of the alternative Natario warp drive equation for a constant speed vs in the original $3+1$ ADM Formalism according to MTW and Alcubierre

General Relativity describes the gravitational field in a fully covariant way using the geometrical line element of a given generic spacetime metric $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ where do not exists a clear difference between space and time.This generical form of the equations using tensor algebra is useful for differential geometry where we can handle the spacetime metric tensor $g_{\mu\nu}$ in a way that keeps both space and time integrated in the same mathematical entity (the metric tensor) and all the mathematical operations do not distinguish space from time under the context of tensor algebra handling mathematically space and time exactly in the same way.

However there are situations in which we need to recover the difference between space and time as for example the evolution in time of an astrophysical system given its initial conditions.

The $3+1$ ADM formalism allows ourselves to separate from the generic equation $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ of a given spacetime the 3 dimensions of space and the time dimension.(see pg 64 in [16])

Consider a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg 65) in [16].

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg 65 in [16])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg 66 in [16])
(see also fig 21.2 pg 506 in [15] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg 507 in [15])[1]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface

- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.

- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

[1] we adopt the Alcubierre notation here

Combining the eqs (21.40),(21.42) and (21.44) pgs 507 and 508 in [15]
with the eqs (2.2.5) and (2.2.6) pg 67 in [16] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{113}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{114}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix} \tag{115}$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{116}$$

But since $dl^2 = \gamma_{ij} dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii} dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{117}$$

$$(dx^i + \beta^i dt)^2 = (dx^i)^2 + 2\beta^i dx^i dt + (\beta^i dt)^2 \tag{118}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\gamma_{ii}\beta^i dx^i dt + \gamma_{ii}(\beta^i dt)^2 \tag{119}$$

$$\beta_i = \gamma_{ii}\beta^i \tag{120}$$

$$\gamma_{ii}(\beta^i dt)^2 = \gamma_{ii}\beta^i\beta^i dt^2 = \beta_i\beta^i dt^2 \tag{121}$$

$$(dx^i)^2 = dx^i dx^i \tag{122}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{123}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{124}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{125}$$

Note that the expression above is exactly the eq (2.2.4) pg 67 in [16].It also appears as eq 1 pg 3 in [14].

With the original equations of the $3+1$ ADM formalism given below:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{126}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{127}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ii} - \frac{\beta^i\beta^i}{\alpha^2} \end{pmatrix} \tag{128}$$

and suppressing the lapse function making $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{129}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{130}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -1 & \beta^i \\ \beta^i & \gamma^{ii} - \beta^i\beta^i \end{pmatrix} \tag{131}$$

changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-1 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{132}$$

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{133}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{134}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{135}$$

Remember that the equations given above corresponds to the generic warp drive metric given below:

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{136}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [1])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{137}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{138}$$

Comparing all these equations

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{139}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{140}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{141}$$

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{142}$$

With

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{143}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i\beta^i = X_i X^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{144}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{145}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{146}$$

Looking to the equation of the alternative Natario vector nWD in Polar Coordinates:

$$nWD = X^r e_r + X^\theta e_\theta \tag{147}$$

With the contravariant shift vector components X^{rs} and X^θ given by:

$$X^r = 2v_s f(r) \cos\theta \tag{148}$$

$$X^\theta = +v_s(2f(r) + (r)f'(r)) \sin\theta \tag{149}$$

But remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_r and X_θ are given by:

$$X_i = \gamma_{ii} X^i \tag{150}$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s f(r) \cos\theta = X^r \tag{151}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = +r^2 v_s(f(r) + (r)f'(r)) \sin\theta \tag{152}$$

The equations of the alternative Natario warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{153}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{154}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{155}$$

Then the equation of the alternative Natario warp drive spacetime in Polar Coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{156}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{157}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{158}$$

Considering now the alternative Natario warp drive vector in $3D$ Spherical Coordinates with a constant speed vs nX given by::

$$nWD = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{159}$$

With the contravariant shift vector components X^{rs}, X^θ and X^ϕ given by:

