Hilbert manifold structures on path spaces

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July 4, 2025

Abstract

In Floer theory one has to deal with two-level manifolds like for instance the space of $W^{2,2}$ loops and the space of $W^{1,2}$ loops. Gradient flow lines in Floer theory are then trajectories in a two-level manifold. Inspired by our endeavor to find a general setup to construct Floer homology we therefore address in this paper the question if the space of paths on a two-level manifold has itself the structure of a Hilbert manifold. In view of the two topologies on a two-level manifold it is unclear how to define the exponential map on a general two-level manifold. We therefore study a different approach how to define charts on path spaces of two-level manifolds. To make this approach work we need an additional structure on a two-level manifold which we refer to as *tameness*. We introduce the notion of tame maps and show that the composition of tame is tame again. Therefore it makes sense to introduce the notion of a tame two-level manifold. The main result of this paper shows that the path spaces on tame two-level manifolds have the structure of a Hilbert manifold.

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1 Introduction

Suppose that $H_2 \subset H_1$ are Hilbert spaces with a dense and compact inclusion of H_2 into H_1. The fact that a dense and compact inclusion exists immediately implies that both Hilbert spaces are separable, see [FW24, Cor. A.5]. It follows that there are two cases, either $H_1 = H_2$ is finite dimensional, or $H_1 \neq H_2$ and both Hilbert spaces are infinite dimensional and separable.

By considering C^2 diffeomorphisms between open subsets of H_1 which restrict to C^2 diffeomorphisms between the intersection of these subsets with H_2 we obtain charts of C^2 **two-level manifolds** $X = X_1 \supset X_2$.

For two points x_- and x_+ we consider $W^{1,2}(\mathbb{R}, H_1) \cap L^2(\mathbb{R}, H_2)$-paths from x_- to x_+. By $\mathcal{P}_{x_-x_+}$ we denote the set of such paths. Over this space of paths there is a bundle, called the **weak tangent bundle**, namely

$$\mathcal{E} \to \mathcal{P}, \qquad \mathcal{E} = \mathcal{E}_{x_-x_+}, \quad \mathcal{P} = \mathcal{P}_{x_-x_+},$$

where the fiber over a path $x \in \mathcal{P}$ consists of $L^2(\mathbb{R}, H_1)$-vector fields along x.

In this article we are interested in the question whether $\mathcal{P}$ and $\mathcal{E}$ have the structure of a C^1 Hilbert manifold. We do not require that the tangent bundle $T\mathcal{P}$ is a C^1 Hilbert manifold, but only its L^2 completion $\mathcal{E}$. We don't know if this is true for general two-level manifolds. However, we introduce a condition for C^2 diffeomorphisms called *tameness*. We then show that the composition of two tame C^2 diffeomorphisms is tame as well (Theorem 2.5). Therefore by considering atlases all whose transition functions are tame we obtain the notion of a *tame two-level manifold*.

Under these assumption we are able to show that the space of paths $\mathcal{P}$ and its L^2 completion $\mathcal{E}$ of its tangent bundle are C^1 Hilbert manifolds. Namely, the main result of this article is the following theorem.

Theorem A. *Let X be a tame two-level manifold. Then for every pair of points $x_\pm \in X$ the space of paths $\mathcal{P}_{x_-x_+}$ and the L^2 completion $\mathcal{E}_{x_-x_+}$ of its tangent bundle have the structure of C^1 Hilbert manifolds.*

The main technical result to prove Theorem A is the following theorem on Hilbert space valued Sobolev spaces. We abbreviate

$$W^{1,2}_{H_1} := W^{1,2}(\mathbb{R}, H_1), \qquad L^2_{H_i} := L^2(\mathbb{R}, H_i). \tag{1.1}$$

Theorem B. *Suppose that $\phi\colon H_1 \to H_1$ is tame. Then the map*

$$\begin{aligned} T\Phi\colon (W^{1,2}_{H_1} \cap L^2_{H_2}) \times L^2_{H_1} &\to (W^{1,2}_{H_1} \cap L^2_{H_2}) \times L^2_{H_1} \\ (\xi, \eta) &\mapsto (\phi \circ \xi, d\phi|_\xi \eta) \end{aligned}$$

is well defined and continuously differentiable.

In order to prove Theorem A we also will need a parametrized version of Theorem B in which ϕ is additionally allowed to depend on time.

Different from other approaches to manifold structures on mapping spaces between finite dimensional manifolds, see e.g. [Elĭ67, Sch93], we do not use an exponential map in the infinite dimensional case since such seems difficult to incorporate in the case where one has several levels of manifolds.

Remark 1.1 (Dense inclusion, not necessarily compact)**.**

a) If we only assume that H_1 and H_2 are Hilbert spaces with a dense inclusion of H_2 into H_1, not necessarily compact, then we do not know if Theorem A holds true.

b) However, for the notion of tameness we do not need that the inclusion is compact and Theorem B continues to hold in this more general case.

c) For Hilbert manifolds a complementary approach to our proof could probably be carried out with the help of the exponential map. The exponential map for C^3 Hilbert manifolds is studied in Lang's book [Lan01, IV §3].

Acknowledgements. UF acknowledges support by DFG grant FR 2637/4-1.

1.1 Outlook

The Hilbert manifold structure of the space of maps $\mathcal{P}$ and the bundle $\mathcal{E}$ play an important role in Floer theory. There one considers a functional $\mathcal{A}$ on special types of two-level manifolds X and is interested in the gradient flow equation

$$\mathcal{F}(u) := \partial_s u + \nabla \mathcal{A}(u) = 0$$

between critical points $x_-, x_+ \in X_2$. The gradient is unregularized in the sense of Floer [Flo88]. Namely, if x is in X_2, then $\nabla \mathcal{A}(x)$ lies only in $T_x X_1$, not in $T_x X_2$. We can interpret Floer gradient flow lines as the zeroes of the section in the bundle

$$\begin{array}{c} \mathcal{E} \\ \big\downarrow \big)\!\uparrow \mathcal{F} \\ \mathcal{P} \end{array}$$

If $\mathcal{E}$ and $\mathcal{P}$ have the structure of C^1 manifolds we can look at the differential of the section at the point $u \in \mathcal{P}$, in symbols

$$d\mathcal{F}_u \colon T_u \mathcal{P} \to T_{\mathcal{F}(u)} \mathcal{E}.$$

If u is a zero of the section, i.e. a gradient flow line, we have a canonical splitting

$$T_u \mathcal{E} = T_u \mathcal{P} \oplus \mathcal{E}_u.$$

We denote by $\pi_u \colon T_u \mathcal{E} \to \mathcal{E}_u$ the projection along $T_u \mathcal{P}$. Hence we obtain the vertical differential

$$D\mathcal{F}_u := \pi_u \circ d\mathcal{F}_u \colon T_u \mathcal{P} \to \mathcal{E}_u.$$

A crucial property in a Floer theory is that this vertical differential is a Fredholm operator. In view of our endeavor to construct abstract Floer homologies which can also be applied to quite general Hamiltonian delay equations, as outlined in our articles [FW24, FW25], the property that the space of paths and the L^2 bundle are both Hilbert manifolds <u>of class C^1</u> seems to be crucial and this is the reason for the current study.

1.2 Outline

This text is organized as follows.

In Section 2 we introduce the notion of *tame maps*. The main result about tame maps is that the composition is again tame. This the content of Theorem 2.5. Thanks to this result one can introduce the notion of *tame manifolds*. Namely, these are manifolds all of whose transitions maps are tame. An example of a tame manifold is the loop space of a smooth finite dimensional manifold. We also introduce the notion of *parametrized tameness* which will be used later to show that the path spaces of tame manifolds are Hilbert manifolds.

In Section 3 we prove a purely analytical result. Namely, we show in Theorem 3.1 that composing a class of Hilbert space valued Sobolev maps with a tame map gives rise to a C^1 map. This result, in particular, implies that for constant paths in a tame manifold the change of chart gives rise to a C^1 diffeomorphism. To extend this result to non-constant paths we describe as well a parametrized version of Theorem 3.1, namely Theorem 3.5.
We further show in this section that the weak differentials of the above maps are themselves C^1 maps – which would not hold for the differentials. This then leads to the proof of Theorem B and its parametrized version Theorem 3.4.

In Section 4 we prove that the space of paths of a tame manifold, as well as its weak tangent bundle, are Hilbert manifolds. For these purposes we construct charts for the space of paths of a tame manifold and show that their transition maps are C^1 diffeomorphisms. In case that the path can be covered by one single chart in the tame manifold the proof of this is a rather straightforward application of the parametrized version of Theorem B. In the case that the path cannot be covered by a single chart the argument gets quite subtle and in particular uses the compact inclusion of H_2 into H_1. Since it is not clear how to implement an exponential map on a two-level manifold, our construction is based on an interpolation procedure. The intricate part is to show that this interpolation procedure preserves tameness.

In Appendix C we provide the reader with a self-contained construction of Hilbert space valued Sobolev spaces used in this work. The (automatic) fact that our Hilbert spaces are separable simplifies various parts of this construction. In particular, we give a detailed proof of Pettis' Theorem in the separable case. We also show how Hilbert space valued Sobolev maps can be approximated by differentiable Hilbert space valued maps.

In Appendix B we recall a quantitative version of the implicit function theorem which plays an important role in the interpolation procedure to produce our charts.

In Appendix C we recall that for finite dimensional manifolds the spaces of paths can be endowed with the structure of Hilbert manifolds using the exponential map. This is to illustrate the difference between exponential map and our construction based on interpolation.

Notation. We often write ξ_s instead of $\xi(s)$ and $df|_x$ instead of $df(x)$ for better readability, avoiding excessive use of parentheses. For Hilbert space norms we often write $|\cdot|$ for distinction of the norm $\|\cdot\|$ in mapping spaces.

2 Tameness

Assume that $H_2 \subset H_1$ are two separable Hilbert spaces with a dense inclusion of H_2 into H_1. To avoid annoying constants we assume that $|\cdot|_1 \le |\cdot|_2$. We abbreviate the norms in H_1 and H_2 by

$$|\cdot|_1 := \|\cdot\|_{H_1}, \qquad |\cdot|_2 := \|\cdot\|_{H_2}, \qquad |\cdot|_1 \le |\cdot|_2. \tag{2.2}$$

2.1 Tameness (unparametrized)

Let U_1 be an open subset of H_1 and let $U_2 := U_1 \cap H_2$ be its part in H_2.

Definition 2.1. A C^2 function $\phi\colon U_1 \to H_1$ is called **tame**, or more precisely ***(H_1, H_2)*-tame**, if if it satisfies two conditions. (i) Its restriction to U_2 takes values in H_2 and is C^2 as a map $\phi|_{U_2}\colon U_2 \to H_2$. (ii) For every $x \in U_1$ there exists an H_1-open neighborhood $W_x \subset U_1$ of x and a constant $\kappa = \kappa(x) > 0$ such that for every $y \in W_x \cap H_2$ and all $\xi, \eta \in H_2$ we have the estimate

$$\left|d^2\phi|_y(\xi,\eta)\right|_2 \le \kappa\bigl(|\xi|_1|\eta|_2 + |\xi|_2|\eta|_1 + |y|_2|\xi|_1|\eta|_1\bigr). \tag{2.3}$$

Remark 2.2 (Vector space)**.** Assume that $\phi, \psi\colon U_1 \to H_1$ are both tame. Then their sum is tame as well. Indeed for $x \in U_1$ take the intersection of the two neighborhoods and the sum of their constants.

Remark 2.3 (Constant scales $H_2 = H_1 =: H$ are tame)**.** We claim that in this special case every C^2-function $\phi\colon U \to H$ is automatically tame. To see this pick $x \in U$. Since ϕ is two times continuously differentiable there exists an open neighborhood W_x of x in U such that for every $y \in W_x$ the following holds

$$\left\|d^2\phi|_y\right\|_{\mathcal{L}(H,H;H)} \le \left\|d^2\phi|_x\right\|_{\mathcal{L}(H,H;H)} + 1.$$

For $y \in W_x$ and $\xi, \eta \in H$ we estimate

$$\begin{aligned}
\left|d^2\phi|_y(\xi,\eta)\right| &\le \left\|d^2\phi|_y\right\|_{\mathcal{L}(H,H;H)} |\xi||\eta| \\
&\le \underbrace{\Bigl(\left\|d^2\phi|_x\right\|_{\mathcal{L}(H,H;H)} + 1\Bigr)}_{=:\kappa}|\xi||\eta| \\
&\le \kappa\left(|\xi||\eta| + |\xi||\eta| + |y||\xi||\eta|\right)
\end{aligned}$$

and this confirms the taming condition (2.3).

Remark 2.4. In the definition of tame the condition that the restriction $\phi|_{U_2}\colon U_2 \to H_2$ is of class C^2 is not used, neither in Theorem 2.5, nor in Theorem B. In both theorems one gets away with C^1.

The next theorem tells that tameness is invariant under coordinate change.

Theorem 2.5. *Given open subsets $U_1, V_1 \subset H_1$, consider (H_1,H_2)-tame maps*

$$\phi\colon U_1 \to V_1, \qquad \psi\colon V_1 \to H_1.$$

Then the composition $\psi\circ\phi\colon U_1 \to H_1$ is (H_1,H_2)-tame as well.

The proof needs the following two technical lemmas.

Lemma 2.6. *Assume that $\phi\colon U_1 \to H_1$ is tame. Then for every $x \in U_1$ there exists an H_1-open neighborhood $W_x' \subset U_1$ of x and a constant $\kappa_1 > 0$ such that for every $y \in W_x' \cap H_2$ and every $\eta \in H_2$ it holds the estimate*

$$\left|d\phi|_y \eta\right|_2 \le \kappa_1 \left(|\eta|_2 + |y|_2|\eta|_1\right).$$

Proof. For $x \in U_1$ let W_x and $\kappa > 0$ be such that for every $y \in W_x \cap H_2$ the estimate (2.3) holds. Since H_2 is dense in H_1 we can pick x_0 in $W_x \cap H_2$ and $\delta > 0$ such that the radius-δ ball in H_1 about x_0 is a subset of W_x containing x, in symbols $x \in B^1_\delta(x_0) \subset W_x$. Pick $y \in B^1_\delta(x_0) \cap H_2$ and $\eta \in H_2$, then by the fundamental theorem of calculus and monotonicity of the integral we obtain

$$\begin{aligned}
&\left|d\phi|_y \eta\right|_2 \\
&\le \int_0^1 \left|\tfrac{d}{dt} d\phi|_{x_0+t(y-x_0)}\eta\right|_2 dt + \left|d\phi|_{x_0}\eta\right|_2 \\
&= \int_0^1 \left|d^2\phi|_{x_0+t(y-x_0)}(y-x_0,\eta)\right|_2 dt + \left|d\phi|_{x_0}\eta\right|_2 \\
&\stackrel{3}{\le} \kappa \int_0^1 \Bigl(|y-x_0|_1|\eta|_2 + |y-x_0|_2|\eta|_1 + \underbrace{|x_0+t(y-x_0)|_2}_{\le |x_0|_2 + t|y-x_0|_2}|y-x_0|_1|\eta|_1\Bigr) dt \\
&\quad + \left\|d\phi|_{x_0}\right\|_{\mathcal{L}(H_2)} |\eta|_2 \\
&\stackrel{4}{\le} \kappa\Bigl(\delta|\eta|_2 + |y-x_0|_2|\eta|_1 + \delta|x_0|_2|\eta|_1 + \tfrac{1}{2}\delta|y-x_0|_2|\eta|_1\Bigr) \\
&\quad + \left\|d\phi|_{x_0}\right\|_{\mathcal{L}(H_2)} |\eta|_2 \\
&\stackrel{5}{\le} \kappa\Bigl(\delta|\eta|_2 + |y|_2|\eta|_1 + |x_0|_2|\eta|_1 + \tfrac{3}{2}\delta|x_0|_2|\eta|_1 + \tfrac{1}{2}\delta|y|_2|\eta|_1\Bigr) \\
&\quad + \left\|d\phi|_{x_0}\right\|_{\mathcal{L}(H_2)} |\eta|_2 \\
&\stackrel{6}{\le} \kappa\Bigl(\delta|\eta|_2 + |y|_2|\eta|_1 + |x_0|_2|\eta|_2 + \tfrac{3}{2}\delta|x_0|_2|\eta|_2 + \tfrac{1}{2}\delta|y|_2|\eta|_1\Bigr) \\
&\quad + \left\|d\phi|_{x_0}\right\|_{\mathcal{L}(H_2)} |\eta|_2 \\
&\stackrel{7}{\le} \kappa_1\left(|\eta|_2 + |y|_2|\eta|_1\right).
\end{aligned}$$

Step 3 is by tameness (2.3), step 4 by integration and since $y \in B^1_\delta(x_0)$. Step 5 is the triangle inequality and step 6 the constant-avoiding assumption $|\cdot|_1 \le |\cdot|_2$. In step 7 we used the constant defined by

$$\kappa_1 := \max\Bigl\{\kappa\left(\delta + (1+\tfrac{3}{2}\delta)|x_0|_2\right) + \left\|d\phi|_{x_0}\right\|_{\mathcal{L}(H_2)}, \kappa\left(1+\tfrac{1}{2}\delta\right)\Bigr\}.$$

Hence Lemma 2.6 follows with $W'_x = B^1_\delta(x_0)$. $\square$

Lemma 2.7. *Assume that $\phi\colon U_1 \to H_1$ is tame. Then for every $x \in U_1$ there exists an H_1-open neighborhood $W''_x \subset U_1$ of x and a constant $\kappa_0 > 0$ such that for every $y \in W''_x \cap H_2$ it holds the estimate*

$$|\phi(y)|_2 \le \kappa_0 \left(1 + |y|_2\right).$$

Proof. For $x \in U_1$ let W'_x and $\kappa_1 > 0$ be as in Lemma 2.6. Since H_2 is dense in H_1 we can pick x_0 in $W'_x \cap H_2$ and $\delta > 0$ such that the radius-δ ball in H_1 about x_0 is a subset of W'_x containing x, in symbols $x \in B^1_\delta(x_0) \subset W'_x$. Pick $y \in B^1_\delta(x_0) \cap H_2$, then by the fundamental theorem of calculus we calculate

$$\begin{aligned}
&|\phi(y)|_2 \\
&\le |\phi(x_0)|_2 + \int_0^1 \left|\tfrac{d}{dt}\phi(x_0 + t(y - x_0))\right|_2 dt \\
&= |\phi(x_0)|_2 + \int_0^1 \left|d\phi|_{x_0+t(y-x_0)}(y - x_0)\right|_2 dt \\
&\stackrel{3}{\le} |\phi(x_0)|_2 + \kappa_1 \int_0^1 \Big(|y - x_0|_2 + \underbrace{|x_0 + t(y - x_0)|_2}_{\le |x_0|_2 + t|y-x_0|_2} |y - x_0|_1\Big) dt \qquad (2.4)\\
&\stackrel{4}{\le} |\phi(x_0)|_2 + \kappa_1 \Big(1 + \tfrac{1}{2}|y - x_0|_1\Big) \underbrace{|y - x_0|_2}_{\le |y|_2 + |x_0|_2} + \kappa_1 |x_0|_2 |y - x_0|_1 \\
&\stackrel{5}{\le} |\phi(x_0)|_2 + \kappa_1 |x_0|_2 \left(1 + \tfrac{3}{2}\delta\right) + \kappa_1 \left(1 + \tfrac{1}{2}\delta\right) |y|_2 .
\end{aligned}$$

Step 3 is by Lemma 2.6, step 4 by integration, step 5 by $|y - x_0|_1 \le \delta$. Define

$$\kappa_0 := \max\Big\{ |\phi(x_0)|_2 + \kappa_1 |x_0|_2 \left(1 + \tfrac{3}{2}\delta\right), \kappa_1 \left(1 + \tfrac{1}{2}\delta\right) \Big\}.$$

Hence Lemma 2.7 follows with $W''_x = B^1_\delta(x_0)$. $\square$

We summarize the two lemmas and (2.3) in the following corollary.

