Deformed Lie Products and Involution

Second Part: in four dimensional spaces

©Thierry PERIAT - no affiliation

8 July 2025

A previous document laid the foundations for the study of involution in any three-dimensional spaces, [c]. This exploration continues the exploration of the topic in focusing now attention on four-dimensional spaces. The repetition of a deformed Lie product on a given argument carries two concepts with it: (i) the eventual invariance of this argument and (ii) the existence of an involution. The work discovers two distinct classes of decomposition without residual part for each deformed Lie product. It explains why only one of both (the simplest one) can characterize the involution when the deforming cube is anti-symmetric and anti-reduced. The document also starts a confrontation between the simplest representation and the electromagnetic duality in Maxwell's vacuum; © Thierry PERIAT: Deformed Lie products and involution - second part: in four-dimensional spaces.

Contents

1 The mathematical context

1.1 Introduction

This document is the second part of an exploration focusing attention on the concept of involution when the latter is applied to deformed Lie products.
The discussion concerns elements in a four-dimensional space $\mathrm{V}_4 = \{\mathbb{C} \otimes \mathrm{E}(4, \mathbb{R}), \mathrm{A} \in \boxplus^-(\mathbb{C})\}$.
With different words: the diverse Lie products (i) are deformed by anti-symmetric cubes with knots in $\mathbb{C}$ and (ii) involve vectors with four components in $\mathbb{C}$.This work is looking for patterns and rules characterizing the repeated action of a given but generic function $\mathrm{f}(\mathrm{A}, {}^{(4)}\mathbf{a}) = [{}^{(4)}\mathbf{a}, {}^{(4)}\mathbf{...}]_A$.

Definition 1.1. *The function f(A, $^{(4)}\boldsymbol{a}$).*

Let consider a given element $^{(4)}\mathbf{a}$ in V_4; per definition, the function $[{}^{(4)}\mathbf{a}, {}^{(4)}\mathbf{...}]_A$ is an element in $\mathrm{End}(\mathrm{V}_4)$ such that:

$$\forall\, {}^{(4)}\mathbf{x}_0 \in V_4 \xrightarrow{[\mathbf{a},\, \dots]_A} {}^{(4)}\mathbf{x}_1 = [{}^{(4)}\mathbf{a}, {}^{(4)}\mathbf{x}_0]_A \in V_4$$

Due to the definition of deformed Lie products, each of the four components of $^{(4)}\mathbf{x}_1$ is represented in $\mathrm{V^*}_4$, the dual space of V_4, by the generic relation:

$$\forall\, \alpha,\, \beta,\, \chi \in Ind_4 = \{0,\, 1,\, 2,\, 3\} : x_1^\chi = \sum_{\alpha<\beta} A^\chi_{\alpha\beta} \,.\, (a^\alpha \,.\, x_0^\beta - a^\beta \,.\, x_0^\alpha)$$

Remark 1.1. *Concerning the components of any deformed Lie product acting on elements in V_4.*

Each component is a linear combination of six terms, due to the fact that one can only form six pairs of indices (α, β) such that $\alpha < \beta$ when α and β are elements in Ind_4.

Concretely, the allowed pairs are labeled: $(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)$.
Hence, already at this stage, one can guess the existence of an indirect link between the components of the deformed Lie product $[{}^{(4)}\mathbf{a}, {}^{(4)}\mathbf{x}_0]_A$ and the wedge product ${}^{(4)}\mathbf{a} \wedge {}^{(4)}\mathbf{x}_0$. At least at a first glance, each component of a given deformed Lie product looks like a weighted sum of the components of the corresponding wedge product. The knots of the deforming cube are responsible for this weighting action.

Remark 1.2. *Concerning the number of knots in any anti-symmetric deforming cube A in $\boxplus^{-}(\mathbb{C})$.*

A cube is a set of mathematical objects (e.g.: scalars) which are disposed along knots, a little bit like atoms in a crystal with a cubic structure. This construction can be decomposed in three different manners:

1. from front to back
2. from the left to the right.
3. from the bottom to the top.

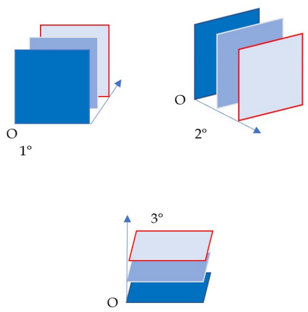

Figure 1: A cube as a set of matrices - in three different manners

Any given anti-symmetric cube A in $\boxplus^{-}(\mathbb{C})$ contains only at most $4 \times 6 = 24$ different non-necessarily vanishing complex numbers. Therefore, they can be regrouped inside an element in M(4 × 6, ℂ) or in M(4 × 6, ℂ) which - in opposition with what can be done when the discussion is developed with elements in a three-dimensional space- are no more square matrices.

1.2 The representations of deformed Lie products

Proposition 1.1. *There always exists a representation in $M(4 \times 6, \mathbb{C})$ and another one in $M(6 \times 4, \mathbb{C})$ for any given vector ${}^{(4)}\boldsymbol{a}$ in V_4. These representations give the opportunity to respectively represent any classical wedge product ${}^{(4)}\boldsymbol{a} \wedge {}^{(4)}\boldsymbol{x}_0$ either in M(1× 6, ℂ) or in M(6× 1, ℂ).*

Proof. As a matter of mathematical facts, any given vector ${}^{(4)}\mathbf{a}$ in V_4 has a dual representation in the dual space $V^*{}_4$ and one can always decide to write:

$$\forall\, {}^{(4)}\mathbf{a} \in M(1 \times 4, \mathbb{C}) \equiv V_4^*, \exists\, [M^*(\mathbf{a})] \in M(4 \times 6, \mathbb{C}) :$$

$$\underbrace{< \mathbf{x}_0|}_{\in M(1\times 4,\mathbb{C})} \quad . \quad \underbrace{[M^*(\mathbf{a})]}_{\in M(4\times 6,\mathbb{C})}$$

$$=$$

$$\left[\begin{array}{cccc} x_0^0 & x_0^1 & x_0^2 & x_0^3 \end{array}\right] . \left[\begin{array}{cccccc} 0 & 0 & 0 & -a^3 & a^2 & -a^1 \\ 0 & a^3 & -a^2 & 0 & 0 & a^0 \\ -a^3 & 0 & a^1 & 0 & -a^0 & 0 \\ a^2 & -a^1 & 0 & a^0 & 0 & 0 \end{array}\right]$$

$$=$$

$$\left[\begin{array}{cccccc} (a^2 . x_0^3 - a^3 . x_0^2) & (a^3 . x_0^1 - a^1 . x_0^3) & (a^1 . x_0^2 - a^2 . x_0^1) & (a^0 . x_0^3 - a^3 . x_0^0) & -(a^0 . x_0^2 - a^2 . x_0^0) & (a^0 . x_0^1 - a^1 . x_0^0) \end{array}\right]$$

$$=$$

$$<^{(4)} \mathbf{a} \wedge {}^{(4)}\mathbf{x}_0| \in M(1 \times 6, \mathbb{C}) \equiv \mathbb{C}^6 \equiv \mathbb{C} \otimes E(6.\mathbb{R})$$

It is evident that one can also make another conventional choice for the representation of ${}^{(4)}\mathbf{a}$; precisely, one can prefer the writing:

$$\forall\, {}^{(4)}\mathbf{a} \in M(4 \times 1, \mathbb{C}) \equiv V_4^*, \exists\, [M(\mathbf{a})] \in M(6 \times 4, \mathbb{C}) :$$

$$\underbrace{[M(\mathbf{a})]}_{\in M(6\times 4,\mathbb{C})} \quad . \quad \underbrace{|\mathbf{x}_0 >}_{\in M(4\times 1,\mathbb{C})}$$

$$=$$

$$\left[\begin{array}{cccc} 0 & 0 & -a^3 & a^2 \\ 0 & a^3 & 0 & -a^1 \\ 0 & -a^2 & a^1 & 0 \\ -a^3 & 0 & 0 & a^0 \\ a^2 & 0 & -a^0 & 0 \\ -a^1 & a^0 & 0 & 0 \end{array}\right] . \left[\begin{array}{c} x_0^0 \\ x_0^1 \\ x_0^2 \\ x_0^3 \end{array}\right]$$

$$=$$

$$\left[\begin{array}{c} (a^2 . x_0^3 - a^3 . x_0^2) \\ (a^3 . x_0^1 - a^1 . x_0^3) \\ (a^1 . x_0^2 - a^2 . x_0^1) \\ (a^0 . x_0^3 - a^3 . x_0^0) \\ -(a^0 . x_0^2 - a^2 . x_0^0) \\ (a^0 . x_0^1 - a^1 . x_0^0) \end{array}\right]$$

$$=$$

$$|^{(4)}\mathbf{a} \wedge {}^{(4)}\mathbf{x}_0 > \in M(6 \times 1, \mathbb{C}) \equiv \mathbb{C}^6 \equiv \mathbb{C} \otimes E(6.\mathbb{R})$$

This alternative representation doesn't change the message contained in these short demonstrations. □

Definition 1.2. *The dilatation.*

The dilatation is a conventional representation in $\mathrm{M}(4 \times 6, \mathbb{C})$ for any given element in V_4:

$$\forall\, \mathbf{a} \in V_4 \xrightarrow{[\odot]} [\odot(\mathbf{a})] = \left[\begin{array}{cccccc} 0 & 0 & 0 & -a^3 & a^2 & -a^1 \\ 0 & a^3 & -a^2 & 0 & 0 & a^0 \\ -a^3 & 0 & a^1 & 0 & -a^0 & 0 \\ a^2 & -a^1 & 0 & a^0 & 0 & 0 \end{array}\right] \in M(4\times 6, \mathbb{C})$$

This function obviously builds an isomorphic representation and:

$$\forall\, \mathbf{a}, \mathbf{x}_0 \in V_4 : < \mathbf{x}_0| \,.\, [\odot(\mathbf{a})] = < \mathbf{a} \wedge \mathbf{x}_0| \in M(1 \times 6, \mathbb{C})$$

To be noted: if one prefers, the classical wedge product can also be written as:

$$|^{(4)}\mathbf{a} \wedge {}^{(4)}\mathbf{x}_0 > = {}^{(6\times 4)}[\odot(\mathbf{a})]^t \,.\, |^{(4)}\mathbf{x}_0 >$$

Proposition 1.2. *Any deformed Lie product is a special kind of deformation for a wedge product with which it can be related.*

Proof. In observing the definition 1.1 characterizing any deformed Lie product, one visually states that the action of the function $\mathrm{f}(\mathrm{A}, {}^{(4)}\mathbf{a}) = [{}^{(4)}\mathbf{a}, {}^{(4)}\mathbf{...}]_A$ on ${}^{(4)}\mathbf{x}_0$ can be represented with the help of the dual representation of the wedge product ${}^{(4)}\mathbf{a} \wedge {}^{(4)}\mathbf{x}_0$ with the relation:

$$^{(4\times 1)}|^{(4)}\mathbf{x}_1 >= {}^{(4\times 6)}[A]\,.\,{}^{(6\times 1)}|^{(4)}\mathbf{a} \wedge {}^{(4)}\mathbf{x}_0 >$$

... which obviously exhibits the fact that the anti-symmetric cube A deforms the classical wedge product:

$$^{(4\times 6)}[A] = \begin{bmatrix} A^0_{01} & A^0_{02} & A^0_{03} & A^0_{12} & A^0_{13} & A^0_{23} \\ A^1_{01} & A^1_{02} & A^1_{03} & A^1_{12} & A^1_{13} & A^1_{23} \\ A^2_{01} & A^2_{02} & A^2_{03} & A^2_{12} & A^2_{13} & A^2_{23} \\ A^3_{01} & A^3_{02} & A^3_{03} & A^3_{12} & A^3_{13} & A^3_{23} \end{bmatrix}$$

Indeed, recall that any element in the exterior algebra $\mathrm{V}^{*\wedge}{}_4$ has a representation in $\mathrm{M}(6 \times 1, \mathbb{C})$ -see proposition 1.1:

$$|^{(4)}\mathbf{a} \wedge {}^{(4)}\mathbf{x}_0 >= \begin{bmatrix} a^2\,.\,x^3_0 - a^3\,.\,x^2_0 \\ a^3\,.\,x^1_0 - a^1\,.\,x^3_0 \\ a^1\,.\,x^2_0 - a^2\,.\,x^1_0 \\ a^0\,.\,x^3_0 - a^3\,.\,x^0_0 \\ -a^0\,.\,x^2_0 + a^2\,.\,x^0_0 \\ a^0\,.\,x^1_0 - a^1\,.\,x^0_0 \end{bmatrix} \in M(6 \times 1, \mathbb{R})$$

Hence, the deforming cube A can effectively be represented with the help of the matrix ${}^{(4\times6)}[\mathrm{A}]$ and its main action is to project the elements in $\mathrm{V}^{*\wedge}{}_4$ into $\mathrm{V}^*{}_4$, the dual of V_4. The function one is studying here must now be written more precisely as:

$$f(A, {}^{(4)}\mathbf{a}) = f({}^{(4\times 6)}[A], {}^{(4)}\mathbf{a}) = [{}^{(4)}\mathbf{a}, {}^{(4)}...]_{{}^{(4\times 6)}[A]}$$

... to avoid confusions with the writings appearing in others contexts; e.g.: when the cube A is not anti-symmetric. □

Remark 1.3. *Concerning the action of f(A, ${}^{(4)}\mathbf{a}$) when A is an anti-symmetric cube.*

Assembling the results which have been obtained in proposition 1.1 and 1.2, one is now in a mental position allowing to understand how $\mathrm{f}(\mathrm{A}, {}^{(4)}\mathbf{a})$ acts on any element $\mathbf{...} = \mathbf{x}_0$ in V_4 when A is an anti-symmetric cube; precisely:

$$^{(4\times 1)}|^{(4)}\mathbf{x}_1 >= {}^{(4\times 6)}[A]\,.\,{}^{(6\times 4)}[\odot(\mathbf{a})]^t\,.\,|^{(4)}\mathbf{x}_0 >$$

Therefore, there is an element [P] in $\mathrm{M}(4, \mathbb{C})$ representing this action:

$$[P]$$

...

$$=$$

$$^{(4\times 6)}[A]\,.\,^{(6\times 4)}[\odot(\mathbf{a})]^{t}$$

$$=$$

$$\begin{bmatrix} A^{0}_{01} & A^{0}_{02} & A^{0}_{03} & A^{0}_{12} & A^{0}_{13} & A^{0}_{23} \\ A^{1}_{01} & A^{1}_{02} & A^{1}_{03} & A^{1}_{12} & A^{1}_{13} & A^{1}_{23} \\ A^{2}_{01} & A^{2}_{02} & A^{2}_{03} & A^{2}_{12} & A^{2}_{13} & A^{2}_{23} \\ A^{3}_{01} & A^{3}_{02} & A^{3}_{03} & A^{3}_{12} & A^{3}_{13} & A^{3}_{23} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -a^{3} & a^{2} \\ 0 & a^{3} & 0 & -a^{1} \\ 0 & -a^{2} & a^{1} & 0 \\ -a^{3} & 0 & 0 & a^{0} \\ a^{2} & 0 & -a^{0} & 0 \\ -a^{1} & a^{0} & 0 & 0 \end{bmatrix}$$

$$=$$

$$\left[\underbrace{\begin{matrix} -A^{0}_{23}\,.\,a^{1} + A^{0}_{13}\,.\,a^{2} - A^{0}_{12}\,.\,a^{3} \\ -A^{1}_{23}\,.\,a^{1} + A^{1}_{13}\,.\,a^{2} - A^{1}_{12}\,.\,a^{3} \\ -A^{2}_{23}\,.\,a^{1} + A^{2}_{13}\,.\,a^{2} - A^{2}_{12}\,.\,a^{3} \\ -A^{3}_{23}\,.\,a^{1} + A^{3}_{13}\,.\,a^{2} - A^{3}_{12}\,.\,a^{3} \end{matrix}}_{a^{0}\ is\ absent} \quad \underbrace{\begin{matrix} A^{0}_{23}\,.\,a^{0} - A^{0}_{03}\,.\,a^{2} + A^{0}_{02}\,.\,a^{3} \\ A^{1}_{23}\,.\,a^{0} - A^{1}_{03}\,.\,a^{2} + A^{1}_{02}\,.\,a^{3} \\ A^{2}_{23}\,.\,a^{0} - A^{2}_{03}\,.\,a^{2} + A^{2}_{02}\,.\,a^{3} \\ A^{3}_{23}\,.\,a^{0} - A^{3}_{03}\,.\,a^{2} + A^{3}_{02}\,.\,a^{3} \end{matrix}}_{a^{1}\ is\ absent} \quad \underbrace{\begin{matrix} -A^{0}_{13}\,.\,a^{0} + A^{0}_{03}\,.\,a^{1} - A^{0}_{01}\,.\,a^{3} \\ -A^{1}_{13}\,.\,a^{0} + A^{1}_{03}\,.\,a^{1} - A^{1}_{01}\,.\,a^{3} \\ -A^{2}_{13}\,.\,a^{0} + A^{2}_{03}\,.\,a^{1} - A^{2}_{01}\,.\,a^{3} \\ -A^{3}_{13}\,.\,a^{0} + A^{3}_{03}\,.\,a^{1} - A^{3}_{01}\,.\,a^{3} \end{matrix}}_{a^{2}\ is\ absent} \quad \underbrace{\begin{matrix} A^{0}_{12}\,.\,a^{0} - A^{0}_{02}\,.\,a^{1} + A^{0}_{01}\,.\,a^{2} \\ A^{1}_{12}\,.\,a^{0} - A^{1}_{02}\,.\,a^{1} + A^{1}_{01}\,.\,a^{2} \\ A^{2}_{12}\,.\,a^{0} - A^{2}_{02}\,.\,a^{1} + A^{2}_{01}\,.\,a^{2} \\ A^{3}_{12}\,.\,a^{0} - A^{3}_{02}\,.\,a^{1} + A^{3}_{01}\,.\,a^{2} \end{matrix}}_{a^{3}\ is\ absent} \right]$$

Proposition 1.3. *Each given deformed Lie product has a dual representation in* $V^{*}{}_{4} \equiv M(4 \times 1, \mathbb{R})$*. This dual representation has at least one trivial decomposition in* $M(4, \mathbb{R})$ *without residual part.*

Proof. Recall that, per definition, any given deformed Lie product is the half of an alternate tensor product deformed by an anti-symmetric cube [a]; in a four-dimensional space (D = 4), it has four components which are:

$$A \in \boxplus^{-}(4, \mathbb{C}), \forall \chi \in Ind_{4} :$$

$$x^{\chi}_{1} = \{[\mathbf{a}, \mathbf{x}_{0}]_{A}\}^{\chi} = \frac{1}{2} \cdot \sum_{\alpha, \beta} A^{\chi}_{\alpha\beta} \,.\, (a^{\alpha} \,.\, x^{\beta}_{0} - a^{\beta} \,.\, x^{\alpha}_{0})$$