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{160}$$

$$X^\theta = +vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{161}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{162}$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$ with $\gamma_{rr} = 1$, $\gamma_{\theta\theta} = r^2$ and $\gamma_{\phi\phi} = r^2\sin^2\theta$. Then the covariant shift vector components X_r,X_θ and X_ϕ are given by:

$$X_i = \gamma_{ii}X^i \tag{163}$$

$$X_r = \gamma_{rr}X^r = X_r = \gamma_{rr}X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] = X^r \tag{164}$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = r^2 X^\theta = +r^2 vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{165}$$

$$X_\phi = \gamma_{\phi\phi}X^\phi = r^2 sin^2\theta X^\phi = r^2 sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{166}$$

Then the equation of the alternative Natario warp drive spacetime in $3D$ Spherical Coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{167}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr dt + X_\theta d\theta dt + X_\phi d\phi dt) - dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{168}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{169}$$

11 Appendix B:mathematical demonstration of the alternative Natario warp drive equation for a constant speed vs in the original $3+1$ ADM Formalism according to MTW and Alcubierre using a lapse function α

This Appendix is a continuation of the Appendix A except for the fact that we do not suppress the lapse function here.Combining the eqs (21.40),(21.42) and (21.44) pgs $[507, 508]$ in [15]
with the eqs (2.2.5) and (2.2.6) pgs [67] in [16] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{170}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{171}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{172}$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{173}$$

But since $dl^2 = \gamma_{ij}dx^i dx^j$ is the ADM induced metric and must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{174}$$

Expanding the square term and recombining all the terms we have:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{175}$$

Note that the expression above is exactly the eq (2.2.4) pgs [67] in [16].It also appears as eq 1 pg 3 in [14].

Changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-\alpha^2 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{176}$$

$$ds^2 = (\alpha^2 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{177}$$

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{178}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i\beta^i = X_i X^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{179}$$

Then the equation of the alternative Natario warp drive spacetime with a constant speed vs in the original $3+1$ ADM formalism with a lapse function is given by:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{180}$$

Inserting the components of the alternative Natario vector nWD in Polar Coordinates:

$$nWD = X^r e_r + X^\theta e_\theta \tag{181}$$

We have:

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{182}$$

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{183}$$

The lapse function is equal to 1 inside and outside the Natario warp bubble while having large values in the Natario warped region.See [3]

Considering now the alternative Natario warp drive vector in $3D$ Spherical Coordinates with a constant speed vs nX given by::

$$nWD = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{184}$$

We have:

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr dt + X_\theta d\theta dt + X_\phi d\phi dt) - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{185}$$

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{186}$$

12 Appendix C:Generic quadratic forms in the $3 + 1$ ADM spacetime without the lapse function.

The alternative Natario warp drive equations with signature $(+, -, -, -)$ that obeys the original $3 + 1$ ADM formalism are given below:

in Polar Coordinates:(see Appendix A).

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{187}$$

in $3D$ Spherical Coordinates:(see also Appendix A).

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{188}$$

Using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity,Horizons can be easily computed for the dimensionally reduced $1 + 1$ spacetime versions of these equations because only the quadratic form dr^2 exists but in the $3+1$ spacetime we have the presence of 3 quadratic forms respectively dr^2,$r^2 d\theta^2$ and $r^2 \sin^2\theta d\phi^2$.Algebraic solutions for the null-like geodesics $ds^2 = 0$ of General Relativity of the $3+1$ equations above are extremely difficult due to the presence of these 3 quadratic forms considering solutions for each quadratic form dr^2 or $r^2 d\theta^2$ or $r^2 \sin^2\theta d\phi^2$ isolated.

