THE EXISTENTIAL PROFILE WARNING ON ADDITION CHAINS

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Abstract. Let

$$\mathcal{C}_n : s_o = 1 < s_1 = 2 < \cdots < s_h = n$$

be an addition chain leading to n. Define the normalized profile

$$v_n(x) := \frac{s_{\lfloor xh \rfloor}}{n}$$

for any $x \in [0,1]$ and set $x_i := \frac{i}{h}$. We show that for any fixed $x \in [0,1]$ there exists an index i with $0 \leq i \leq h$ such that

$$x\frac{h}{n} \leq v_n(x_i) \leq x + \frac{1}{h} = x + o(1).$$

This implies that no matter how an addition chain is built, at each fraction $x \in [0,1]$, there is some term whose normalized size is in the interval $[x\frac{h}{n}, x + o(1)]$.

1. Introduction

Let

$$\mathcal{C}_n : s_o = 1 < s_1 = 2 < \cdots < s_h = n$$

be an addition chain leading to n and normalize

$$\frac{\mathcal{C}_n}{n} : \frac{s_o}{n} < \frac{s_1}{n} < \cdots < 1$$

which is a non-decreasing sequence of real numbers in the interval $[0,1]$. Define the normalized profile

$$v_n(x) := \frac{s_{\lfloor xh \rfloor}}{n}$$

as a function of x for any $x \in [0,1]$ and set $x_i := \frac{i}{h}$. For $0 \leq i \leq h$ and with $x_i := \frac{i}{h}$, we can write

$$v_n(x_i) := v_n(\frac{i}{h}) = \frac{s_i}{n} \geq \frac{i+1}{n}.$$

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The following intermediate observation tracks the growth of terms in addition chain at each step.

Lemma 1.1 (Local linear control)**.** *For each term s_j in an addition chain leading to n of length h, we have*

$$s_j \leq \frac{n-1}{h}(j+1)$$

for all $0 \leq j \leq h-1$.

Proof. Let s_j $(0 \leq j \leq h)$ be a term indexed by the j^{th} step in an addition chain leading to n. We note that in an addition chain $s_j \leq 2^j$ holds for all $0 \leq j \leq h$. It follows that $\frac{s_j h}{j+1} \leq \frac{2^j h}{j+1}$. Now, put

$$f(j) := \frac{2^j h}{j+1}.$$

Since $\frac{f(j+1)}{f(j)} = 2(\frac{j+1}{j+2}) \geq 1$ for each $j \in [0, h-1]$, it follows that $f(j)$ is monotone increasing on $[0, h-1]$ and therefore must reach a maximum at $j = h-1$. This implies that

$$\max_{0 \leq j \leq h-1} f(j) = 2^{h-1} = \max(s_{h-1}).$$

Consequently, we have

$$\frac{s_j h}{j+1} \leq \max_{0 \leq j \leq h-1} f(j) = 2^{h-1} = \max(s_{h-1}) \leq n-1$$

which proves the inequality stated. □

2. Main result

In this section, we show that for each addition chain and for every $x \in [0,1]$, there is at least an index $0 \leq i \leq h$ for which the normalized profile $v_n(x_i)$ as a function of x_i is restricted in an interval defined by x.

Theorem 2.1 (The existential profile warning)**.** *Let*

$$\mathcal{C}_n : s_o = 1 < s_1 = 2 < \cdots < s_h = n$$

be an addition chain leading to n. Define the normalized profile

$$v_n(x) := \frac{s_{\lfloor xh \rfloor}}{n}$$

for any $x \in [0,1]$ and set $x_i := \frac{i}{h}$. Then for any fixed $x \in [0,1]$ there exists an index i with $0 \leq i \leq h$ such that

$$x\frac{h}{n} \leq v_n(x_i) \leq x + \frac{1}{h} = x + o(1).$$

Proof. For $0 \leq i \leq h$ and with $x_i := \frac{i}{h}$, we can write

$$v_n(x_i) := v_n(\frac{i}{h}) = \frac{s_i}{n}.$$

Invoking Lemma 1.1, we deduce

$$v_n(x_i) = \frac{s_i}{n} \leq \frac{i+1}{h} - \frac{i+1}{nh} < \frac{i}{h} + \frac{1}{h}.$$

Now for a fixed $x \in [0,1]$, put $i = \lfloor xh \rfloor$. Then we can write

$$xh - 1 \leq i \leq xh \iff x - \frac{1}{h} \leq x_i := \frac{i}{h} \leq x.$$

It follows

$$v_n(x_i) \leq x + \frac{1}{h}.$$

With $\lceil \log_2 n \rceil \leq h \leq n$, we get

$$v_n(x_i) \leq x + \frac{1}{\log_2 n} = x + o(1).$$

Using the inequality $\frac{xh-1}{n} \leq \frac{i}{n} \leq \frac{xh}{n}$, we write

$$v_n(x_i) := v_n(\frac{i}{h}) = \frac{s_i}{n} \geq \frac{i+1}{n} \geq x\frac{h}{n}$$

□

Remark 2.2. The existential profile warning implies that no matter how an addition chain is built, at each fraction $x \in [0,1]$, there is some term in the chain whose normalized size must remain in the interval $[x\frac{h}{n}, x + o(1)]$.

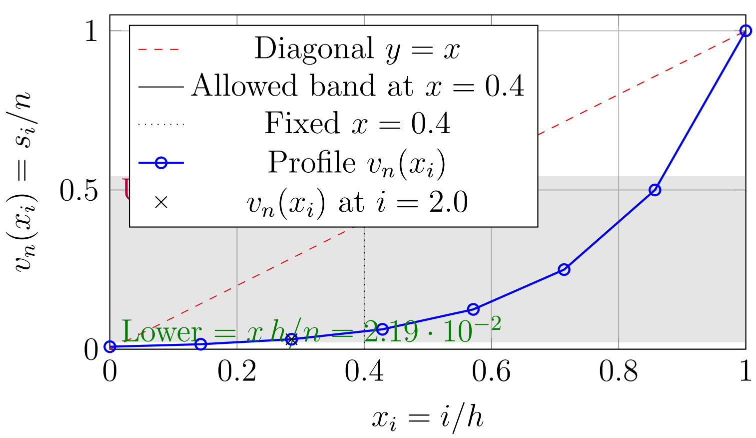

FIGURE 1. Illustration of the existential profile warning at $x = 0.4$ (doubling chain, $n = 128$).

Corollary 2.3 (No uniform exotic profile above the diagonal)**.** *Let* $f : [0,1] \longrightarrow \mathbb{R}$ *be any function that satisfies* $f(x) > x$ *for all* $x \in [0,1]$. *Then for every addition chain*

$$s_o = 1 < s_1 = 2 < \cdots < s_h = n$$

with normalized profile v_n *and all sufficiently large* n *it is impossible that*

$$v_n(x_i) \geq f(x)$$

for all $x_i := \frac{i}{h}$.

Proof. Fix $x \in [0,1]$. With $f(x) - x > 0$, we choose N sufficiently large so that for all $n \geq N$ we have

$$x + \frac{1}{h} < f(x)$$

with $h := h(n)$. By the existential profile warning (Theorem 2.1), at $i = \lfloor xh \rfloor$, we must have

$$v_n(x_i) < x + \frac{1}{h} < f(x)$$

for all $n \geq N$. This shows that it is impossible to have $v_n(x_i) \geq f(x)$ for all $x_i := \frac{i}{h}$. □

References

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