They describe the vector $\mathbf{x}_{1}$ in $V^{*}{}_{4}$. Each of these components is seemingly a sum of (4 × 4) minus 4 = 12 terms because the anti-symmetry implies that each knot with a repeated subscript vanishes ($A^{\chi}{}_{\alpha\alpha} = 0$):

$$2 \,.\, x^{\chi}_{1}$$

$$=$$

$$\begin{aligned} & A^{\chi}_{01} \,.\, (a^{0} \,.\, x^{1}_{0} - a^{1} \,.\, x^{0}_{0}) + A^{\chi}_{02} \,.\, (a^{0} \,.\, x^{2}_{0} - a^{2} \,.\, x^{0}_{0}) + A^{\chi}_{03} \,.\, (a^{0}_{0} \,.\, x^{3} - a^{3} \,.\, x^{0}_{0}) \\ & + A^{\chi}_{10} \,.\, (a^{1} \,.\, x^{0}_{0} - a^{0} \,.\, x^{1}_{0}) + A^{\chi}_{12} \,.\, (a^{1} \,.\, x^{2}_{0} - a^{2} \,.\, x^{1}_{0}) + A^{\chi}_{13} \,.\, (a^{1} \,.\, x^{3}_{0} - a^{3} \,.\, x^{1}_{0}) \\ & + A^{\chi}_{20} \,.\, (a^{2} \,.\, x^{0}_{0} - a^{0} \,.\, x^{2}_{0}) + A^{\chi}_{21} \,.\, (a^{2} \,.\, x^{1}_{0} - a^{1} \,.\, x^{2}_{0}) + A^{\chi}_{23} \,.\, (a^{2} \,.\, x^{3}_{0} - a^{3} \,.\, x^{2}_{0}) \\ & + A^{\chi}_{30} \,.\, (a^{2} \,.\, x^{0} - a^{0} \,.\, x^{2}) + A^{\chi}_{21} \,.\, (a^{2} \,.\, x^{1} - a^{1} \,.\, x^{2}) + A^{\chi}_{23} \,.\, (a^{2} \,.\, x^{3} - a^{3} \,.\, x^{2}) \end{aligned}$$

But since the anti-symmetry also implies that:

$$A^{\chi}_{\alpha\beta} = -A^{\chi}_{\beta\alpha} \, \forall \, \alpha \neq \beta$$

One must state that each component is a sum of only six terms:

$$x^{\chi}{}_{1}$$

...

$$=$$

$$A^{\chi}_{01} . (a^0 . x_0^1 - a^1 . x_0^0) + A^{\chi}_{02} . (a^0 . x_0^2 - a^2 . x_0^0) + A^{\chi}_{03} . (a^0 . x_0^3 - a^3 . x_0^0)$$
$$+ A^{\chi}_{12} . (a^1 . x_0^2 - a^2 . x_0^1) + A^{\chi}_{13} . (a^1 . x_0^3 - a^3 . x_0^1) + A^{\chi}_{23} . (a^2 . x_0^3 - a^3 . x_0^2)$$

These six terms can be regrouped inside a linear combination depending on the components of vector $\mathbf{x}_0$:

$$x_1^{\chi}$$
$$=$$
$$(-A^{\chi}_{01} . a^1 - A^{\chi}_{02} . a^2 - A^{\chi}_{03} . a^3) . x_0^0$$
$$+ (A^{\chi}_{01} . a^0 - A^{\chi}_{12} . a^2 - A^{\chi}_{13} . a^3) . x_0^1$$
$$+ (A^{\chi}_{02} . a^0 - A^{\chi}_{12} . a^1 - A^{\chi}_{23} . a^3) . x_0^2$$
$$+ (A^{\chi}_{03} . a^0 + A^{\chi}_{13} . a^1 + A^{\chi}_{23} . a^2) . x_0^3$$

This fact allows the construction of a (4-4) matrix such that:

$$\begin{bmatrix} x_1^0 \\ x_1^1 \\ x_1^2 \\ x_1^3 \end{bmatrix}$$
$$=$$
$$\left[\underbrace{\begin{matrix} (-A^0_{01} . a^1 - A^0_{02} . a^2 - A^0_{03} . a^3) \\ (-A^1_{01} . a^1 - A^1_{02} . a^2 - A^1_{03} . a^3) \\ (-A^2_{01} . a^1 - A^2_{02} . a^2 - A^2_{03} . a^3) \\ (-A^3_{01} . a^1 - A^3_{02} . a^2 - A^3_{03} . a^3) \end{matrix}}_{a^0\, is\, absent} \quad \underbrace{\begin{matrix} (A^0_{01} . a^0 - A^0_{12} . a^2 - A^0_{13} . a^3) \\ (A^1_{01} . a^0 - A^1_{12} . a^2 - A^1_{13} . a^3) \\ (A^2_{01} . a^0 - A^2_{12} . a^2 - A^2_{13} . a^3) \\ (A^3_{01} . a^0 - A^3_{12} . a^2 - A^3_{13} . a^3) \end{matrix}}_{a^1\, is\, absent} \quad \underbrace{\begin{matrix} (A^0_{02} . a^0 - A^0_{12} . a^1 - A^0_{23} . a^3) \\ (A^1_{02} . a^0 - A^1_{12} . a^1 - A^1_{23} . a^3) \\ (A^2_{02} . a^0 - A^2_{12} . a^1 - A^2_{23} . a^3) \\ (A^3_{02} . a^0 - A^3_{12} . a^1 - A^3_{23} . a^3) \end{matrix}}_{a^2\, is\, absent} \quad \underbrace{\begin{matrix} (A^0_{03} . a^0 + A^0_{13} . a^1 + A^0_{23} . a^2) \\ (A^1_{03} . a^0 + A^1_{13} . a^1 + A^1_{23} . a^2) \\ (A^2_{03} . a^0 + A^2_{13} . a^1 + A^2_{23} . a^2) \\ (A^3_{03} . a^0 + A^3_{13} . a^1 + A^3_{23} . a^2) \end{matrix}}_{a^3\, is\, absent} \right]$$
$$.$$
$$\begin{bmatrix} x_0^0 \\ x_0^1 \\ x_0^2 \\ x_0^3 \end{bmatrix}$$

Once again, the anti-symmetry of cube A allows to state that one can condense this writing with:

$$\begin{bmatrix} x_1^0 \\ x_1^1 \\ x_1^2 \\ x_1^3 \end{bmatrix} = [\sum_{\beta} A^{line}_{\beta\, row} . a^{\beta}] . \begin{bmatrix} x_0^0 \\ x_0^1 \\ x_0^2 \\ x_0^3 \end{bmatrix}$$

One can condense this relation in writing per convention:

$$\underbrace{|\mathbf{x}_1 >}_{\in V_4^*} = \underbrace{{}_{(4 \times 6)[A]}\Phi(\mathbf{a})}_{\in M(4, \mathbb{C})} . \underbrace{|\mathbf{x}_0 >}_{\in V_4^*}$$

One says that the deformed Lie product at hand has a dual representation in $\mathrm{V^*}_4$ accepting a trivial representation of its action on the dual representation of $\mathbf{x}_0$ in $\mathrm{V^*}_4$. This trivial representation lies in $\mathrm{M}(4, \mathbb{C})$ and it is called the *simplest decomposition* of the deformed Lie product at hand because there is no residual part in this decomposition. □

...

Remark 1.4. *Introducing the (E) question.*

As a matter of facts, there are at least two different trivial representations in $\mathrm{M}(4,\mathbb{C})$ for the action of the dual representation in $\mathrm{V^*}_4$ of any given deformed Lie product because both matrices exhibit some similarities but they are obviously different:

$$[P] \neq {}_{(4\times 6)[A]}\Phi(\mathbf{a})$$

... and this is introducing the so-called enigmatic (E) question. Why is it possible to describe the action of any given deformed Lie product in at least two different manners?

Remark 1.5. *Concerning the conditions of coincidence between both trivial representations.*

Due to the fact that both trivial representations have a similar structure, it is possible to envisage a coincidence between them. This coincidence can only exist, whatever the natural value of χ taken in Ind_4 is, if:

$$A^{\chi}_{01} = A^{\chi}_{23}$$
$$A^{\chi}_{02} = -A^{\chi}_{13}$$
$$A^{\chi}_{03} = A^{\chi}_{12}$$

The simple fact that the coincidence can eventually occur is equivalent to the affirmation that there exists a subset of anti-symmetric cube insuring this coincidence. This subset is represented by the generic matrix:

$$\begin{bmatrix} A^0_{23} & -A^0_{13} & A^0_{12} & A^0_{12} & A^0_{13} & A^0_{23} \\ A^1_{23} & -A^1_{13} & A^1_{12} & A^1_{12} & A^1_{13} & A^1_{23} \\ A^2_{23} & -A^2_{13} & A^2_{12} & A^2_{12} & A^2_{13} & A^2_{23} \\ A^3_{23} & -A^3_{13} & A^3_{12} & A^3_{12} & A^3_{13} & A^3_{23} \end{bmatrix}$$

One indirectly states that each element in this subset of special anti-symmetric cube can form an element in $\mathrm{M}(4,\mathbb{C})$ containing at most 16 different complex numbers; it may eventually be written, per convention:

$$\begin{bmatrix} A^0_{23} & -A^0_{13} & A^0_{12} & A^0_{13} \\ A^1_{23} & -A^1_{13} & A^1_{12} & A^1_{13} \\ A^2_{23} & -A^2_{13} & A^2_{12} & A^2_{13} \\ A^3_{23} & -A^3_{13} & A^3_{12} & A^3_{13} \end{bmatrix}$$

Lemma 1.1. *Concerning the difference between both trivial representations of $f(A, {}^{(4)}\mathbf{a})$ when A is an anti-symmetric cube.*

Anyway, there is here no argument insuring the systematic coincidence between both trivial representations and one can in general admit the existence of a difference:

$$\exists\,\{[P] - {}_{(4\times 6)[A]}\Phi(\mathbf{a})\} \in M(4,\mathbb{C})$$

This difference vanishes when the cube A is an element in the special subset of $\boxplus^-(4,\mathbb{C})$ such that:

$$\forall\,\chi \in Ind_4 : A^{\chi}_{01} = A^{\chi}_{23},\, A^{\chi}_{02} = -A^{\chi}_{13},\, A^{\chi}_{03} = A^{\chi}_{12}$$

Proposition 1.4. *For any deformed Lie product, there is an alternative representation involving Pythagorean tables and their respective traces.*

Proof. The deforming matrix ${}^{(4\times 6)}[\mathrm{A}]$ representing the anti-symmetric cube A is the superposition of four elements which are taken in a six-dimensional space. Each element contributes to the construction of one component of a deformed Lie product related to a given wedge product. With this image behind the head, one can consider that each given deformed product involves five classical wedge products: four of them are indirectly represented by the deforming matrix ${}^{(4\times 6)}[\mathrm{A}]$ and the fifth one is for example ${}^{(4)}\mathbf{a} \wedge {}^{(4)}\mathbf{x}_0$. Let write per convention:

$$\exists\, {}^{(4\times 6)}[A] \,\Rightarrow\, \exists\, {}^{(1\times 6)} < \mathbf{w}_\chi| \,:\, (A^\chi_{01},\, A^\chi_{02},\, A^\chi_{03},\, A^\chi_{12},\, A^\chi_{13},\, A^\chi_{23}),\, \forall \chi \in Ind_4$$

This situation offers the possibility to build four Pythagorean tables:

$$\forall \chi \in Ind_4,\, \exists T_2(\otimes)({}^{(1\times 6)}\!< \mathbf{w}_\chi|,\, {}^{(6\times 1)}|{}^{(4)}\mathbf{a} \wedge {}^{(4)}\mathbf{x}_0 >) \in M(6,\, \mathbb{C})$$

... and to easily state that:

$$\forall \chi \in Ind_4 :$$

$$x^\chi_1 = \{[\mathbf{a},\, \mathbf{x}_0]_{{}^{(4\times 6)}[A]}\}^\chi = Trace\{T_2(\otimes)({}^{(1\times 6)}\!< \mathbf{w}_\chi|,\, {}^{(6\times 1)}|{}^{(4)}\mathbf{a} \wedge {}^{(4)}\mathbf{x}_0 >)\} \in \mathbb{C}$$

□

Remark 1.6. *Concerning the anti-symmetric cubes which are also anti-reduced.*

The set containing the cubes of which the knots respect simultaneously two conditions, the one of anti-symmetry and the other one of anti-reduction is not empty because both conditions are compatible:

$$\forall \alpha,\, \beta \in Ind_4 :$$

$$A \in \boxplus^-(4,\, \mathbb{C}) : A^\delta_{\alpha\beta} + A^\delta_{\beta\alpha} = 0$$

$$\boxplus^\downarrow(4,\, \mathbb{C}) : A^\delta_{\alpha\beta} + A^\beta_{\alpha\delta} = 0$$

$$\Downarrow$$

$$A^\chi_{\alpha\beta} = -A^\chi_{\beta\alpha} = A^\alpha_{\beta\chi} = -A^\alpha_{\chi\beta} = A^\beta_{\chi\alpha} = -A^\beta_{\alpha\chi} = A^\chi_{\alpha\beta} = etc...$$

In this case, all knots with repeated indices or subscripts vanish, the total number of non-vanishing knots is reduced and the deforming matrix ${}^{(4\times 6)}[\mathrm{A}]$ initially representing the anti-symmetric cube A must be modified in the following manner:

$${}^{(4\times 6)}[A] = \begin{bmatrix} 0 & 0 & 0 & A^0_{12} & A^0_{13} & A^0_{23} \\ 0 & A^1_{02} & A^1_{03} & 0 & 0 & A^1_{23} \\ A^2_{01} & 0 & A^2_{03} & 0 & A^2_{13} & 0 \\ A^3_{01} & A^3_{02} & 0 & A^3_{12} & 0 & 0 \end{bmatrix}$$

Let now write per convention:

$$A^0 = A^3_{12}$$

$$A^1 = A^2_{03}$$

$$A^2 = A^0_{13}$$
$$A^3 = A^1_{02}$$

... and state that the representation of any anti-symmetric and anti-reduced cube A can be written as:

$$^{(4\times 6)}[A] = \begin{bmatrix} 0 & 0 & 0 & -A^3 & A^2 & -A^1 \\ 0 & A^3 & -A^2 & 0 & 0 & A^0 \\ -A^3 & 0 & A^1 & 0 & -A^0 & 0 \\ A^2 & -A^1 & 0 & A^0 & 0 & 0 \end{bmatrix} = [\odot(\mathbf{A})]$$

This matrix has exactly the same formalism than the matrix representing the vector:

$$^{(4)}\mathbf{A} : (A^0, A^1, A^2, A^3) \in V_4$$

Recall the result of proposition 1.1 and the definition 1.2 for confirmation. This statement allows the logical sentences:

$$A \in \boxplus(4, \mathbb{C}) \xrightarrow{anti-symmetry} [A] \in M(4 \times 6, \mathbb{C}) \xrightarrow{anti-reduction} {}^{(4)}\mathbf{A} \in V_4$$

$$A \to {}^{(4)}\mathbf{A} \equiv (A^1_{23}, A^0_{23}, A^0_{13}, A^1_{02}) = (A^0, A^1, A^2, A^3) \in V_4^*$$

Theorem 1.1. *Any anti-symmetric and anti-reduced cube A in $\boxplus(4, \mathbb{C})$ is equivalent to the dual representation in $V^*{}_4$ of some element $^{(4)}\boldsymbol{A}$ in V_4 and conversely.*

Corollary 1.1. *Concerning the action of $[^{(4)}\boldsymbol{a}, {}^{(4)}...]_A$ when A is equivalent to a vector $^{(4)}\boldsymbol{A}$.*

A) Studying this action with the approach which has been proposed in remark 1.3, one gets:

$$[P]$$
$$=$$
$$^{(4\times 6)}[\odot(\mathbf{A})] \,.\, ^{(6\times 4)}[\odot(\mathbf{a})]^t$$
$$=$$
$$\begin{bmatrix} 0 & 0 & 0 & -A^3 & A^2 & -A^1 \\ 0 & A^3 & -A^2 & 0 & 0 & A^0 \\ -A^3 & 0 & A^1 & 0 & -A^0 & 0 \\ A^2 & -A^1 & 0 & A^0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 0 & -a^3 & a^2 \\ 0 & a^3 & 0 & -a^1 \\ 0 & -a^2 & a^1 & 0 \\ -a^3 & 0 & 0 & a^0 \\ a^2 & 0 & -a^0 & 0 \\ -a^1 & a^0 & 0 & 0 \end{bmatrix}$$
$$=$$
$$\begin{bmatrix} A^1.a^1 + A^2.a^2 + A^3.a^3 & -A^1.a^0 & -A^2.a^0 & -A^3.a^0 \\ -A^0.a^1 & A^0.a^0 + A^2.a^2 + A^3.a^3 & -A^2.a^1 & -A^3.a^1 \\ -A^0.a^2 & -A^1.a^2 & A^0.a^0 + A^1.a^1 + A^3.a^3 & -A^3.a^2 \\ -A^0.a^3 & -A^1.a^3 & -A^2.a^3 & A^0.a^0 + A^1.a^1 + A^2.a^2 \end{bmatrix}$$
$$=$$
$$<^{(4)} \mathbf{A}, {}^{(4)}\mathbf{a} >_{Id_4} .Id_4 - T_2(\otimes)(^{(4)}\mathbf{A}, {}^{(4)}\mathbf{a})$$

B) Studying this action with the approach which has been proposed in proposition 1.3, it is useful to introduce the:

Definition 1.3. *The correspondence ${}_A\Phi$ for a given cube A.*

The correspondence ${}_A\Phi$ is an application with sources in V_4 and targets in $M(4, \mathbb{C})$:

$$A \in \boxplus(4,\, \mathbb{R}),\, \forall \mathbf{a} \in V_4 \xrightarrow{{}_A\Phi} {}_A\Phi(\mathbf{a}) \in M(4,\, \mathbb{C})$$

It is such that, per definition:

$$\begin{gathered} {}_A\Phi(\mathbf{a}) \\ = \\ [\sum_{\beta=0}^{\beta=3} A^{line}_{\beta\, row} \,.\, a^\beta] \\ = \end{gathered}$$

$$\begin{bmatrix} A^0_{00} . a^0 + A^0_{10} . a^1 + A^0_{20} . a^2 + A^0_{30} . a^3 & A^0_{01} . a^0 + A^0_{11} . a^1 + A^0_{21} . a^2 + A^0_{31} . a^3 & . \\ A^1_{00} . a^0 + A^1_{10} . a^1 + A^1_{20} . a^2 + A^1_{30} . a^3 & A^1_{01} . a^0 + A^1_{11} . a^1 + A^1_{21} . a^2 + A^1_{31} . a^3 & . \\ A^2_{00} . a^0 + A^2_{10} . a^1 + A^2_{20} . a^2 + A^2_{30} . a^3 & A^2_{01} . a^0 + A^2_{11} . a^1 + A^2_{21} . a^2 + A^2_{31} . a^3 & . \\ A^3_{00} . a^0 + A^3_{10} . a^1 + A^3_{20} . a^2 + A^3_{30} . a^3 & A^3_{01} . a^0 + A^3_{11} . a^1 + A^3_{21} . a^2 + A^3_{31} . a^3 & . \end{bmatrix}$$

$$\begin{bmatrix} . & A^0_{02} . a^0 + A^0_{12} . a^1 + A^0_{22} . a^2 + A^0_{32} . a^3 & A^0_{03} . a^0 + A^0_{13} . a^1 + A^0_{23} . a^2 + A^0_{33} . a^3 \\ . & A^1_{02} . a^0 + A^1_{12} . a^1 + A^1_{22} . a^2 + A^1_{32} . a^3 & A^1_{03} . a^0 + A^1_{13} . a^1 + A^1_{23} . a^2 + A^1_{33} . a^3 \\ . & A^2_{02} . a^0 + A^2_{12} . a^1 + A^2_{22} . a^2 + A^2_{32} . a^3 & A^2_{03} . a^0 + A^2_{13} . a^1 + A^2_{23} . a^2 + A^2_{33} . a^3 \\ . & A^3_{02} . a^0 + A^3_{12} . a^1 + A^3_{22} . a^2 + A^3_{32} . a^3 & A^3_{03} . a^0 + A^3_{13} . a^1 + A^3_{23} . a^2 + A^3_{33} . a^3 \end{bmatrix}$$

Example 1.1. *When the cube A is anti-symmetric.*

In that case:

$$A \in \boxplus^-(4,\, \mathbb{R}) \equiv [A] \in M(4 \times 6,\, \mathbb{C})$$

$$[A] \in M(4 \times 6,\, \mathbb{C}),\, \forall \mathbf{a} \in V_4 \xrightarrow{{}_{[A]}\Phi} {}_{[A]}\Phi(\mathbf{a}) \in M(4,\, \mathbb{C})$$

With:

$$\begin{gathered} {}_{[A]}\Phi(\mathbf{a}) \\ = \end{gathered}$$

$$\begin{bmatrix} -A^0_{01} . a^1 - A^0_{02} . a^2 - A^0_{03} . a^3 & A^0_{01} . a^0 - A^0_{12} . a^2 - A^0_{13} . a^3 & A^0_{02} . a^0 + A^0_{12} . a^1 - A^0_{23} . a^3 & A^0_{03} . a^0 + A^0_{13} . a^1 + A^0_{23} . a^2 \\ -A^1_{01} . a^1 - A^1_{02} . a^2 - A^1_{03} . a^3 & A^1_{01} . a^0 - A^1_{12} . a^2 - A^1_{13} . a^3 & A^1_{02} . a^0 + A^1_{12} . a^1 - A^1_{23} . a^3 & A^1_{03} . a^0 + A^1_{13} . a^1 + A^1_{23} . a^2 \\ -A^2_{01} . a^1 - A^2_{02} . a^2 - A^2_{03} . a^3 & A^2_{01} . a^0 - A^2_{12} . a^2 - A^2_{13} . a^3 & A^2_{02} . a^0 + A^2_{12} . a^1 - A^2_{23} . a^3 & A^2_{03} . a^0 + A^2_{13} . a^1 + A^2_{23} . a^2 \\ -A^3_{01} . a^1 - A^3_{02} . a^2 - A^3_{03} . a^3 & A^3_{01} . a^0 - A^3_{12} . a^2 - A^3_{13} . a^3 & A^3_{02} . a^0 + A^3_{12} . a^1 - A^3_{23} . a^3 & A^3_{03} . a^0 + A^3_{13} . a^1 + A^3_{23} . a^2 \end{bmatrix}$$

$$\in M(4,\, \mathbb{C})$$

One fortunately recovers the result already obtained at the end of proposition 1.3.