The best effort to solve the null-like geodesics $ds^2 = 0$ in the case of the $3 + 1$ spacetime equations given above is to find out a solution that encompasses all the 3 quadratic forms dr^2 and $r^2 d\theta^2$ and $r^2 \sin^2\theta d\phi^2$ grouped together.

We will demonstrate all the required mathematics step by step.

Back to the $3 + 1$ ADM formalism compact generic equation given below:(see Appendix A)

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{189}$$

Expanding the equation above we have:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{190}$$

The null-like geodesics $ds^2 = 0$ is:

$$0 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{191}$$

Dividing by dt^2 we have:

$$0 = (1 - X_i X^i) + 2X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{dx^i dx^i}{dt^2} \tag{192}$$

$$0 = (1 - X_i X^i) + 2X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{(dx^i)^2}{dt^2} \tag{193}$$

$$0 = (1 - X_i X^i) + 2X_i \frac{dx^i}{dt} - \gamma_{ii}(\frac{dx^i}{dt})^2 \quad (194)$$

Introducing the term U^i as being:

$$U^i = \frac{dx^i}{dt} \quad (195)$$

We have now a generic quadratic form in the term U^i:

$$0 = (1 - X_i X^i) + 2X_i U^i - \gamma_{ii}(U^i)^2 \quad (196)$$

Rearranging the terms in this quadratic form we have:

$$\gamma_{ii}(U^i)^2 - 2X_i - (1 - X_i X^i) = 0 \quad (197)$$

$$\gamma_{ii}(U^i)^2 - 2X_i + (X_i X^i - 1) = 0 \quad (198)$$

The solution of this generic quadratic form in the term U^i is given by:

$$U^i = \frac{2X_i \pm \sqrt{[-2X_i]^2 - 4[\gamma_{ii}(X_i X^i - 1)]}}{2\gamma_{ii}} = \frac{2X_i \pm \sqrt{4[X_i]^2 - 4[\gamma_{ii}(X_i X^i] + 4[\gamma_{ii}]}}{2\gamma_{ii}} \quad (199)$$

But since:

$$X_i = \gamma_{ii} X^i \quad (200)$$

We have:

$$U^i = \frac{2X_i \pm \sqrt{4[X_i]^2 - 4[X_i]^2 + 4[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2X_i \pm \sqrt{+4[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2X_i \pm 2\sqrt{[\gamma_{ii}]}}{2\gamma_{ii}} \quad (201)$$

$$U^i = \frac{2X_i \pm 2\sqrt{\gamma_{ii}}}{2\gamma_{ii}} = \frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}} \quad (202)$$

At last we have the final solution of this generic quadratic form in the term U^i given by:

$$U^i = \frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}} \quad (203)$$

But this expression actually means:

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{\sum_{i=1}^{3} X_i \pm \sum_{i=1}^{3} \sqrt{\gamma_{ii}}}{\sum_{i=1}^{3} \gamma_{ii}} = \frac{\sum_{i=1}^{3} X_i \pm \sqrt{\sum_{i=1}^{3} \gamma_{ii}}}{\sum_{i=1}^{3} \gamma_{ii}} \quad (204)$$

The subscript γ_{ii} is inside the root $\sqrt{\gamma_{ii}}$ so the sum must be taken also inside the root.(see pg 5,pg 227 section 7.3 and pg 241 section 7.10 in [27]).Then $\sum_{i=1}^{3} \sqrt{\gamma_{ii}}$ actually must be $\sqrt{\sum_{i=1}^{3} \gamma_{ii}}$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 + U^3 = \frac{X_1 + X_2 + X_3 \pm \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \quad (205)$$

The generic quadratic form in the term U^i for the null-like geodesics $ds^2 = 0$ is given by:

$$U^i = \frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{206}$$

Expanding the terms in the expression above we have:

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 + U^3 = \frac{X_1 + X_2 + X_3 \pm \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{207}$$

The line element in the $3+1$ ADM spacetime without the lapse function is:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{208}$$