Corollary 2.8. *Assume that $\phi\colon U_1 \to H_1$ is tame. Then for every $x \in U_1$ there exists an H_1-open neighborhood $W_x \subset U_1$ of x and a constant $\kappa > 0$ such that for every $y \in W_x \cap H_2$ and all $\xi, \eta \in H_2$ there are the three estimates*

$$\begin{aligned}
\left|\phi(y)\right|_2 &\le \kappa \left(1 + |y|_2\right) \\
\left|d\phi|_y \eta\right|_2 &\le \kappa \left(|\eta|_2 + |y|_2 |\eta|_1\right) \qquad (2.5)\\
\left|d^2\phi|_y(\xi, \eta)\right|_2 &\le \kappa \big(|\xi|_1 |\eta|_2 + |\xi|_2 |\eta|_1 + |y|_2 |\xi|_1 |\eta|_1\big).
\end{aligned}$$

Proof of Theorem 2.5. Assume that $x \in U_1$. Concerning ϕ pick an open neighborhood W_x of x in U_1 such that all three estimates in Corollary 2.8 hold true.

Since $\phi\colon U_1 \to V_1$ is C^2 we can additionally assume that for every $y \in W_x$ and all $\xi, \eta \in H_1$ the following three estimates hold true as well

$$\begin{aligned} \left|\phi(y)\right|_1 &\le \kappa \\ \left|d\phi|_y \eta\right|_1 &\le \kappa |\eta|_1 \\ \left|d^2\phi|_y(\xi,\eta)\right|_1 &\le \kappa |\xi|_1 |\eta|_1. \end{aligned} \tag{2.6}$$

Concerning $\psi\colon V_1 \to H_1$ pick an open neighborhood $W_{\phi(x)}$ of $\phi(x)$ in V_1 such that, maybe after enlarging the constant κ, the six estimates hold as well for ϕ replaced by ψ. Maybe after shrinking the open neighborhood W_x we can additionally assume that $W_x \subset \phi^{-1}(W_{\phi(x)})$.

Using that $d(\psi\circ\phi)|_y \xi = d\psi|_{\phi(y)} d\phi|_y \xi$ we calculate

$$\begin{aligned} &\left|d^2(\psi\circ\phi)|_y(\xi,\eta)\right|_2 \\ &\le \left|d^2\psi|_{\phi(y)}\left(d\phi|_y\xi, d\phi|_y\eta\right)\right|_2 + \left|d\psi|_{\phi(y)} d^2\phi|_y(\xi,\eta)\right|_2 \\ &\stackrel{2}{\le} \kappa\left(\left|d\phi|_y\xi\right|_1 \left|d\phi|_y\eta\right|_2 + \left|d\phi|_y\xi\right|_2 \left|d\phi|_y\eta\right|_1 + |\phi(y)|_2 \left|d\phi|_y\xi\right|_1 \left|d\phi|_y\eta\right|_1\right) \\ &\quad + \kappa\left(\left|d^2\phi|_y(\xi,\eta)\right|_2 + |\phi(y)|_2 \left|d^2\phi|_y(\xi,\eta)\right|_1\right) \\ &\stackrel{3}{\le} \kappa\Big(\kappa|\xi|_1\kappa\left(|\eta|_2 + |y|_2|\eta|_1\right) + \kappa|\eta|_1\kappa\left(|\xi|_2 + |y|_2|\xi|_1\right)\Big) \\ &\quad + \kappa(1+|y|_2)\kappa^2|\xi|_1|\eta|_1 + \kappa^2\left(|\xi|_1|\eta|_2 + |\xi|_2|\eta|_1 + |y|_2|\xi|_1|\eta|_1\right) \\ &\quad + \kappa(1+|y|_2)\kappa|\xi|_1|\eta|_1 \\ &= (\kappa^3+\kappa^2)|\xi|_1|\eta|_2 + (\kappa^3+\kappa^2)|\xi|_2|\eta|_1 \\ &\quad + \underbrace{(3\kappa^3+2\kappa^2)}_{=:\kappa'}|y|_2|\xi|_1|\eta|_1 + (\kappa^3+\kappa^2)|\xi|_1 \underbrace{|\eta|_1}_{\le|\eta|_2} \\ &\le \kappa'\left(|\xi|_1|\eta|_2 + |\xi|_2|\eta|_1 + |y|_2|\xi|_1|\eta|_1\right). \end{aligned}$$

In step 2 we used (2.5) for ψ, namely for summand one the third estimate and for summand two the second estimate. In step 3 we used (2.5) and (2.6) for ϕ. This finishes the proof of Theorem 2.5. □

For later reference we state the following lemma.

Lemma 2.9 (Differences)**.** *Assume that $\phi\colon U_1 \to H_1$ is tame. Then for every $x \in U_1$ there exists an H_1-open neighborhood $W_x \subset U_1$ of x and a constant $\kappa > 0$ such that for all $y_0, y_1 \in W_x \cap H_2$ and $\eta \in H_2$ there is the estimate*

$$\begin{aligned} &\left|\left(d\phi|_{y_1} - d\phi|_{y_0}\right)\eta\right|_2 \\ &\le \kappa\left(|y_1-y_0|_1|\eta|_2 + |y_1-y_0|_2|\eta|_1\right) + \tfrac{\kappa}{2}\left(|y_1|_2+|y_0|_2\right)|y_1-y_0|_1|\eta|_1. \end{aligned} \tag{2.7}$$

Proof. For $x \in U_1$ let W_x and κ be as in Definition 2.1. Without loss of generality we can additionally assume that W_x is convex. Hence we estimate using the fundamental theorem of calculus and (2.3) in step 3 to get

$$
\begin{aligned}
&\left|(d\phi|_{y_1} - d\phi|_{y_0})\,\eta\right|_2 \\
&= \left|\int_0^1 \tfrac{d}{dt} d\phi|_{ty_1+(1-t)y_0}\eta\, dt\right|_2 \\
&\le \int_0^1 \left|d^2\phi|_{ty_1+(1-t)y_0}\,(y_1-y_0,\eta)\right|_2 dt \\
&\overset{3}{\le} \kappa \int_0^1 \left(|y_1-y_0|_1|\eta|_2 + |y_1-y_0|_2|\eta|_1\right) + \left(t|y_1|_2 + (1-t)|y_0|_2\right)|y_1-y_0|_1|\eta|_1\, dt.
\end{aligned}
$$

Integration then concludes the proof. □

2.2 Tame two-level manifolds

Definition 2.10. Let $H_2 \subset H_1$ be separable Hilbert spaces with a dense inclusion of H_2 into H_1. Let $X = X_1$ be a C^2 Hilbert manifold modeled on H_1.

a) A **tame two-level structure on** X_1, more precisely a **tame $(\boldsymbol{H_1}, \boldsymbol{H_2})$-two-level structure on $\boldsymbol{X_1}$**, is a C^2 atlas $\mathcal{A} = \{\psi\colon H_1 \supset U_1 \to V_1 \subset X_1\}$ of X_1 with the property that all transition maps

$$\phi := \tilde\psi^{-1} \circ \psi\colon H_1 \supset \psi^{-1}(V_1 \cap \tilde V_1) \to \tilde\psi^{-1}(V_1 \cap \tilde V_1) \subset H_1$$

are (H_1, H_2)-tame. We refer to such an atlas as a **tame two-level atlas**.

b) For an open subset $U_1 \subset H_1$ let $U_2 = U_1 \cap H_2$ be its part in H_2. If $\mathcal{A}$ is a tame two-level atlas of X_1 we define a subset $X_2 \subset X_1$ by

$$X_2 := \bigcup_{\psi\in\mathcal{A}} \psi(U_2).$$

Then X_2 is itself a C^2 Hilbert manifold with C^2 atlas $\mathcal{A}_2 = \{\psi|_{U_2} \mid \psi \in \mathcal{A}\}$ and X_2 is dense in X_1.

c) The pair (X_1, X_2) is referred to as a **tame two-level manifold**, more precisely as a **tame $(\boldsymbol{H_1}, \boldsymbol{H_2})$-two-level manifold**.

Remark 2.11 (Maximal tame two-level atlas)**.** Two tame two-level atlases are **compatible** if their union is itself a tame two-level atlas. Compatibility is an equivalence relation. That compatibility is reflexive and symmetric is obvious. That compatibility is transitive follows from Theorem 2.5 which tells us that a composition of tame maps is tame, together with the fact that tameness is a local condition (the restriction of a tame map to an open subset of its domain is itself tame).

Since compatibility is an equivalence relation, for each tame two-level manifold there exists a unique **maximal** tame two-level atlas obtained by taking the union of all tame two-level atlases compatible to a given one.

2.3 Parametrized tameness

Let O_1 be an open subset of $\mathbb{R} \times H_1$ and let O_2 be its part in $\mathbb{R} \times H_2$.

Definition 2.12. A C^2 map $\phi\colon O_1 \to H_1$ is called **parametrized tame**, if it satisfies two conditions. Firstly, its restriction to $O_2 = O_1 \cap (\mathbb{R} \times H_2)$ takes values in H_2 and is C^2 as a map $\phi|_{O_2}\colon O_2 \to H_2$. Secondly, for every point $(s,x) \in O_1$ there exists an open neighborhood $W_{s,x}$ of (s,x) in O_1 and a constant $\kappa_{s,x} > 0$ such that for every $(t,y) \in W_{s,x} \cap (\mathbb{R} \times H_2)$ it holds the estimate

$$\left|d^2\phi_t|_y(\xi,\eta)\right|_2 \le \kappa_{s,x}\big(|\xi|_1|\eta|_2 + |\xi|_2|\eta|_1 + |y|_2|\xi|_1|\eta|_1\big). \tag{2.8}$$

Definition 2.13. The ***s*-slices** of O_1 and O_2 are defined by

$$U_1^s := \{x \in H_1 \mid (s,x) \in O_1\} = O_1 \cap (\{s\} \times H_1),$$
$$U_2^s := \{x \in H_2 \mid (s,x) \in O_2\} = O_2 \cap (\{s\} \times H_2) = U_1^s \cap H_2.$$

A parametrized tame map is called **asymptotically constant** if there exists $T > 0$ such that, firstly, $\phi_s := \phi(s,\cdot)\colon U_1^s \to H_1$ is independent of s and, secondly, $\phi_s(0) = 0$, both whenever $|s| \ge T$.

2.4 Loop space is a tame two-level manifold

The loop space $X = \Lambda\mathbb{R}$ of $M = \mathbb{R}$ has the form of a Hilbert space pair

$$X_1 := H_1 := W^{1,2}(\mathbb{S}^1, \mathbb{R}), \qquad X_2 := H_2 := W^{2,2}(\mathbb{S}^1, \mathbb{R}).$$

A C^∞ diffeomorphism $\varphi\colon \mathbb{R} \to \mathbb{R}$ induces the loop space **transition map**

$$\phi\colon U_1 := H_1 \to H_1, \quad x \mapsto [t \mapsto \varphi(x(t))].$$

Lemma 2.14. *The induced loop space transition map ϕ is tame.*

Proof. Condition (i). By assumption $\varphi\colon \mathbb{R} \to \mathbb{R}$ is C^∞. For $x \in H_1$ and $y \in H_2$ and abbreviating $x_t := x(t)$ we get

$$\left(\tfrac{d}{dt}\phi(x)\right)_t = \varphi'(x_t)\dot{x}_t, \qquad \left(\tfrac{d^2}{dt^2}\phi(y)\right)_t = \varphi''(y_t)\dot{y}_t\dot{y}_t + \varphi'(y_t)\ddot{y}_t$$

for every $t \in \mathbb{S}^1$. This shows that ϕ on H_1 takes values in H_1 and on H_2 it takes values in H_2. For $x, \xi, \eta \in H_1$ we calculate

$$(d\phi|_x\xi)_t := \left(\tfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}\phi(x+\varepsilon\xi)\right)_t = \tfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}\phi(x_t+\varepsilon\xi_t) = \varphi'|_{x_t}\xi_t$$

whenever $t \in \mathbb{S}^1$ and similarly

$$\left(d^2\phi|_x(\xi,\eta)\right)_t = \varphi''(x_t)\xi_t\eta_t.$$

At $x \in H_1$ and $y \in H_2$ the first two derivatives with respect to time $t \in \mathbb{S}^1$ are

$$\begin{aligned}
\tfrac{d}{dt}\left(d^2\phi|_x(\xi,\eta)\right) &= \varphi'''(x)\dot{x}\xi\eta + \varphi''(x)\dot{\xi}\eta + \varphi''(x)\xi\dot{\eta} \\
\tfrac{d^2}{dt^2}\left(d^2\phi|_y(\xi,\eta)\right) &= \varphi''''(y)\dot{y}^2\xi\eta + \varphi'''(y)\ddot{y}\xi\eta + 2\varphi'''(y)\dot{y}\dot{\xi}\eta + 2\varphi'''(y)\dot{y}\xi\dot{\eta} \\
&\quad + 2\varphi''(y)\dot{\xi}\dot{\eta} + \varphi''(y)\ddot{\xi}\eta + \varphi''(y)\xi\ddot{\eta}.
\end{aligned}$$

The first identity holds for all $\xi, \eta \in H_1$ and the second for all $\xi, \eta \in H_2$. Using the Sobolev embedding $W^{1,2}(\mathbb{S}^1, \mathbb{R}) \hookrightarrow C^0(\mathbb{S}^1, \mathbb{R})$ with constant 1 shows that both right hand sides are in $L^2(\mathbb{S}^1, \mathbb{R})$. Thus $d^2\phi|_x(\xi, \eta) \in H_1$ and $d^2\phi|_y(\xi, \eta) \in H_2$. Moreover, these formulae show that both maps

$$d^2\phi\colon H_1 \to \mathcal{L}(H_1, H_1; H_1), \qquad d^2\phi\colon H_2 \to \mathcal{L}(H_2, H_2; H_2),$$

are continuous.
Condition (ii). By the Sobolev embedding $W^{1,2}(\mathbb{S}^1, \mathbb{R}) \hookrightarrow C^0(\mathbb{S}^1, \mathbb{R})$ with constant 1 we have for each $y \in H_2$ the estimate

$$\begin{aligned} \|d^2\phi|_y(\xi, \eta)\|_{2,2}^2 &= \|\varphi''(y)\xi\eta\|_2^2 + \|\tfrac{d}{dt}\varphi''(y)\xi\eta\|_2^2 + \|\tfrac{d^2}{dt^2}\varphi''(y)\xi\eta\|_2^2 \\ &\le c_y\Big(\|\xi\|_{2,2}\|\eta\|_{1,2} + \|\xi\|_{1,2}\|\eta\|_{2,2} + \|y\|_{2,2}\|\xi\|_{1,2}\|\eta\|_{1,2}\Big) \end{aligned}$$

for some constant c_y which depends continuously on $|y|_1 = \|y\|_{1,2}$.
This proves Lemma 2.14. □

Lemma 2.14 shows that the loop space is a tame two-level manifold. Strictly speaking we only showed this for $M = \mathbb{R}$ to simply notation. The same argument should, however, work for general smooth finite dimensional manifolds M.

3 Differentiability in Hilbert space valued Sobolev spaces

In this section we assume that H_1 and H_2 are separable Hilbert spaces together with a dense incusion of H_2 in H_1. In Section 3 we do not need that the inclusion is compact. For $i = 1, 2$ we abbreviate

$$W^{1,2}_{H_1} := W^{1,2}(\mathbb{R}, H_1), \qquad L^2_{H_i} := L^2(\mathbb{R}, H_i).$$

The intersection

$$W^{1,2}_{H_1} \cap L^2_{H_2}$$

is itself a Hilbert space with inner product

$$\langle\langle\langle \xi, \eta \rangle\rangle\rangle := \int_{-\infty}^{\infty} \left\langle \dot{\xi}(s), \dot{\eta}(s) \right\rangle_{H_1} ds + \int_{-\infty}^{\infty} \langle \xi(s), \eta(s) \rangle_{H_2} ds.$$

For $\xi, \eta \in \mathcal{L}(W^{1,2}_{H_1} \cap L^2_{H_2})$ we denote

$$|||\, \eta |||^2_{1,2} := \|\eta\|^2_{L^2_{H_2}} + \|\dot{\eta}\|^2_{L^2_{H_1}}, \qquad \|\eta\|^2_{1,2} = \|\eta\|^2_{L^2_{H_1}} + \|\dot{\eta}\|^2_{L^2_{H_1}}. \tag{3.9}$$

3.1 Theorem B

3.1.1 Base component Φ

Theorem 3.1. *Let $U_1 \subset H_1$ be open and $\phi \colon U_1 \to H_1$ be tame. Assume in addition that $0 \in U_1$ and $\phi(0) = 0$. With $W^{1,2}_{H_1}$ and $L^2_{H_i}$ as in (1.1) the map*

$$\begin{aligned} \Phi \colon W^{1,2}_{H_1} \cap L^2_{H_2} \supset \mathcal{U} &\to W^{1,2}_{H_1} \cap L^2_{H_2} \\ \xi &\mapsto [s \mapsto \phi \circ \xi_s] \end{aligned}$$

is well defined and continuously differentiable where

$$\mathcal{U} := \{\xi \in W^{1,2}_{H_1} \cap L^2_{H_2} \mid \xi_s \in U_1,\ \forall s \in \mathbb{R}\}. \tag{3.10}$$

Proof. The proof is in four steps. We abbreviate

$$|\cdot|_1 := \|\cdot\|_{H_1}, \quad |\cdot|_2 := \|\cdot\|_{H_2}, \quad \xi_s := \xi(s), \quad \dot{\xi} := \tfrac{d}{ds}\xi. \tag{3.11}$$

Step 1 (Well defined).

Proof. By (A.27) there is an embedding

$$W^{1,2}_{H_1} := W^{1,2}(\mathbb{R}, H_1) \hookrightarrow C^0(\mathbb{R}, H_1), \qquad \|v\|_\infty \le \|v\|_{1,2} \tag{3.12}$$

with constant 1 and this implies that $\mathcal{U}$ is open in $W^{1,2}_{H_1} \cap L^2_{H_2}$. There exists an H_1-open neighborhood W_0 of 0 in U_1 and a constant $\kappa_\infty > 0$ such that

$$|\phi(y)|_2 \le \kappa_\infty |y|_2 \tag{3.13}$$

for every $y \in W_0$. To see this observe that, since $\phi(0) = 0$, we can in the proof of Lemma 2.7 choose $x_0 = 0$ and then the estimate follows from (2.4).

STEP A. Let $\xi \in \mathcal{U}$, then $\Phi(\xi) \in L^2_{H_2} := L^2(\mathbb{R}, H_2) \subset L^2(\mathbb{R}, H_1)$.

• Asymptotic part: To see this we claim that there exists $T = T(\xi) > 0$ such that $\xi_s \in W_0$ whenever $|s| \geq T$. Since W_0 is an H_1-open neighborhood of 0 there exists a $\delta > 0$ such that the closed δ-ball about 0 is contained in W_0. Since $\xi \in W^{1,2}(\mathbb{R}, H_1)$ there exists $T > 0$ such that the restrictions satisfy $\|\xi|_{(-\infty,T)}\|_{1,2} \leq \delta$ and $\|\xi|_{(T,\infty)}\|_{1,2} \leq \delta$. Hence, by Remark A.28, we have $\|\xi|_{(-\infty,T)}\|_\infty \leq \delta$ and $\|\xi|_{(T,\infty)}\|_\infty \leq \delta$ and therefore $\xi_s \in W_0$ whenever $|s| \geq T$.
• Compact part $[-T, T]$: Since ϕ is tame, for every $s \in [-T, T]$ there exists, by Lemma 2.7, an H_1-open neighborhood W_s of $\xi_s := \xi_s$ in U_1 and a constant $\kappa_s > 0$ such that for every $y \in W_s \cap H_2$ it holds $|\phi(y)|_2 \leq \kappa_s(1+|y|_2)$. The family $\{W_s\}_{s\in[-T,T]}$ is an open cover of the image $\xi([-T, T])$. Since ξ is in $W^{1,2}(\mathbb{R}, H_1)$ it is in particular a continuous map $\xi\colon [-T, T] \to H_1$ and therefore this image is compact in H_1. Therefore there exist finitely many times $s_1, \dots, s_N \in [-T, T]$ such that the image $\xi([-T, T])$ already lies in the finite union $\cup_{k=1}^N W_{s_k}$.
• Let $\kappa := \max\{\kappa_{s_1}, \dots, \kappa_{s_N}, \kappa_\infty\}$. Then we have the estimate

$$|\phi(\xi_s)|_2 \leq \kappa(1 + |\xi_s|_2), \quad \forall s \in [-T, T].$$

In the asymptotic case $|s| > T$, by (3.13), we have the estimate

$$|\phi(\xi_s)|_2 \leq \kappa|\xi_s|_2, \quad \forall |s| > T.$$

Using these two displayed estimates we obtain

$$\begin{aligned}
\|\Phi(\xi)\|^2_{L^2_{H_2}} &= \int_{-\infty}^{\infty} |\phi(\xi_s)|_2^2 \, ds \\
&= \int_{-\infty}^{-T} |\phi(\xi_s)|_2^2 \, ds + \int_{-T}^{T} |\phi(\xi_s)|_2^2 \, ds + \int_{T}^{\infty} |\phi(\xi_s)|_2^2 \, ds \\
&\leq \kappa^2 \int_{-\infty}^{-T} |\xi_s|_2^2 \, ds + \kappa^2 \int_{-T}^{T} \underbrace{(1 + |\xi_s|_2)^2}_{\leq 2(1+|\xi_s|_2^2)} \, ds + \kappa^2 \int_{T}^{\infty} |\xi_s|_2^2 \, ds \\
&\leq 2\kappa^2 \int_{-\infty}^{\infty} |\xi_s|_2^2 \, ds + 4\kappa^2 T \\
&= 2\kappa^2 \|\xi\|^2_{L^2_{H_2}} + 4\kappa^2 T < \infty.
\end{aligned}$$

This proves Step A.