Example 1.2. *When the cube A is anti-symmetric and anti-reduced.*

This case concerns the corollary and one can now verify that:

$$\begin{gathered} {}_{\mathbf{A}}\Phi(\mathbf{a}) \\ = \end{gathered}$$

$$\begin{bmatrix} 0 & -A^0_{12} . a^2 - A^0_{13} . a^3 & A^0_{12} . a^1 - A^0_{23} . a^3 & A^0_{13} . a^1 + A^0_{23} . a^2 \\ -A^1_{02} . a^2 - A^1_{03} . a^3 & 0 & A^1_{02} . a^0 - A^1_{23} . a^3 & A^1_{03} . a^0 + A^1_{23} . a^2 \\ -A^2_{01} . a^1 - A^2_{03} . a^3 & A^2_{01} . a^0 - A^2_{13} . a^3 & 0 & A^2_{03} . a^0 + A^2_{13} . a^1 \\ -A^3_{01} . a^1 - A^3_{02} . a^2 & A^3_{01} . a^0 - A^3_{12} . a^2 & A^3_{02} . a^0 + A^3_{12} . a^1 & 0 \end{bmatrix}$$

...

$$=$$

$$\begin{bmatrix} 0 & -A^0_{12}\,.\,a^2 - A^0_{13}\,.\,a^3 & A^0_{12}\,.\,a^1 - A^0_{23}\,.\,a^3 & A^0_{13}\,.\,a^1 + A^0_{23}\,.\,a^2 \\ A^0_{12}\,.\,a^2 + A^0_{13}\,.\,a^3 & 0 & -A^0_{12}\,.\,a^0 - A^1_{23}\,.\,a^3 & -A^0_{13}\,.\,a^0 + A^1_{23}\,.\,a^2 \\ -A^0_{12}\,.\,a^1 + A^0_{23}\,.\,a^3 & A^0_{12}\,.\,a^0 + A^1_{23}\,.\,a^3 & 0 & -A^0_{23}\,.\,a^0 - A^1_{23}\,.\,a^1 \\ -A^0_{13}\,.\,a^1 - A^0_{23}\,.\,a^2 & A^0_{13}\,.\,a^0 - A^1_{23}\,.\,a^2 & A^0_{23}\,.\,a^0 + A^1_{23}\,.\,a^1 & 0 \end{bmatrix}$$

This matrix is an element in $\mathrm{M}^-(4, \mathbb{C})$ which allows the definition of six complex numbers:

$$\Phi_{01} = -A^0_{12}\,.\,a^2 - A^0_{13}\,.\,a^3$$
$$\Phi_{02} = A^0_{12}\,.\,a^1 - A^0_{23}\,.\,a^3$$
$$\Phi_{03} = A^0_{13}\,.\,a^1 + A^0_{23}\,.\,a^2$$
$$\Phi_{12} = -A^0_{12}\,.\,a^0 - A^1_{23}\,.\,a^3$$
$$\Phi_{13} = -A^0_{13}\,.\,a^0 + A^1_{23}\,.\,a^2$$
$$\Phi_{23} = -A^0_{23}\,.\,a^0 - A^1_{23}\,.\,a^1$$

If one now writes per convention:

$$^{(4)}\mathbf{A} \equiv (A^0_{12}, A^0_{13}, A^0_{23}, A^1_{23}) = (-A^3, A^2, -A^1, A^0)$$

One states that:

$$\Phi_{01} = A^3\,.\,a^2 - A^2\,.\,a^3$$
$$\Phi_{02} = -A^3\,.\,a^1 + A^1\,.\,a^3$$
$$\Phi_{03} = A^2\,.\,a^1 - A^1\,.\,a^2$$
$$\Phi_{12} = A^3\,.\,a^0 - A^0\,.\,a^3$$
$$\Phi_{13} = -A^2\,.\,a^0 + A^0\,.\,a^2$$
$$\Phi_{23} = A^1\,.\,a^0 - A^0\,.\,a^1$$

Recall here that:

$$|^{(4)}\mathbf{a} \wedge {}^{(4)}\mathbf{A} >= \begin{bmatrix} a^2\,.\,A^3 - a^3\,.\,A^2 \\ a^3\,.\,A^1 - a^1\,.\,A^3 \\ a^1\,.\,A^2 - a^2\,.\,A^1 \\ a^0\,.\,A^3 - a^3\,.\,A^0 \\ -(a^0\,.\,A^2 - a^2\,.\,A^0) \\ a^0\,.\,A^1 - a^1\,.\,A^0 \end{bmatrix} \in M(6 \times 1, \mathbb{C})$$

It is now obvious that, within the specific context of this example, $_{\mathbf{A}}\Phi$ represents a classical wedge product $\wedge$ acting on elements in V_4; precisely:

$$_{\mathbf{A}}\Phi(\mathbf{a}) \equiv \mathbf{a} \wedge \mathbf{A}$$

Recall the result which has been obtained in proposition 1.1 for confirmation.

At this stage, the matrices $_A\Phi(\mathbf{a}), {}_{[A]}\Phi(\mathbf{a})$ and $_{\mathbf{A}}\Phi(\mathbf{a})$ representing the simplest decomposition for the functions $[\mathbf{a}, ...]_A, [\mathbf{a}, ...]_{[A]}$ and $[\mathbf{a}, ...]_{\mathbf{A}}$ have been

...

introduced into this discussion. What is meant here, is the fact that one should be able to write for any given vector $\mathbf{x}_0$:

$$|[\mathbf{a},\, \mathbf{x}_0]_A >= {}_A\Phi(\mathbf{a})\,.\,|\mathbf{x}_0 >$$

$$|[\mathbf{a},\, \mathbf{x}_0]_{[A]} >= {}_{[A]}\Phi(\mathbf{a})\,.\,|\mathbf{x}_0 >$$

$$|[\mathbf{a},\, \mathbf{x}_0]_{\mathbf{A}} >= {}_{\mathbf{A}}\Phi(\mathbf{a})\,.\,|\mathbf{x}_0 >$$

... each specific writing depending on the properties of cube A; respectively: (i) anyone, (ii) anti-symmetric, or (iii) anti-symmetric and anti-reduced. This having been said, in the latter case, one is forced to observe the existence of a difference:

$$\exists <^{(4)}\mathbf{A},\, {}^{(4)}\mathbf{a} >_{Id_4}\, .Id_4 - T_2(\otimes)({}^{(4)}\mathbf{A},\, {}^{(4)}\mathbf{a}) - {}_{\mathbf{A}}\Phi(\mathbf{a})$$

Definition 1.4. *The Perian matrix*

Per convention, the formalism of the previous difference is a so-called Perian matrix:

$$[M({}^{(4)}\mathbf{A},\, {}^{(4)}\mathbf{a})] = <^{(4)}\mathbf{A},\, {}^{(4)}\mathbf{a} >_{Id_4}\, .Id_4 - T_2(\otimes)({}^{(4)}\mathbf{A},\, {}^{(4)}\mathbf{a}) - {}_{\mathbf{A}}\Phi(\mathbf{a})$$

The corollary 1.1 brings no explanation for the existence of this type of matrices within the context of this discussion. It only gives the visage of the difference between two approaches proposing a representation in $\mathrm{M}(4, \mathbb{C})$ for the action of f when the cube A is equivalent to a vector $\mathbf{A}$ in V_4.

One only observes that the approach A) which has been proposed in remark 1.3 realizes a short journey in another set than $\mathrm{M}(4, \mathbb{C})$ to obtain a representation for the action of f. This is not the case for the approach B) which has been exposed in proposition 1.3.

Is the existence of Perian matrices related to the difference of methodology between both approaches? It is at least legitimate to suspect it.

2 The repeated action of $\mathbf{f}(\mathrm{A},\, {}^{(4)}\mathbf{a}) = [{}^{(4)}\mathbf{a},\, {}^{(4)}\mathbf{...}]_A$ when A is an anti-symmetric cube

2.1 Justification

The main purpose of this document is a deepening of the concept of involution when the latter is represented by the generic function $\mathrm{f}(\mathrm{A},\, {}^{(4)}\mathbf{a}) = [{}^{(4)}\mathbf{a},\, {}^{(4)}\mathbf{...}]_A$. To reach the goal, it is useful to study what happens when the action of this generic function is reiterated.

Definition 2.1. *Involution based on the function $[\mathbf{a}, ...]_A$.*

The function $[\mathbf{a}, \mathbf{...}]_A$ is an involution for **...** when:

$$[\mathbf{a},\, [\mathbf{a},\, ...]_A]_A = ...$$

...

2.2 Basics when the cube A is anti-symmetric

In this case, one knows (see at the end of proposition 1.2) that A can be represented with the help of an element [A] in $\mathrm{M}(4 \times 6, \mathbb{C})$:

$$f(A,\, {}^{(4)}\mathbf{a}) \;=\; f({}^{(4\times 6)}[A],\, {}^{(4)}\mathbf{a}) \;=\; [{}^{(4)}\mathbf{a},\, {}^{(4)}\ldots]_{(4\times 6)[A]}$$

Hence, when the action of the generic function is reiterated, one must calculate:

$$\forall\, \mathbf{x}_0 \,\in\, V_4 \,:\; [\mathbf{a},\, \underbrace{[\mathbf{a},\, \mathbf{x}_0]_{(4\times 6)[A]}}_{=\,\mathbf{x}_1}]_{(4\times 6)[A]}$$

Proposition 2.1. *The repetition of the action of the function $[\mathbf{a}, \ldots]_{[A]}$ is equivalent to the action of a function $[\mathbf{a}, \ldots]_{B([A],\mathbf{a})}$ where $B([A],\, \mathbf{a})$ is a cube depending on the characteristics of the generic function.*

Proof. Let calculate the components of $[\mathbf{a}, \mathbf{x}_1]_{[A]}$:

$$\forall\, \epsilon \,\in\, Ind_4 \,:$$

$$\{[\mathbf{a}, \mathbf{x}_1]_{(4\times 6)[A]}\}^{\epsilon}$$

$$=$$

$$\sum_{\chi<\delta} A^{\epsilon}_{\chi\delta} \,.\, (a^{\chi} \,.\, x_1^{\delta} \,-\, a^{\delta} \,.\, x_1^{\chi})$$

$$=$$

$$\sum_{\chi<\delta} A^{\epsilon}_{\chi\delta} \,.\, \{a^{\chi} \,.\, \{\sum_{\alpha<\beta} A^{\delta}_{\alpha\beta} \,.\, (a^{\alpha} \,.\, x_0^{\beta} - a^{\beta} \,.\, x_0^{\alpha})\} - a^{\delta} \,.\, \{\sum_{\alpha<\beta} A^{\chi}_{\alpha\beta} \,.\, (a^{\alpha} \,.\, x_0^{\beta} - a^{\beta} \,.\, x_0^{\alpha})\}\}$$

$$=$$

$$\sum_{\chi<\delta} A^{\epsilon}_{\chi\delta} \,.\, \sum_{\alpha<\beta}\{\underbrace{(a^{\chi} \,.\, A^{\delta}_{\alpha\beta} \,-\, a^{\delta} \,.\, A^{\chi}_{\alpha\beta})}_{=\,T_{\chi\delta\alpha\beta}} \,.\, (a^{\alpha} \,.\, x_0^{\beta} \,-\, a^{\beta} \,.\, x_0^{\alpha})\}$$

$$=$$

$$\sum_{\chi<\delta} A^{\epsilon}_{\chi\delta} \,.\, \sum_{\alpha<\beta} T_{\chi\delta\alpha\beta} \,.\, (a^{\alpha} \,.\, x_0^{\beta} \,-\, a^{\beta} \,.\, x_0^{\alpha})$$

In a four-dimensional space, the pairs $(\alpha,\, \beta)$ such that $\alpha < \beta$ can only be the pairs: (0, 1), (0, 2), (0, 3), (1, 2), (1, 3) and (2, 3). One can conventionally decide to label them with six subscripts in Ind_6:

$$\{\forall\, \alpha,\, \beta \,\in\, Ind_4 \,|\, \alpha \,<\, \beta\} \,\Rightarrow\, \{\exists\, \mu \,\in\, Ind_6 \,|\, \mu \,\equiv\, (\alpha,\, \beta)\}$$

Hence, the calculation can be continued with:

$$\{[\mathbf{a}, \mathbf{x}_1]_{(4\times 6)[A]}\}^{\epsilon}$$

$$=$$

$$\sum_{\mu\in\, Ind_6} A^{\epsilon}_{\mu} \,.\, \sum_{\alpha<\beta} T_{\mu\alpha\beta} \,.\, (a^{\alpha} \,.\, x_0^{\beta} \,-\, a^{\beta} \,.\, x_0^{\alpha})$$

$$=$$

$$\sum_{\alpha<\beta} \sum_{\mu\in Ind_6} \underbrace{A^{\epsilon}_{\mu} \,.\, T_{\mu\alpha\beta}}_{= B^{\epsilon}_{\alpha\beta}} \,.\, (a^{\alpha} \,.\, x^{\beta}_0 \;-\; a^{\beta} \,.\, x^{\alpha}_0)$$

$$=$$

$$\sum_{\alpha<\beta} B^{\epsilon}_{\alpha\beta} \,.\, (a^{\alpha} \,.\, x^{\beta}_0 \;-\; a^{\beta} \,.\, x^{\alpha}_0)$$

$$=$$

$$\{[\mathbf{a}, \mathbf{x}_0]_{B([A],\, \mathbf{a})}\}^{\epsilon}$$

As announced in the proposition, the repetition of the action of the function at hand is a Lie product which is deformed through some cube B with, precisely:

$$B^{\epsilon}_{\alpha\beta}(A,\, \mathbf{a}) \;=\; \sum_{\chi<\delta} A^{\epsilon}_{\chi\delta} \,.\, (a^{\chi} \,.\, A^{\delta}_{\alpha\beta} \;-\; a^{\delta} \,.\, A^{\chi}_{\alpha\beta})$$

□

Proposition 2.2. *The cube B([A], **a**) which has been obtained in repeating the action of the function [**a**, ...]*$_{[A]}$ *is anti-symmetric and can therefore be represented by an element [B([A], **a**)] in M(4 × 6, ℂ).*

Proof. Because A is an element in $\boxplus^{-}(4, \mathbb{C})$, let state without difficulty that:

$$B^{\epsilon}_{\beta\alpha}(A,\, \mathbf{a})$$

$$=$$

$$\sum_{\chi<\delta} A^{\epsilon}_{\chi\delta} \,.\, (a^{\chi} \,.\, A^{\delta}_{\beta\alpha} \;-\; a^{\delta} \,.\, A^{\chi}_{\beta\alpha})$$

$$=$$

$$-\sum_{\chi<\delta} A^{\epsilon}_{\chi\delta} \,.\, (a^{\chi} \,.\, A^{\delta}_{\alpha\beta} \;-\; a^{\delta} \,.\, A^{\chi}_{\alpha\beta})$$

$$=$$

$$-B^{\epsilon}_{\alpha\beta}([A],\, \mathbf{a})$$

Hence:

$$B([A],\, \mathbf{a}) \,\in\, \boxplus^{-}(4,\, \mathbb{C})$$

Concretely, the cube B can be condensed in an element [B] in M(4 × 6, ℂ) and one must write:

$$\forall\, \mathbf{x}_0 \,\in\, V_4 :\; [\mathbf{a},\, [\mathbf{a},\, \mathbf{x}_0]_{(4\times 6)[A]}]_{(4\times 6)[A]} \;=\; [\mathbf{a},\, \mathbf{x}_0]_{(4\times 6)[B({}^{(4\times 6)}[A],\, {}^{(4)}\mathbf{a})]}$$

□

Proposition 2.3. *The repetition of the action of the function $[\boldsymbol{a}, \ldots]_{[A]}$ introduces a mathematical object with four subscripts. This object exhibits three properties mimicking the ones of the components of the Riemann-Christoffel tensor.*

Proof. Let remark that the previous calculations have introduced:

$$\forall\, \chi,\, \delta,\, \alpha,\, \beta \,=\, 0,\, 1,\, 2,\, 3,\, \alpha \,<\, \beta,\, \chi \,<\, \delta :$$

$$T_{\chi\delta\alpha\beta} \,=\, a^{\chi} \,.\, A^{\delta}_{\alpha\beta} \,-\, a^{\delta} \,.\, A^{\chi}_{\alpha\beta}$$

Since each of both pairs $(\alpha,\, \beta)$ and $(\chi,\, \delta)$ can only be in six different configurations because one must have $\alpha < \beta$ and $\chi < \delta$, this object has $6 \times 6 = 36$ hyper-knots. It might be represented with an element [T] in $\mathrm{M}(6, \mathbb{C})$; this is a (6-6) square matrix.