Expanding the terms in the expression above we have:

$$ds^2 = (1 - X_1 X^1 - X_2 X^2 - X_3 X^3)dt^2 + 2(X_1 dx^1 + X_2 dx^2 + X_3 dx^3)dt - \gamma_{11} dx^1 dx^1 - \gamma_{22} dx^2 dx^2 - \gamma_{33} dx^3 dx^3 \tag{209}$$

$$ds^2 = (1 - X_1 X^1 - X_2 X^2 - X_3 X^3)dt^2 + 2(X_1 dx^1 + X_2 dx^2 + X_3 dx^3)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 - \gamma_{33}(dx^3)^2 \tag{210}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{211}$$

$$U^i = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{212}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 + U^3 = \frac{X_1 + X_2 + X_3 + \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{213}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 + U^3 = \frac{X_1 + X_2 + X_3 - \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{214}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 + \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{215}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 - \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{216}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $3+1$ spacetime equations given above with the solution that encompasses all the 3 quadratic forms $(dx^1)^2$,$(dx^2)^2$ and $(dx^3)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt}$.

The line element in the $2+1$ ADM spacetime without the lapse function is:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{217}$$

Expanding the terms in the expression above we have:

$$ds^2 = (1 - X_1 X^1 - X_2 X^2)dt^2 + 2(X_1 dx^1 + X_2 dx^2)dt - \gamma_{11} dx^1 dx^1 - \gamma_{22} dx^2 dx^2 \tag{218}$$

$$ds^2 = (1 - X_1 X^1 - X_2 X^2)dt^2 + 2(X_1 dx^1 + X_2 dx^2)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 \tag{219}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{220}$$

$$U^i = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{221}$$

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 = \frac{X_1 + X_2 + \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{222}$$

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 = \frac{X_1 + X_2 - \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{223}$$

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 + \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{224}$$

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 - \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{225}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $2+1$ spacetime equations given above with the solution that encompasses all the 2 quadratic forms $(dx^1)^2$ and $(dx^2)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt}$.

The line element in the $1+1$ ADM spacetime without the lapse function is:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{226}$$

Expanding the terms in the expression above we have:

$$ds^2 = (1 - X_1 X^1)dt^2 + 2(X_1 dx^1)dt - \gamma_{11} dx^1 dx^1 \tag{227}$$

$$ds^2 = (1 - X_1 X^1)dt^2 + 2(X_1 dx^1)dt - \gamma_{11}(dx^1)^2 \tag{228}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{229}$$

$$U^i = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{230}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 = \frac{X_1 + \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{231}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 = \frac{X_1 - \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{232}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} = \frac{X_1 + \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{233}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} = \frac{X_1 - \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{234}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $1+1$ spacetime equations given above with the solution that encompasses the single quadratic forms $(dx^1)^2$.The solution is given in function of $\frac{dx^1}{dt}$.

13 Appendix D:Generic quadratic forms in the $3+1$ ADM spacetime with the lapse function.

This Appendix is a continuation of the Appendix C but this time we consider the lapse function.We provide all the step by step mathematical calculations.

Back to the $3+1$ ADM formalism compact generic equation with the lapse function given below:(see Appendix B)

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{235}$$

Expanding the equation above we have:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{236}$$

The null-like geodesics $ds^2 = 0$ is:

$$0 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{237}$$

Dividing by dt^2 we have:

$$0 = (\alpha^2 - X_i X^i) + 2X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{dx^i dx^i}{dt^2} \tag{238}$$

$$0 = (\alpha^2 - X_i X^i) + 2X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{(dx^i)^2}{dt^2} \tag{239}$$

$$0 = (\alpha^2 - X_i X^i) + 2X_i \frac{dx^i}{dt} - \gamma_{ii} (\frac{dx^i}{dt})^2 \tag{240}$$