STEP B. Let $\xi \in \mathcal{U}$, then $\frac{d}{ds}\Phi(\xi) = \frac{d}{ds}[s \mapsto \phi \circ \xi_s] \in L^2_{H_1} := L^2(\mathbb{R}, H_1)$.

• By definition of Φ we have $\frac{d}{ds}(\Phi(\xi))_s = d\phi|_{\xi_s} \frac{d}{ds}\xi_s$. As explained in Step A (asymptotic part), in view of Remark A.28 it holds $\lim_{s\to\mp\infty} \xi_s = 0 \in H_1$. Moreover, since $\xi \in W^{1,2}_{H_1}$, it is continuous as a map $\mathbb{R} \to H_1$ and therefore the image of ξ in H_1 is compact. Hence, since ϕ is continuously differentiable, there exists a constant κ such that $\|d\phi|_{\xi_s}\|_{\mathcal{L}(H_1)} \leq \kappa$ for every $s \in \mathbb{R}$. Therefore $\|\frac{d}{ds}\Phi(\xi)\|_{L^2_{H_1}} \leq \kappa\|\frac{d}{ds}\xi\|_{L^2_{H_1}}$. This finishes the proof of Step B and Step 1. □

In the following three steps 2–4 we show that Φ is differentiable.

Step 2 (Candidate). Given $\xi \in \mathcal{U}$, a natural candidate for the derivative is

$$\left(d\Phi|_\xi \hat{\xi}\right)(s) = d\phi|_{\xi_s} \hat{\xi}_s$$

whenever $s \in \mathbb{R}$ and $\hat{\xi} \in W^{1,2}_{H_1} \cap L^2_{H_2}$. The map $d\Phi|_\xi$ lies in $\mathcal{L}(W^{1,2}_{H_1} \cap L^2_{H_2})$.

Proof. The proof has two steps A and B.

Step A. We show that $d\Phi|_\xi \hat{\xi}$ lies in $W^{1,2}_{H_1} \cap L^2_{H_2}$. To see this we first show that $d\Phi|_\xi \hat{\xi} \in L^2_{H_2}$. Since the image of ξ in H_1 is compact and ϕ is tame, by the second inequality in (2.5), there exists $\kappa > 0$ such that

$$|d\phi|_{\xi_s} \eta|_2 \le \kappa \left(|\eta|_2 + |\xi_s|_2 |\eta|_1\right)$$

for all $s \in \mathbb{R}$ and $\eta \in H_2$. Use this estimate to obtain step 2 in what follows

$$\begin{aligned}
\|d\Phi|_\xi \hat{\xi}\|^2_{L^2_{H_2}} &= \int_{\mathbb{R}} |d\phi|_{\xi_s} \hat{\xi}_s|_2^2 ds \\
&\stackrel{2}{\le} \kappa^2 \int_{\mathbb{R}} \left(|\hat{\xi}_s|_2 + |\xi_s|_2 |\hat{\xi}_s|_1\right)^2 ds \\
&\le 2\kappa^2 \int_{\mathbb{R}} \left(|\hat{\xi}_s|_2^2 + |\xi_s|_2^2 |\hat{\xi}_s|_1^2\right) ds \\
&\le 2\kappa^2 \|\hat{\xi}\|^2_{L^2_{H_2}} + 2\kappa^2 \|\hat{\xi}\|^2_{L^\infty_{H_1}} \int_{\mathbb{R}} |\xi_s|_2^2 \, ds \\
&\stackrel{5}{\le} \kappa^2 \underline{\|\hat{\xi}\|^2_{L^2_{H_2}}} + 2\kappa^2 \underline{\|\hat{\xi}\|^2_{W^{1,2}_{H_1}}} \|\xi\|^2_{L^2_{H_2}} \\
&\le \kappa^2 \left(1 + 2\|\xi\|^2_{W^{1,2}_{H_1} \cap L^2_{H_2}}\right) \underline{\|\hat{\xi}\|^2_{W^{1,2}_{H_1} \cap L^2_{H_2}}}.
\end{aligned} \tag{3.14}$$

In step 5 we used the embedding Theorem A.27.

Step B. Next we show that $\frac{d}{ds} d\Phi_\xi \hat{\xi} \in L^2_{H_1}$. To see this we calculate

$$\int_{\mathbb{R}} |\tfrac{d}{ds}(d\phi|_{\xi_s} \hat{\xi}_s)|_1^2 ds = \int_{\mathbb{R}} |d^2\phi|_{\xi_s}(\hat{\xi}_s, \dot{\xi}_s) + d\phi|_{\xi_s} \dot{\hat{\xi}}_s|_1^2 ds. \tag{3.15}$$

Term 1. As $d^2\phi$ is continuous and ξ_s asymptotically converges to zero, there is a constant $c = c(\xi)$ such that $\max\{\|d^2\phi|_{\xi_s}\|_{\mathcal{L}(H_1,H_1;H_1)}, \|d\phi|_{\xi_s}\|_{\mathcal{L}(H_1)}\} \le c$ for every $s \in \mathbb{R}$. By (A.82) we have $\|\hat{\xi}\|_{L^\infty_{H_1}} \le \|\hat{\xi}\|_{W^{1,2}_{H_1}}$. So we get

$$\int_{\mathbb{R}} |d^2\phi|_{\xi_s}(\hat{\xi}_s, \dot{\xi}_s)|_1^2 ds \le c^2 \|\dot{\xi}\|^2_{L^2_{H_1}} \|\hat{\xi}\|^2_{W^{1,2}_{H_1}}.$$

Term 2. We have $\int_{\mathbb{R}} |d\phi|_{\xi_s} \dot{\hat{\xi}}_s|_1^2 ds \le c^2 \|\dot{\hat{\xi}}\|^2_{L^2_{H_1}} \le c^2 \|\hat{\xi}\|^2_{W^{1,2}_{H_1}}$. Together we get

$$\|\tfrac{d}{ds} d\Phi|_\xi \hat{\xi}\|_{L^2_{H_1}} \le c\sqrt{2} \sqrt{1 + \|\dot{\xi}\|^2_{L^2_{H_1}}} \|\hat{\xi}\|_{W^{1,2}_{H_1}}. \tag{3.16}$$

By (3.14) and (3.16) there is a constant C_ξ such that

$$\|d\Phi|_\xi \hat{\xi}\|_{W^{1,2}_{H_1}\cap L^2_{H_2}} \le C_\xi \|\hat{\xi}\|_{W^{1,2}_{H_1}\cap L^2_{H_2}}.$$

Hence the operator norm is bounded by

$$\|d\Phi|_\xi\|_{\mathcal{L}(W^{1,2}_{H_1}\cap L^2_{H_2})} \le C_\xi.$$

Thus the candidate is well defined and this proves Step 2. □

Step 3 (Candidate from Step 2 is derivative of Φ).

Proof. To $\xi, \eta \in \mathcal{L}(W^{1,2}_{H_1} \cap L^2_{H_2})$ applies the triple norm 3.9. Note that, in view of (2.2), we have

$$\boxed{\|\eta\|_{1,2} \le |||\eta\, |||_{1,2}\, .} \tag{3.17}$$

To see that our candidate actually is the derivative we show that the limit

$$\begin{aligned}
&\lim_{h\to 0} \sup_{|||\eta|||_{1,2}\le 1} \frac{|||\Phi(\xi+h\eta)-\Phi(\xi)-h\, d\phi|_\xi\eta|||^2_{1,2}}{h^2} \\
&= \lim_{h\to 0} \sup_{|||\eta|||_{1,2}\le 1} \frac{\int_{\mathbb{R}} |\phi(\xi_s+h\eta_s)-\phi(\xi_s)-h\, d\phi|_{\xi_s}\eta_s|_2^2\, ds}{h^2} \\
&\quad + \lim_{h\to 0} \sup_{|||\eta|||_{1,2}\le 1} \frac{\int_{\mathbb{R}} \left|\frac{d}{ds}\left(\phi(\xi_s+h\eta_s)-\phi(\xi_s)-h\, d\phi|_{\xi_s}\eta_s\right)\right|_1^2 ds}{h^2}
\end{aligned} \tag{3.18}$$

exists and vanishes. We treat each of the two summands separately.

Summand 1. To see that summand 1 vanishes we use the fundamental theorem of calculus to write it in the form

$$\begin{aligned}
&\sup_{|||\eta|||_{1,2}\le 1} \frac{1}{h^2} \int_{\mathbb{R}} \left| \int_0^1 \Big(\underbrace{\tfrac{d}{dt}\phi(\xi_s+th\eta_s)}_{=d\phi|_{\xi_s+th\eta_s} h\eta_s} - h\, d\phi|_{\xi_s}\eta_s \Big) dt \right|_2^2 ds \\
&= \sup_{|||\eta|||_{1,2}\le 1} \int_{\mathbb{R}} \left| \int_0^1 \left(d\phi|_{\xi_s+th\eta_s} - d\phi|_{\xi_s}\right)\eta_s dt \right|_2^2 ds \\
&\le \sup_{|||\eta|||_{1,2}\le 1} \int_{\mathbb{R}} \left(\int_0^1 \left|\left(d\phi|_{\xi_s+th\eta_s} - d\phi|_{\xi_s}\right)\eta_s\right|_2 dt \right)^2 ds.
\end{aligned} \tag{3.19}$$

By Lemma 2.9 for every $s \in \mathbb{R}$ there exists an H_1-open neighborhood W_s of ξ_s in H_1 and a constant $\kappa_s > 0$ such that for all $y_0, y_1 \in W_s$ estimate (2.7) holds with κ replaced by κ_s. Since the image of ξ in H_1 is compact we can find a uniform constant κ independent of s such that estimate (2.7) holds. Since $W^{1,2}_{H_1}$

embeds in $L^\infty_{H_1}$ with constant 1, see (A.82), there exists $h_0 \in (0,1]$ such that $\xi_s + h\eta_s$ is element of W_s for every η with $\|\eta\|_{1,2} \le h_0$. We estimate

$$
\begin{aligned}
&\sup_{\|\eta\|_{1,2}\le 1} \int_{\mathbb{R}} \left(\int_0^1 \left|\left(d\phi|_{\xi_s+th\eta_s} - d\phi|_{\xi_s}\right)\eta_s\right|_2 dt \right)^2 ds \\
&\stackrel{1}{\le} \sup_{\|\eta\|_{1,2}\le 1} \int_{\mathbb{R}} \left(\int_0^1 2\kappa th|\eta_s|_2|\eta_s|_1 + \tfrac{\kappa}{2}\left(|\xi_s+th\eta_s|_2 + |\xi_s|_2\right) th|\eta_s|_1^2\, dt \right)^2 ds \\
&\stackrel{2}{\le} \sup_{\|\eta\|_{1,2}\le 1} \int_{\mathbb{R}} \left(\int_0^1 2\kappa th|\eta_s|_2|\eta_s|_1 + \kappa th|\xi_s|_2|\eta_s|_1^2 + \tfrac{\kappa}{2}t^2h^2|\eta_s|_2|\eta_s|_1^2\, dt \right)^2 ds \\
&\stackrel{3}{\le} \sup_{\|\eta\|_{1,2}\le 1} \int_{\mathbb{R}} \left(\kappa h|\eta_s|_2|\eta_s|_1 + \kappa\tfrac{1}{2}h|\xi_s|_2|\eta_s|_1^2 + \tfrac{\kappa}{6}h^2|\eta_s|_2|\eta_s|_1^2 \right)^2 ds \\
&\stackrel{4}{\le} \sup_{\|\eta\|_{1,2}\le 1} h^2\kappa^2 \max\{\|\eta\|^2_{L^\infty_{H_1}}, \|\eta\|^4_{L^\infty_{H_1}}\} \int_{\mathbb{R}} \underbrace{\left(|\eta_s|_2 + |\xi_s|_2 + |\eta_s|_2\right)^2}_{\le 8|\eta_s|_2^2 + 2|\xi_s|_2} ds \\
&\stackrel{5}{\le} \sup_{\|\eta\|_{1,2}\le 1} h^2\kappa^2 \max\{\|\eta\|^2_{1,2}, \|\eta\|^4_{1,2}\} \left(8\|\eta\|^2_{L^2_{H_2}} + 2\|\xi\|_{L^2_{H_2}} \right) \\
&\stackrel{6}{\le} h^2\kappa^2 \left(8 + 2\|\xi\|_{L^2_{H_2}} \right) \longrightarrow 0, \text{ as } h \to 0.
\end{aligned}
$$

Step 1 is by (2.7. Step 2 is by the triangle inequality. Step 3 is by integrating t. Step 4 pulls out the supremum norm of $|\eta_s|_1$ and uses that $h \ge 1$. Step 5 is by the Sobolev estimate $\|\eta\|_{L^\infty_{H_1}} \le \|\eta\|_{1,2}$ from (A.82). Step 6 is by (3.17).

The previous estimate, which starts at (3.19), tells that summand 1 is zero.

Summand 2. To see that summand 2 vanishes, we abbreviate and compute

$$
\begin{aligned}
G(s) :&= \tfrac{d}{ds}\left(\phi(\xi_s + h\eta_s) - \phi(\xi_s) - h\, d\phi|_{\xi_s}\eta_s\right) \\
&= d\phi|_{\xi_s+h\eta_s}(\dot\xi_s + h\dot\eta_s) \\
&\quad - d\phi|_{\xi_s}\dot\xi_s \\
&\quad - h\, d^2\phi|_{\xi_s}(\dot\xi_s, \eta_s) - h\, d\phi|_{\xi_s}\dot\eta_s \\
&= d\phi|_{\xi_s+h\eta_s}\dot\xi_s - d\phi|_{\xi_s}\dot\xi_s - h\, d^2\phi|_{\xi_s}(\dot\xi_s, \eta_s) \\
&\quad + d\phi|_{\xi_s+h\eta_s}h\dot\eta_s - h\, d\phi|_{\xi_s}\dot\eta_s \\
&= h\int_0^1 \left(d^2\phi|_{\xi_s+th\eta_s} - d^2\phi|_{\xi_s}\right)(\dot\xi_s, \eta_s)\, dt \\
&\quad + h\left(d\phi|_{\xi_s+h\eta_s} - d\phi|_{\xi_s}\right)\dot\eta_s
\end{aligned}
$$

for every $s \in \mathbb{R}$. Square this identity and integrate to obtain

$$\begin{aligned}
&\sup_{|||\eta|||_{1,2}\le 1} \tfrac{1}{h^2}\int_{\mathbb{R}} |G(s)|_1^2 ds \\
&\le \sup_{|||\eta|||_{1,2}\le 1} \int_{\mathbb{R}} 4 \left| \int_0^1 \left(d^2\phi|_{\xi_s+th\eta_s} - d^2\phi|_{\xi_s}\right)(\dot\xi_s,\eta_s)\, dt \right|_1^2 ds \\
&\quad + \sup_{|||\eta|||_{1,2}\le 1} \int_{\mathbb{R}} 4 \left| \left(d\phi|_{\xi_s+h\eta_s} - d\phi|_{\xi_s}\right)\dot\eta_s \right|_1^2 ds.
\end{aligned}$$

There are two terms in the sum.
Term 1. We first claim that for every $\varepsilon > 0$ there exists $h_0 = h_0(\varepsilon) > 0$ such that whenever $h \in [0, h_0]$, then

$$\begin{aligned}
\|d\phi|_{\xi_s+th\eta_s} - d\phi|_{\xi_s}\|_{\mathcal{L}(H_1)} &< \varepsilon, \\
\|d^2\phi|_{\xi_s+th\eta_s} - d^2\phi|_{\xi_s}\|_{\mathcal{L}(H_1,H_1;H_1)} &< \varepsilon.
\end{aligned} \tag{3.20}$$

To see this note that by (A.82) and (3.17) we have that

$$\boxed{|\eta(s)|_1 \le \|\eta\|_{L^\infty_{H_1}} \le \|\eta\|_{1,2} \le |||\eta|||_{1,2} \le 1}$$

whenever $s \in \mathbb{R}$. Hence the claim follows since ϕ is C^2 on H_1, by Definition 2.1 of tameness. Now suppose that $h \le h_0$, then we estimate term 1 by

$$\begin{aligned}
&\sup_{|||\eta|||_{1,2}\le 1} \int_{\mathbb{R}} 4 \left| \int_0^1 \left(d^2\phi|_{\xi_s+th\eta_s} - d^2\phi|_{\xi_s}\right)(\dot\xi_s,\eta_s)\, dt \right|_1^2 ds \\
&\le \sup_{|||\eta|||_{1,2}\le 1} \int_{\mathbb{R}} 4 \left(\int_0^1 \left| \left(d^2\phi|_{\xi_s+th\eta_s} - d^2\phi|_{\xi_s}\right)(\dot\xi_s,\eta_s) \right|_1 dt \right)^2 ds \\
&\le \sup_{|||\eta|||_{1,2}\le 1} \int_{\mathbb{R}} 4 \left(\int_0^1 \|d^2\phi|_{\xi_s+th\eta_s} - d^2\phi|_{\xi_s}\|_{\mathcal{L}(H_1,H_1;H_1)} \cdot |\dot\xi_s|_1 |\eta_s|_1 dt \right)^2 ds \\
&\overset{3}{\le} 4\varepsilon^2 \int_{\mathbb{R}} \left(\int_0^1 |\dot\xi_s|_1 dt \right)^2 ds \\
&\le 4\varepsilon^2 \|\xi\|_{1,2}^2.
\end{aligned}$$

In step 3 we used that $|\eta_s|_1 \le 1$, as shown after (3.20), and we also used estimate two in (3.20). Since $\varepsilon > 0$ was arbitrary, the limit of term 1, as $h \to 0$, is zero.

Term 2. We estimate

$$\begin{aligned}
&\sup_{|||\eta|||_{1,2}\le 1} \int_{\mathbb{R}} 4 \left|(d\phi|_{\xi_s+h\eta_s} - d\phi|_{\xi_s})\,\dot\eta_s\right|_1^2 ds\\
&\le 4 \sup_{|||\eta|||_{1,2}\le 1} \int_{\mathbb{R}} \|d\phi|_{\xi_s+h\eta_s} - d\phi|_{\xi_s}\|^2_{\mathcal{L}(H_1)} |\dot\eta_s|_1^2 ds\\
&\le 4\varepsilon^2 \sup_{|||\eta|||_{1,2}\le 1} \underbrace{\int_{\mathbb{R}} |\dot\eta_s|_1^2 ds}_{=\|\dot\eta\|^2_{L^2_{H_1}}\le |||\eta|||^2_{1,2}\le 1}\\
&\le 4\varepsilon^2.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the limit of term 2, as $h \to 0$, is zero as well.