Because A is an element in $\boxplus^{-}(4, \mathbb{R})$, it is easy to remark that:

$$T_{\delta\chi\alpha\beta} \,=\, a^{\delta} \,.\, A^{\chi}_{\alpha\beta} \,-\, a^{\chi} \,.\, A^{\delta}_{\alpha\beta} \,=\, -T_{\chi\delta\alpha\beta}$$

$$T_{\chi\delta\beta\alpha} \,=\, a^{\chi} \,.\, A^{\delta}_{\beta\alpha} \,-\, a^{\delta} \,.\, A^{\chi}_{\beta\alpha} \,=\, -T_{\chi\delta\alpha\beta}$$

It follows:

$$T_{\chi\delta\alpha\beta} \,=\, T_{\delta\chi\beta\alpha}$$

These relations are mimicking three properties of the components of the Riemann-Christoffel curvature tensor. Nevertheless, both objects are in general totally different, at least due to their respective numbers of hyper-knots: 36 for T and 20 for the latter. ☐

Proposition 2.4. *One can build elements in $M(4, \mathbb{C})$ with the help of [A] and [T].*

Proof. This is a simple technical evidence; since:

$$[A] \,.\, [T]$$

$$=$$

$$\begin{bmatrix} A^0_{01} & A^0_{02} & A^0_{03} & A^0_{12} & A^0_{13} & A^0_{23} \\ A^1_{01} & A^1_{02} & A^1_{03} & A^1_{12} & A^1_{13} & A^1_{23} \\ A^2_{01} & A^2_{02} & A^2_{03} & A^2_{12} & A^2_{13} & A^2_{23} \\ A^3_{01} & A^3_{02} & A^3_{03} & A^3_{12} & A^3_{13} & A^3_{23} \end{bmatrix} . \begin{bmatrix} T_{0101} & T_{0102} & T_{0103} & T_{0112} & T_{0113} & T_{0123} \\ T_{0201} & T_{0202} & T_{0203} & T_{0212} & T_{0213} & T_{0223} \\ T_{0301} & T_{0302} & T_{0303} & T_{0312} & T_{0313} & T_{0323} \\ T_{1201} & T_{1202} & T_{1203} & T_{1212} & T_{1213} & T_{1223} \\ T_{1301} & T_{1302} & T_{1303} & T_{1312} & T_{1313} & T_{1323} \\ T_{2301} & T_{2302} & T_{2303} & T_{2312} & T_{2313} & T_{2323} \end{bmatrix}$$

$$\in\, M(4 \,\times\, 6,\, \mathbb{C})$$

One easily gets an element in $\mathrm{M}(4, \mathbb{C})$ with the matrices [A] and [T] in calculating:

$$[A] \,.\, [T] \,.\, [A]^{t} \,\in\, M(4,\, \mathbb{C})$$

☐

Proposition 2.5. *The repetition of the action of* $[\mathbf{a}, \ldots]_{[A]}$ *on* $\boldsymbol{x}_0$ *can be represented with an element in* $M(4 \times 6, \mathbb{C})$ *acting on a representation in* $M(6 \times 1, \mathbb{C})$ *of the classical wedge product* $\boldsymbol{a} \wedge \boldsymbol{x}_0$.

Proof. In observing the definition of B (proposition 2.1), it is easy to remark that:

$$[B] = [A] \,.\, [T] \in M(4 \times 6, \mathbb{C})$$

Reconsidering the results previously obtained for any Lie product which is deformed through an anti-symmetric cube (proposition 1.2), one can write:

$$|[\mathbf{a}, [\mathbf{a}, \mathbf{x}_0]_{[A]}]_{[A]}] >= [B] \,.\, |\mathbf{a} \wedge \mathbf{x}_0 >$$

And, recalling the information which is contained in remark 1.3:

$$|[\mathbf{a}, [\mathbf{a}, \mathbf{x}_0]_{[A]}]_{[A]}] >= \underbrace{[A] \,.\, [T] \,.\, [\odot(\mathbf{a})]^t}_{\in M(4, \mathbb{C}),\, see\, prop.\, 2.4} \,.\, |\mathbf{x}_0 >$$

□

2.3 When the cube A is anti-symmetric and anti-reduced

In this circumstance, one must speak of the repeated action of the function $[\mathbf{a}, \ldots]_{\mathbf{A}}$ on some vector $\mathbf{x}_0$ and reformulate it as:

$$|[\mathbf{a}, [\mathbf{a}, \mathbf{x}]_{\mathbf{A}}]_{\mathbf{A}}] >= [\odot(\mathbf{A})] \,.\, [T] \,.\, [\odot(\mathbf{a})]^t \,.\, |\mathbf{x}_0 >$$

And one must also rewrite the matrix [T] more precisely.

Remark 2.1. *Concerning the formalism of the matrix [T] when the deforming cube is anti-symmetric and anti-reduced.*

In a first step, one can easily state that its diagonal is null because:

$$\forall \alpha, \beta : T_{\alpha\beta\alpha\beta} = a^\alpha \,.\, A^\beta_{\alpha\beta} - a^\beta \,.\, A^\alpha_{\alpha\beta} = 0$$

In a second step, one is obliged to calculate:

$$T_{0101} = a^0 \,.\, A^1_{01} - a^1 \,.\, A^0_{01} = 0$$

$$T_{0102} = a^0 \,.\, A^1_{02} - a^1 \,.\, A^0_{02} = -a^0 \,.\, A^1_{20} = a^0 \,.\, A^0_{21} = -a^0 \,.\, A^0_{12} = a^0 \,.\, A^3$$

$$T_{0103} = a^0 \,.\, A^1_{03} - a^1 \,.\, A^0_{03} = -a^0 \,.\, A^1_{30} = a^0 \,.\, A^0_{31} = -a^0 \,.\, A^0_{13} = -a^0 \,.\, A^2$$

$$T_{0112} = a^0 \,.\, A^1_{12} - a^1 \,.\, A^0_{12} = a^1 \,.\, A^3$$

$$T_{0113} = a^0 \,.\, A^1_{13} - a^1 \,.\, A^0_{13} = -a^1 \,.\, A^2$$

$$T_{0123} = a^0 \,.\, A^1_{23} - a^1 \,.\, A^0_{23} = a^0 \,.\, A^0 + a^1 \,.\, A^1$$

$$T_{0201} = a^0 \,.\, A^2_{01} - a^2 \,.\, A^0_{01} = a^0 \,.\, A^0_{12} = -a^0 \,.\, A^3$$

$$T_{0202} = a^0 \,.\, A^2_{02} - a^2 \,.\, A^0_{02} = 0$$

$$T_{0203} = a^0 \,.\, A^2_{03} - a^2 \,.\, A^0_{03} = -a^0 \,.\, A^2_{30} = a^0 \,.\, A^0_{32} = -a^0 \,.\, A^0_{23} = a^0 \,.\, A^1$$

$$T_{0212} = a^0 . A^2_{12} - a^2 . A^0_{12} = a^2 . A^3$$
$$T_{0213} = a^0 . A^2_{13} - a^2 . A^0_{13} = -(a^0 . A^0 + a^2 . A^2)$$
$$T_{0223} = a^0 . A^2_{23} - a^2 . A^0_{23} = a^2 . A^1$$
$$T_{0301} = a^0 . A^3_{01} - a^3 . A^0_{01} = a^0 . A^0_{13} = a^0 . A^2$$
$$T_{0302} = a^0 . A^3_{02} - a^3 . A^0_{02} = -a^0 . A^3_{20} = a^0 . A^0_{23} = -a^0 . A^1$$
$$T_{0303} = a^0 . A^3_{03} - a^3 . A^0_{03} = 0$$
$$T_{0312} = a^0 . A^3_{12} - a^3 . A^0_{12} = a^0 . A^0 + a^3 . A^3$$
$$T_{0313} = a^0 . A^3_{13} - a^3 . A^0_{13} = -a^3 . A^2$$
$$T_{0323} = a^0 . A^3_{23} - a^3 . A^0_{23} = a^3 . A^1$$
$$T_{1201} = a^1 . A^2_{01} - a^2 . A^1_{01} = a^1 . A^0_{12} = -a^1 . A^3$$
$$T_{1202} = a^1 . A^2_{02} - a^2 . A^1_{02} = a^2 . A^1_{20} = -a^2 . A^0_{21} = a^2 . A^0_{12} = -a^2 . A^3$$
$$T_{1203} = a^1 . A^2_{03} - a^2 . A^1_{03} = -a^1 . A^0_{23} + a^2 . A^0_{13} = a^1 . A^1 + a^2 . A^2$$
$$T_{1212} = a^1 . A^2_{12} - a^2 . A^1_{12} = 0$$
$$T_{1213} = a^1 . A^2_{13} - a^2 . A^1_{13} = -a^1 . A^2_{31} = a^1 . A^1_{32} = -a^1 . A^1_{23} = -a^1 . A^0$$
$$T_{1223} = a^1 . A^2_{23} - a^2 . A^1_{23} = -a^2 . A^0$$
$$T_{1301} = a^1 . A^3_{01} - a^3 . A^1_{01} = a^1 . A^0_{13} = a^1 . A^2$$
$$T_{1302} = a^1 . A^3_{02} - a^3 . A^1_{02} = -a^1 . A^3_{20} + a^3 . A^1_{20} = a^1 . A^0_{23} + a^3 . A^0_{12} = -(a^1 . A^1 + a^3 . A^3)$$
$$T_{1303} = a^1 . A^3_{03} - a^3 . A^1_{03} = a^3 . A^1_{30} = -a^3 . A^0_{31} = a^3 . A^0_{13} = a^3 . A^2$$
$$T_{1312} = a^1 . A^3_{12} - a^3 . A^1_{12} = -a^1 . A^3_{21} = a^1 . A^1_{23} = a^1 . A^0$$
$$T_{1313} = a^1 . A^3_{13} - a^3 . A^1_{13} = 0$$
$$T_{1323} = a^1 . A^3_{23} - a^3 . A^1_{23} = -a^3 . A^0$$
$$T_{2301} = a^2 . A^3_{01} - a^3 . A^2_{01} = a^2 . A^0_{13} - a^3 . A^0_{12} = a^2 . A^2 + a^3 . A^3$$
$$T_{2302} = a^2 . A^3_{02} - a^3 . A^2_{02} = -a^2 . A^3_{20} = a^2 . A^0_{23} = -a^2 . A^1$$
$$T_{2303} = a^2 . A^3_{03} - a^3 . A^2_{03} = a^3 . A^2_{30} = -a^3 . A^0_{32} = a^3 . A^0_{23} = -a^3 . A^1$$
$$T_{2312} = a^2 . A^3_{12} - a^3 . A^2_{12} = -a^2 . A^3_{21} = a^2 . A^1_{23} = a^2 . A^0$$
$$T_{2313} = a^2 . A^3_{13} - a^3 . A^2_{13} = a^3 . A^1_{23} = a^3 . A^0$$
$$T_{2323} = a^2 . A^3_{23} - a^3 . A^2_{23} = 0$$

As consequence:

$$[T] = \begin{bmatrix} 0 & a^0 . A^3 & -a^0 . A^2 & a^1 . A^3 & -a^1 . A^2 & a^0 . A^0 + a^1 . A^1 \\ -a^0 . A^3 & 0 & a^0 . A^1 & a^2 . A^3 & -(a^0 . A^0 + a^2 . A^2) & a^2 . A^1 \\ a^0 . A^2 & -a^0 . A^1 & 0 & a^0 . A^0 + a^3 . A^3 & -a^3 . A^2 & a^3 . A^1 \\ -a^1 . A^3 & -a^2 . A^3 & a^1 . A^1 + a^2 . A^2 & 0 & -a^1 . A^0 & -a^2 . A^0 \\ a^1 . A^2 & -(a^1 . A^1 + a^3 . A^3) & a^3 . A^2 & a^1 . A^0 & 0 & -a^3 . A^0 \\ a^2 . A^2 + a^3 . A^3 & -a^2 . A^1 & -a^3 . A^1 & a^2 . A^0 & a^3 . A^0 & 0 \end{bmatrix}$$

...

The matrix [T] is the sum of two matrices, the first one is *anti-diagonal* and the second of them is anti-symmetric.

$$[T]$$

$$=$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a^0 . A^0 + a^1 . A^1 \\ 0 & 0 & 0 & 0 & -(a^0 . A^0 + a^2 . A^2) & 0 \\ 0 & 0 & 0 & a^0 . A^0 + a^3 . A^3 & 0 & 0 \\ 0 & 0 & a^1 . A^1 + a^2 . A^2 & 0 & 0 & 0 \\ 0 & -(a^1 . A^1 + a^3 . A^3) & 0 & 0 & 0 & 0 \\ a^2 . A^2 + a^3 . A^3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{=[T^{/}]}$$

$$+$$

$$\underbrace{\begin{bmatrix} 0 & a^0 . A^3 & -a^0 . A^2 & a^1 . A^3 & -a^1 . A^2 & 0 \\ -a^0 . A^3 & 0 & a^0 . A^1 & a^2 . A^3 & 0 & a^2 . A^1 \\ a^0 . A^2 & -a^0 . A^1 & 0 & 0 & -a^3 . A^2 & a^3 . A^1 \\ -a^1 . A^3 & -a^2 . A^3 & 0 & 0 & -a^1 . A^0 & -a^2 . A^0 \\ a^1 . A^2 & 0 & a^3 . A^2 & a^1 . A^0 & 0 & -a^3 . A^0 \\ 0 & -a^2 . A^1 & -a^3 . A^1 & a^2 . A^0 & a^3 . A^0 & 0 \end{bmatrix}}_{=[T^-]}$$

$$=$$

$$[T^{/}] + [T^-]$$

The sum of the entries of the anti-diagonal matrix is the Euclidean scalar product between the vectors **a** and **A**:

$$[T^{/}]^{\oplus} = < \mathbf{a}, \mathbf{A} >_{Id_4}$$

The sum of the entries of the anti-symmetric matrix is null:

$$[T^-]^{\oplus} = 0$$

Remark 2.2. *Concerning the matrix [B] when the deforming cube is anti-symmetric and anti-reduced.*

If one now wants to know what the matrix [B] is, one must yet calculate:

$$[B]$$

$$=$$

$$[\odot(\mathbf{A})] . [T]$$

$$=$$

$$\begin{bmatrix} 0 & 0 & 0 & -A^3 & A^2 & -A^1 \\ 0 & A^3 & -A^2 & 0 & 0 & A^0 \\ -A^3 & 0 & A^1 & 0 & -A^0 & 0 \\ A^2 & -A^1 & 0 & A^0 & 0 & 0 \end{bmatrix}$$

$$.$$

$$\begin{bmatrix} 0 & a^0 . A^3 & -a^0 . A^2 & a^1 . A^3 & -a^1 . A^2 & a^0 . A^0 + a^1 . A^1 \\ -a^0 . A^3 & 0 & a^0 . A^1 & a^2 . A^3 & -(a^0 . A^0 + a^2 . A^2) & a^2 . A^1 \\ a^0 . A^2 & -a^0 . A^1 & 0 & a^0 . A^0 + a^3 . A^3 & -a^3 . A^2 & a^3 . A^1 \\ -a^1 . A^3 & -a^2 . A^3 & a^1 . A^1 + a^2 . A^2 & 0 & -a^1 . A^0 & -a^2 . A^0 \\ a^1 . A^2 & -(a^1 . A^1 + a^3 . A^3) & a^3 . A^2 & a^1 . A^0 & 0 & -a^3 . A^0 \\ a^2 . A^2 + a^3 . A^3 & -a^2 . A^1 & -a^3 . A^1 & a^2 . A^0 & a^3 . A^0 & 0 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} a^1 . \{(A^2)^2 + (A^3)^2\} - A^1 . (a^2 . A^2 + a^3 . A^3) & a^2 . \{(A^1)^2 + (A^3)^2\} - A^2 . (a^1 . A^1 + a^3 . A^3) & . \\ -a^0 . \{(A^2)^2 + (A^3)^2\} + A^0 . (a^2 . A^2 + a^3 . A^3) & A^1 . (a^0 . A^2 - a^2 . A^0) & . \\ A^2 . (a^0 . A^1 - a^1 . A^0) & A^0 . (a^1 . A^1 + a^3 . A^3) - a^0 . \{(A^1)^2 + (A^3)^2\} & . \\ A^3 . (a^0 . A^1 - a^1 . A^0) & A^3 . (a^0 . A^2 - a^2 . A^0) & . \end{bmatrix}$$

$$\begin{bmatrix} . & a^3 . \{(A^1)^2 + (A^2)^2\} - A^3 . (a^1 . A^1 + a^2 . A^2) & A^0 . (a^1 . A^2 - a^2 . A^1) & . \\ . & A^1 . (a^0 . A^3 - a^3 . A^0) & a^2 . \{(A^0)^2 + (A^3)^2\} - A^2 . (a^0 . A^0 + a^3 . A^3) & . \\ . & A^2 . (a^0 . A^3 - a^3 . A^0) & A^1 . (a^0 . A^0 + a^3 . A^3) - a^1 . \{(A^0)^2 + (A^3)^2\} & . \\ . & A^0 . (a^1 . A^1 + a^2 . A^2) - a^0 . \{(A^1)^2 + (A^2)^2\} & A^3 . (a^1 . A^2 - a^2 . A^1) & . \end{bmatrix}$$

$$\begin{bmatrix} . & A^0 . (a^1 . A^3 - a^3 . A^1) & A^0 . (a^2 . A^3 - a^3 . A^2) \\ . & a^3 . \{(A^0)^2 + (A^2)^2\} - A^3 . (a^0 . A^0 + a^2 . A^2) & A^1 . (a^2 . A^3 - a^3 . A^2) \\ . & A^2 . (a^1 . A^3 - a^3 . A^1) & -A^3 . (a^0 . A^0 + a^1 . A^1) + a^3 . \{(A^1)^2 + (A^0)^2\} \\ . & -a^1 . \{(A^0)^2 + (A^2)^2\} + A^1 . (a^0 . A^0 + a^2 . A^2) & A^2 . (a^0 . A^0 + a^1 . A^1) - a^2 . \{(A^0)^2 + (A^1)^2\} \end{bmatrix}$$

At this stage, if one wants to exactly know the result of the repeated action of $[\mathbf{a}, \mathbf{...}]_\mathbf{A}$ on $\mathbf{x}_0$, one must multiply this specific matrix [B] on its right side by the matrix $[\odot(\mathbf{a})]^t$; see its expression in remark 1.3. A brute force calculation proves that:

$$[\odot(\mathbf{A})] . [T] . [\odot(\mathbf{a})]^t$$

$$=$$

$$\begin{bmatrix} 0 & . & . & . \\ A^2 . A^3 . \{(a^3)^2 - (a^2)^2\} - a^2 . a^3 . \{(A^3)^2 - (A^2)^2\} - (a^0 . A^0 + a^1 . A^1) . (a^2 . A^3 - a^3 . A^2) & 0 & . & . \\ -\{A^1 . A^3 . \{(a^3)^2 - (a^1)^2\} - a^1 . a^3 . \{(A^3)^2 - (A^1)^2\} - (a^0 . A^0 + a^2 . A^2) . (a^1 . A^3 - a^3 . A^1)\} & . & 0 & . \\ A^1 . A^2 . \{(a^2)^2 - (a^1)^2\} - a^1 . a^2 . \{(A^2)^2 - (A^1)^2\} - (a^0 . A^0 + a^3 . A^3) . (a^1 . A^2 - a^2 . A^1) & . & . & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\{A^2 . A^3 . \{(a^3)^2 - (a^2)^2\} - a^2 . a^3 . \{(A^3)^2 - (A^2)^2\} - (a^0 . A^0 + a^1 . A^1) . (a^2 . A^3 - a^3 . A^2)\} & . & . \\ . & 0 & . & . \\ . & A^0 . A^3 . \{(a^3)^2 - (a^0)^2\} - a^0 . a^3 . \{(A^3)^2 - (A^0)^2\} - (a^1 . A^1 + a^2 . A^2) . (a^0 . A^3 - a^3 . A^0) & 0 & . \\ . & -\{A^0 . A^2 . \{(a^2)^2 - (a^0)^2\} - a^0 . a^2 . \{(A^2)^2 - (A^0)^2\} - (a^1 . A^1 + a^3 . A^3) . (a^0 . A^2 - a^2 . A^0)\} & . & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & . & A^1 . A^3 . \{(a^3)^2 - (a^1)^2\} - a^1 . a^3 . \{(A^3)^2 - (A^1)^2\} - (a^0 . A^0 + a^2 . A^2) . (a^1 . A^3 - a^3 . A^1) & . \\ . & 0 & -\{A^0 . A^3 . \{(a^3)^2 - (a^0)^2\} - a^0 . a^3 . \{(A^3)^2 - (A^0)^2\} - (a^1 . A^1 + a^2 . A^2) . (a^0 . A^3 - a^3 . A^0)\} & . \\ . & . & 0 & . \\ . & . & A^0 . A^1 . \{(a^1)^2 - (a^0)^2\} - a^0 . a^1 . \{(A^1)^2 - (A^0)^2\} - (a^2 . A^2 + a^3 . A^3) . (a^0 . A^1 - a^1 . A^0) & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & . & . & -\{A^1 . A^2 . \{(a^2)^2 - (a^1)^2\} - a^1 . a^2 . \{(A^2)^2 - (A^1)^2\} - (a^0 . A^0 + a^3 . A^3) . (a^1 . A^2 - a^2 . A^1)\} \\ . & 0 & . & A^0 . A^2 . \{(a^2)^2 - (a^0)^2\} - a^0 . a^2 . \{(A^2)^2 - (A^0)^2\} - (a^1 . A^1 + a^3 . A^3) . (a^0 . A^2 - a^2 . A^0) \\ . & . & 0 & -\{A^0 . A^1 . \{(a^1)^2 - (a^0)^2\} - a^0 . a^1 . \{(A^1)^2 - (A^0)^2\} - (a^2 . A^2 + a^3 . A^3) . (a^0 . A^1 - a^1 . A^0)\} \\ . & . & . & 0 \end{bmatrix}$$

At this stage, one remarks that:

- The diagonal vanishes.
- There is a pattern describing the non-vanishing entries resembling the relation below, up to a minus sign depending on the position (to be clarified):

$$m_{\alpha\beta}^{\chi\delta}$$

$$=$$

$$(a^\alpha . A^\alpha + a^\beta . A^\beta) . (a^\chi . A^\delta - a^\delta . A^\chi)$$
$$+ A^\chi . A^\delta . \{(a^\delta)^2 - (a^\chi)^2\} - a^\chi . a^\delta . \{(A^\delta)^2 - (A^\chi)^2\}$$

  With: α = row, β = column, $\alpha \neq \beta$, $\chi \neq \delta$, $\chi < \delta$, χ and $\delta \neq \alpha$, χ and $\delta \neq \beta$.