Introducing the term U^i as being:

$$U^i = \frac{dx^i}{dt} \tag{241}$$

We have now a generic quadratic form in the term U^i:

$$0 = (\alpha^2 - X_i X^i) + 2X_i U^i - \gamma_{ii}(U^i)^2 \tag{242}$$

Rearranging the terms in this quadratic form we have:

$$\gamma_{ii}(U^i)^2 - 2X_i - (\alpha^2 - X_i X^i) = 0 \tag{243}$$

$$\gamma_{ii}(U^i)^2 - 2X_i + (X_i X^i - \alpha^2) = 0 \tag{244}$$

The solution of this generic quadratic form in the term U^i is given by:

$$U^i = \frac{2X_i \pm \sqrt{[-2X_i]^2 - 4[\gamma_{ii}(X_i X^i - \alpha^2)]}}{2\gamma_{ii}} = \frac{2X_i \pm \sqrt{4[X_i]^2 - 4[\gamma_{ii}(X_i X^i] + 4\alpha^2[\gamma_{ii}]}}{2\gamma_{ii}} \tag{245}$$

But since:

$$X_i = \gamma_{ii} X^i \tag{246}$$

We have:

$$U^i = \frac{2X_i \pm \sqrt{4[X_i]^2 - 4[X_i]^2 + 4\alpha^2[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2X_i \pm \sqrt{+4\alpha^2[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2X_i \pm 2\alpha\sqrt{[\gamma_{ii}]}}{2\gamma_{ii}} \tag{247}$$

$$U^i = \frac{2X_i \pm 2\alpha\sqrt{\gamma_{ii}}}{2\gamma_{ii}} = \frac{X_i \pm \alpha\sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{248}$$

At last we have the final solution of this generic quadratic form for the null-like geodesics $ds^2 = 0$ in the term U^i given by:

$$U^i = \frac{X_i \pm \alpha\sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{249}$$

The solution have two roots:

$$U^i = \frac{X_i + \alpha\sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{250}$$

$$U^i = \frac{X_i - \alpha\sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{251}$$

The subscript γ_{ii} is inside the root $\sqrt{\gamma_{ii}}$ so the sum must be taken also inside the root.(see pg 5,pg 227 section 7.3 and pg 241 section 7.10 in [27]).Then $\sum_{i=1}^{3} \sqrt{\gamma_{ii}}$ actually must be $\sqrt{\sum_{i=1}^{3} \gamma_{ii}}$

Adapting the results from the previous section we have for the equation of the $3+1$ spacetime in the ADM formalism:

$$ds^2 = (\alpha^2 - X_1X^1 - X_2X^2 - X_3X^3)dt^2 + 2(X_1dx^1 + X_2dx^2 + X_3dx^3)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 - \gamma_{33}(dx^3)^2 \tag{252}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 + \alpha\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{253}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 - \alpha\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{254}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $3+1$ spacetime equations given above with the solution that encompasses all the 3 quadratic forms $(dx^1)^2$,$(dx^2)^2$ and $(dx^3)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt}$.

Adapting the results from the previous section we have for the equation of the $2+1$ spacetime in the ADM formalism:

$$ds^2 = (\alpha^2 - X_1 X^1 - X_2 X^2)dt^2 + 2(X_1 dx^1 + X_2 dx^2)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 \tag{255}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 + \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{256}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 - \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{257}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $2+1$ spacetime equations given above with the solution that encompasses all the 2 quadratic forms $(dx^1)^2$ and $(dx^2)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt}$.

Adapting the results from the previous section we have for the equation of the $1+1$ spacetime in the ADM formalism:

$$ds^2 = (\alpha^2 - X_1 X^1)dt^2 + 2(X_1 dx^1)dt - \gamma_{11}(dx^1)^2 \tag{258}$$

$$\frac{dx^1}{dt} = \frac{X_1 + \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} \tag{259}$$

$$\frac{dx^1}{dt} = \frac{X_1 - \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} \tag{260}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $1+1$ spacetime equations given above with the solution that encompasses the single quadratic form $(dx^1)^2$.The solution is given in function of $\frac{dx^1}{dt}$.