We proved that summand 2 vanishes as well. Therefore our candidate is the derivative and this concludes the proof of Step 3. □

Step 4 (Differential is continuous). The differential

$$d\Phi \colon \mathcal{U} \to \mathcal{L}(W^{1,2}_{H_1} \cap L^2_{H_2}), \quad \xi \mapsto d\Phi|_\xi$$

is continuous.

Proof. Recall $|||\cdot|||_{1,2}$ in (3.9). For $\xi, \tilde\xi \in \mathcal{U}$ and $\eta \in W^{1,2}_{H_1} \cap L^2_{H_2}$ we estimate

$$\begin{aligned}
&|||\, d\Phi|_\xi \eta - d\Phi|_{\tilde\xi}\eta|||^2_{1,2}\\
&= \int_{\mathbb{R}} \left|\big(d\phi|_{\xi_s} - d\phi|_{\tilde\xi_s}\big)\eta_s\right|_2^2 ds + \int_{\mathbb{R}} \left|\tfrac{d}{ds}\big[\big(d\phi|_{\xi_s} - d\phi|_{\tilde\xi_s}\big)\eta_s\big]\right|_1^2 ds\\
&\le \int_{\mathbb{R}} \left|\big(d\phi|_{\xi_s} - d\phi|_{\tilde\xi_s}\big)\eta_s\right|_2^2 ds\\
&\quad + \int_{\mathbb{R}} \left|\big(d^2\phi|_{\xi_s}\dot\xi_s - d^2\phi|_{\tilde\xi_s}\dot\xi_s + d^2\phi|_{\tilde\xi_s}\dot\xi_s - d^2\phi|_{\tilde\xi_s}\dot{\tilde\xi}_s\big)\eta_s\right|_1^2 ds\\
&\quad + \int_{\mathbb{R}} |\big(d\phi|_{\xi_s} - d\phi|_{\tilde\xi_s}\big)\dot\eta_s|_1^2 ds\\
&\le \int_{\mathbb{R}} \left|\big(d\phi|_{\xi_s} - d\phi|_{\tilde\xi_s}\big)\eta_s\right|_2^2 ds\\
&\quad + \int_{\mathbb{R}} 2\left|\big(d^2\phi|_{\xi_s}\dot\xi_s - d^2\phi|_{\tilde\xi_s}\dot\xi_s\big)\eta_s\right|_1^2 ds\\
&\quad + \int_{\mathbb{R}} 2\left|d^2\phi|_{\tilde\xi_s}\big(\dot\xi_s - \dot{\tilde\xi}_s, \eta_s\big)\right|_1^2 ds\\
&\quad + \int_{\mathbb{R}} \left|\big(d\phi|_{\xi_s} - d\phi|_{\tilde\xi_s}\big)\dot\eta_s\right|_1^2 ds.
\end{aligned} \tag{3.21}$$

Term 1. By Lemma 2.9 for every $s \in \mathbb{R}$ there exists an H_1-open neighborhood W_s of ξ_s in H_1 and a constant $\kappa_s > 0$ such that for all $y_0, y_1 \in W_s$ estimate (2.7)

holds with κ replaced by κ_s. Since the image of ξ in H_1 is compact we can find a uniform constant κ independent of s such that estimate (2.7) holds.
If $\tilde\xi$ lies in a sufficiently small $|||\cdot|||_{1,2}$-neighborhood around ξ, we can assume that $\tilde\xi_s \in W_s$ for every $s \in \mathbb{R}$. We use this in inequality 1 in the estimate

$$
\begin{aligned}
&\int_{\mathbb{R}} \left|\left(d\phi|_{\xi_s} - d\phi|_{\tilde\xi_s}\right)\eta_s\right|_2^2 ds \\
&\overset{1}{\le} \kappa^2 \int_{\mathbb{R}} \Big(|\xi_s-\tilde\xi_s|_1|\eta_s|_2 + |\xi_s-\tilde\xi_s|_2|\eta_s|_1 + \tfrac12 \underbrace{\left(|\xi_s|_2+|\tilde\xi_s|_2\right)}_{\le 2|\xi_s|_2+|\xi_s-\tilde\xi_s|_2} |\xi_s-\tilde\xi_s|_1|\eta_s|_1\Big)^2 ds \\
&\overset{2}{\le} 4\kappa^2 \int_{\mathbb{R}} \Big(|\xi_s-\tilde\xi_s|_1^2|\eta_s|_2^2 + |\xi_s-\tilde\xi_s|_2^2|\eta_s|_1^2 + \left(|\xi_s|_2^2+|\xi_s-\tilde\xi_s|_2^2\right) |\xi_s-\tilde\xi_s|_1^2|\eta_s|_1^2\Big) ds \\
&\overset{3}{\le} 4\kappa^2 \left(\|\xi-\tilde\xi\|_\infty^2 \|\eta\|_{L^2_{H_2}}^2 + \|\eta\|_\infty^2 \|\xi-\tilde\xi\|_{L^2_{H_2}}^2 \right) \\
&\quad + 4\kappa^2 \left(\|\xi\|_{L^2_{H_2}}^2 + \|\xi-\tilde\xi\|_{L^2_{H_2}}^2 \right) \|\xi-\tilde\xi\|_\infty^2 \|\eta\|_\infty^2 \\
&\le 4\kappa^2 \, ||| \, \xi-\tilde\xi \, |||_{1,2}^2 \left(2 + |||\xi \, |||_{1,2}^2 + ||| \, \xi-\tilde\xi |||_{1,2}^2 \right) ||| \, \eta \, |||_{1,2}^2 \, .
\end{aligned}
\tag{3.22}
$$

In step 2 we used the inequality $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$. In step 3 we pulled out L^∞ norms which in step 4 we estimated by the $|||\cdot|||_{1,2}$ norms (3.9) with constant 1.
Term 2. Since tame maps are C^2 on level one, the second differential $d^2\phi$ is continuous. Hence, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$
\|\tilde\xi - \xi\|_{1,2} \le \delta \quad \Rightarrow \quad \|d^2\phi|_{\xi_s} - d^2\phi|_{\tilde\xi_s}\|_{\mathcal{L}(H_1,H_1;H_1)} \le \varepsilon.
$$

In particular, for every $\tilde\xi$ in the $W^{1,2}$ δ-ball around ξ it holds that

$$
\begin{aligned}
\int_{\mathbb{R}} 2\left|\left(d^2\phi|_{\xi_s}\dot\xi_s - d^2\phi|_{\tilde\xi_s}\dot\xi_s\right)\eta_s\right|_1^2 ds &\le 2\varepsilon^2 \left\| |\dot\xi|_1 \cdot |\eta|_1 \right\|_{L^2_{H_1}}^2 \\
&\le 2\varepsilon^2 \, ||| \, \xi \, |||_{1,2}^2 \, |||\eta \, |||_{1,2}^2 \, .
\end{aligned}
$$

Term 3. By continuity of $d^2\phi$ there exists a constant $c > 0$, only depending on ξ but not $\tilde\xi$, such that

$$
\|\tilde\xi - \xi\|_{1,2} \le \delta \quad \Rightarrow \quad \|d^2\phi|_{\tilde\xi_s}\|_{\mathcal{L}(H_1,H_1;H_1)} \le c.
$$

Hence we estimate term 3, similarly as term 2, by

$$
\begin{aligned}
\int_{\mathbb{R}} 2|d^2\phi|_{\tilde\xi_s}\left(\dot\xi_s - \dot{\tilde\xi}_s, \eta_s\right)|_1^2 ds &\le 2c^2 \left\| |\dot\xi - \dot{\tilde\xi}|_1 \cdot |\eta|_1 \right\|_{L^2_{H_1}}^2 \\
&\le 2c^2 \, ||| \, \dot\xi - \dot{\tilde\xi} \, |||_{1,2}^2 \cdot ||| \, \eta \, |||_{1,2}^2 \, .
\end{aligned}
$$

Term 4. Given $\varepsilon > 0$, by continuity of the map $d\phi$, there exists $\delta > 0$ with

$$
\|\tilde\xi - \xi\|_{1,2} \le \delta \quad \Rightarrow \quad \|d\phi|_{\xi_s} - d\phi|_{\tilde\xi_s}\|_{\mathcal{L}(H_1)} \le \varepsilon.
$$

In particular, for every $\tilde{\xi}$ in the $W^{1,2}$ δ-ball around ξ it holds that

$$\int_{\mathbb{R}} |\big(d\phi|_{\xi_s} - d\phi|_{\tilde{\xi}_s}\big)\dot{\eta}_s|_1^2 ds \le \varepsilon^2 \int_{\mathbb{R}} |\dot{\eta}_s|_1^2\, ds \le \varepsilon^2 \,|||\, \eta \,|||_{1,2}^2 .$$

Conclusion. The term by term analysis above shows that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\begin{aligned}
&\lim_{|||\tilde{\xi}-\xi|||_{1,2}\to 0} \|d\Phi|_\xi - d\Phi|_{\tilde{\xi}}\|^2_{\mathcal{L}(W^{1,2}_{H_1}\cap L^2_{H_2})}\\
&= \lim_{|||\tilde{\xi}-\xi|||_{1,2}\to 0} \sup_{|||\eta|||_{1,2}\le 1} |||d\Phi|_\xi \eta - d\Phi|_{\tilde{\xi}}\eta|||^2_{1,2} \qquad (3.23)\\
&= 0.
\end{aligned}$$

This proves Step 4. ☐

The proof of Theorem 3.1 is complete. ☐

3.1.2 Weak tangent map $T\Phi$

Theorem 3.2. *Let $U_1 \subset H_1$ be open and $\phi\colon U_1 \to H_1$ be tame. Assume in addition that $0 \in U_1$ and $\phi(0) = 0$. With $W^{1,2}_{H_1}$ and $L^2_{H_i}$ as in (1.1) and $\mathcal{U}$ as in (3.10) the weak tangent map*

$$\begin{aligned}
T\Phi\colon \mathcal{U} \times L^2_{H_1} &\to (W^{1,2}_{H_1} \cap L^2_{H_2}) \times L^2_{H_1}\\
(\xi, \eta) &\mapsto (\phi \circ \xi, d\phi|_\xi \eta) = (\Phi(\xi), d\Phi|_\xi \eta)
\end{aligned}$$

is well defined and continuously differentiable.

Proof. The first component map $\xi \mapsto \phi \circ \xi$ is of class C^1 as we proved in Theorem 3.1. We use the abbreviations from (3.1). The proof is in four steps.

Step 1 (Well defined).

Proof. In view of Theorem 3.1 it suffices to show that the map $s \mapsto d\phi|_{\xi_s}\eta_s$ is in $L^2_{H_1}$. Since ξ is in $W^{1,2}_{H_1}$, it is continuous (with embedding constant 1) and converges asymptotically to zero; see Section A.6. Therefore $\|d\phi|_{\xi_s}\|_{\mathcal{L}(H_1)}$ is uniformly bounded by a constant $c_1(\xi)$ depending on ξ. Hence the L^2 norm of $d\phi|_{\xi_s}\eta_s$ can be estimated above by this constant times the $L^2_{H_1}$ norm of η_s. ☐

Step 2 (Candidate). A natural candidate for the derivative of $T\Phi$ at (ξ, η) is

$$\begin{aligned}
d(T\Phi)|_{(\xi,\eta)}\colon \Big(W^{1,2}_{H_1} \cap L^2_{H_2}\Big) \times L^2_{H_1} &\to \Big(W^{1,2}_{H_1} \cap L^2_{H_2}\Big) \times L^2_{H_1}\\
\Big(d(T\Phi)|_{(\xi,\eta)}(\hat{\xi},\hat{\eta})\Big)(s) &= \Big(d\Phi|_\xi \hat{\xi}, d\,(d\Phi|_\xi \eta)\,(\hat{\xi},\hat{\eta})\Big)(s) \qquad (3.24)\\
&:= \Big(d\phi|_{\xi_s}\hat{\xi}_s, d^2\phi|_{\xi_s}(\hat{\xi}_s, \eta_s) + d\phi|_{\xi_s}\hat{\eta}_s\Big)
\end{aligned}$$

for all $s \in \mathbb{R}$ and $(\xi, \eta) \in \mathcal{U} \times L^2_{H_1}$ and where $d(d\Phi|_\xi \eta)$ means $d\,[(\xi, \eta) \mapsto d\Phi|_\xi \eta]$. The map $d(T\Phi)|_{(\xi,\eta)}$ is in $\mathcal{L}((W^{1,2}_{H_1} \cap L^2_{H_2}) \times L^2_{H_1})$.

Proof. Component one is the subject of Theorem 3.1. Concerning component two, first we show that $s \mapsto d^2\phi|_{\xi_s}(\hat{\xi}_s, \eta_s)$ is in $L^2_{H_1}$. As ξ is in $W^{1,2}_{H_1}$, it is continuous and converges asymptotically to zero. Therefore, since ϕ is C^2, there is a constant $c_2(\xi)$ such that $\|d^2\phi|_{\xi_s}\|_{\mathcal{L}(H_1,H_1;H_1)} \le c_2(\xi)$ for any $s \in \mathbb{R}$. We get

$$\begin{aligned}\|d^2\phi|_\xi(\hat{\xi}, \eta)\|_{L^2_{H_1}} &\le c_2(\xi)\|\eta\|_{L^2(\mathbb{R},\mathbb{R}^n)}\|\hat{\xi}\|_{C^0(\mathbb{R},\mathbb{R}^n)} \\ &\le c_2(\xi)\|\eta\|_{L^2_{H_1}}\|\hat{\xi}\|_{W^{1,2}(\mathbb{R},\mathbb{R}^n)}.\end{aligned}$$

As explained in Step 1, we further have that $\|d\phi|_\xi\hat{\eta}\|_{L^2_{H_1}} \le c_1(\xi)\|\hat{\eta}\|_{L^2_{H_1}}$. These estimates together with Step 2 from Theorem 3.1 show that $d(T\Phi)|_{(\xi,\eta)}$ is a bounded linear map on $W^{1,2}_{H_1} \cap L^2_{H_2}$. This proves Step 2. ∎

Remark 3.3. To show that $T\Phi$, as a map

$$T\Phi\colon \underbrace{\mathcal{U} \times \mathcal{W}}_{T_{\mathcal{U}}\mathcal{W}} \to \underbrace{\mathcal{W} \times \mathcal{W}}_{T\mathcal{W}}, \qquad \mathcal{U} \subset \mathcal{W} := (W^{1,2}_{H_1} \cap L^2_{H_2}),$$

is C^1 would fail right away in Step 2: It would require that both maps $s \mapsto d^2\phi|_{\xi_s}(\hat{\xi}_s, \eta_s)$ and $s \mapsto \frac{d}{ds}d^2\phi|_{\xi_s}(\hat{\xi}_s, \eta_s)$ are in $L^2_{H_1}$. But the second one would require three continuous derivatives of ϕ.

Step 3 (Candidate from Step 2 is derivative of $T\Phi$).

Proof. Let $\xi, \hat{\xi} \in W^{1,2}_{H_1}$ and $\eta, \hat{\eta} \in L^2_{H_1}$. Recall that the norm $|||\cdot|||_{1,2}$ on $W^{1,2}_{H_1} \cap L^2_{H_2}$ is given by (3.9). To see that our candidate actually is the derivative we show that the limit

$$\lim_{h\to 0} \sup_{|||\hat{\xi}|||^2_{1,2} + \|\hat{\eta}\|^2_{L^2_{H_1}} \le 1} \frac{|||T\Phi(\xi + h\hat{\xi}, \eta + h\hat{\eta}) - T\Phi(\xi, \eta) - h\, dT\Phi(\xi, \eta)(\hat{\xi}, \hat{\eta})|||^2_{1,2}}{h^2}$$

exists and vanishes. The map $T\Phi = (\Phi, d\Phi)$ has two components. Step 2 in Theorem 3.1 shows for the first component that the limit exists and vanishes. Hence it suffices to show this for the second component, namely

$$\begin{aligned}&\lim_{h\to 0} \sup_{|||\hat{\xi}|||^2_{1,2} + \|\hat{\eta}\|^2_{L^2_{H_1}} \le 1} \frac{\|d\phi|_{\xi + h\hat{\xi}}(\eta + h\hat{\eta}) - d\phi|_\xi\eta - h\, d^2\phi|_\xi(\hat{\xi}, \eta) - h\, d\phi|_\xi\hat{\eta}\|^2_{L^2_{H_1}}}{h^2} \\ &\le \lim_{h\to 0} \sup_{|||\hat{\xi}|||^2_{1,2} + \|\hat{\eta}\|^2_{L^2_{H_1}} \le 1} \frac{\|d\phi|_{\xi + h\hat{\xi}}\eta - d\phi|_\xi\eta - h\, d^2\phi|_\xi(\hat{\xi}, \eta)\|^2_{L^2_{H_1}}}{h^2} \\ &\quad + \lim_{h\to 0} \sup_{|||\hat{\xi}|||^2_{1,2} + \|\hat{\eta}\|^2_{L^2_{H_1}} \le 1} \frac{h^2\|d\phi|_{\xi + h\hat{\xi}}\hat{\eta} - d\phi|_\xi\hat{\eta}\|^2_{L^2_{H_1}}}{h^2}\end{aligned}$$

We treat each of the two summands separately.
Summand 1. To see that summand 1 vanishes note the following. Since ϕ is tame it is C^2. Hence for any given $\varepsilon > 0$ there exists $h_0 > 0$ such that

$$h \in (0, h_0) \quad \Rightarrow \quad \|d^2\phi|_{\xi_s + th\hat{\xi}_s} - d^2\phi|_{\xi_s}\|_{\mathcal{L}(H_1,H_1;H_1)} \le \varepsilon.$$

whenever $|||\hat{\xi}|||_{1,2} \le 1$, $s \in \mathbb{R}$, and $t \in [0,1]$. Pick $h \in (0, h_0)$ and use the fundamental theorem of calculus to estimate

$$\begin{aligned}
&\sup_{|||\hat{\xi}|||_{1,2}^2 + \|\hat{\eta}\|_{L^2_{H_1}}^2 \le 1} \frac{1}{h^2} \int_{\mathbb{R}} \left| \int_0^1 \Bigl(\underbrace{\tfrac{d}{dt} d\phi|_{\xi_s + th\hat{\xi}_s} \eta_s}_{h d^2\phi|_{\xi_s+th\hat{\xi}_s}(\hat{\xi}_s, \eta_s)} - h\, d^2\phi|_{\xi_s}(\hat{\xi}_s, \eta_s) \Bigr) dt \right|_{H_1}^2 ds \\
&\le \sup_{|||\hat{\xi}|||_{1,2}^2 + \|\hat{\eta}\|_{L^2_{H_1}}^2 \le 1} \underbrace{\int_{\mathbb{R}} \int_0^1 \left| d^2\phi|_{\xi_s + th\hat{\xi}_s}(\hat{\xi}_s, \eta_s) - d^2\phi|_{\xi_s}(\hat{\xi}_s, \eta_s) \right|_{H_1}^2 dt\, ds}_{\le \varepsilon^2 |||\hat{\xi}|||_{1,2}^2 \|\eta\|_{L^2_{H_1}}^2} \\
&\le \varepsilon^2 \|\eta\|_{L^2_{H_1}}^2 .
\end{aligned}$$

Since ε was arbitrary the limit, as $h \to 0$, vanishes.
Summand 2. To see that summand 2 vanishes note that since $\phi \in C^1$ for any given $\varepsilon > 0$ there exists $h_0 > 0$ such that $\|d\phi|_{\xi_s + h\hat{\xi}_s} - d\phi|_{\xi_s}\|_{\mathcal{L}(H_1)} \le \varepsilon$ whenever $h \in (0, h_0)$, $|||\hat{\xi}|||_{1,2} \le 1$, and $s \in \mathbb{R}$. Thus

$$\sup_{|||\hat{\xi}|||_{1,2}^2 + \|\hat{\eta}\|_{L^2_{H_1}}^2 \le 1} \|d\phi|_{\xi + h\hat{\xi}} \hat{\eta} - d\phi|_{\xi} \hat{\eta}\|_{L^2_{H_1}}^2 \le \sup_{\|\hat{\eta}\|_{L^2_{H_1}} \le 1} \varepsilon^2 \|\hat{\eta}\|_{L^2_{H_1}}^2 \le \varepsilon^2.$$

Since ε was arbitrary the limit, as $h \to 0$, vanishes. This proves Step 3. □

Step 4 (Differential is continuous). The differential

$$\begin{gathered}
d(T\Phi) \colon \mathcal{U} \times L^2_{H_1} \to \mathcal{L}((W^{1,2}_{H_1} \cap L^2_{H_2}) \times L^2_{H_1}) \\
(\xi, \eta) \mapsto d(T\Phi)|_{(\xi,\eta)}
\end{gathered}$$

is continuous.