- The result is an anti-symmetric matrix. Therefore, it can never be identified with the identity matrix Id_4.

These statements have very important consequences; for example, it can never represent an involution. For more details, please see the analysis at the end of the document.

Example 2.1. *When* $\boldsymbol{a} = k.\boldsymbol{A}, \forall\ k \in \mathbb{C}$.

In this case, the matrix [B] vanishes:

$$[B] \;=\; [0]$$

This is not really a surprise since one knows that (see example 1.2 above):

$${}_{\mathbf{A}}\Phi(\mathbf{A}) \;=\; [0]$$

... because this matrix represents the vanishing wedge product $\mathbf{A} \wedge \mathbf{A} = \mathbf{0}$.

2.4 Another representation for the matrix [B([A], a)]

Remark 2.3. *Another formulation for the matrix [B([A],* **a***)] when the cube A is only anti-symmetric.*

One is looking for a generalization of results which have been obtained in treating this topic in a three-dimensional space; this purpose justifies the:

Proposition 2.6. *There exists a set of four matrices* ${}_{\lambda}[D]$ *in M(4 × 6,* $\mathbb{C}$*) such that one can write:*

$$\eta \;=\; (01),\ (02),\ (03),\ (12),\ (13),\ (23) \;\equiv\; 0,\ 1,\ 2,\ 3,\ 4,\ 5$$

$$[B^{\epsilon}_{\eta}({}^{(4\times 6)}[A],\ {}^{(4)}\mathbf{a})] \;=\; \sum_{\lambda} a^{\lambda} \,.\, {}_{\lambda}[D^{\epsilon}_{\eta}]$$

Proof. Recall that the matrix [B([A], **a**)] is an element in M(4 × 6, $\mathbb{C}$):

$$B^{\epsilon}_{\eta}({}^{(4\times 6)}[A],\ {}^{(4)}\mathbf{a})$$

$$=$$

$$A^{\epsilon}_{01} \,.\, (a^0 \,.\, A^1_{\alpha\beta} - a^1 \,.\, A^0_{\alpha\beta}) + A^{\epsilon}_{02} \,.\, (a^0 \,.\, A^2_{\alpha\beta} - a^2 \,.\, A^0_{\alpha\beta}) + A^{\epsilon}_{03} \,.\, (a^0 \,.\, A^3_{\alpha\beta} - a^3 \,.\, A^0_{\alpha\beta})$$

$$+ A^{\epsilon}_{12} \,.\, (a^1 \,.\, A^2_{\alpha\beta} - a^2 \,.\, A^1_{\alpha\beta}) + A^{\epsilon}_{13} \,.\, (a^1 \,.\, A^3_{\alpha\beta} - a^3 \,.\, A^1_{\alpha\beta}) + A^{\epsilon}_{23} \,.\, (a^2 \,.\, A^3_{\alpha\beta} - a^3 \,.\, A^2_{\alpha\beta})$$

$$=$$

$$(A^{\epsilon}_{01} \,.\, A^1_{\alpha\beta} \;+\; A^{\epsilon}_{02} \,.\, A^2_{\alpha\beta} \;+\; A^{\epsilon}_{03} \,.\, A^3_{\alpha\beta}) \,.\, a^0$$

$$+\,(-A^{\epsilon}_{01} \,.\, A^0_{\alpha\beta} \;+\; A^{\epsilon}_{12} \,.\, A^2_{\alpha\beta} \;+\; A^{\epsilon}_{13} \,.\, A^3_{\alpha\beta}) \,.\, a^1$$

$$+\,(-A^{\epsilon}_{02} \,.\, A^0_{\alpha\beta} \;-\; A^{\epsilon}_{12} \,.\, A^1_{\alpha\beta} \;+\; A^{\epsilon}_{23} \,.\, A^3_{\alpha\beta}) \,.\, a^2$$

$$+\,(-A^{\epsilon}_{03} \,.\, A^0_{\alpha\beta} \;-\; A^{\epsilon}_{13} \,.\, A^1_{\alpha\beta} \;-\; A^{\epsilon}_{23} \,.\, A^2_{\alpha\beta}) \,.\, a^3$$

Because of the anti-symmetry:

$$=$$

$$(A^{\epsilon}_{01} \,.\, A^1_{\alpha\beta} \;+\; A^{\epsilon}_{02} \,.\, A^2_{\alpha\beta} \;+\; A^{\epsilon}_{03} \,.\, A^3_{\alpha\beta}) \,.\, a^0$$

...

$$+ (A^{\epsilon}_{10} \,.\, A^{0}_{\alpha\beta} \,+\, A^{\epsilon}_{12} \,.\, A^{2}_{\alpha\beta} \,+\, A^{\epsilon}_{13} \,.\, A^{3}_{\alpha\beta}) \,.\, a^1$$
$$+ (A^{\epsilon}_{20} \,.\, A^{0}_{\alpha\beta} \,+\, A^{\epsilon}_{21} \,.\, A^{1}_{\alpha\beta} \,+\, A^{\epsilon}_{23} \,.\, A^{3}_{\alpha\beta}) \,.\, a^2$$
$$+ (A^{\epsilon}_{30} \,.\, A^{0}_{\alpha\beta} \,+\, A^{\epsilon}_{31} \,.\, A^{1}_{\alpha\beta} \,+\, A^{\epsilon}_{32} \,.\, A^{2}_{\alpha\beta}) \,.\, a^3$$

These relations can be condensed in:

$$B^{\epsilon}_{\eta}$$
$$=$$
$$\underbrace{\sum_{\lambda} \sum_{\theta} A^{\epsilon}_{\lambda\theta} \,.\, \underbrace{A^{\theta}_{\alpha\beta}}_{= A^{\theta}_{\eta}}}_{= D^{\epsilon}_{\lambda\eta}} \,.\, a^{\lambda}$$
$$=$$
$$\sum_{\lambda} D^{\epsilon}_{\lambda\eta} \,.\, a^{\lambda}$$

This formalism must be manipulated with precaution because the cube D appearing here is not an element in $\boxplus(4, \mathbb{C})$. Instead of that, it is the superposition of four elements in M(4 × 6, $\mathbb{C}$). Therefore, to avoid a misinterpretation of this relation, one must absolutely note that:

$$[B] \,\neq\, {}_{D}\Phi(\mathbf{a})$$

... and highly prefer the writing:

$$[B^{\epsilon}_{\eta}] \,=\, \sum_{\lambda} a^{\lambda} \,.\, {}_{\lambda}[D^{\epsilon}_{\eta}]$$

... where the ${}_{\lambda}$[D] for λ = 0, 1, 2, 3, are four elements in M(4 × 6, $\mathbb{R}$). □

Proposition 2.7. *Each element ${}_{\lambda}$[D] in M(4 × 6, $\mathbb{C}$) is the product of the λ^{th} matrix, [${}_{\lambda}$A] in M(4, $\mathbb{C}$), implicitly contained in the anti-symmetric cube A by the matrix [A] in M(4 × 6, $\mathbb{C}$) containing all remaining non-vanishing knots of this cube.*

Proof. Let try to write each given element ${}_{\lambda}$[D] precisely:

$$\forall\, \lambda,\, \epsilon \,=\, 0,\, 1,\, 2,\, 3;\, \forall\, \eta \,=\, 0,\, 1,\, 2,\, 3,\, 4,\, 5:$$

$${}_{\lambda}[D^{\epsilon}_{\eta}] \,=\, [\sum_{\theta=0}^{\theta=3} A^{\epsilon}_{\lambda\theta} \,.\, A^{\theta}_{\eta}]$$

In general, one must start with:

$$\lambda\, given$$

$${}_{\lambda}[D^{\epsilon}_{\eta}]$$
$$=$$

...

$$[D^{\epsilon}_{\lambda\eta}]$$

$$=$$

$$\begin{bmatrix} D^0_{\lambda 0} & D^0_{\lambda 1} & D^0_{\lambda 2} & D^0_{\lambda 3} & D^0_{\lambda 4} & D^0_{\lambda 5} \\ D^1_{\lambda 0} & D^1_{\lambda 1} & D^1_{\lambda 2} & D^1_{\lambda 3} & D^1_{\lambda 4} & D^1_{\lambda 5} \\ D^2_{\lambda 0} & D^2_{\lambda 1} & D^2_{\lambda 2} & D^2_{\lambda 3} & D^2_{\lambda 4} & D^2_{\lambda 5} \\ D^3_{\lambda 0} & D^3_{\lambda 1} & D^3_{\lambda 2} & D^3_{\lambda 3} & D^3_{\lambda 4} & D^3_{\lambda 5} \end{bmatrix}$$

In a next step, taking care of the definition of the entries:

$$=$$

$$\begin{bmatrix} \sum_{\theta=0}^{\theta=3} A^0_{\lambda\theta}\,.\,A^\theta_0 & \sum_{\theta=0}^{\theta=3} A^0_{\lambda\theta}\,.\,A^\theta_1 & \sum_{\theta=0}^{\theta=3} A^0_{\lambda\theta}\,.\,A^\theta_2 & \sum_{\theta=0}^{\theta=3} A^0_{\lambda\theta}\,.\,A^\theta_3 & \sum_{\theta=0}^{\theta=3} A^0_{\lambda\theta}\,.\,A^\theta_4 & \sum_{\theta=0}^{\theta=3} A^0_{\lambda\theta}\,.\,A^\theta_5 \\ \sum_{\theta=0}^{\theta=3} A^1_{\lambda\theta}\,.\,A^\theta_0 & \sum_{\theta=0}^{\theta=3} A^1_{\lambda\theta}\,.\,A^\theta_1 & \sum_{\theta=0}^{\theta=3} A^1_{\lambda\theta}\,.\,A^\theta_2 & \sum_{\theta=0}^{\theta=3} A^1_{\lambda\theta}\,.\,A^\theta_3 & \sum_{\theta=0}^{\theta=3} A^1_{\lambda\theta}\,.\,A^\theta_4 & \sum_{\theta=0}^{\theta=3} A^1_{\lambda\theta}\,.\,A^\theta_5 \\ \sum_{\theta=0}^{\theta=3} A^2_{\lambda\theta}\,.\,A^\theta_0 & \sum_{\theta=0}^{\theta=3} A^2_{\lambda\theta}\,.\,A^\theta_1 & \sum_{\theta=0}^{\theta=3} A^2_{\lambda\theta}\,.\,A^\theta_2 & \sum_{\theta=0}^{\theta=3} A^2_{\lambda\theta}\,.\,A^\theta_3 & \sum_{\theta=0}^{\theta=3} A^2_{\lambda\theta}\,.\,A^\theta_4 & \sum_{\theta=0}^{\theta=3} A^2_{\lambda\theta}\,.\,A^\theta_5 \\ \sum_{\theta=0}^{\theta=3} A^3_{\lambda\theta}\,.\,A^\theta_0 & \sum_{\theta=0}^{\theta=3} A^3_{\lambda\theta}\,.\,A^\theta_1 & \sum_{\theta=0}^{\theta=3} A^3_{\lambda\theta}\,.\,A^\theta_2 & \sum_{\theta=0}^{\theta=3} A^3_{\lambda\theta}\,.\,A^\theta_3 & \sum_{\theta=0}^{\theta=3} A^3_{\lambda\theta}\,.\,A^\theta_4 & \sum_{\theta=0}^{\theta=3} A^3_{\lambda\theta}\,.\,A^\theta_5 \end{bmatrix}$$

With an attentive observation of the diverse sums, one recognize the product of two matrices. The first one is in some way a cut in the cube A when the former is understood as a superposition of four elements [$_\lambda$A] in M(4, $\mathbb{C}$). The second is the matrix [A] containing the remaining non-vanishing knots of the anti-symmetric cube A:

$$=$$

$$\underbrace{\begin{bmatrix} A^0_{\lambda 0} & A^0_{\lambda 1} & A^0_{\lambda 2} & A^0_{\lambda 3} \\ A^1_{\lambda 0} & A^1_{\lambda 1} & A^1_{\lambda 2} & A^1_{\lambda 3} \\ A^2_{\lambda 0} & A^2_{\lambda 1} & A^2_{\lambda 2} & A^2_{\lambda 3} \\ A^3_{\lambda 0} & A^3_{\lambda 1} & A^3_{\lambda 2} & A^3_{\lambda 3} \end{bmatrix}}_{[_\lambda A]} \,.\, \underbrace{\begin{bmatrix} A^0_0 & A^0_1 & A^0_2 & A^0_3 & A^0_4 & A^0_5 \\ A^1_0 & A^1_1 & A^1_2 & A^1_3 & A^1_4 & A^1_5 \\ A^2_0 & A^2_1 & A^2_2 & A^2_3 & A^2_4 & A^2_5 \\ A^3_0 & A^3_1 & A^3_2 & A^3_3 & A^3_4 & A^3_5 \end{bmatrix}}_{=[A]}$$

$$=$$

$$[_\lambda A]\,.\,[A]$$

□

At the end of the day, one may propose an alternative and easily understandable formalism for the matrix:

$$[B^{\epsilon}_{\eta}(^{(4\times 6)}[A],\ ^{(4)}\mathbf{a})] \;=\; \{\sum_{\lambda=0}^{\lambda=3} a^{\lambda}\,.\,[_\lambda A]\}\,.\,[A]$$

Remark 2.4. *The formalism of the [$_\lambda$A] matrices.*

For the completeness of this exploration, let give the precise formulation of the [$_\lambda$A] matrices when A is an anti-symmetric cube; recall that this kind of anti-symmetry means:

$$A^{\epsilon}_{\lambda\eta} + A^{\epsilon}_{\eta\lambda} \;=\; 0$$

Hence:

- When $\lambda = 0$

$$[{}_0A] = \begin{bmatrix} 0 & A^0_{01} & A^0_{02} & A^0_{03} \\ 0 & A^1_{01} & A^1_{02} & A^1_{03} \\ 0 & A^2_{01} & A^2_{02} & A^2_{03} \\ 0 & A^3_{01} & A^3_{02} & A^3_{03} \end{bmatrix} = \begin{bmatrix} 0 & A^0_{0} & A^0_{1} & A^0_{2} \\ 0 & A^1_{0} & A^1_{1} & A^1_{2} \\ 0 & A^2_{0} & A^2_{1} & A^2_{2} \\ 0 & A^3_{0} & A^3_{1} & A^3_{2} \end{bmatrix}$$

- When $\lambda = 1$

$$[{}_1A] = \begin{bmatrix} A^0_{10} & 0 & A^0_{12} & A^0_{13} \\ A^1_{10} & 0 & A^1_{12} & A^1_{13} \\ A^2_{10} & 0 & A^2_{12} & A^2_{13} \\ A^3_{10} & 0 & A^3_{12} & A^3_{13} \end{bmatrix} = \begin{bmatrix} -A^0_{0} & 0 & A^0_{3} & A^0_{4} \\ -A^1_{0} & 0 & A^1_{3} & A^1_{4} \\ -A^2_{0} & 0 & A^2_{3} & A^2_{4} \\ -A^3_{0} & 0 & A^3_{3} & A^3_{4} \end{bmatrix}$$

- When $\lambda = 2$

$$[{}_2A] = \begin{bmatrix} A^0_{20} & A^0_{21} & 0 & A^0_{23} \\ A^1_{20} & A^1_{21} & 0 & A^1_{23} \\ A^2_{20} & A^2_{21} & 0 & A^2_{23} \\ A^3_{20} & A^3_{21} & 0 & A^3_{23} \end{bmatrix} = \begin{bmatrix} -A^0_{1} & -A^0_{3} & 0 & A^0_{5} \\ -A^1_{1} & -A^1_{3} & 0 & A^1_{5} \\ -A^2_{1} & -A^2_{3} & 0 & A^2_{5} \\ -A^3_{1} & -A^3_{3} & 0 & A^3_{5} \end{bmatrix}$$

- When $\lambda = 3$

$$[{}_3A] = \begin{bmatrix} A^0_{30} & A^0_{31} & A^0_{32} & 0 \\ A^1_{30} & A^1_{31} & A^1_{32} & 0 \\ A^2_{30} & A^2_{31} & A^2_{32} & 0 \\ A^3_{30} & A^3_{31} & A^3_{32} & 0 \end{bmatrix} = \begin{bmatrix} -A^0_{2} & -A^0_{4} & -A^0_{5} & 0 \\ -A^1_{2} & -A^1_{4} & -A^1_{5} & 0 \\ -A^2_{2} & -A^2_{4} & -A^2_{5} & 0 \\ -A^3_{2} & -A^3_{4} & -A^3_{5} & 0 \end{bmatrix}$$

3 Involution and deformed Lie products in a four dimensional space

3.1 First representation

Remark 3.1. *When the action of f is described with the approach proposed in remark 1.3.*

Let go back to remark 1.3, apply it here and get:

$$|[\mathbf{a}, \mathbf{x}_1]_{[A]} >= \{{}_{[A]}\Phi(\mathbf{a})\} \,.\, |\mathbf{x}_1 >$$

And then:

$$|[\mathbf{a},\, [\mathbf{a},\, \mathbf{x}_0]_{[A]}]_{[A]} >= \{{}_{[A]}\Phi(\mathbf{a})\}^2 \,.\, |\mathbf{x}_0 >$$

Therefore, the simplest signature for the existence of an involution is:

$$\{{}_{[A]}\Phi(\mathbf{a})\}^2 \,=\, Id_4$$

3.2 Second representation

Remark 3.2. *When the action of f is written with the spirit exposed in proposition 1.3.*