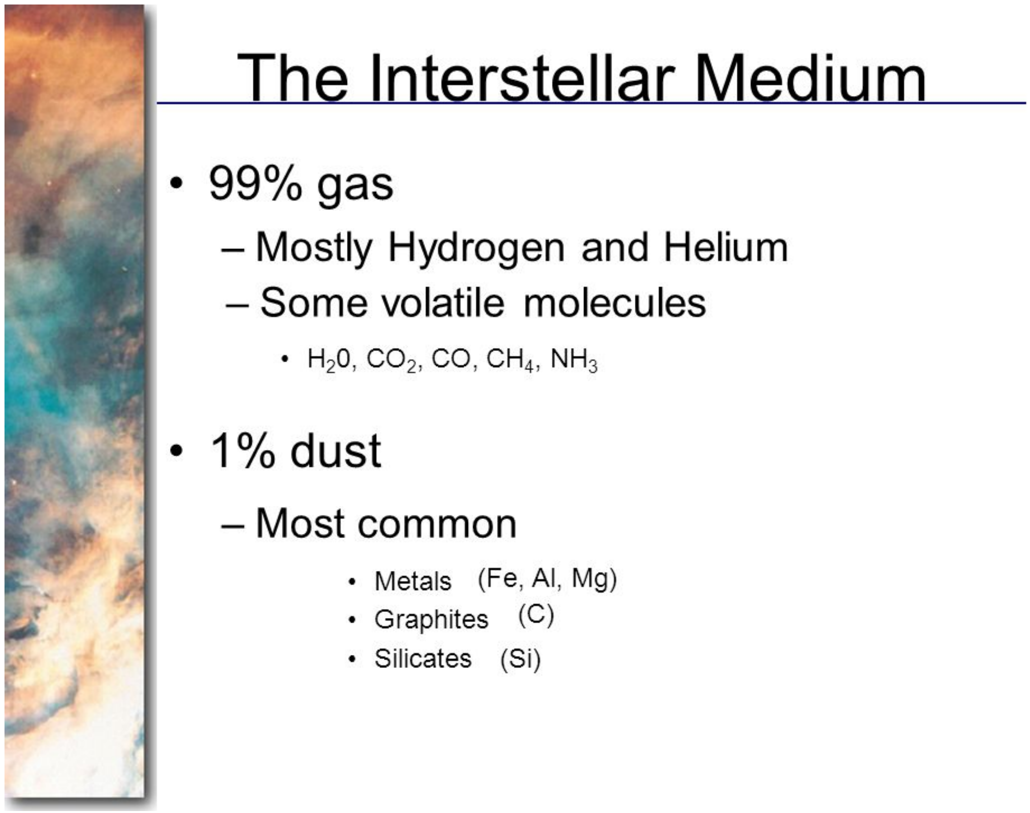

Figure 1: Composition of the Interstellar Medium IM(Source:Internet)

14 Appendix E:Composition of the Interstellar Medium IM

The problem of collisions between a warp drive spaceship moving at superluminal velocity and the potentially dangerous particles from the Interstellar Medium IM is not new.

It was first noticed in 1999 in the work of Chad Clark,Will Hiscock and Shane Larson(see [32]).Later on in 2010 it appeared again in the work of Carlos Barcelo,Stefano Finazzi and Stefano Liberatti(see [33]).In 2012 the same problem of collisions against hazardous IM particles appeared in the work of Brendan McMonigal,Geraint Lewis and Philip O'Byrne(see [29]).

The last work addressing interstellar collisions was the work in ([30]) in 2022.It covers the analysis of Siyu Bian ,Yi Wang,Zun Wang and Mian Zhu.

All these works uses the geometry of the original Alcubierre warp drive 1994 paper in [14] and the results outlined in these works are completely correct.