Proof. The map $d(T\Phi)$, see (3.24), has two components. Continuity of the first component is shown in Step 4 in the proof of Theorem 3.1. Hence it suffices to show continuity of the second component, namely

Pick $(\xi, \eta) \in \mathcal{U} \times L^2_{H_1}$. Let (v, w) and $(\hat{\xi}, \hat{\eta})$ be elements of $(W^{1,2}_{H_1} \cap L^2_{H_2}) \times L^2_{H_1}$. Since $\phi \in C^2$ the following holds. Given $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$, such that $|||v|||_{1,2} < \delta_\varepsilon$ implies

$$\|d\phi|_{\xi_s + v_s} - d\phi|_{\xi_s}\|_{\mathcal{L}(H_1)}, \|d^2\phi|_{\xi_s + v_s} - d^2\phi|_{\xi_s}\|_{\mathcal{L}(H_1,H_1;H_1)} < \varepsilon$$

whenever $s \in \mathbb{R}$. Moreover, maybe after shrinking $\delta > 0$, we can assume that there is a constant c_ξ such that $\|d^2\phi|_{\xi_s + v_s}\|_{\mathcal{L}(H_1,H_1;H_1)} \le c_\xi$. We assume that

$$|||v|||_{1,2} \le \delta_\varepsilon, \qquad \|w\|_{L^2_{H_1}} < \varepsilon.$$

We estimate

$$
\begin{aligned}
&\|d^2\phi|_{\xi+v}(\hat{\xi},\eta+w)+d\phi|_{\xi+v}\hat{\eta}-d^2\phi|_{\xi}(\hat{\xi},\eta)-d\phi|_{\xi}\hat{\eta}\|^2_{L^2_{H_1}}\\
&=\int_{\mathbb{R}}\left|d^2\phi|_{\xi_s+v_s}(\hat{\xi}_s,\eta_s+w_s)+d\phi|_{\xi_s+v_s}\hat{\eta}_s-d^2\phi|_{\xi_s}(\hat{\xi}_s,\eta_s)-d\phi|_{\xi_s}\hat{\eta}_s\right|^2_{H_1}ds\\
&\le 2\int_{\mathbb{R}}\left|d^2\phi|_{\xi_s+v_s}(\hat{\xi}_s,\eta_s)-d^2\phi|_{\xi_s}(\hat{\xi}_s,\eta_s)\right|^2_{H_1}ds\\
&\quad+2\int_{\mathbb{R}}2\left|d\phi|_{\xi_s+v_s}\hat{\eta}_s-d\phi|_{\xi_s}\hat{\eta}_s\right|^2_{H_1}+2\left|d^2\phi|_{\xi_s+v_s}(\hat{\xi}_s,w_s)\right|^2_{H_1}ds\\
&\le 2\varepsilon^2\,|||\,\hat{\xi}\,|||^2_{1,2}\,\|\eta\|^2_{L^2_{H_1}}+4\varepsilon^2\|\hat{\eta}\|^2_{L^2_{H_1}}+4c_\xi^2\,|||\,\hat{\xi}\,|||^2_{1,2}\,\|w\|^2_{L^2_{H_1}}\\
&\le\varepsilon^2\left(2\|\eta\|^2_{L^2_{H_1}}+4c_\xi^2+4\right)\left(|||\hat{\xi}\,|||^2_{1,2}+\|\hat{\eta}\|^2_{L^2_{H_1}}\right).
\end{aligned}
$$

This proves continuity of the map $d(T\Phi)$. This proves Step 4. ☐

This concludes the proof of Theorem 3.2. ☐

3.2 Theorem B parametrized

Let O_1 be an open subset of $\mathbb{R}\times H_1$ and let O_2 be its part in $\mathbb{R}\times H_2$.

Theorem 3.4. *Let $\phi\colon O_1\to H_1$ be asymptotically constant parametrized tame. Let $\mathcal{U}:=\{\xi\in W^{1,2}_{H_1}\cap L^2_{H_2}\mid (s,\xi_s)\in O_1,\ \forall s\in\mathbb{R}\}$. Then the weak tangent map*

$$
\begin{aligned}
T\Phi\colon\mathcal{U}\times L^2_{H_1}&\to(W^{1,2}_{H_1}\cap L^2_{H_2})\times L^2_{H_1}\\
(\xi,\eta)&\mapsto(\Phi(\xi),d\Phi|_\xi\eta)
\end{aligned}
\tag{3.25}
$$

defined by

$$
\Phi(\xi)(s):=\phi_s(\xi_s),\qquad\left(d\Phi|_\xi\eta\right)(s):=d\phi_s|_{\xi_s}\eta_s,
$$

is well defined and continuously differentiable where $\phi_s(x):=\phi(s,x)$.

3.2.1 Base component Φ

Theorem 3.5. *The base component $\Phi\colon\mathcal{U}\to W^{1,2}_{H_1}\cap L^2_{H_2}$, $\Phi(\xi)(s):=\phi_s(\xi_s)$, of the map $T\Phi$ in Theorem 3.4 is well defined and continuously differentiable.*

Proof. The proof follows the same scheme as the proof of Theorem 3.1, just replace ϕ by ϕ_s. Due to the time dependency of our maps ϕ_s, some additional terms arise which we explain how to treat.
We discuss the four steps of the proof of Theorem 3.1. We use the abbreviations (3.11).

Step 1 (Well defined).

Proof. The proof consists of two sub-steps A and B. Namely, one has to show A) that $\Phi(\xi)$ lies in $L^2_{H_2}\subset L^2_{H_1}$ and B) that its derivative $\frac{d}{ds}\Phi(\xi)$ lies in $L^2_{H_1}$.
Step A. This is precisely the same argument as in Theorem 3.1 by using that ϕ

is parametrized tame and therefore asymptotically constant, in particular zero is mapped to zero asymptotically.
Step B. Here in $\frac{d}{ds}(\Phi(\xi))_s = d\phi_s|_{\xi_s}\frac{d}{ds}\xi_s + \underline{\dot\phi_s(\xi_s)}$ a new term arises. But $\dot\phi_s \equiv 0$ whenever $|s| > T$, by parametrized tameness. Since ξ is in $W^{1,2}_{H_1}$ it is continuous, and therefore the composed map $s \mapsto \dot\phi_s(\xi_s)$ is continuous as well. Since this map vanishes for $|s| \geq T$ it has compact image. Therefore there exists a constant $c > 0$, independent of s, such that $|\dot\phi_s(\xi_s)|_1 \leq c$. Hence we estimate $\int_{-\infty}^{\infty} |\dot\phi_s(\xi_s)|_1^2\, ds = \int_{-T}^{T} |\dot\phi_s(\xi_s)|_1^2\, ds \leq 2Tc^2$. The first term in the sum is estimated precisely as in the unparametrized case Theorem 3.1.

This finishes the proof of Step B and Step 1. □

Step 2 (Candidate). Given $\xi \in \mathcal{U}$, a natural candidate for the derivative is

$$\left(d\Phi|_\xi \hat\xi\right)(s) = d\phi_s|_{\xi_s}\hat\xi_s$$

whenever $s \in \mathbb{R}$ and $\hat\xi \in W^{1,2}_{H_1} \cap L^2_{H_2}$. The map $d\Phi|_\xi$ lies in $\mathcal{L}(W^{1,2}_{H_1} \cap L^2_{H_2})$.

Proof. The proof consists of two sub-steps A and B. Step A. This is precisely the same argument as in Theorem 3.1.
Step B. In (3.15) now arises a third term, namely $\int_{\mathbb{R}} \left|d\dot\phi_s|_{\xi_s}\hat\xi_s\right|_1^2 ds$. Since ϕ_s is asymptotically constant we have $d\dot\phi_s \equiv 0$ for $|s| \geq T$. Since $\xi\colon \mathbb{R} \to H_1$ is continuous there exists a constant $c > 0$ such that $\|d\dot\phi_s|_{\xi_s}\|_{\mathcal{L}(H_1)} \leq c$. Therefore $\int_{\mathbb{R}} \left|d\dot\phi_s|_{\xi_s}\hat\xi_s\right|_1^2 ds \leq c^2 \int_{\mathbb{R}} |\hat\xi_s|_1^2\, ds \leq c^2 \|\hat\xi\|^2_{W^{1,2}_{H_1}\cap L^2_{H_2}}$. Thus in the parametrized case the candidate for the derivative $d\Phi|_\xi$ still lies in $\mathcal{L}(W^{1,2}_{H_1} \cap L^2_{H_2})$. This proves Step 2. □

Step 3 (Candidate from Step 2 is derivative of Φ).

Proof. The norm $|||\cdot|||_{1,2}$ on $W^{1,2}_{H_1} \cap L^2_{H_2}$ is given by (3.9).

As in (3.18) there are two summands. In summand 1 there are no s-derivatives. In summand 2 we get the following additional term in $G(s)$, namely

$$\begin{aligned} G_2(s) :&= \dot\phi_s(\xi_s + h\eta_s) - \dot\phi_s(\xi_s) - h\, d\dot\phi_s|_{\xi_s}\eta_s \\ &= \int_0^1 \tfrac{d}{ds}\dot\phi_s(\xi_s + th\eta_s)\, dt - h\, d\dot\phi_s|_{\xi_s}\eta_s \\ &= h\int_0^1 \left(d\dot\phi_s|_{\xi_s + th\eta_s} - d\dot\phi_s|_{\xi_s}\right)\eta_s\, dt \end{aligned}$$

where we used the fundamental theorem of calculus. Square this identity and integrate to get

$$\begin{aligned} &\sup_{|||\eta|||_{1,2}\leq 1} \tfrac{1}{h^2}\int_{\mathbb{R}} |G_2(s)|_1^2 ds \\ &\leq \sup_{|||\eta|||_{1,2}\leq 1} \int_{\mathbb{R}} \left|\left(d\dot\phi_s|_{\xi_s+h\eta_s} - d\dot\phi_s|_{\xi_s}\right)\eta_s\right|_1^2 ds. \end{aligned}$$

As in (3.20), for every $\varepsilon > 0$ there exists $h_0 = h_0(\varepsilon) > 0$ such that

$$\|d\dot{\phi}_s|_{\xi_s + th\eta_s} - d\dot{\phi}_s|_{\xi_s}\|_{\mathcal{L}(H_1)} < \varepsilon \tag{3.26}$$

whenever $h \in [0, h_0]$. Therefore, if $h \in [0, h_0]$, we estimate

$$\begin{aligned}
&\sup_{\||\eta\||_{1,2}\le 1} \int_{\mathbb{R}} \left| \left(d\dot{\phi}_s|_{\xi_s + h\eta_s} - d\dot{\phi}_s|_{\xi_s} \right) \eta_s \right|_1^2 ds \\
&\le \sup_{\||\eta\||_{1,2}\le 1} \int_{\mathbb{R}} \|d\dot{\phi}_s|_{\xi_s + h\eta_s} - d\dot{\phi}_s|_{\xi_s}\|_{\mathcal{L}(H_1)}^2 |\eta_s|_1^2 ds \\
&\le \varepsilon^2 \sup_{\||\eta\||_{1,2}\le 1} \int_{\mathbb{R}} |\eta_s|_1^2 ds \\
&\le \varepsilon^2.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the limit of the extra term, as $h \to 0$, is zero as well. This proves Step 3. □

Step 4 (Differential is continuous). The differential

$$d\Phi \colon \mathcal{U} \to \mathcal{L}(W^{1,2}_{H_1} \cap L^2_{H_2}), \quad \xi \mapsto d\Phi|_\xi$$

is continuous.

Proof. In (3.21) there arises the new term $\int_{\mathbb{R}} |\big(d\dot{\phi}_s|_{\xi_s} - d\dot{\phi}_s|_{\tilde{\xi}_s}\big)\eta_s|_1^2 ds$ which we call term 5. To estimate term 5 note that by the continuity of the map $d\dot{\phi}$, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\tilde{\xi} - \xi\|_{1,2} \le \delta \quad \Rightarrow \quad \|d\dot{\phi}_s|_{\xi_s} - d\dot{\phi}_s|_{\tilde{\xi}_s}\|_{\mathcal{L}(H_1)} \le \varepsilon.$$

In particular, for every $\tilde{\xi}$ in the $W^{1,2}$ δ-ball around ξ it holds that

$$\int_{\mathbb{R}} |\big(d\dot{\phi}_s|_{\xi_s} - d\dot{\phi}_s|_{\tilde{\xi}_s}\big)\eta_s|_1^2 ds \le \varepsilon^2 \int_{\mathbb{R}} |\eta_s|_1^2\, ds \le \varepsilon^2 \, \||\, \eta \,\||_{1,2}^2 \, .$$

So the conclusion (3.23), hence Step 4, also holds in the parametrized case. □

This concludes the proof of Theorem 3.5. □

3.2.2 Weak tangent map $T\Phi$

Proof of Theorem 3.4. The first component map $\xi \mapsto \phi \circ \xi$ is of class C^1 by Theorem 3.5. The second component is a map $\mathcal{U} \times L^2_{H_1} \to L^2_{H_1}$, $(\xi, \eta) \mapsto d\Phi|_\xi \eta$. Note that the target space $L^2_{H_1}$ does not involve any s-derivative and takes values in level one H_1, and not in the analytically tricky level two H_2.

Line-by-line inspection of the proof of Theorem 3.2 shows that no new terms arise. Thus the proof of Theorem 3.2 also proves Theorem 3.4. □

4 Coordinate charts

Let $H_2 \subset H_1$ be separable Hilbert spaces with a compact dense inclusion of H_2 in H_1. Assume that (X_1, X_2) is a tame (H_1, H_2)-two-level manifold; see Section 2.2. We denote the unique maximal tame two-level atlas by

$$\mathcal{A} = \{\psi \colon H_1 \supset U_1 \to V_1 \subset X_1\}$$

see Remark 2.11. Suppose further that x_-, x_+ are elements of X_2. The space of paths $\mathcal{P}_{x_-x_+}$ will be defined as a subset of the space of continuous paths

$$\mathcal{C}_{x_-x_+} := \left\{ x \in C^0(\mathbb{R}, X_1) \mid \lim_{s \to \mp\infty} x(s) = x_\mp \right\}. \tag{4.27}$$

In Section 4.1 we endow the space of paths $\mathcal{P}_{x_-x_+}$ with the structure of a C^1 Hilbert manifold modeled on the Hilbert space $W^{1,2}_{H_1} \cap L^2_{H_2}$ where we abbreviate

$$W^{1,2}_{H_1} := W^{1,2}(\mathbb{R}, H_1), \qquad L^2_{H_2} := L^2(\mathbb{R}, H_2).$$

In Subsection 4.1.1 we define local parametrizations for $\mathcal{P}_{x_-x_+}$. In Subsection 4.1.2 we provide an atlas for $\mathcal{P}_{x_-x_+}$. In Section 4.1.3 we show that the transition maps are C^1 diffeomorphisms.

In Section 4.2 we define the weak tangent bundle $\mathcal{E}_{x_-x_+} \to \mathcal{P}_{x_-x_+}$ and show that it is a C^1 Hilbert manifold.

4.1 Path spaces

4.1.1 Local parametrizations

Basic paths and basic coverings

Definition 4.1. Given $x_-, x_+ \in X_2$, consider a C^2 path $x \colon \mathbb{R} \to X_2$ such that

$$x(s) = \begin{cases} x_- & , s \le -T, \\ x_+ & , s \ge T, \end{cases}$$

for some $T > 0$. Such paths are called **basic paths**.

We will construct local charts for $\mathcal{P}_{x_-x_+}$ around basic paths.

Definition 4.2 (Basic covering)**.** Assume that $x \colon \mathbb{R} \to X_2$ from x_- to x_+ is a basic path.[1] A **basic covering of x** is an ordered finite collection

$$\{\psi_i \colon H_1 \supset U^i \to V^i \subset X_1\}_{i=1}^k \subset \mathcal{A}$$

of a finite number k of local parametrizations (homeomorphisms) $\psi_i \colon U^i \to V^i$ whose images cover $x(\mathbb{R})$ and such that the following is true.

[1] Note that the closure of the path image $x(\mathbb{R}) \subset X_2$ is compact and that a basic path x arrives at its endpoints $x_\mp$ already in finite time $\mp T$.

1) There are times $t_1 < t_2 < \cdots < t_{k-1}$ with the following properties, firstly,

$$\begin{aligned} (-\infty, t_1] &\subset x^{-1}(V^1), \\ [t_1, t_2] &\subset x^{-1}(V^2), \\ \vdots \quad &\vdots \; \vdots \\ [t_{k-2}, t_{k-1}] &\subset x^{-1}(V^{k-1}), \\ [t_{k-1}, \infty) &\subset x^{-1}(V^k), \end{aligned}$$

and, secondly, $x(t_j) \neq x(t_{j+1})$ for every $j = 1, \ldots, k-2$.

2) It holds that

$$d(\psi_{i+1}^{-1} \circ \psi_i)|_{\psi_i(x(t_i))} = \mathrm{Id}, \quad i = 1, \ldots, k-1.$$

Remark 4.3 (Case $k = 1$)**.** There is just one map $\psi_1 \colon U^1 \to V^1$ and, by 1), we have $-\infty =: t_0 < t_1 := \infty$ and $x^{-1}(V^1) \supset (-\infty, t_1] \cup [t_0, \infty) = \mathbb{R}$. Thus one coordinate patch V already covers the entire path x. Part 2) is void.

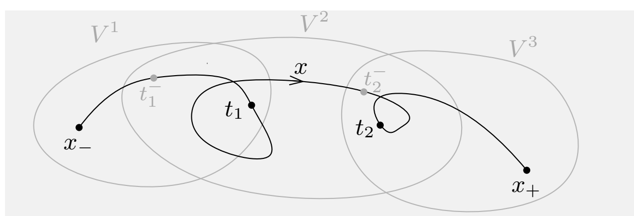

Figure 1: Basic covering of x by $k = 3$ local parametrizations $\psi_i \colon U^i \to V^i$

Lemma 4.4. *For each basic path x a basic covering exists.*

Proof. Choose a finite collection of local parametrizations $\{\psi_i \colon H_1 \supset U^i \to V^i \subset X_1\}_{i \in I} \subset \mathcal{A}$ which satisfies condition 1). If $k = 1$ we are done. Let $k \geq 2$.

To see that condition 2) can be achieved, too, we modify our charts inductively as follows. We replace U^2 by $\tilde{U}^2$ defined as

$$\tilde{U}^2 := d(\psi_2^{-1} \circ \psi_1)|_{\psi_1(x(t_1))}^{-1} U^1$$

and consider the modified chart $\tilde{\psi}_2 \colon \tilde{U}^2 \to V^2$ defined by

$$\tilde{\psi}_2 := \psi_2 \circ d(\psi_2^{-1} \circ \psi_1)|_{\psi_1(x(t_1))}.$$

Note that $\tilde{\psi}_2$ is indeed a chart, i.e. $\tilde{\psi}_2 \in \mathcal{A}$. To see this observe that by tameness the linear map $d(\psi_2^{-1} \circ \psi_1)|_{\psi_1(x(t_1))}$ is element of $\mathcal{L}(H_1) \cap \mathcal{L}(H_2)$. Since this map is linear its second derivative vanishes, hence condition (2.3) is void. In particular, the composed map $\tilde{\psi}_2$ is tame since by Theorem 2.5 tameness is preserved under composition. We compute

$$\begin{aligned} d(\tilde{\psi}_2^{-1} \circ \psi_1)|_{\psi_1(x(t_1))} &= d\left(d(\psi_2^{-1} \circ \psi_1)|^{-1}_{\psi_1(x(t_1))} \psi_2^{-1} \circ \psi_1 \right) |_{\psi_1(x(t_1))} \\ &= d(\psi_2^{-1} \circ \psi_1)|^{-1}_{\psi_1(x(t_1))} d(\psi_2^{-1} \circ \psi_1)|_{\psi_1(x(t_1))} \\ &= \mathrm{Id}. \end{aligned}$$

Then we modify U^3 accordingly and so on. □

Convexity setup on target manifold X_1

Given a basic path $x\colon \mathbb{R} \to X_2$ from x_- to x_+, let $\{\psi_i\colon H_1 \supset U^i \to V^i \subset X_1\}_{i=1}^k$ be a basic covering of x. In case $k \geq 2$ we construct for every overlap $V^j \cap V^{j+1} \subset X_1$ a subset which, under both coordinate charts ψ_j^{-1} and ψ_{j+1}^{-1}, has a convex image in H_1. Convexity will be needed for convex interpolation (4.38). In case $k = 1$ there is no overlap and the following is void.