If in acting on $\mathbf{x}_0$ the function $[\mathbf{a}, ...]_{[A]}$ is an involution, then one must write:

$$\mathbf{x}_2 \,=\, [\mathbf{a},\, \mathbf{x}_0]_{[B]} \,=\, \mathbf{x}_0$$

In this specific case, one spontaneously believe that there exists a representation of $[\mathbf{a}, ...]_{[B]}$ which is equivalent to the identity matrix Id_4. Unfortunately, although the matrix [B] is a linear combination depending on coefficients in $\mathrm{M}(4,\mathbb{C})$, is is not itself an element in $\mathrm{M}(4,\mathbb{C})$; recall:

$$[B] \,=\, \underbrace{\{\sum_{\lambda=0}^{\lambda=3} a^{\lambda} \,.\, [{}_{\lambda}A]\}}_{\in M(4,\mathbb{C})} \,.\, \underbrace{[A]}_{\in M(4\times 6,\mathbb{C})} \;\in\, M(4\,\times\,6,\,\mathbb{C})$$

Therefore, the intuitive belief lies on a misleading though. On the other hand, the matrix [A] plays obviously an important role, both (i) as the representation of the source of the deformation and (ii) in the process of iteration. Let now look for elements in $\mathrm{M}(4,\mathbb{C})$ which are equivalent to the repeated actions of $[\mathbf{a}, ...]_{[A]}$ on $\mathbf{x}_0$:

- In absence of this action, it is obvious that:

$$|^{(4)}\mathbf{x}_0 > = \underbrace{Id_4}_{\in M(4,\mathbb{C})} \,.\, |^{(4)}\mathbf{x}_0 >$$

- The representation of the first action on $\mathbf{x}_0$ is equivalent to the action of [A] on $|\mathbf{a} \wedge ...>$. But one can also go a step further and write:

$$|^{(4)}\mathbf{x}_1 > = {}^{(4\times 6)}[A] \,.\, |^{(4)}\mathbf{a} \wedge {}^{(4)}\mathbf{x}_0 > = \underbrace{{}^{(4\times 6)}[A] \,.\, {}^{(6\times 4)}[\odot(\mathbf{a})]^t}_{\in M(4,\mathbb{C})} \,.\, |^{(4)}\mathbf{x}_0 >$$

- The representation of the first repetition of the action is the matrix [B] which is proportional to the matrix [A]; but in reintroducing what has been proved previously:

$$|^{(4)}\mathbf{x}_2 > = \underbrace{{}^{(4\times 6)}[A] \,.\, {}^{(6\times 6)}[T] \,.\, {}^{(6\times 4)}[\odot(\mathbf{a})]^t}_{\in M(4,\mathbb{C})} \,.\, |^{(4)}\mathbf{x}_0 >$$

This enumeration starts a list of which one ignores the next terms. What happens for the next iterations? For now, one has the sensation to have a second criterion characterizing the involution:

$$[A] \,.\, [T] \,.\, [\odot(\mathbf{a})]^t \,=\, Id_4$$

But the problematic which has been exposed in remark 1.4 appears here again (the so-called (E) question) because, at this stage, one is not in a mental position allowing to decide which approach is the correct one: proposition 1.3 or remark 1.3?

4 The enigmatic question

One is obviously facing a question of representation; but not only.

4.1 Analysis for an anti-symmetric and anti-reduced cube

Let start softly in considering only what happens when the cube A is anti-symmetric and anti-reduced. As it has already been stated, each deformed Lie product has many possible representations; e.g: in M(1 × 6, ℂ), M(6 × 1, ℂ), M(4, ℂ), etc. For example here, in involving the Dirac's convention, it has been possible to find two different representations for the action of f.

A) Following the approach exposed in remark 1.3:

$$\underbrace{|[\mathbf{a},\, \mathbf{x}_0]_{\mathbf{A}} >}_{\in M(4\times 1,\mathbb{C})\equiv V_4^*} \;=\; \underbrace{{}_{\mathbf{A}}\Phi(\mathbf{a})}_{\in M^-(4,\mathbb{C})} \;.\; \underbrace{|\mathbf{x}_0 >}_{\in M(4\times 1,\mathbb{C})\equiv V_4^*}$$

B) Following the approach exposed in proposition 1.3:

$$\underbrace{|[\mathbf{a},\, \mathbf{x}_0]_{\mathbf{A}} >}_{\in V_4^*} = \underbrace{[\odot(\mathbf{A})]}_{\in M(4\times 6,\mathbb{C})} \;.\; \underbrace{|\mathbf{a}\wedge \mathbf{x}_0 >}_{\in M(6\times 1,\mathbb{C})} \underbrace{=}_{?} \underbrace{[\odot(\mathbf{A})]\;.\; \underbrace{[\odot(\mathbf{a})]^t}_{\in M(6\times 4,\mathbb{C})}}_{\in M(4,\mathbb{C})} \;.\; \underbrace{|\mathbf{x}_0 >}_{\in V_4^*}$$

- The first line is a representation of this product with the help of elements in V^*_4 and M(4, ℂ). The action of f is represented through an element ${}_{\mathbf{A}}\Phi(\mathbf{a})$ in $M^-(4,\mathbb{C})$; this is an anti-symmetric matrix. With different words, it can never be identified with the identity matrix, except when the argument **a** vanishes. Therefore, f is unable to leave the vector $\mathbf{x}_0$ unchanged. In opposition, the square of ${}_{\mathbf{A}}\Phi(\mathbf{a})$ is a symmetric matrix and one may sometimes envisage the existence of circumstances characterized by the relation signing the presence of an involution:

$$\{{}_{\mathbf{A}}\Phi(\mathbf{a})\}^2 \,=\, Id_4$$

- The second line is a representation of this product with the intervening of elements in M(4 × 6, ℂ), M(6 × 4, ℂ), M(6 × 1, ℂ) and V^*_4. Multiplying the dilatation of **A** which is an element in M(4 × 6, ℂ) on its right side by the transposed of the dilatation of **a** yields an element in M(4, ℂ). In multiplying it on its right side by any element $|\mathbf{x}_0>$ in V^*_4, one gets a representation of the deformed Lie product $[\mathbf{a}, \mathbf{x}_0]_{\mathbf{A}}$ in V^*_4 as well:

$${}^{(4\times 6)}[\odot(\mathbf{A})]\,.\,{}^{(6\times 4)}[\odot(\mathbf{a})]^t \;=<^{(4)}\mathbf{A},\, {}^{(4)}\mathbf{a}>_{Id_4}\;.\,Id_4 \,-\, T_2(\otimes)({}^{(4)}\mathbf{A},\, {}^{(4)}\mathbf{a})$$

The case $\mathbf{a} = \mathbf{A}$ confirms the difference between both approaches:

$$[\odot(\mathbf{A})]\,.\,[\odot(\mathbf{A})]^t \;=<^{(4)}\mathbf{A},\, {}^{(4)}\mathbf{A}>_{Id_4}\;.\,Id_4 \,-\, T_2(\otimes)({}^{(4)}\mathbf{A},\, {}^{(4)}\mathbf{A})$$

$${}_{\mathbf{A}}\Phi(\mathbf{A}) \,=\, {}^{(4)}[0]$$

Both matrices can only coincide when the deforming vector $\mathbf{A}$ vanishes; but this eventuality is totally meaningless in this theory.

This representation can be anyone; with different words: it is not necessarily anti-symmetric or totally symmetric. Hence, there is perhaps situations leaving the vector $\mathbf{x}_0$ unchanged. Nevertheless and unexpectedly, the repetition of the action of f on $\mathbf{x}_0$ with this representation systematically yields an anti-symmetric matrix:

$$[\odot(\mathbf{A})]\,.\,[T]\,.\,[\odot(\mathbf{A})]^t \in M^-(4,\,\mathbb{C})$$

It can never be identified with the identity matrix. Therefore, the approach which has been exposed in proposition 1.3 is discarded for a representation of the involution.

Remark 4.1. *A diagram to visualize both representations.*

As a matter of facts, the analysis of this approach introduces this diagram for any pair $(\mathbf{a}, \mathbf{...})$ on which a deforming vector $\mathbf{A}$ acts:

$$\begin{array}{ccc}
\mathbf{a} & \xrightarrow{\quad [\odot] \quad} & [\odot(\mathbf{a})] \\
\downarrow {\scriptstyle {}_{\mathbf{A}}\Phi} & & \downarrow {\scriptstyle \pi_{\mathbf{A}}} \\
{}_{\mathbf{A}}\Phi(\mathbf{a}) \in M(4,\,\mathbb{C}) & \underset{?}{\overset{=}{\longleftrightarrow}} & \pi_{\mathbf{A}}([\odot(\mathbf{a})]) = [\odot(\mathbf{A})]\,.\,[\odot(\mathbf{a})]^t \in M(4,\,\mathbb{C}) \\
\downarrow {\scriptstyle .|...>} & & \downarrow {\scriptstyle .|...>} \\
[\mathbf{a},\,...]_{\mathbf{A}} & \overset{=}{\longleftrightarrow} & [\mathbf{a},\,...]_{\mathbf{A}}
\end{array}$$

... and suggests to introduce:

Definition 4.1. *The function $\pi_{\mathbf{A}}$*

The function $\pi_{\mathbf{A}}$ is defined through the sentence:

$$\pi_{\mathbf{A}} : \forall [\odot(\mathbf{a})] \in M(4 \times 6, \mathbb{C}) \xrightarrow{\pi_{\mathbf{A}}} \pi_{\mathbf{A}}([\odot(\mathbf{a})]) = [\odot(\mathbf{A})]\,.\,[\odot(\mathbf{a})]^t \in M(4,\,\mathbb{C})$$

The matrix $[\odot(\mathbf{A})]$ is sufficient to describe the action of $[\mathbf{a}, \mathbf{...}]_{\mathbf{A}}$ on $|\mathbf{a} \wedge \mathbf{...}>$ when the latter is written in $\mathrm{M}(6 \times 1,\,\mathbb{C})$. One must multiply this matrix on its right side by the transposed of the matrix $[\odot(\mathbf{a})]$ to describe the action of $[\mathbf{a}, \mathbf{...}]_{\mathbf{A}}$ on $|\mathbf{...}>$ when the latter is written in $\mathrm{M}(4 \times 1,\,\mathbb{C}) \equiv \mathrm{V^*}_4$.

Hence, despite appearances, this multiplication is softly changing the context and the purpose of calculations whilst the use of the simplest decomposition induces no complication. This evidence is perhaps the beginning of a reasonable explanation for the difference between the simplest decomposition and the product $[\odot(\mathbf{A})]\,.\,[\odot(\mathbf{a})]^t$: they have not the same role, not the same function.

4.2 Application in physics for an anti-symmetric and anti-reduced cube

Proposition 4.1. *When the cube A is anti-symmetric and anti-reduced, the product $[\odot(\boldsymbol{A})] . [\odot(\boldsymbol{a})]^t$ can be related to the simplest decomposition ${}_{\boldsymbol{A}}\Phi(\boldsymbol{a})$.*

Proof. One remarks by the way that:

$$[\odot(\mathbf{A})] . [\odot(\mathbf{a})]^t - \{[\odot(\mathbf{A})] . [\odot(\mathbf{a})]^t\}^t$$

$$=$$

$$T_2(\otimes)({}^{(4)}\mathbf{A}, {}^{(4)}\mathbf{a}) - T_2(\otimes)({}^{(4)}\mathbf{a}, {}^{(4)}\mathbf{A})$$

$$=$$

$$\begin{bmatrix} 0 & A^0 . a^1 - A^1 . a^0 & A^0 . a^2 - A^2 . a^0 & A^0 . a^3 - A^3 . a^0 \\ -(A^0 . a^1 - A^1 . a^0) & 0 & -(A^2 . a^1 - A^1 . a^2) & (A^1 . a^3 - A^3 . a^1) \\ -(A^0 . a^2 - A^2 . a^0) & (A^2 . a^1 - A^1 . a^2) & 0 & -(A^3 . a^2 - A^2 . a^3) \\ -(A^0 . a^3 - A^3 . a^0) & -(A^1 . a^3 - A^3 . a^1) & (A^3 . a^2 - A^2 . a^3) & 0 \end{bmatrix}$$

Let recall that:

$$\Phi_{01} = A^3 . a^2 - A^2 . a^3$$

$$\Phi_{02} = -A^3 . a^1 + A^1 . a^3$$

$$\Phi_{03} = A^2 . a^1 - A^1 . a^2$$

$$\Phi_{12} = A^3 . a^0 - A^0 . a^3$$

$$\Phi_{13} = -A^2 . a^0 + A^0 . a^2$$

$$\Phi_{23} = A^1 . a^0 - A^0 . a^1$$

Hence, the calculation continues with:

$$T_2(\otimes)({}^{(4)}\mathbf{A}, {}^{(4)}\mathbf{a}) - T_2(\otimes)({}^{(4)}\mathbf{a}, {}^{(4)}\mathbf{A})$$

$$=$$

$$\begin{bmatrix} 0 & -\Phi_{23} & \Phi_{13} & -\Phi_{12} \\ \Phi_{23} & 0 & -\Phi_{03} & \Phi_{02} \\ -\Phi_{13} & \Phi_{03} & 0 & -\Phi_{01} \\ \Phi_{12} & -\Phi_{02} & \Phi_{01} & 0 \end{bmatrix}$$

The goal of this proposition is to preove that this result can be related to:

$${}_{\mathbf{A}}\Phi(\mathbf{a}) = \begin{bmatrix} 0 & \Phi_{01} & \Phi_{02} & \Phi_{03} \\ -\Phi_{01} & 0 & \Phi_{12} & \Phi_{13} \\ -\Phi_{02} & -\Phi_{12} & 0 & \Phi_{23} \\ -\Phi_{03} & -\Phi_{13} & -\Phi_{23} & 0 \end{bmatrix}$$

The formalism of these matrices suggests that one is facing a dual representation of ${}_{\mathbf{A}}\Phi(\mathbf{a})$. Therefore, the discovery of at least one matrix [W] insuring the passage between both representations becomes a legitimate purpose; let envisage the relation:

$$\begin{bmatrix} 0 & -\Phi_{23} & \Phi_{13} & -\Phi_{12} \\ \Phi_{23} & 0 & -\Phi_{03} & \Phi_{02} \\ -\Phi_{13} & \Phi_{03} & 0 & -\Phi_{01} \\ \Phi_{12} & -\Phi_{02} & \Phi_{01} & 0 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} w_{00} & w_{01} & w_{02} & w_{03} \\ w_{10} & w_{11} & w_{12} & w_{13} \\ w_{20} & w_{21} & w_{22} & w_{23} \\ w_{30} & w_{31} & w_{32} & w_{33} \end{bmatrix} . \begin{bmatrix} 0 & \Phi_{01} & \Phi_{02} & \Phi_{03} \\ -\Phi_{01} & 0 & \Phi_{12} & \Phi_{13} \\ -\Phi_{02} & -\Phi_{12} & 0 & \Phi_{23} \\ -\Phi_{03} & -\Phi_{13} & -\Phi_{23} & 0 \end{bmatrix}$$

Because of a very strong analogy with the representations of the electromagnetic fields, two three-dimensional vectors will be introduced into this discussion:

$$\frac{1}{c} \,.\, {}^{(3)}\mathbf{E} \,:\, (\Phi_{01},\, \Phi_{02},\, \Phi_{03})$$

$${}^{(3)}\mathbf{B} \,:\, (\Phi_{23},\, -\Phi_{13},\, \Phi_{12})$$

Although these vectors are actually any ones, this decision will later simplify an eventual application to the theory of electromagnetism.

To accelerate the calculations, all elements in $\mathrm{M}(4, \mathbb{C})$ will also be decoded as a corner χ in $\mathbb{C}$, a wing $|\mathbf{W}>$ on the West side in $\mathrm{M}(3 \times 1, \mathbb{C})$, a wing $<\mathbf{N}|$ on the North side in $\mathrm{M}(1 \times 3, \mathbb{C})$ and a heart [H] in $\mathrm{M}(3, \mathbb{C})$. Concretely:

$$T_2(\otimes)({}^{(4)}\mathbf{A},\, {}^{(4)}\mathbf{a}) \,-\, T_2(\otimes)({}^{(4)}\mathbf{a},\, {}^{(4)}\mathbf{A}) \,=\, -\begin{bmatrix} 0 & <^{(3)}\,\mathbf{B}| \\ -|^{(3)}\mathbf{B}> & -\frac{1}{c} \cdot {}_{[J]}\Phi({}^{(3)}\mathbf{E}) \end{bmatrix}$$

$$[W] \,=\, \begin{bmatrix} \chi & <^{(3)}\,\mathbf{N}| \\ -|^{(3)}\mathbf{W}> & {}^{(3)}[H] \end{bmatrix}$$

$${}_{\mathbf{A}}\Phi(\mathbf{a}) \,=\, \begin{bmatrix} 0 & \frac{1}{c} \,.\, <^{(3)}\,\mathbf{E}| \\ -\frac{1}{c} \,.\, |^{(3)}\mathbf{E}> & {}_{[J]}\Phi({}^{(3)}\mathbf{B}) \end{bmatrix}$$

This demonstration will only be completed when the matrix [W] will have received a visage. There is neither a difficulty to decode any element in $\mathrm{M}(\mathrm{D}, \mathbb{C})$ as already explained nor to prove that:

$${}^{(D)}[M] \,.\, {}^{(D)}[M']$$

$$=$$

$$\begin{bmatrix} m_1 & <^{(D-1)}\,\mathbf{M}_1| \\ |^{(D-1)}\mathbf{N}_1> & {}^{(D-1)}[M_1] \end{bmatrix} . \begin{bmatrix} m_2 & <^{(D-1)}\,\mathbf{M}_2| \\ |^{(D-1)}\mathbf{N}_2> & {}^{(D-1)}[M_2] \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} m_1 \,.\, m_2 \,+\, <^{(D-1)}\,\mathbf{M}_1| \,.\, |^{(D-1)}\mathbf{N}_2> & m_1 . <^{(D-1)}\,\mathbf{M}_2| \,+\, <^{(D-1)}\,\mathbf{M}_1| . {}^{(D-1)}[M_2] \\ m_2 \,.\, |^{(D-1)}\mathbf{N}_1> \,+\, {}^{(D-1)}[M_1] \,.\, |^{(D-1)}\mathbf{N}_2> & {}^{(D-1)}[M_1] \,.\, {}^{(D-1)}[M_2] \,+\, T_2(\otimes)({}^{(D-1)}\mathbf{N}_1,\, {}^{(D-1)}\mathbf{M}_2) \end{bmatrix}$$

This general rule can be applied to the matrices one is studying here:

$${}^{(4)}[W] \,.\, {}^{(4)}_{\mathbf{A}}\Phi(\mathbf{a})$$

$$=$$

$$\begin{bmatrix} \chi & <^{(3)}\,\mathbf{N}| \\ |^{(3)}\mathbf{W}> & {}^{(3)}[H] \end{bmatrix} . \begin{bmatrix} 0 & \frac{1}{c} \,.\, <^{(3)}\,\mathbf{E}| \\ -\frac{1}{c} \,.\, |^{(3)}\mathbf{E}> & {}_{[J]}\Phi({}^{(3)}\mathbf{B}) \end{bmatrix}$$

$$=$$

...