From the picture above we all can see that the Interstellar Medium IM is not empty and a collision at superluminal speeds is highly dangerous.

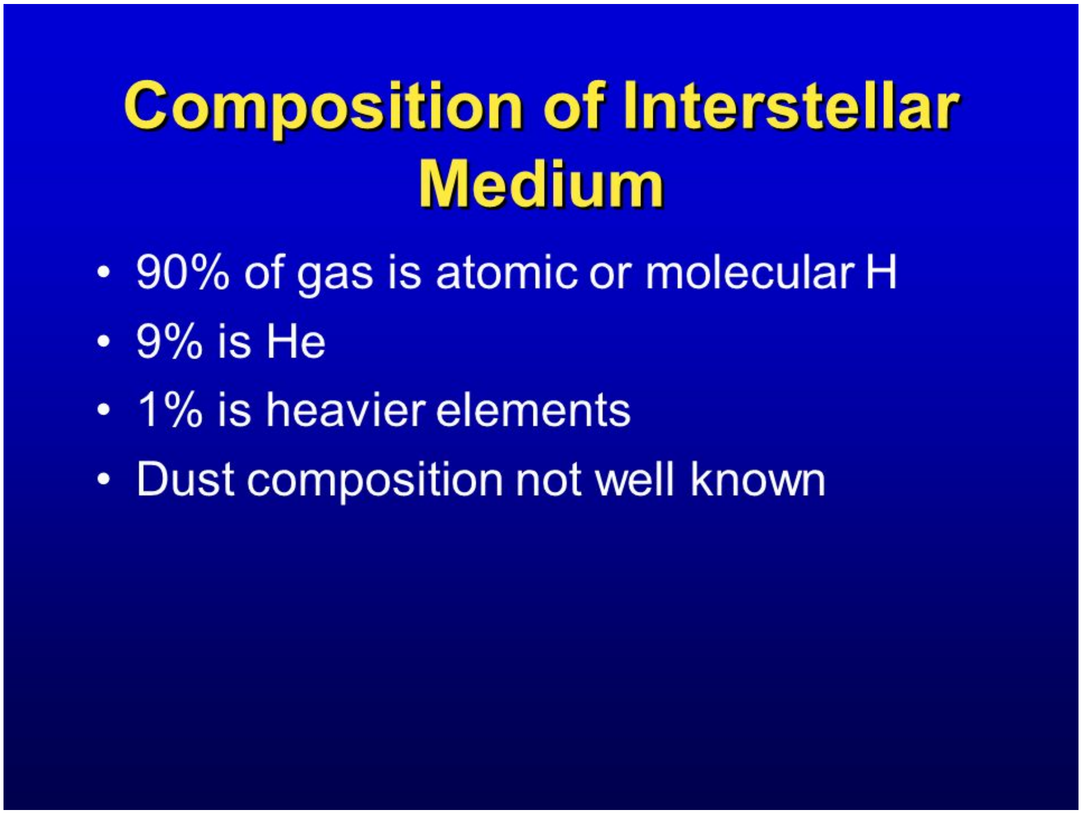

Figure 2: Composition of the Interstellar Medium IM(Source:Internet)

15 Appendix F:Composition of the Interstellar Medium IM

The original Natario warp drive is probably the best candidate(known until now) for an interstellar space travel considering the fact that a spaceship in a real superluminal interstellar spaceflight will encounter(or collide against) hazardous objects(asteroids,comets,interstellar dust and debris etc) and due to a different distribution of the negative energy in front of the ship with repulsive gravitational behavior(see pg 116 in [17]) deflecting all the incoming hazardous particles of the Interstellar Medium.The Natario spacetime offers an excellent protection to the crew members as depicted in the works [7],[8] and specially [9],[10] and [31].

The alternative Natario warp drive have the negative energy also in front of the ship even in a $1+1$ spacetime protecting the ship and the crew members from collisions against the hazardous components of the Interstellar Medium IM that according with the picture above is not empty.

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