We use the following abbreviations. For $j = 1, \dots, k-1$ we define sets

$$U_+^j := \psi_j^{-1}(V^j \cap V^{j+1}) \subset U^j, \qquad U_-^{j+1} := \psi_{j+1}^{-1}(V^j \cap V^{j+1}) \subset U^{j+1},$$

as illustrated by Figure 2.

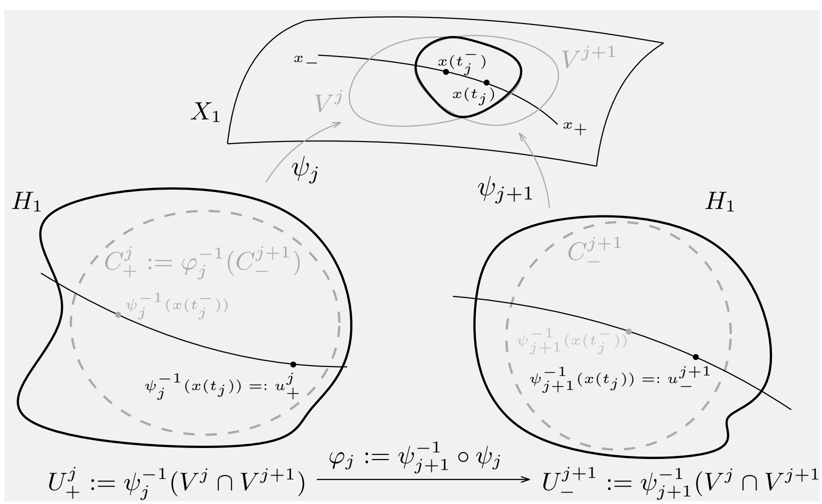

Figure 2: Manifold X_1 and convex parametrization domains $C_+^j, C_-^{j+1} \subset H_1$

We define **basic covering transition maps** (these are C^2 since X_1 is)

$$\varphi_j := \psi_{j+1}^{-1} \circ \psi_j|_{U_+^j} \colon H_1 \supset U_+^j \to U_-^{j+1} \subset H_1, \tag{4.28}$$

and points

$$u_+^j := \psi_j^{-1}(x(t_j)) \in U_+^j \cap H_2, \qquad u_-^{j+1} := \psi_{j+1}^{-1}(x(t_j)) \in U_-^{j+1} \cap H_2.$$

Note that basic paths lie in X_2, hence the intersection with H_2 above. Note that by definition of a basic covering we have that

$$d\varphi_j|_{u_+^j} = \mathrm{Id}. \tag{4.29}$$

Lemma 4.5. *For each $j = 1, \dots, k-1$ there exists an open* ball C_-^{j+1} *in $U_-^{j+1} \subset H_1$ centered at u_-^{j+1} satisfying the following conditions*

a) the pre-image $C_+^j := \varphi_j^{-1}(C_-^{j+1}) \subset U_+^j \subset H_1$ is convex;

b) at every point q of the ball C_-^{j+1} it holds the estimate

$$\|d(\varphi_j^{-1})|_q - \mathrm{Id}\|_{\mathcal{L}(H_1)} \le \tfrac{1}{2}; \tag{4.30}$$

c) for $j \ge 2$ the closures of the subsets $C_-^j, C_+^j \subset U^j \subset H_1$ are disjoint;

d) there exists a constant $c_j > 0$ such that for all $\eta \in H_2$ and $y \in C_-^{j+1} \cap H_2$ there holds the inequality

$$|d\varphi_j^{-1}|_y \eta - \eta|_2 \le \tfrac{1}{2}|\eta|_2 + c_j\left(|y|_2 + 1\right)|\eta|_1. \tag{4.31}$$

Proof. By choosing the ball C_-^{j+1} small enough, condition a) (and c)) can always be achieved since φ_j^{-1} is in C^2. Therefore its derivative is in particular locally Lipschitz and, moreover, since the derivative is invertible at every point, it is in particular injective. Thus the two conditions in [Pol01, Thm. 2.1] are satisfied and φ_j^{-1} maps a sufficiently small convex set to a convex set. b) Note that from (4.29) it follows that

$$d(\varphi_j^{-1})|_{u_-^{j+1}} = \mathrm{Id}. \tag{4.32}$$

Hence b) follows by continuity of $d(\varphi_j^{-1})$ possibly after shrinking the ball again. d) This follows from Lemma 2.9 as follows. Since $\varphi_j^{-1}\colon U_-^{j+1} \to U_+^j$ is tame and using (4.32) there exists an open neighborhood W of u_-^{j+1} and a constant $\kappa = \kappa(j) > 0$ such that for all $y \in W \cap H_2$ and $\eta \in H_2$ there holds the estimate

$$\begin{aligned} |d\varphi_j^{-1}|_y \eta - \eta|_2 &\le \kappa\left(|y - u_-^{j+1}|_1 |\eta|_2 + |y - u_-^{j+1}|_2 |\eta|_1\right) \\ &\quad + \tfrac{\kappa}{2}\left(|y|_2 + |u_-^{j+1}|_2\right)|y - u_-^{j+1}|_1 |\eta|_1. \end{aligned}$$

Maybe after shrinking the ball C_-^{j+1} centered at u_-^{j+1} again, we can assume that $C_-^{j+1} \subset W$. Denote by ε the radius of the ball C_-^{j+1}. Then for every $y \in C_-^{j+1} \cap H_2$, and by the triangle inequality, the above estimate simplifies to

$$\begin{aligned} |d\varphi_j^{-1}|_y \eta - \eta|_2 &\le \kappa\varepsilon|\eta|_2 + \kappa|y|_2|\eta|_1 + \kappa|u_-^{j+1}|_2|\eta|_1 + \varepsilon\tfrac{\kappa}{2}\left(|y|_2 + |u_-^{j+1}|_2\right)|\eta|_1 \\ &= \varepsilon\kappa|\eta|_2 + \left(\kappa + \varepsilon\tfrac{\kappa}{2}\right)|y|_2|\eta|_1 + \kappa|u_-^{j+1}|_2\left(1 + \tfrac{\varepsilon}{2}\right)|\eta|_1. \end{aligned}$$

Maybe after shrinking the ball C_-^{j+1} a last time, we can further assume that $\varepsilon\kappa < 1/2$. Under this assumption the assertion (4.31) follows from the above inequality for $c_j = \frac{5}{4}\kappa\max\{1, |u_-^{j+1}|_2\}$. □

Interpolation

Let $\{\psi_i\colon H_1 \supset U^i \to V^i \subset X_1\}_{i=1}^k$ be a basic covering of a basic path x from x_- to x_+. Our goal is to define, on a small neighborhood $\mathcal{U}$ of 0, an injection

$$\Psi\colon W^{1,2}_{H_1} \cap L^2_{H_2} \supset \mathcal{U} \to \mathcal{C}_{x_- x_+}, \qquad \Psi = \Psi^x, \mathcal{U} = \mathcal{U}^x, \tag{4.33}$$

which takes the zero map to the basic path x. We abbreviate $x_s := x(s)$.

Definition 4.6 (Domain $\mathcal{U}^x$)**.** Let $\{\psi_i\colon H_1 \supset U^i \to V^i \subset X_1\}_{i=1}^k$ be a basic covering of a path x from x_- to x_+. If $k = 1$ set $t_0 := -\infty$ and $t_1^- := \infty$. If $k \geq 2$ choose times t_j^- for $j = 1, \dots, k-1$ (see Figure 2) such that (i) it holds

$$\underbrace{-\infty}_{=:t_0} < -T < \overbrace{t_1^- < t_1}^{\text{interpolate}} < \overbrace{t_2^- < t_2}^{\text{interpolate}} < \dots < \overbrace{t_{k-1}^- < t_{k-1}}^{\text{interpolate}} < T < \underbrace{+\infty}_{=:t_k^-}$$

and such that (ii) the path x along any **interpolation interval** $\overline{[t_j^-, t_j]}$ is via the local coordinate chart ψ_j^{-1} taken into the convex set $C_+^j \subset H_1$, in symbols

$$s \in [t_j^-, t_j] \qquad \Rightarrow \qquad \psi_j^{-1}(x_s) \in C_+^j,$$

or equivalently (since by definition $C_+^j = \psi_j^{-1} \circ \psi_{j+1}(C_-^{j+1})$)

$$s \in [t_j^-, t_j] \qquad \Rightarrow \qquad \psi_{j+1}^{-1}(x_s) \in C_-^{j+1}.$$

(iii) Let $B_R(y)$ be the open radius-R ball in H_1 centered at y. Fix $R > 0$ with

$$s \in [t_j^-, t_j] \qquad \Rightarrow \qquad B_R(\psi_{j+1}^{-1}(x_s)) \subset C_-^{j+1}, \tag{4.34}$$

whenever $j \in \{1, \dots, k-1\}$ and

$$s \in [t_j, t_{j+1}^-] \qquad \Rightarrow \qquad B_R(\psi_{j+1}^{-1}(x_s)) \subset U^{j+1},$$

whenever $j \in \{0, \dots, k-1\}$. Here use the convention $x(\pm\infty) := x_{\pm\infty}$. Such R exists since the time intervals are compact, hence their images in H_1 under $\psi_{j+1}^{-1} \circ x$ are compact, now we use an elementary consideration in set theoretic topology.[2] (Here we profit from our choice to work with basic paths, thus x is constant outside the *compact* time interval $[-T, T]$.)

(iv) Using R from (iii) we define the domain of the local parametrization by

$$\boxed{\mathcal{U}^x = \mathcal{U}_R^x := \left\{\xi \in W^{1,2}_{H_1} \cap L^2_{H_2} \text{ such that } |\xi_s|_1 < R \;\forall s \in \mathbb{R}\right\}.} \tag{4.35}$$

By definition of R for any $\xi \in \mathcal{U}^x$ the following is true

[2] For a compact subset K of an open set U there exists $R > 0$ such that $B_R(x) \subset U$ for every $x \in K$. To see this pick $y \in U$, then by openness there exists $R_y > 0$ such that $B_{R_y}(y) \subset U$. Hence, by the triangle inequality, $z \in B_{R_y/2}(y) \Rightarrow B_{R_y/2}(z) \subset U$. By compactness the open cover $\{B_{R_y/2}(y) \mid y \in K\}$ of K admits a finite subcover, let R be the smallest of these radii.

1. for $s \in [t_j^-, t_j]$ we have $\psi_j^{-1}(x_s) + \xi_s \in C_+^j$ $\quad j = 1, \dots, k-1$
2. for $s \in [t_j^-, t_j]$ we have $\psi_{j+1}^{-1}(x_s) + \xi_s \in C_-^{j+1}$ $\quad j = 1, \dots, k-1$
3. for $s \in [t_j, t_{j+1}^-]$ we have $\psi_{j+1}^{-1}(x_s) + \xi_s \in U^{j+1}$ $\quad j = 0, \dots, k-1$

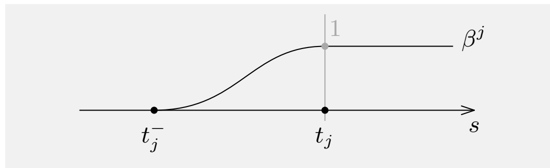

Figure 3: Cutoff function β^j along interpolation interval $[t_j^-, t_j]$

Remark 4.7 (Case $k = 1$)**.** There is just one map $\psi_1 \colon U^1 \to V^1$. Only 3. is non-void. It yields $\psi_1^{-1}(x_s) + \xi_s \in U^1$ $\forall s \in (-\infty, \infty)$.

Definition 4.8 (Local parametrization Ψ^x near basic path x)**.** In Definition 4.6 for each $j = 1, \dots, k-1$ pick a monotone smooth cutoff function $\beta^j \colon \mathbb{R} \to [0, 1]$ such that $\beta^j \equiv 0$ on $(-\infty, t_j^-]$ and $\beta^j \equiv 1$ on $[t_j, \infty)$; see Figure 3. For $\xi \in \mathcal{U}^x$ we define a **local parametrization**

$$\Psi \colon W^{1,2}_{H_1} \cap L^2_{H_2} \supset \mathcal{U} \to \mathcal{C}_{x_- x_+}, \qquad \Psi = \Psi^x, \mathcal{U} = \mathcal{U}^x, \tag{4.36}$$

centered at the basic path x from x_- to x_+ by

$$\boxed{(\Psi(\xi))(s) := \begin{cases} \chi_j(s; \xi) & , s \in [t_j^-, t_j],\ j = 1, \dots, k-1 \\ \psi_{j+1}\bigl(\psi_{j+1}^{-1}(x_s) + \xi_s\bigr) & , s \in [t_j, t_{j+1}^-],\ j = 0, \dots, k-1 \end{cases}} \tag{4.37}$$

for every $s \in \mathbb{R}$. Here the map $\chi_j(s; \xi)$, for $j = 1, \dots, k-1$, is defined by convex interpolation in the convex sets C_+^j and C_-^{j+1} and for $s \in \mathbb{R}$ as follows

$$\begin{aligned} &\chi_j(s; \xi) \\ &:= \psi_j\Biggl((1 - \beta_s^j)\Bigl(\underbrace{\psi_j^{-1}(x_s) + \xi_s}_{\in C_+^j}\Bigr) + \beta_s^j \underbrace{\overbrace{\psi_j^{-1} \circ \psi_{j+1}}^{\overset{(4.28)}{=} \varphi_j^{-1}}\Bigl(\overbrace{\psi_{j+1}^{-1}(x_s) + \xi_s}^{\in C_-^{j+1}}\Bigr)}_{\in \varphi_j^{-1}(C_-^{j+1}) \subset C_+^j}\Biggr) \\ &= \psi_j \circ \mathcal{S}_s^j(\xi_s) \end{aligned} \tag{4.38}$$

where we abbreviated the map (B.85) for $\varphi = \varphi_j^{-1} \in C^2$ and $x_0 = \psi_{j+1}^{-1}(x_s)$ by

$$\mathcal{S}_s^j(\xi_s) := \mathcal{S}^{\varphi_j^{-1}}_{\beta_s^j, \psi_{j+1}^{-1}(x_s)}(\xi_s). \tag{4.39}$$

By Remark B.4, cf. Figure 5, the **interpolation map**

$$\mathcal{S}_s^j \colon H_1 \supset B_R(0) =: B_R \to H_1 \tag{4.40}$$

is a C^2 diffeomorphism onto its image, in particular it is injective.

Remark 4.9 (Case $k=1$)**.** There is just one map $\psi_1 \colon U^1 \to V^1$ and

$$(\Psi(\xi))(s) = \psi_1 \underbrace{\left(\psi_1^{-1}(x_s) + \xi_s\right)}_{\in U^1} \in V^1, \quad \forall s \in (-\infty, \infty).$$

Proposition 4.10. *The map $\Psi \colon \mathcal{U} \to \mathcal{C}_{x_-x_+}$ is injective and $\Psi(0) = x$ reproduces the basic path x used in the construction of Ψ.*

Proof. We first check that Ψ is injective. Hence assume that there exist ξ and $\tilde{\xi}$ such that $\Psi(\xi) = \Psi(\tilde{\xi})$. In particular $\Psi(\xi)(s) = \Psi(\tilde{\xi})(s)$ for every $s \in \mathbb{R}$. We show that this implies $\xi(s) = \tilde{\xi}(s)$ for every $s \in \mathbb{R}$. The only times where this is not obvious is when s lies in an interpolation interval $[t_j^-, t_j]$ for $j = 1, \dots, k-1$. In this case by (4.38) there are the identities

$$\psi_j \circ \mathcal{S}_s^j(\xi_s) = \chi_j(s;\xi) = \Psi(\xi)(s) = \Psi(\tilde{\xi})(s) = \chi_j(s;\tilde{\xi}) = \psi_j \circ \mathcal{S}_s^j(\tilde{\xi}_s).$$

But ψ_j is injective, since it is a chart map, and the map $\mathcal{S}_s^j$ is a diffeomorphism, as explained in Appendix B.1. Hence $\xi_s = \tilde{\xi}_s$ for every $s \in \mathbb{R}$, that is $\xi = \tilde{\xi}$. This finishes the proof that Ψ is injective.

Now we check that $\Psi(0)$ is a basic path. We call $I = \cup_{j=1}^{k-1} [t_j^-, t_j]$ the **interpolation region**. If $s \in \mathbb{R} \setminus$ I, i.e. s is in the non-interpolation region, then $\Psi(0)(s) = x(s)$ is immediate from (4.37). If $s \in I$, i.e. there exists a $j \in \{1, \dots, k-1\}$ such that $s \in [t_j^-, t_j]$, then by definition of $\mathcal{S}_s^j$, see (4.38), we get

$$\mathcal{S}_s^j(0) = \left((1-\beta_s^j)\psi_j^{-1}(x_s) + \beta_s^j \psi_j^{-1} \circ \psi_{j+1} \circ \psi_{j+1}^{-1}(x_s) \right) = \psi_j^{-1}(x_s). \tag{4.41}$$

Therefore the interpolation map for $\xi_s = 0$ reproduces the basic path, namely

$$\chi_j(s;0) = \psi_j \circ \mathcal{S}_s^j(0) = x_s$$

for every $s \in \mathbb{R}$. This proves Proposition 4.10. $\square$

4.1.2 Definition of path space

Definition 4.11. Given $x_-, x_+ \in X_2$, we define the path space $\mathcal{P}_{x_-x_+}$ as the subset of $\mathcal{C}_{x_-x_+}$ consisting of the images of all local parametrizations Ψ constructed in (4.36) above. We denote the set of all local parametrizations centered at a basic path $x \colon \mathbb{R} \to X_2$ from x_- to x_+ by

$$\mathcal{AP}_{x_-x_+} := \{\text{local parametrizations } \Psi^x \colon W^{1,2}_{H_1} \cap L^2_{H_2} \supset \mathcal{U}^x \to \mathcal{C}_{x_-x_+}\}. \tag{4.42}$$

Then the **space of paths** in $X = X_1$ from x_- to x_+ is defined by

$$\mathcal{P}_{x_-x_+} := \bigcup_{\Psi^x \in \mathcal{AP}_{x_-x_+}} \Psi(\mathcal{U}), \qquad \Psi = \Psi^x, \mathcal{U} = \mathcal{U}^x.$$

Note that $\mathcal{P}_{x_-x_+} \subset \mathcal{C}_{x_-x_+}$. We define a **topology** on the set $\mathcal{P}_{x_-x_+}$ as follows. A subset $\mathcal{V}$ of $\mathcal{P}_{x_-x_+}$ is **open** if and only if $\Psi^{-1}(\mathcal{V} \cap \Psi(\mathcal{U}))$ is open in $\mathcal{U}$ for all $\Psi \in \mathcal{AP}_{x_-x_+}$.