$$\begin{bmatrix} \chi\,.\,0\ -\ \frac{1}{c}\,.\ <^{(3)}\mathbf{N}|\,.\,|^{(3)}\mathbf{E}> & \frac{\chi}{c}\,.\ <^{(3)}\mathbf{E}|\ +\ <^{(3)}\mathbf{N}|\,.\,_{[J]}\Phi(^{(3)}\mathbf{B}) \\ 0\,.\,|^{(3)}\mathbf{W}>\ -\ \frac{1}{c}\,.\,^{(3)}[H]\,.\,|^{(3)}\mathbf{E}> & ^{(3)}[H]\,.\,_{[J]}\Phi(^{(3)}\mathbf{B})\ +\ \frac{1}{c}\,.\,T_2(\otimes)(^{(3)}\mathbf{W},\,^{(3)}\mathbf{E}) \end{bmatrix}$$
$$=$$
$$\begin{bmatrix} -\frac{1}{c}\,.\ <^{(3)}\mathbf{N}|\,.\,|^{(3)}\mathbf{E}> & \frac{\chi}{c}\,.\ <^{(3)}\mathbf{E}|\ +\ <^{(3)}\mathbf{N}|\,.\,_{[J]}\Phi(^{(3)}\mathbf{B}) \\ -\frac{1}{c}\,.\,^{(3)}[H]\,.\,|^{(3)}\mathbf{E}> & ^{(3)}[H]\,.\,_{[J]}\Phi(^{(3)}\mathbf{B})\ +\ \frac{1}{c}\,.\,T_2(\otimes)(^{(3)}\mathbf{W},\,^{(3)}\mathbf{E}) \end{bmatrix}$$
$$=$$
$$-\begin{bmatrix} 0 & <^{(3)}\mathbf{B}| \\ -|^{(3)}\mathbf{B}> & -\frac{1}{c}\,.\,_{[J]}\Phi(^{(3)}\mathbf{E}) \end{bmatrix}$$

As consequence one disposes of four equations with which one must discover what the matrix [W] is:

$$<^{(3)}\mathbf{N}|\,.\,|^{(3)}\mathbf{E}>=0 \iff {}^{(3)}\mathbf{N}\perp{}^{(3)}\mathbf{E}$$

$$\frac{\chi}{c}\,.\ <^{(3)}\mathbf{E}|\ +\ <^{(3)}\mathbf{N}|\,.\,_{[J]}\Phi(^{(3)}\mathbf{B})\ =\ -<^{(3)}\mathbf{B}|$$

$$-\frac{1}{c}\,.\,^{(3)}[H]\,.\,|^{(3)}\mathbf{E}>=|^{(3)}\mathbf{B}>$$

$$^{(3)}[H]\,.\,_{[J]}\Phi(^{(3)}\mathbf{B})\ +\ \frac{1}{c}\,.\,T_2(\otimes)(^{(3)}\mathbf{W},\,^{(3)}\mathbf{E})\ =\ \frac{1}{c}\,.\,_{[J]}\Phi(^{(3)}\mathbf{E})$$

Remark 4.2. *Concerning an eventual application to the electromagnetic duality in vacuum.*

If this discussion would effectively focus attention on electromagnetic fields, one immediately would propose to identify the vector in the North wing with the Poynting vector, usually denoted **S** in the scientific literature [07; p.76, (31.2)]. This choice validates the first equation without calculation. It transforms the second equation into:

$$\frac{\chi}{c}\,.\,^{(3)}\mathbf{E}\ +\ ^{(3)}\mathbf{S}\wedge{}^{(3)}\mathbf{B}\ =\ -^{(3)}\mathbf{B}$$

Since one can write that:

$$^{(3)}\mathbf{S}\ =\ \frac{c}{4\,.\,\pi}\,.\,(^{(3)}\mathbf{E}\wedge{}^{(3)}\mathbf{B})$$

And since:

$$\frac{c}{4\,.\,\pi}\,.\,(^{(3)}\mathbf{E}\wedge{}^{(3)}\mathbf{B})\wedge{}^{(3)}\mathbf{B}\ =\ \frac{c}{4\,.\,\pi}\,.\,\{<^{(3)}\mathbf{E},\,^{(3)}\mathbf{B}>\,.\,^{(3)}\mathbf{B}\ -\ ||^{(3)}\mathbf{B}||^2\,.\,^{(3)}\mathbf{E}\}$$

The second equation is coherent when the physical conditions are such that:

$$<^{(3)}\mathbf{E},\,^{(3)}\mathbf{B}>=-\frac{4\,.\,\pi}{c}\sim 0^-\ \ ^{(3)}\mathbf{E}\sim\perp{}^{(3)}\mathbf{B}$$

... provided one writes that χ is the total energy density of the electromagnetic field:

$$\chi\ =\ \frac{c^2}{4\,.\,\pi}\,.\,||^{(3)}\mathbf{B}||^2$$

The third equation might appear to be a little bit mysterious except if one keeps in mind the fact that the electromagnetic duality probably concerns electromagnetic plane waves slightly polarized moving at constant spatial speed in vacuum.

This underlying hypothesis allows an immediate physical interpretation for the matrix [H]; precisely:

$$^{(3)}[H] = -_{[J]}\Phi(^{(3)}\mathbf{v})$$

... where $\mathbf{v}$ is the constant spatial speed of the wave. Indeed, with this interpretation, the third equation is nothing but [07; §38, p.93, (38.9)]:

$$\frac{1}{c} \cdot {}_{[J]}\Phi(^{(3)}\mathbf{v}) \,.\, |^{(3)}\mathbf{E}> = \frac{1}{c} \cdot (^{(3)}\mathbf{v} \wedge {}^{(3)}\mathbf{E}) = {}^{(3)}\mathbf{B}$$

And the fourth equation can now be managed because it is:

$$-_{[J]}\Phi(^{(3)}\mathbf{v}) \,.\, {}_{[J]}\Phi(^{(3)}\mathbf{B}) + \frac{1}{c} \,.\, T_2(\otimes)(^{(3)}\mathbf{W}, {}^{(3)}\mathbf{E}) = \frac{1}{c} \cdot {}_{[J]}\Phi(^{(3)}\mathbf{E})$$

And because this formulation can be transposed, giving:

$$-_{[J]}\Phi(^{(3)}\mathbf{B}) \,.\, {}_{[J]}\Phi(^{(3)}\mathbf{v}) + \frac{1}{c} \,.\, T_2(\otimes)(^{(3)}\mathbf{E}, {}^{(3)}\mathbf{W}) = -\frac{1}{c} \cdot {}_{[J]}\Phi(^{(3)}\mathbf{E})$$

From these two lines (i.e.: line 1 minus line 2) one gets:

$$\{_{[J]}\Phi(^{(3)}\mathbf{B}) \,.\, {}_{[J]}\Phi(^{(3)}\mathbf{v}) - {}_{[J]}\Phi(^{(3)}\mathbf{v}) \,.\, {}_{[J]}\Phi(^{(3)}\mathbf{B})\}$$

$$+$$

$$\frac{1}{c} \,.\, \{T_2(\otimes)(^{(3)}\mathbf{W}, {}^{(3)}\mathbf{E}) - T_2(\otimes)(^{(3)}\mathbf{E}, {}^{(3)}\mathbf{W})\}$$

$$=$$

$$\frac{2}{c} \cdot {}_{[J]}\Phi(^{(3)}\mathbf{E})$$

A short travel in the algebra-land proves that:

$$_{[J]}\Phi(^{(3)}\mathbf{B}) \,.\, {}_{[J]}\Phi(^{(3)}\mathbf{v}) = T_2(\otimes)(^{(3)}\mathbf{B}, {}^{(3)}\mathbf{v}) - <^{(3)} \mathbf{B}, {}^{(3)}\mathbf{v} > \,.\, Id_3$$

$$T_2(\otimes)(^{(3)}\mathbf{W}, {}^{(3)}\mathbf{E}) - T_2(\otimes)(^{(3)}\mathbf{E}, {}^{(3)}\mathbf{W}) = {}_{[J]}\Phi(^{(3)}\mathbf{W} \wedge {}^{(3)}\mathbf{E})$$

With these tools, the fourth equation contains the information:

$$_{[J]}\Phi(^{(3)}\mathbf{B} \wedge {}^{(3)}\mathbf{v}) + \frac{1}{c} \cdot {}_{[J]}\Phi(^{(3)}\mathbf{W} \wedge {}^{(3)}\mathbf{E}) = \frac{2}{c} \cdot {}_{[J]}\Phi(^{(3)}\mathbf{E})$$

Since $_{[J]}\Phi$ is an isomorphism, this information is equivalent to:

$$(^{(3)}\mathbf{B} \wedge {}^{(3)}\mathbf{v}) + \frac{1}{c} \,.\, (^{(3)}\mathbf{W} \wedge {}^{(3)}\mathbf{E}) = \frac{2}{c} \,.\, {}^{(3)}\mathbf{E}$$

It should be sufficient to discover the vector $\mathbf{W}$; for this purpose, let calculate:

$$(^{(3)}\mathbf{B} \wedge {}^{(3)}\mathbf{v}) \wedge {}^{(3)}\mathbf{E} + \frac{1}{c} \,.\, (^{(3)}\mathbf{W} \wedge {}^{(3)}\mathbf{E}) \wedge {}^{(3)}\mathbf{E} = \mathbf{0}$$

At this stage, supposing that the classical rules governing the cross product apply because one believes that Maxwell's vacuum is related to the spatial Euclidean three-dimensional geometry:

$$<^{(3)} \mathbf{B}, {}^{(3)}\mathbf{E} > \,.\, {}^{(3)}\mathbf{v} - <^{(3)} \mathbf{B}, {}^{(3)}\mathbf{v} > \,.\, {}^{(3)}\mathbf{E}$$

...

$$+$$

$$\frac{1}{c} . \{<^{(3)} \mathbf{W}, \, ^{(3)}\mathbf{E} > \, . \, ^{(3)}\mathbf{E} \, - \, ||^{(3)}\mathbf{E})||^2 \, . \, ^{(3)}\mathbf{W}\}$$

$$=$$

$$\mathbf{0}$$

Due to the second equation and due to the underlying hypothesis pretending that one is working with electromagnetic plane waves slightly polarized, one may write:

$$<^{(3)} \mathbf{B}, \, ^{(3)}\mathbf{E} > \sim \, 0^-$$

$$<^{(3)} \mathbf{B}, \, ^{(3)}\mathbf{v} > \sim \, 0$$

These conditions yield:

$$\frac{1}{c} . \{<^{(3)} \mathbf{W}, \, ^{(3)}\mathbf{E} > \, . \, ^{(3)}\mathbf{E} \, - \, ||^{(3)}\mathbf{E})||^2 \, . \, ^{(3)}\mathbf{W}\} \, \sim \, \mathbf{0}$$

... suggesting that the vector **W** is proportional to the electric field and that the matrix one is looking for is:

$$[W] \, \sim \, \begin{bmatrix} \frac{c^2}{4 \, . \, \pi} . ||^{(3)}\mathbf{B}||^2 \, \sim \, \rho_{em} & <^{(3)} \mathbf{S}| \\ -\psi \, . \, |^{(3)}\mathbf{E} > & -_{[J]}\Phi(^{(3)}\mathbf{v}) \end{bmatrix}$$

Now, one should look for a matrix [W'] insuring the inverse passage:

$$[W'] \, . \, \{T_2(\otimes)(^{(4)}\mathbf{A}, \, ^{(4)}\mathbf{a}) \, - \, T_2(\otimes)(^{(4)}\mathbf{a}, \, ^{(4)}\mathbf{A})\} \, = \, _{\mathbf{A}}\Phi(\mathbf{a})$$

If this second matrix exists, then:

$$\{[W'] \, . \, [W] \, - \, Id_4\} \, . \, _{\mathbf{A}}\Phi(\mathbf{a}) \, = \, ^{(4)}[0]$$

Provided the simplest decomposition does not vanish (This is at least the case when $\mathbf{a} \neq \mathbf{A}$), there exists at least one matrix [W']: it is the inverse of the matrix [W]. Warning: this eventuality is not the unique possibility but it is the simplest one:

$$\begin{bmatrix} \chi' & <^{(3)} \mathbf{N'}| \\ |^{(3)}\mathbf{W'} > & ^{(3)}[H'] \end{bmatrix} . \begin{bmatrix} \rho_{em} & <^{(3)} \mathbf{S}| \\ -\psi \, . \, |^{(3)}\mathbf{E} > & -_{[J]}\Phi(^{(3)}\mathbf{v}) \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \chi' \cdot \rho_{em} \, - \, \psi \, . \, <^{(3)} \mathbf{N'}| \, . \, |^{(3)}\mathbf{E} > & \chi' \, . \, <^{(3)} \mathbf{S}| \, - \, <^{(3)} \mathbf{N'}| \, . \, _{[J]}\Phi(^{(3)}\mathbf{v}) \\ \rho_{em} \, . \, |^{(3)}\mathbf{W'} > -\psi \, . \, ^{(3)}[H'] \, . \, |^{(3)}\mathbf{E} > & -^{(3)}[H'] \, . \, _{[J]}\Phi(^{(3)}\mathbf{v}) \, + \, T_2(\otimes)(^{(3)}\mathbf{W'}, \, ^{(3)}\mathbf{S}) \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} 1 & <^{(3)} \mathbf{0}| \\ |^{(3)}\mathbf{0} > & Id_3 \end{bmatrix}$$

One gets four equations:

$$\chi' \cdot \rho_{em} \, - \, \psi \, . \, <^{(3)} \mathbf{N'}| \, . \, |^{(3)}\mathbf{E} >= \, 1$$

$$\chi' \, . \, ^{(3)}\mathbf{S} \, - \, ^{(3)}\mathbf{N'} \, \wedge \, ^{(3)}\mathbf{v} \, = \, ^{(3)}\mathbf{0}$$

...

$$\rho_{em} \,.\, |^{(3)}\mathbf{W'} > -\psi \,.\, ^{(3)}[H'] \,.\, |^{(3)}\mathbf{E} >= |^{(3)}\mathbf{0} >$$
$$T_2(\otimes)(^{(3)}\mathbf{W'},\, ^{(3)}\mathbf{S}) \;-\; ^{(3)}[H'] \,.\,_{[J]}\Phi(^{(3)}\mathbf{v}) \;=\; Id_4$$

They are not very helpful and it will perhaps be easier to find [W'] in following the same logic than the one which has been used to discover [W].

At this stage, one must also add a supplementary constraint on [W] to be certain that one is treating the electromagnetic duality.

Maxwell's work on electricity and magnetism in vacuum [14] led to four remarkably simple equations. They can be written [15; p.264, (7.88)][1]:

$$rot_{\mathbf{x}}\mathbf{E} \;=\; -\frac{\partial \mathbf{B}}{\partial t},\; div_{\mathbf{x}}\frac{\mathbf{E}}{c^2} \;=\; \mathbf{0}$$
$$rot_{\mathbf{x}}\mathbf{B} \;=\; \frac{\partial \frac{\mathbf{E}}{c^2}}{\partial t},\; div_{\mathbf{x}}\mathbf{B} \;=\; \mathbf{0}$$

In Planck's units, c = 1 and they can be rewritten as [16; p.12, (1.1)]:

$$rot_{\mathbf{x}}\mathbf{E} \;=\; -\frac{\partial \mathbf{B}}{\partial t},\; div_{\mathbf{x}}\mathbf{E} \;=\; \mathbf{0}$$
$$rot_{\mathbf{x}}\mathbf{B} \;=\; \frac{\partial \mathbf{E}}{\partial t},\; div_{\mathbf{x}}\mathbf{B} \;=\; \mathbf{0}$$

As a matter of facts, the transformation $(\mathbf{E}, \mathbf{B}) \to (c.\mathbf{B}, -\mathbf{E}/c)$ preserves the formalism of these equations. This strange property is called: *electromagnetic duality*. Hence, that property is a direct by-product of Maxwell's laws governing the electromagnetic fields in vacuum [05; §3.4 and §4.5]. It attracts much attention in mathematical physics because it is often considered as the simplest example of a deeper symmetry dispatched in many other domains.

Definition 4.2. *The Hodge's star operator acting on electromagnetic 2-forms in a four dimensional space.*

The Hodge's star operator acting on 2-forms in an oriented four dimensional smooth manifold E endowed with a metric g, short: in E_4 = (E, g), is a linear function with source and target in $\Omega^2(E_4)$, the vector space of 2-forms on E_4; it is denoted "*" and is unique.

The electromagnetic fields are represented by 2-forms and any of them can be represented by a (4-4) square matrix in M(4, $\mathbb{C}$); see any book explaining the basics of linear algebra.

Therefore, any representation for that operator when it is acting on electromagnetic 2-forms is an application having its source in $\Omega^2(E_4)$ and fulfilling the three criterion:

[1] Here, a reference to the coordinates system has been given and the relation [15; annex, p.450, (8)] has been involved:

$$\epsilon_0 \,.\, \mu_0 \,.\, c^2 \;=\; 1$$

1. The representation must be an element in M(4, ℂ), for example denoted [O]:

$$\begin{array}{ccccc} F \in \Omega^2(E) & \longrightarrow & * & \longrightarrow & {}^*F \in \Omega^2(E) \\ & & \downarrow & & \\ & & [O] \in M(4,\mathbb{C}) & & \end{array}$$

2. The representation of the operator, e.g.: [O], acts on the left side of elements in M(4, ℂ) in a manner which reproduces the action of the completely anti-symmetric tensor on 2-forms:

$$\begin{array}{ccccc} F \in \Omega^2(E) & \longrightarrow & * & \longrightarrow & {}^*F \in \Omega^2(E) \\ & & \downarrow & & \\ [F] \in M(4,\mathbb{C}) & \longrightarrow & [O] \in M(4,\mathbb{C}) & \longrightarrow & [{}^*F] = [O]\,.\,[F] \in M(4,\mathbb{C}) \end{array}$$

3. The square of the representation [O] must be minus the identity matrix:

$$[O]^2 = -Id_4$$

Within the context of this example, it can easily be verified that the matrix [W] is insuring the passage between the (2, 0) version of the electromagnetic tensor and *minus the dual form* of the latter:

$$[F^{\mu\nu}] = \begin{bmatrix} 0 & \frac{1}{c}\,.<^{(3)}\mathbf{E}| \\ -\frac{1}{c}\,.\,|^{(3)}\mathbf{E}> & \Phi_{[J]}(\mathbf{B}) \end{bmatrix} \xrightarrow{[W]} -[{}^*F_{\mu\nu}] = \begin{bmatrix} 0 & -<^{(3)}\mathbf{B}| \\ |^{(3)}\mathbf{B}> & \Phi_{[J]}(\mathbf{E}) \end{bmatrix}$$

Therefore, the representation of the dual operator is:

$$[O] = -[W]$$

... and one must yet obligatorily verify that:

$$\begin{bmatrix} \rho_{em} & <^{(3)}\mathbf{S}| \\ -\psi\,.\,|^{(3)}\mathbf{E}> & -_{[J]}\Phi(^{(3)}\mathbf{v}) \end{bmatrix} . \begin{bmatrix} \rho_{em} & <^{(3)}\mathbf{S}| \\ -\psi\,.\,|^{(3)}\mathbf{E}> & -_{[J]}\Phi(^{(3)}\mathbf{v}) \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \rho_{em}^2 - \psi\,.\,\underbrace{<^{(3)}\mathbf{S}|\,.\,|^{(3)}\mathbf{E}>}_{=\mathbf{0}} & \rho_{em}\,.<^{(3)}\mathbf{S}| - {}^{(3)}\mathbf{S}\wedge{}^{(3)}\mathbf{v}| \\ \psi\,.\,\rho_{em}\,.\,|^{(3)}\mathbf{E}> + \psi\,.\,|^{(3)}\mathbf{v}\wedge{}^{(3)}\mathbf{E}> & _{[J]}\Phi^2(^{(3)}\mathbf{v}) + \psi\,.\,T_2(\otimes)(^{(3)}\mathbf{E},\,^{(3)}\mathbf{S}) \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} -1 & <^{(3)}\mathbf{0}| \\ |^{(3)}\mathbf{0}> & -Id_3 \end{bmatrix}$$

This system is yielding four equations:

$$\rho_{em}^2 = -1$$

$$\rho_{em}\,.\,^{(3)}\mathbf{S} - (^{(3)}\mathbf{S}\wedge{}^{(3)}\mathbf{v}) = {}^{(3)}\mathbf{0}$$

$$\rho_{em}\,.\,^{(3)}\mathbf{E} + (^{(3)}\mathbf{v}\wedge{}^{(3)}\mathbf{E}) = {}^{(3)}\mathbf{0}$$

$$_{[J]}\Phi^2(^{(3)}\mathbf{v}) + \psi\,.\,T_2(\otimes)(^{(3)}\mathbf{E},\,^{(3)}\mathbf{S}) = -Id_3$$

... and, unfortunately, they introduce really unpleasant technical difficulties:

1. The electromagnetic densities of energy must be either pure imaginary complex numbers or any imaginary generator for the quaternions (resp. for the octonions);

2. Two cross products have a result which is proportional to one of their arguments. This is impossible in a strictly classical real three-dimensional space. One can only envisage to go further with vectors having complex components.