Theorem 4.12 (C^1 atlas)**.** *The set of local parametrizations $\mathcal{AP}_{x_-x_+}$ is a C^1 atlas for the path space $\mathcal{P}_{x_-x_+}$, in particular $\mathcal{P}_{x_-x_+}$ is a C^1 Hilbert manifold modeled on the Hilbert space $W^{1,2}_{H_1} \cap L^2_{H_2}$ with inner product*

$$\langle\langle\langle \xi, \eta \rangle\rangle\rangle := \int_{-\infty}^{\infty} \left\langle \dot{\xi}(s), \dot{\eta}(s) \right\rangle_{H_1} ds + \int_{-\infty}^{\infty} \langle \xi(s), \eta(s) \rangle_{H_2} ds.$$

Proof. To prove Theorem 4.12 we need to show that the transition maps are C^1 diffeomorphisms. This will be carried out in the next section and Theorem 4.12 follows from Theorem 4.14. □

4.1.3 Transition maps

We show that transition maps are C^1 diffeomorphisms on the Hilbert space $W^{1,2}_{H_1} \cap L^2_{H_2}$. Assume that x and $\tilde{x}$ are basic paths from x_- to x_+. Let

$$T := \max\{T_x, T_{\tilde{x}}\} > 0$$

be the maximum of the two times that come with the basic paths x and $\tilde{x}$, respectively; see Definition 4.1.

Pick a basic covering of x, notation $\{\psi_i \colon H_1 \supset U^i \to V^i \subset X_1\}_{i=1}^k$, and a basic covering of $\tilde{x}$, notation $\{\tilde{\psi}_i \colon H_1 \supset \tilde{U}^i \to \tilde{V}^i \subset X_1\}_{i=1}^{\tilde{k}}$.

For the basic coverings of x and $\tilde{x}$ choose $t_1, \dots, t_{k-1}$, respectively $\tilde{t}_1, \dots, \tilde{t}_{\tilde{k}-1}$ according to 1) and 2) in Definition 4.2 for the common T. For $j = 1, \dots, k-1$, respectively $\tilde{j} = 1, \dots, \tilde{k}-1$, choose open neighborhoods C^j_+ of u^j_+ in U^j_+, respectively $\tilde{C}^{\tilde{j}}_+$ of $\tilde{u}^{\tilde{j}}_+$ in $\tilde{U}^{\tilde{j}}_+$, satisfying a) b) c) right after (4.29). In addition, choose disjoint interpolation intervals $[t_1^-, t_1], \dots, [t_{k-1}^-, t_{k-1}]$, respectively $[\tilde{t}_1^-, \tilde{t}_1], \dots, [\tilde{t}_{\tilde{k}-1}^-, \tilde{t}_{\tilde{k}-1}]$, satisfying (i) and (ii) in Definition 4.6. Now choose open subsets $\mathcal{U} = \mathcal{U}^x_R$, respectively $\tilde{\mathcal{U}} = \mathcal{U}^{\tilde{x}}_{\tilde{R}}$, of $W^{1,2}_{H_1} \cap L^2_{H_2}$ as in (4.35).

After choosing cutoff functions $\beta^1, \dots, \beta^{k-1}$, respectively $\tilde{\beta}^1, \dots, \tilde{\beta}^{\tilde{k}-1}$, for the interpolation intervals, see Figure 3, we define local parametrizations

$$\Psi = \Psi^x \colon \mathcal{U} \to \mathcal{P}_{x_-x_+}, \quad \tilde{\Psi} = \Psi^{\tilde{x}} \colon \tilde{\mathcal{U}} \to \mathcal{P}_{x_-x_+}, \quad 0 \in \mathcal{U}, \tilde{\mathcal{U}} \subset W^{1,2}_{H_1} \cap L^2_{H_2},$$

by formula (4.37) for these choices. In particular, definition (4.37) for $\Psi^{\tilde{x}}$ involves $\tilde{\chi}_{\tilde{j}} = \chi^{\tilde{x}}_{\tilde{j}}$ defined by (4.38) with $\sim$-quantities on the right hand side, e.g.

$$\tilde{\mathcal{S}}^{\tilde{j}}_s := \mathcal{S}^{\tilde{\varphi}_{\tilde{j}}^{-1}}_{\tilde{\beta}^{\tilde{j}}_s, \tilde{\psi}^{-1}_{\tilde{j}+1}(\tilde{x}_s)}, \quad \text{with } \mathcal{S} \text{ as in (B.85).} \tag{4.43}$$

Definition 4.13 (Path space transition map)**.** We abbreviate $\mathcal{V} := \Psi(\mathcal{U})$ and $\tilde{\mathcal{V}} := \tilde{\Psi}(\tilde{\mathcal{U}})$ and $\mathcal{U}_0 := \Psi^{-1}(\mathcal{V} \cap \tilde{\mathcal{V}})$ and $\tilde{\mathcal{U}}_0 := \tilde{\Psi}^{-1}(\mathcal{V} \cap \tilde{\mathcal{V}})$. As illustrated by Figure 4, we define the corresponding **path space transition map** by

$$\boxed{\Phi := \tilde{\Psi}^{-1} \circ \Psi|_{\mathcal{U}_0} : \mathcal{U}_0 \to \tilde{\mathcal{U}}_0,} \qquad \mathcal{U}_0, \tilde{\mathcal{U}}_0 \subset W^{1,2}_{H_1} \cap L^2_{H_2} . \tag{4.44}$$

In view of Proposition 4.10 the map $\Phi \colon \mathcal{U}_0 \to \tilde{\mathcal{U}}_0$ is a bijection with inverse

$$\boxed{\Phi^{-1} = \Psi^{-1} \circ \tilde{\Psi}|_{\mathcal{U}_0} : \tilde{\mathcal{U}}_0 \to \mathcal{U}_0.} \tag{4.45}$$

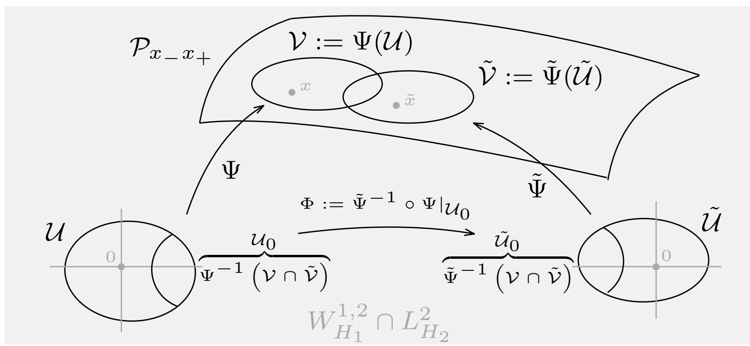

Figure 4: Transition map Φ for path space $\mathcal{P}_{x_-x_+}$ modeled on $W^{1,2}_{H_1} \cap L^2_{H_2}$

Theorem 4.14. *The transition map $\Phi \colon \mathcal{U}_0 \to \tilde{\mathcal{U}}_0$ is a C^1 diffeomorphism.*

Proof. The proof has ten steps.

Denote the **interpolation region** for x, respectively $\tilde{x}$, by

$$I = \bigcup_{j=1}^{k-1} [t_j^-, t_j], \qquad \tilde{I} = \bigcup_{\tilde{j}=1}^{\tilde{k}-1} [\tilde{t}_{\tilde{j}}^-, \tilde{t}_{\tilde{j}}].$$

As in Definition 4.6 we use the conventions $t_0 := -\infty =: \tilde{t}_0$ and $t_k^- := \infty =: \tilde{t}_{\tilde{k}}^-$.

Step 1. Pick $\xi \in \mathcal{U}_0$. Set $\eta := \Phi(\xi)$. By definition of Φ we have the identity

$$\tilde{\Psi}(\eta)(s) = \Psi(\xi)(s) \tag{4.46}$$

for every $s \in \mathbb{R}$. We solve (4.46) for $\eta_s := \eta(s) = \Phi(\xi)(s)$ considering four cases.

Proof. To prove Step 1 we consider four cases.

Case 1. Suppose $s \in \mathbb{R} \setminus (I \cup \tilde{I})$, that is s is a non-interpolation time for both. Then there is $j \in \{1, \dots, k\}$ such that s lies in the non-interpolation interval (t_{j-1}, t_j^-). Moreover, there is $\tilde{j} \in \{1, \dots, \tilde{k}\}$ such that s lies in the non-interpolation interval $(\tilde{t}_{\tilde{j}-1}, \tilde{t}_{\tilde{j}}^-)$. The two sides of (4.46) take on the form

$$\tilde{\psi}_{\tilde{j}} \left(\tilde{\psi}_{\tilde{j}}^{-1}(\tilde{x}_s) + \eta_s \right) = \psi_j \left(\psi_j^{-1}(x_s) + \xi_s \right)$$

which we resolve for

$$\eta_s = \tilde{\psi}_{\tilde{j}}^{-1} \circ \psi_j \left(\psi_j^{-1}(x_s) + \xi_s\right) - \tilde{\psi}_{\tilde{j}}^{-1}(\tilde{x}_s).$$

Case 2. Suppose $s \in I \cap (\mathbb{R} \setminus \tilde{I})$.
There is $j \in \{1, \dots, k-1\}$ such that s is in the interpolation interval $[t_j^-, t_j]$. There is $\tilde{j} \in \{1, \dots, \tilde{k}\}$ such that s lies in the non-interpolation interval $(\tilde{t}_{\tilde{j}-1}, \tilde{t}_{\tilde{j}}^-)$. The two sides of (4.46), using (4.39) in the end, take on the form

$$\begin{aligned}
&\tilde{\psi}_{\tilde{j}} \left(\tilde{\psi}_{\tilde{j}}^{-1}(\tilde{x}_s) + \eta_s\right) \\
&= \chi_j(s; \xi) \\
&:= \psi_j\Big((1 - \beta_s^j)\Big(\psi_j^{-1}(x_s) + \xi_s\Big) + \beta_s^j \underline{\psi_j^{-1} \circ \psi_{j+1}} \Big(\underline{\psi_{j+1}^{-1}}(x_s) + \xi_s\Big)\Big) \\
&= \psi_j\Big((1 - \beta_s^j)\Big(\psi_j^{-1}(x_s) + \xi_s\Big) + \beta_s^j \varphi_j^{-1}\Big(\varphi_j \circ \psi_j^{-1}(x_s) + \xi_s\Big)\Big) \\
&= \psi_j \circ \mathcal{S}_s^j(\xi_s).
\end{aligned}$$

Observe that $\psi_j^{-1}\psi_{j+1} = \varphi_j^{-1}$ is the basic covering transition map (4.28). Then

$$\eta_s = \tilde{\psi}_{\tilde{j}}^{-1} \circ \psi_j \circ \mathcal{S}_s^j(\xi_s) - \tilde{\psi}_{\tilde{j}}^{-1}(\tilde{x}_s).$$

Case 3. Suppose $s \in (\mathbb{R} \setminus I) \cap \tilde{I}$.
There is $j \in \{1, \dots, k\}$ such that s lies in the non-interpolation interval (t_{j-1}, t_j^-). There is $\tilde{j} \in \{1, \dots, \tilde{k}-1\}$ such that s lies in the interpolation interval $[\tilde{t}_{\tilde{j}}^-, \tilde{t}_{\tilde{j}}]$. The two sides of (4.46) take on the form

$$\begin{aligned}
\tilde{\psi}_{\tilde{j}}\Big((1 - \tilde{\beta}_s^{\tilde{j}})\Big(\tilde{\psi}_{\tilde{j}}^{-1}(\tilde{x}_s) + \eta_s\Big) + \tilde{\beta}_s^{\tilde{j}} \tilde{\varphi}_{\tilde{j}}^{-1}\Big(\tilde{\varphi}_{\tilde{j}} \circ \tilde{\psi}_{\tilde{j}}^{-1}(\tilde{x}_s) + \eta_s\Big)\Big) &=: \tilde{\chi}_{\tilde{j}}(s; \eta) \\
&= \psi_j \left(\psi_j^{-1}(x_s) + \xi_s\right)
\end{aligned}$$

where $\tilde{\varphi}_{\tilde{j}}$ is the basic covering transition map from (4.28). Equivalently we have

$$(1 - \tilde{\beta}_s^{\tilde{j}})\Big(\tilde{\psi}_{\tilde{j}}^{-1}(\tilde{x}_s) + \eta_s\Big) + \tilde{\beta}_s^{\tilde{j}} \tilde{\varphi}_{\tilde{j}}^{-1}\Big(\tilde{\varphi}_{\tilde{j}} \circ \tilde{\psi}_{\tilde{j}}^{-1}(\tilde{x}_s) + \eta_s\Big) = \tilde{\psi}_{\tilde{j}}^{-1} \circ \psi_j \left(\psi_j^{-1}(x_s) + \xi_s\right).$$

We wish to resolve for η_s. With $\tilde{\mathcal{S}}_s^{\tilde{j}}$ as in (4.43) we obtain the identity

$$\tilde{\mathcal{S}}_s^{\tilde{j}}(\eta_s) = \tilde{\psi}_{\tilde{j}}^{-1} \circ \psi_j \left(\psi_j^{-1}(x_s) + \xi_s\right).$$

But $\tilde{\mathcal{S}}_s^{\tilde{j}}$ is invertible, by Remark B.4, and we obtain the formula

$$\eta_s = (\tilde{\mathcal{S}}_s^{\tilde{j}})^{-1} \left(\tilde{\psi}_{\tilde{j}}^{-1} \circ \psi_j \left(\psi_j^{-1}(x_s) + \xi_s\right)\right).$$

Case 4. Suppose $s \in I \cap \tilde{I}$.

There is $j \in \{1, \dots, k-1\}$ such that s lies in the interpolation interval $[t_j^-, t_j]$. There is $\tilde{j} \in \{1, \dots, \tilde{k}-1\}$ such that s lies in the interpolation interval $[\tilde{t}_{\tilde{j}}^-, \tilde{t}_{\tilde{j}}]$. The two sides of (4.46) take on the form

$$\begin{aligned}
&\tilde{\psi}_{\tilde{j}}\left((1-\tilde{\beta}_s^{\tilde{j}})\left(\tilde{\psi}_{\tilde{j}}^{-1}(\tilde{x}_s)+\eta_s\right)+\tilde{\beta}_s^{\tilde{j}}\tilde{\varphi}_{\tilde{j}}^{-1}\left(\tilde{\varphi}_{\tilde{j}}\circ\tilde{\psi}_{\tilde{j}}^{-1}(\tilde{x}_s)+\eta_s\right)\right)\\
&=: \tilde{\chi}_{\tilde{j}}(s;\eta)\\
&= \chi_j(s;\xi)\\
: &= \psi_j\left((1-\beta_s^j)\left(\psi_j^{-1}(x_s)+\xi_s\right)+\beta_s^j\varphi_j^{-1}\left(\varphi_j\circ\psi_j^{-1}(x_s)+\xi_s\right)\right).
\end{aligned}$$

Similarly as in Case 3 with $\tilde{\mathcal{S}}_s^{\tilde{j}}$ as in (4.43) we resolve for η_s, namely

$$\tilde{\mathcal{S}}_s^{\tilde{j}}(\eta_s) = \tilde{\psi}_{\tilde{j}}^{-1}\circ\psi_j\circ\mathcal{S}_s^j(\xi_s), \qquad \eta_s = (\tilde{\mathcal{S}}_s^{\tilde{j}})^{-1}\circ\tilde{\psi}_{\tilde{j}}^{-1}\circ\psi_j\circ\mathcal{S}_s^j(\xi_s).$$

This concludes the proof of Step 1. □

Step 2. We find a subset $O \subset \mathbb{R}\times H_1$ and a map $\varphi\colon \mathbb{R}\times H_1 \supset O \to H_1$ such that we can write

$$\boxed{(\Phi(\xi))(s) = \varphi(s,\xi(s)) =: \varphi_s(\xi(s))} \tag{4.47}$$

for all $s \in \mathbb{R}$ and $\xi \in \mathcal{U}_0$, see Figure 4.

Proof. To prove Step 2 we first describe the slices U^s of O such that

$$O = \bigcup_{s\in\mathbb{R}} \left(\{s\}\times U^s\right).$$

We consider the same four cases as in Step 1. The map $\mathcal{S}_s^j$ in the composition

$$H_1 \supset B_R := B_R(0) \xrightarrow{\mathcal{S}_s^j} U^j \xrightarrow{\psi^j} V^j \subset X_1$$

is defined in (4.40).

Case 1. For $s \in \mathbb{R}\setminus(I\cup\tilde{I})$ let

$$U^s := \psi_j^{-1}\left(\psi_j(B_R(u_s^j))\cap\tilde{\psi}^{\tilde{j}}(B_{\tilde{R}}(\tilde{u}_s^{\tilde{j}}))\right) - u_s^j, \qquad \boxed{u_s^j = \psi_j^{-1}(x_s).}$$

Case 2. For $s \in I\cap(\mathbb{R}\setminus\tilde{I})$ let

$$U^s := (\mathcal{S}_s^j)^{-1}\circ\psi_j^{-1}\left(\left(\psi_j\circ\mathcal{S}_s^j\right)(B_R)\cap\tilde{\psi}^{\tilde{j}}(B_{\tilde{R}}(\tilde{u}_s^{\tilde{j}})\right).$$

Case 3. For $s \in (\mathbb{R}\setminus I)\cap\tilde{I}$ let

$$U^s := \psi_j^{-1}\left(\psi_j(B_R(u_s^j))\cap\left(\tilde{\psi}_{\tilde{j}}\circ\tilde{\mathcal{S}}_s^{\tilde{j}}\right)(B_{\tilde{R}})\right) - u_s^j.$$

Case 4. For $s \in I \cap \tilde{I}$ let

$$U^s := (\mathcal{S}_s^j)^{-1} \circ \psi_j^{-1} \left(\left(\psi_j \circ \mathcal{S}_s^j\right)(B_R) \cap \left(\tilde{\psi}_{\tilde{j}} \circ \tilde{\mathcal{S}}_s^{\tilde{j}}\right)(B_{\tilde{R}}) \right).$$

For $s \in \mathbb{R}$ and $v \in U^s$ we define (juxtaposition means composition)

$$\boxed{\varphi(s,v) := \begin{cases} \tilde{\psi}_{\tilde{j}}^{-1}\psi_j\left(\psi_j^{-1}(x_s)+v\right) - \tilde{\psi}_{\tilde{j}}^{-1}(\tilde{x}_s) & , s \in (t_{j-1}, t_j^-) \cap (\tilde{t}_{\tilde{j}-1}, \tilde{t}_{\tilde{j}}^-), \\ \tilde{\psi}_{\tilde{j}}^{-1}\psi_j\mathcal{S}_s^j(v) - \tilde{\psi}_{\tilde{j}}^{-1}(\tilde{x}_s) & , s \in [t_j^-, t_j] \cap (\tilde{t}_{\tilde{j}-1}, \tilde{t}_{\tilde{j}}^-), \\ (\tilde{\mathcal{S}}_s^{\tilde{j}})^{-1}\left(\tilde{\psi}_{\tilde{j}}^{-1}\psi_j\left(\psi_j^{-1}(x_s)+v\right)\right) & , s \in (t_{j-1}, t_j^-) \cap [\tilde{t}_{\tilde{j}}^-, \tilde{t}_{\tilde{j}}], \\ (\tilde{\mathcal{S}}_s^{\tilde{j}})^{-1}\tilde{\psi}_{\tilde{j}}^{-1}\psi_j\mathcal{S}_s^j(v) & , s \in [t_j^-, t_j] \cap [\tilde{t}_{\tilde{j}}^-, \tilde{t}_{\tilde{j}}]. \end{cases}} \tag{4.48}$$

For $s \le -T$ it holds $s \in (t_0, t_1^-) \cap (\tilde{t}_0, \tilde{t}_1^-)$, so by the first case of (4.48) we get

$$\varphi(s,v) = \tilde{\psi}_1^{-1}\psi_1\left(\psi_1^{-1}(x_-)+v\right) - \tilde{\psi}_1^{-1}(x_-), \qquad \varphi(s,0) = 0. \tag{4.49}$$

Similarly for $s \ge T$ we get

$$\varphi(s,v) = \tilde{\psi}_{\tilde{k}}^{-1}\psi_k\left(\psi_k^{-1}(x_+)+v\right) - \tilde{\psi}_{\tilde{k}}^{-1}(x_+), \qquad \varphi(s,0) = 0. \tag{4.50}$$

Figure 5 illustrates the definition of $\varphi_s := \varphi(s,\cdot)$ in case 2. The other three cases are similar, in the figure one just needs to interchange the interpolation maps $\mathcal{S}_s^j$ and the translation maps T.