An approximate formalism for the matrix [W] can be obtained within the context focusing on the representations of wedge products in a four-dimensional space if one translates the physical objects into their respective twin mathematical representations. This imposes indirectly to suppose that the spatial speed is proportional to the Poynting vector :

$$^{(4)}[W]$$

$$\sim$$

$$\begin{bmatrix} \frac{c^2}{4\,.\,\pi}\,.\,(\Phi_{23}^2\,+\,\Phi_{13}^2\,+\,\Phi_{12}^2) & \frac{c^2}{4.\pi}\,.\,(\Phi_{02}\,.\,\Phi_{12}\,+\,\Phi_{13}\,.\,\Phi_{03}) & \frac{c^2}{4.\pi}\,.\,(\Phi_{03}\,.\,\Phi_{23}\,-\,\Phi_{12}\,.\,\Phi_{01}) & -\frac{c^2}{4.\pi}\,.\,(\Phi_{01}\,.\,\Phi_{13}\,+\,\Phi_{02}\,.\,\Phi_{23}) \\ -\psi\,.\,\Phi_{01} & 0 & -\frac{\Psi\,.\,c^2}{4.\pi}\,.\,(\Phi_{01}\,.\,\Phi_{13}\,+\,\Phi_{02}\,.\,\Phi_{23}) & -\frac{\Psi\,.\,c^2}{4.\pi}\,.\,(\Phi_{03}\,.\,\Phi_{23}\,-\,\Phi_{12}\,.\,\Phi_{01}) \\ -\psi\,.\,\Phi_{02} & \frac{\Psi\,.\,c^2}{4.\pi}\,.\,(\Phi_{01}\,.\,\Phi_{13}\,+\,\Phi_{02}\,.\,\Phi_{23}) & 0 & \frac{\Psi\,.\,c^2}{4.\pi}\,.\,(\Phi_{02}\,.\,\Phi_{12}\,+\,\Phi_{13}\,.\,\Phi_{03}) \\ -\psi\,.\,\Phi_{03} & \frac{\Psi\,.\,c^2}{4.\pi}\,.\,(\Phi_{03}\,.\,\Phi_{23}\,-\,\Phi_{12}\,.\,\Phi_{01}) & -\frac{\Psi\,.\,c^2}{4.\pi}\,.\,(\Phi_{02}\,.\,\Phi_{12}\,+\,\Phi_{13}\,.\,\Phi_{03}) & 0 \end{bmatrix}$$

It is possible to verify that:

$$-\psi\,.\,\Phi_{01}\,.\,\frac{c^2}{4.\pi}\,.\,(\Phi_{02}\,.\,\Phi_{12}\,+\,\Phi_{13}\,.\,\Phi_{03})$$

$$-\psi\,.\,\Phi_{02}\,.\,\frac{c^2}{4.\pi}\,.\,(\Phi_{03}\,.\,\Phi_{23}\,-\,\Phi_{12}\,.\,\Phi_{01})$$

$$+\psi\,.\,\Phi_{03}\,.\,\frac{c^2}{4.\pi}\,.\,(\Phi_{01}\,.\,\Phi_{13}\,+\,\Phi_{02}\,.\,\Phi_{23})$$

$$=$$

$$-\psi\,.\,\Phi_{01}\,.\,\frac{c^2}{4.\pi}\,.\,\Phi_{02}\,.\,\Phi_{12}\,-\,\psi\,.\,\Phi_{01}\,.\,\frac{c^2}{4.\pi}\,.\,\Phi_{13}\,.\,\Phi_{03}$$

$$-\psi\,.\,\Phi_{02}\,.\,\frac{c^2}{4.\pi}\,.\,\Phi_{03}\,.\,\Phi_{23}\,+\,\psi\,.\,\Phi_{02}\,.\,\frac{c^2}{4.\pi}\,.\,\Phi_{12}\,.\,\Phi_{01}$$

$$+\psi\,.\,\Phi_{03}\,.\,\frac{c^2}{4.\pi}\,.\,\Phi_{01}\,.\,\Phi_{13}\,+\,\psi\,.\,\Phi_{03}\,.\,\frac{c^2}{4.\pi}\,.\,\Phi_{02}\,.\,\Phi_{23}$$

$$=$$

$$0$$

This result is obtained whatever the value of ψ is and it is the image inside this discussion of the physical relation:

$$< \mathbf{E}, \mathbf{S} > = 0$$

Following the same spirit, one states that:

$$-\frac{\Psi\,.\,c^2}{4.\pi}\,.\,(\Phi_{03}\,.\,\Phi_{23}\,-\,\Phi_{12}\,.\,\Phi_{01})\,.\,\Phi_{03}\,-\,\frac{\Psi\,.\,c^2}{4.\pi}\,.\,(\Phi_{01}\,.\,\Phi_{13}\,+\,\Phi_{02}\,.\,\Phi_{23})\,.\,\Phi_{02}$$

...

$$=$$

$$-\frac{\Psi\,.\,c^2}{4.\pi}\,.\,\{\Phi_{23}\,.\,(\Phi_{01}^2 + \Phi_{02}^2 + \Phi_{03}^2) + \Phi_{01}\,.\,(\Phi_{23}\,.\,\Phi_{01} + \Phi_{13} + \Phi_{02} + \Phi_{12}\,.\,\Phi_{03})\}$$

If one injects the image **E** and **B** of the physical objects, one finds:

$$-\frac{\Psi}{4.\pi}\,.\,||\mathbf{E}||^2\,.\,B^1 + \underbrace{<\mathbf{B},\,\mathbf{E}>}_{\sim 0}\,.\,E^1$$

If one takes in consideration the second equation associated with the existence of a matrix [P] realizing the passage:

$$-\frac{\Psi}{4.\pi}\,.\,||\mathbf{E}||^2\,.\,B^1$$

Anyway, if these calculations would be the image of the physical relation:

$$\frac{1}{c}\cdot{}_{[J]}\Phi({}^{(3)}\mathbf{v})\,.\,|{}^{(3)}\mathbf{E}> = \frac{1}{c}\,.\,({}^{(3)}\mathbf{v}\wedge{}^{(3)}\mathbf{E}) = {}^{(3)}\mathbf{B}$$

One should have found:

$$B^1$$

The coherence of previous mathematical calculations with their eventual application in electromagnetism is imposing:

$$-\frac{\Psi}{4.\pi}\,.\,||\mathbf{E}||^2 = 1 \iff \Psi = -\frac{4.\pi}{||\mathbf{E}||^2} \equiv -\frac{1}{\rho_{em}}$$

Hence, the optimal formalism of [W] for an eventual application in electromagnetism is:

$${}^{(4)}[W]$$

$$\sim$$

$$\begin{bmatrix} \rho_{em} & <^{(3)}\mathbf{E}\wedge{}^{(3)}\mathbf{B}| \\ \frac{1}{\rho_{em}}\cdot|{}^{(3)}\mathbf{E}> & \frac{1}{\rho_{em}}\cdot{}_{[J]}\Phi({}^{(3)}\mathbf{E}\wedge{}^{(3)}\mathbf{B}) \end{bmatrix}$$

At a technical level, with the notations which have been introduces at the beginning of this discussion, an application to the electromagnetic duality in vacuum imposed to find solutions for the following system:

$$\rho_{em}^2 = -1$$

$$\rho_{em}\cdot({}^{(3)}\mathbf{E}\wedge{}^{(3)}\mathbf{B}) - \frac{1}{\rho_{em}}\cdot(({}^{(3)}\mathbf{E}\wedge{}^{(3)}\mathbf{B})\wedge({}^{(3)}\mathbf{E}\wedge{}^{(3)}\mathbf{B})) = {}^{(3)}\mathbf{0}$$

$$\rho_{em}\,.\,{}^{(3)}\mathbf{E} - \frac{1}{\rho_{em}}\cdot(({}^{(3)}\mathbf{E}\wedge{}^{(3)}\mathbf{B})\wedge{}^{(3)}\mathbf{E}) = {}^{(3)}\mathbf{0}$$

$$\frac{1}{\rho_{em}^2}\cdot{}_{[J]}\Phi^2({}^{(3)}\mathbf{E}\wedge{}^{(3)}\mathbf{B}) - \frac{1}{\rho_{em}}\,.\,T_2(\otimes)({}^{(3)}\mathbf{E},\,{}^{(3)}\mathbf{E}\wedge{}^{(3)}\mathbf{B}) = -Id_3$$

This is impossible within a discussion which is restricted to a classical geometrical context. This particular approach seems to be a *cul-de-sac*. One can also suspect that the origin of this situation is the incorrect hypothesis which has been made to interpret the third equation (see above).

Comments:
The electromagnetic duality in Maxwell's vacuum is only a particular illustration for a quite more general mathematical topic concerning wedge products in a four-dimensional space. The matrix [W] exists and, in this document, it realizes the passage:

$$_{\mathbf{A}}\Phi({}^{(4)}\mathbf{a}) \xrightarrow{[W]} T_2(\otimes)({}^{(4)}\mathbf{A},\, {}^{(4)}\mathbf{a}) \,-\, T_2(\otimes)({}^{(4)}\mathbf{a},\, {}^{(4)}\mathbf{A}) \,=\, [W]\,.\,{}_{\mathbf{A}}\Phi(\mathbf{a})$$

... even if this passage does not (cannot yet because of some approximations?) report on the electromagnetic duality in vacuum. □

Lemma 4.1. *On the electromagnetic duality in Maxwell's vacuum.*

The hypothesis interpreting the Maxwell's vacuum as bath of slightly polarized electromagnetic plane waves moving at constant spatial speed is compatible with the existence of matrix [W] insuring the passage:

$$_{\mathbf{A}}\Phi({}^{(4)}\mathbf{a}) \xrightarrow{[W]} T_2(\otimes)({}^{(4)}\mathbf{A},\, {}^{(4)}\mathbf{a}) \,-\, T_2(\otimes)({}^{(4)}\mathbf{a},\, {}^{(4)}\mathbf{A}) \,=\, [W]\,.\,{}_{\mathbf{A}}\Phi(\mathbf{a})$$

$${}^{(4)}[W] = \begin{bmatrix} \rho_{em} & <^{(3)}\mathbf{E} \wedge {}^{(3)}\mathbf{B}| \\ \frac{1}{\rho_{em}}\,.\,|^{(3)}\mathbf{E}> & \frac{1}{\rho_{em}}\,.\,{}_{[J]}\Phi({}^{(3)}\mathbf{E} \wedge {}^{(3)}\mathbf{B}) \end{bmatrix}$$

$$\rho_{em} = \frac{||\mathbf{E}||^2}{4.\pi} = \frac{||\mathbf{B}||^2}{4.\pi}$$

If $_{\mathbf{A}}\Phi(\mathbf{a})$ would be the (2, 0) representation of some electromagnetic field understood as bi-vector ($\mathbf{E}$, $\mathbf{B}$), then the matrix -[W] would have the same action than a passage to the dual representation of this electromagnetic field.

Unfortunately, within the context of this discussion, this hypothesis and this matrix can never be involved for a coherent description of the electromagnetic duality in Maxwell's vacuum and more work is needed.

4.3 Useful calculations for later

Let also note attentively that the following product is no more an element in $M(4, \mathbb{C})$ but in $M(6, \mathbb{C})$:

$$[\odot(\mathbf{a})]^t\,.\,[\odot(\mathbf{A})]$$

$$=$$

$$\begin{bmatrix} 0 & 0 & -a^3 & a^2 \\ 0 & a^3 & 0 & -a^1 \\ 0 & -a^2 & a^1 & 0 \\ -a^3 & 0 & 0 & a^0 \\ a^2 & 0 & -a^0 & 0 \\ -a^1 & a^0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 0 & 0 & -A^3 & A^2 & -A^1 \\ 0 & A^3 & -A^2 & 0 & 0 & A^0 \\ -A^3 & 0 & A^1 & 0 & -A^0 & 0 \\ A^2 & -A^1 & 0 & A^0 & 0 & 0 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} a^2.A^2 + a^3.A^3 & -a^2.A^1 & -a^3.A^1 & a^2.A^0 & a^3.A^0 & 0 \\ -a^1.A^2 & a^1.A^1 + a^3.A^3 & -a^3.A^2 & -a^1.A^0 & 0 & a^3.A^0 \\ -a^1.A^3 & -a^2.A^3 & a^1.A^1 + a^2.A^2 & 0 & -a^1.A^0 & -a^2.A^0 \\ a^0.A^2 & -a^0.A^1 & 0 & a^0.A^0 + a^3.A^3 & -a^3.A^2 & a^3.A^1 \\ a^0.A^3 & 0 & -a^0.A^1 & -a^2.A^3 & a^0.A^0 + a^2.A^2 & -a^2.A^1 \\ 0 & a^0.A^3 & -a^0.A^2 & a^1.A^3 & -a^1.A^2 & a^0.A^0 + a^1.A^1 \end{bmatrix}$$

...

It is legitimate to look for a link with the [T] matrix. In a first step, one remarks that:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$.$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a^0 . A^0 + a^1 . A^1 \\ 0 & 0 & 0 & 0 & -(a^0 . A^0 + a^2 . A^2) & 0 \\ 0 & 0 & 0 & a^0 . A^0 + a^3 . A^3 & 0 & 0 \\ 0 & 0 & a^1 . A^1 + a^2 . A^2 & 0 & 0 & 0 \\ 0 & -(a^1 . A^1 + a^3 . A^3) & 0 & 0 & 0 & 0 \\ a^2 . A^2 + a^3 . A^3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} a^2 . A^2 + a^3 . A^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & a^1 . A^1 + a^3 . A^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^1 . A^1 + a^2 . A^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^0 . A^0 + a^3 . A^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & a^0 . A^0 + a^2 . A^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & a^0 . A^0 + a^1 . A^1 \end{bmatrix}$$

In a second step, let test the action of the newly introduced matrix on [T$^-$]:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$.$$

$$\begin{bmatrix} 0 & a^0 . A^3 & -a^0 . A^2 & a^1 . A^3 & -a^1 . A^2 & 0 \\ -a^0 . A^3 & 0 & a^0 . A^1 & a^2 . A^3 & 0 & a^2 . A^1 \\ a^0 . A^2 & -a^0 . A^1 & 0 & 0 & -a^3 . A^2 & a^3 . A^1 \\ -a^1 . A^3 & -a^2 . A^3 & 0 & 0 & -a^1 . A^0 & -a^2 . A^0 \\ a^1 . A^2 & 0 & a^3 . A^2 & a^1 . A^0 & 0 & -a^3 . A^0 \\ 0 & -a^2 . A^1 & -a^3 . A^1 & a^2 . A^0 & a^3 . A^0 & 0 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} 0 & -a^2 . A^1 & -a^3 . A^1 & a^2 . A^0 & a^3 . A^0 & 0 \\ -a^1 . A^2 & 0 & -a^3 . A^2 & -a^1 . A^0 & 0 & a^3 . A^0 \\ -a^1 . A^3 & -a^2 . A^3 & 0 & 0 & -a^1 . A^0 & -a^2 . A^0 \\ a^0 . A^2 & -a^0 . A^1 & 0 & 0 & -a^3 . A^2 & a^3 . A^1 \\ a^0 . A^3 & 0 & -a^0 . A^1 & -a^2 . A^3 & 0 & -a^2 . A^1 \\ 0 & a^0 . A^3 & -a^0 . A^2 & a^1 . A^3 & -a^1 . A^2 & 0 \end{bmatrix}$$

Let write:

$$[*] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

These calculations have proved that:

$$[\odot(\mathbf{a})]^t . [\odot(\mathbf{A})] = [*] . [T]$$

4.4 Conclusion

At the end of the day, one states that the *theory of representations* sneakily invited itself in the discussion studying when the deformed Lie products act like an involution does. There is in general no argument justifying the non-existence of several and eventually different representations for the same action.

Until now, especially for the part of these explorations which have been developed in three-dimensional spaces [a] and [b], the multiplicity of the representations has been represented by *the concept of decomposition* (synonym: division). There is no reason forbidding the extrapolation of this way of doing when the discussion concerns vectors in four-dimensional spaces.

Hence, each matrix [P] resulting from the approach exposed in proposition 1.3 presumably represents an alternative decomposition without residual part. If the way of thinking which has been explained in [b] is correct, then - when the cube A is anti-symmetric and anti-reduced- these representations must obligatorily verify the constraint:

$$|M(^{(4)}\mathbf{A},\, ^{(4)}\mathbf{a})| = |<^{(4)} \mathbf{A},\, ^{(4)}\mathbf{a}>_{Id_4} .Id_4 - T_2(\otimes)(^{(4)}\mathbf{A},\, ^{(4)}\mathbf{a}) - {}_{\mathbf{A}}\Phi(\mathbf{a})| = 0_{\mathbb{C}}$$

Any way, at least for anti-symmetric and anti-reduces cubes, the analysis (subsection 4.1) has brought important information and distinctions concerning both classes of representations.

The repetition of the action of f on a given argument $\mathbf{x}_0$ carries two concepts with it: (i) the eventual invariance of this argument and (ii) the existence of an involution. The simplest decomposition cannot leave the argument unchanged but its square is a suitable tool to characterize an involution. The decomposition without residual part resulting from the approach exposed in proposition 1.3 may eventually leave the argument unchanged whilst it cannot be involved for the description of situation associated with an involution.

References

5 Bibliography

5.1 My contributions

[a] The (E) Question in a Three-Dimensional Space - Analyzing a Subset of Linear Systems with the Intrinsic Method; viXra:2503.0200, 32 pages.

[b] The so-called Extrinsic Method; viXra:2406.0127, 12 pages.

[c] Deformed Lie products and involution - first part; viXra:2506.0039, 24 pages.

5.2 International works

[05] M. T. W.: Gravitation; ©1973 by W.H. Freeman and Company.

[07] Landau and Lifshitz: The classical theory of fields; course of theoretical physics, volume 2, third revised English edition, Copyright ©1971 Pergamon Press Ltd. 374 p.

[14] Maxwell, J. C.: A Dynamical Theory of the Electromagnetic Field; Philosophical Transactions of the Royal Society of London, 1865, 155: 459 - 512.

[15] Crawford, S. F. Junior and Léna P. : Ondes ; Berkeley, cours de physique, volume 3, © Librairie Armand Colin, Paris, 1972 (It is the translation of the book titled « Waves » Berkeley Physics Course, volume 3, published in 1968 by the McGraw-Hill Book Company), 603 pages.

[16] Figueroa O'Farill J.M.: Electromagnetic duality for children, draft version of 8 October 1998, 209 pages.