By construction of the map φ the identity (4.47) holds. This concludes the proof of Step 2. $\square$

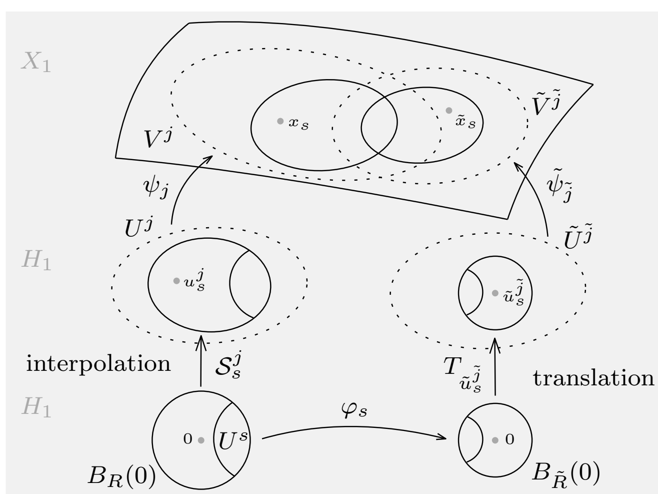

Figure 5: The map $\varphi_s\colon H_1 \supset U^s \to H_1$ in (4.48) in case 2

Outlook on the next few steps. In the next few steps we show that φ is parametrized tame, see Definition 2.12. This then allows us to conclude with the help of Theorem 3.5 that Φ in (4.44) is C^1.

Step 3 (Tameness). The interpolation map $\mathcal{S}_s^j\colon H_1 \supset B_R \to H_1$, see (4.40) and Figure 5, is tame and has a tame inverse $(\mathcal{S}_s^j)^{-1}\colon \mathcal{S}_s(B_R) \to B_R$, $\forall j$.

Proof. The proof of Step 3 is a rather lengthy undertaking.

Tameness of $\mathcal{S}_s^j$. Since φ_j^{-1} is tame, the map $\mathcal{S}_s^j$ is tame as well, in particular of class C^2 on H_1 and on H_2; for the definition of $\mathcal{S}_s^j$ see (4.39) and (B.85).

Tameness of $(\mathcal{S}_s^j)^{-1}$. Let s be in the interpolation interval $[t_j^-, t_j]$ and let $v \in H_1$ be in the ball $B_R \subset H_1$ centered at 0.

Lemma 4.15. *The restriction to H_2 of the linearized interpolation map*

$$T := d\mathcal{S}_s^j|_v|_{H_2}\colon H_2 \to H_2$$

at $v \in B_R \cap H_2$ is an isomorphism on H_2 and there is a constant $\mu > 0$ such that

$$|(d\mathcal{S}_s^j|_v)^{-1}\xi|_2 \le \mu\left(|\xi|_2 + |v|_2|\xi|_1\right) \tag{4.51}$$

for every $\xi \in H_2$.

Proof of Lemma 4.15. Pick $v \in B_R \cap H_2$. By (4.34) it follows that $v + \psi_{j+1}^{-1}(x_s) \in C_-^{j+1}$. By definition (4.40) of $\mathcal{S}_s^j$ and by identity (B.86) we obtain

$$\begin{aligned} d\mathcal{S}_s^j|_v - \mathrm{Id} &= d\mathcal{S}^{\varphi_j^{-1}}_{\beta_s^j, \psi_{j+1}^{-1}(x_s)}|_v - \mathrm{Id} \\ &= \beta_s^j \left(d\varphi_j^{-1}|_{\psi_{j+1}^{-1}(x_s)+v} - \mathrm{Id}\right). \end{aligned} \tag{4.52}$$

• A first consequence of this identity, using (4.30) and $\beta_s^j \le 1$, is the estimate

$$\|d\mathcal{S}_s^j|_v - \mathrm{Id}\|_{\mathcal{L}(H_1)} \le \tfrac{1}{2}.$$

By the Neumann series, see e.g. [FW, Rmk. B.5], we get the uniform bound

$$\|(d\mathcal{S}_s^j|_v)^{-1}\|_{\mathcal{L}(H_1)} \le \sum_{k=0}^\infty \|\mathrm{Id} - d\mathcal{S}_s^j|_v\|^k_{\mathcal{L}(H_1)} \le \sum_{k=0}^\infty \tfrac{1}{2^k} = \tfrac{1}{1-\frac{1}{2}} = 2. \tag{4.53}$$

• A second consequence of (4.52) is the following. Assume that $v \in B_R \cap H_2$. Since basic paths are mapped to H_2, it follows that $\psi_{j+1}^{-1}(x_s) + v \in C_-^{j+1} \cap H_2$. Hence, by (4.31) and using that $\beta_s^j \le 1$, we obtain for every $\eta \in H_2$ the estimate

$$\begin{aligned} |d\mathcal{S}_s^j|_v\eta - \eta|_2 &\le \tfrac{1}{2}|\eta|_2 + c_j\left(|\psi_{j+1}^{-1}(x_s) + v|_2 + 1\right)|\eta|_1 \\ &\le \tfrac{1}{2}|\eta|_2 + c_j\left(|v|_2 + |\psi_{j+1}^{-1}(x_s)|_2 + 1\right)|\eta|_1. \end{aligned}$$

Since the basic path x is continuous as a map to H_2, the following maximum

$$\boxed{C := \max_{j=1,\dots,k-1} \max_{s\in[t_j^-,t_j]} \{c_j(1+|\psi_{j+1}^{-1}(x_s)|_2)\}}$$

is well defined. Therefore we obtain the inequality

$$|d\mathcal{S}_s^j|_v\eta - \eta|_2 \le \tfrac{1}{2}|\eta|_2 + C\,(|v|_2+1)\,|\eta|_1.$$

With this inequality we estimate

$$\begin{aligned}|d\mathcal{S}_s^j|_v\eta|_2 = |\eta + (d\mathcal{S}_s^j|_v - \mathrm{Id})\eta|_2 &\ge |\eta|_2 - |(d\mathcal{S}_s^j|_v - \mathrm{Id})\eta|_2\\ &\ge |\eta|_2 - \tfrac{1}{2}|\eta|_2 - C\,(|v|_2+1)\,|\eta|_1\\ &= \tfrac{1}{2}|\eta|_2 - C\,(|v|_2+1)\,|\eta|_1.\end{aligned}$$

We rewrite this inequality as

$$|\eta|_2 \le 2|d\mathcal{S}_s^j|_v\eta|_2 + 2C\,(|v|_2+1)\,|\eta|_1. \tag{4.54}$$

Note that since the restriction of the linearized transition map $d\varphi_j^{-1}|_{\psi_{j+1}^{-1}(x_s)+v}$ to H_2 takes values in H_2, by (4.52) so does the restriction of $d\mathcal{S}_s^j|_v$ to H_2. The level map $d\mathcal{S}_s^j|_v\colon H_2\to H_2$ is bounded linear again by (4.52). Using that the inclusion $\iota\colon H_2\hookrightarrow H_1$ is compact, estimate (4.54) implies that the level map

$$T := d\mathcal{S}_s^j|_v|_{H_2}\colon H_2\to H_2$$

is a semi-Fredholm operator; see [MS04, Le. A.1.1] for $D = d\mathcal{S}_s^j|_v$, $K=\iota$, $X=Y=H_2$, and $Z=H_1$. In the terminology of [Mül07, §16] it is an upper semi-Fredholm operator.

• Next we compute the semi-Fredholm index of T as an element of $\mathbb{Z}\cup\{-\infty\}$. For this purpose we define the bounded linear operator

$$T_\beta := (1-\beta)\mathrm{Id} + \beta d\varphi_j^{-1}|_{\psi_{j+1}^{-1}(x_s)+v}|_{H_2}\colon H_2\to H_2.$$

for $\beta\in\mathbb{R}$. In view of (4.52) we have $T_{\beta_s^j} = T$ and $T_0 = \mathrm{Id}$. If $\beta\in[0,1]$ the same arguments which established (4.54) show that

$$|\eta|_2 \le 2|T_\beta\eta|_2 + 2C\,(|v|_2+1)\,|\eta|_1$$

for every $\eta\in H_2$. Indeed note that in the derivation of (4.54) we only used that $\beta_s^j\le 1$. In particular T_β is a semi-Fredholm operator for any $\beta\in[0,1]$. Since $T_0=\mathrm{Id}$ we have $\mathrm{index}(T_0)=0$. Since the semi-Fredholm index is locally constant, see [Mül07, §18 Cor. 3], follows that $\mathrm{index}(T_\beta)=0$ for any $\beta\in[0,1]$. In particular $0=\mathrm{index}(T) := \dim\ker T - \dim\operatorname{coker} T$. Since $d\mathcal{S}_s^j|_v\colon H_1\to H_1$ is an isomorphism, it has trivial kernel, hence its restriction T has trivial kernel as well. Since the index of T vanishes it follows that its cokernel is trivial as well, in particular T is bijective and therefore, by the open mapping theorem, it is an isomorphism from H_2 to H_2.

In particular, the operator $(d\mathcal{S}^j_s|_v)^{-1}\colon H_1 \to H_1$ restricts to a bounded linear isomorphism

$$T^{-1} = (d\mathcal{S}^j_s|_v)^{-1}|_{H_2}\colon H_2 \xrightarrow{\cong} H_2. \tag{4.55}$$

Consequently for any $\xi \in H_2$ there exists $\eta \in H_2$ such that $\xi = d\mathcal{S}^j_s|_v\eta$. Using (4.54) we compute

$$\begin{aligned} |(d\mathcal{S}^j_s|_v)^{-1}\xi|_2 &\le 2|\xi|_2 + 2C\left(|v|_2+1\right)|(d\mathcal{S}^j_s|_v)^{-1}\xi|_1 \\ &\le 2|\xi|_2 + 2C\left(|v|_2+1\right)\|(d\mathcal{S}^j_s|_v)^{-1}\|_{\mathcal{L}(H_1)}|\xi|_1 \\ &\le 2|\xi|_2 + 4C\left(|v|_2+1\right)|\xi|_1 \\ &\le \mu\left(|\xi|_2 + |v|_2|\xi|_1\right), \qquad \boxed{\mu := 2(1+2C) > 2} \end{aligned}$$

for every $\xi \in H_2$. Here step three is by (4.53), the final step by (2.2).

This proves (4.51) and concludes the proof of Lemma 4.15. $\square$

We continue the proof of Step 3, more precisely the proof of tameness of the inverse $(\mathcal{S}^j_s)^{-1}$ of the interpolation map.

• Continuity of the second derivative of $(\mathcal{S}^j_s)^{-1}$ with respect to the C^2 topology. The map $\mathcal{S}^j_s\colon B_R \cap H_2 \to \mathcal{S}^j_s(B_R) \cap H_2$ is C^2 since it is tame. It follows from (4.55) by the implicit function theorem, see e.g. [MS04, Thm. A.3.1] that $(\mathcal{S}^j_s)^{-1}\colon \mathcal{S}^j_s(B_R)\cap H_2 \to B_R \cap H_2$ is C^2 as well.

• Recall from (4.40) that $\mathcal{S}^j_s\colon H_1 \supset B_R \to H_1$ is a C^2 diffeomorphism onto its image. Linearize the identity $\mathrm{id} = (\mathcal{S}^j_s)^{-1} \circ \mathcal{S}^j_s$ at $v \in B_R \subset H_1$ to get

$$\mathrm{Id}_{H_1} = d(\mathcal{S}^j_s)^{-1}|_{\mathcal{S}^j_s(v)} d\mathcal{S}^j_s|_v. \tag{4.56}$$

Linearizing the displayed identity, we resolve that

$$d^2(\mathcal{S}^j_s)^{-1}|_{\mathcal{S}^j_s(v)}(\xi,\eta) = -\left(d\mathcal{S}^j_s|_v\right)^{-1} d^2\mathcal{S}^j_s|_v\left(\left(d\mathcal{S}^j_s|_v\right)^{-1}\xi, \left(d\mathcal{S}^j_s|_v\right)^{-1}\eta\right) \tag{4.57}$$

for all $\xi, \eta \in H_1$.

• We show $(\mathcal{S}^j_s)^{-1}\colon H_1 \supset \mathcal{S}^j_s(B_R) \to B_R$ satisfies the tameness estimate (2.3). By (4.55) and since $\mathcal{S}^j_s$ maps $B_R \cap H_2$ to H_2 it follows that $(\mathcal{S}^j_s)^{-1}|_{\mathcal{S}^j_s(B_R)\cap H_2}$ maps $\mathcal{S}^j_s(B_R)\cap H_2$ onto $B_R \cap H_2$. Hence if $\mathcal{S}^j_s(v_0)$ is in $\mathcal{S}^j_s(B_R)\cap H_2$ it follows that $v_0 \in B_R \cap H_2$. Since $\mathcal{S}^j_s$ is tame there exists an H_1-open neighborhood of v_0 and a constant $\kappa > 0$, notation

$$V \subset B_R, \qquad \boxed{\kappa > 0,} \tag{4.58}$$

such that for every $v \in V \cap H_2$ and all $\xi, \eta \in H_2$ it holds the estimate

$$\left|d^2\mathcal{S}^j_s|_v(\xi,\eta)\right|_2 \le \kappa\big(|\xi|_1|\eta|_2 + |\xi|_2|\eta|_1 + |v|_2|\xi|_1|\eta|_1\big). \tag{4.59}$$

Since $\mathcal{S}_s^j$ is C^2 on B_R, maybe after shrinking V and enlarging κ, we can additionally assume that

$$\left|d^2\mathcal{S}_s^j|_v(\xi,\eta)\right|_1 \le \kappa|\xi|_1|\eta|_1. \tag{4.60}$$

Hence for $\mathcal{S}_s^j(v) \in \mathcal{S}_s^j(V) \cap H_2$ we estimate

$$\begin{aligned}
&\left|d^2(\mathcal{S}_s^j)^{-1}|_{\mathcal{S}_s^j(v)}(\xi,\eta)\right|_2 \\
&= \left|\left(d\mathcal{S}_s^j|_v\right)^{-1} d^2\mathcal{S}_s^j|_v \left(\left(d\mathcal{S}_s^j|_v\right)^{-1}\xi, \left(d\mathcal{S}_s^j|_v\right)^{-1}\eta\right)\right|_2 \\
&\le \mu\left|d^2\mathcal{S}_s^j|_v \left(\left(d\mathcal{S}_s^j|_v\right)^{-1}\xi, \left(d\mathcal{S}_s^j|_v\right)^{-1}\eta\right)\right|_2 \\
&\quad + \mu|v|_2\left|d^2\mathcal{S}_s^j|_v \left(\left(d\mathcal{S}_s^j|_v\right)^{-1}\xi, \left(d\mathcal{S}_s^j|_v\right)^{-1}\eta\right)\right|_1 \\
&\le \mu\kappa\left|\left(d\mathcal{S}_s^j|_v\right)^{-1}\xi\right|_1\left|\left(d\mathcal{S}_s^j|_v\right)^{-1}\eta\right|_2 + \mu\kappa\left|\left(d\mathcal{S}_s^j|_v\right)^{-1}\xi\right|_2\left|\left(d\mathcal{S}_s^j|_v\right)^{-1}\eta\right|_1 \\
&\quad + 2\mu\kappa|v|_2\left|\left(d\mathcal{S}_s^j|_v\right)^{-1}\xi\right|_1\left|\left(d\mathcal{S}_s^j|_v\right)^{-1}\eta\right|_1 \\
&\le 2\mu^2\kappa|\xi|_1\left(|\eta|_2+|v|_2|\eta|_1\right) + 2\mu^2\kappa|\eta|_1\left(|\xi|_2+|v|_2|\xi|_1\right) + 8\mu\kappa|v|_2|\xi|_1|\eta|_1 \\
&\le 4\mu\kappa(\mu+2)\left(|\xi|_1|\eta|_2+|\xi|_2|\eta|_1+|v|_2|\xi|_1|\eta|_1\right).
\end{aligned}$$

Here step 1 is by (4.57) and step 2 by (4.51). Step 3 follows by (4.59) and (4.60). Step 4 is by (4.53) and (4.51) again.
To summarize, with

$$\boxed{\kappa_* := 4\mu\kappa(\mu+2) > 32\kappa}$$

and whenever $v \in V \cap H_2$ we have

$$\left|d^2(\mathcal{S}_s^j)^{-1}|_{\mathcal{S}_s^j(v)}(\xi,\eta)\right|_2 \le \kappa_*\left(|\xi|_1|\eta|_2+|\xi|_2|\eta|_1+|v|_2|\xi|_1|\eta|_1\right) \tag{4.61}$$

for all $\xi,\eta \in H_2$. This estimate is not yet the tameness estimate (2.3). The reason is that in place of $|v|_2$ we need $|\mathcal{S}_s^j(v)|_2$. To this end we show that, maybe after shrinking V, there exists a constant C such that

$$\boxed{|v|_2 \le C\left(|\mathcal{S}_s^j(v)|_2+1\right)} \tag{4.62}$$

whenever $v \in V \cap H_2$. For that purpose we use (4.51) inductively.

Special case. To simplify the computation we first derive (4.62) in the special case where $v_0 = 0$ and $\mathcal{S}_s^j(0) = 0$. Maybe after shrinking V we can assume that $\mathcal{S}_s^j(V)$ is convex and contained in the radius-$\frac{1}{2\mu}$ ball in H_1 about the origin. For

$v \in V \cap H_2$ we estimate (juxtaposition means composition)

$$\begin{aligned}
|v|_2 &= |(\mathcal{S}^j_s)^{-1}\mathcal{S}^j_s(v)|_2 \\
&\stackrel{2}{=} \Bigl|\int_0^1 \frac{d}{dt_1}(\mathcal{S}^j_s)^{-1}\left(t_1\mathcal{S}^j_s(v)\right) dt_1\Bigr|_2 \\
&\stackrel{3}{=} \Bigl|\int_0^1 d(\mathcal{S}^j_s)^{-1}|_{t_1\mathcal{S}^j_s(v)}\mathcal{S}^j_s(v)\, dt_1\Bigr|_2 \\
&\stackrel{4}{=} \Bigl|\int_0^1 (d\mathcal{S}^j_s|_{(\mathcal{S}^j_s)^{-1}(t_1\mathcal{S}^j_s(v))})^{-1}\mathcal{S}^j_s(v)\, dt_1\Bigr|_2 \\
&\stackrel{5}{\le} \int_0^1 \Bigl|(d\mathcal{S}^j_s|_{(\mathcal{S}^j_s)^{-1}(t_1\mathcal{S}^j_s(v))})^{-1}\mathcal{S}^j_s(v)\Bigr|_2\, dt_1 \\
&\stackrel{6}{\le} \int_0^1 \mu\Bigl(\bigl|\mathcal{S}^j_s(v)\bigr|_2 + \bigl|(\mathcal{S}^j_s)^{-1}(t_1\mathcal{S}^j_s(v))\bigr|_2 \underbrace{\bigl|\mathcal{S}^j_s(v)\bigr|_1}_{\le 1/2\mu}\Bigr)\, dt_1 \\
&\le \mu\bigl|\mathcal{S}^j_s(v)\bigr|_2 + \tfrac{1}{2}\int_0^1 \bigl|(\mathcal{S}^j_s)^{-1}(t_1\mathcal{S}^j_s(v))\bigr|_2\, dt_1.
\end{aligned}$$

Step 2 is by the fundamental theorem of calculus. Step 3 is by the chain rule. Step 4 is by (4.57). Step 5 is by monotonicity of the integral. Step 6 is by (4.51).

The integrand in the second summand we similarly estimate further

$$\begin{aligned}
&\bigl|(\mathcal{S}^j_s)^{-1}(t_1\mathcal{S}^j_s(v))\bigr|_2 \\
&= \Bigl|\int_0^1 \underbrace{\overbrace{d(\mathcal{S}^j_s)^{-1}|_{t_1t_2\mathcal{S}^j_s(v)}t_1\mathcal{S}^j_s(v)}^{\frac{d}{dt_2}(\mathcal{S}^j_s)^{-1}\left(t_1t_2\mathcal{S}^j_s(v)\right)}}_{(d\mathcal{S}^j_s|_{(\mathcal{S}^j_s)^{-1}(t_1t_2\mathcal{S}^j_s(v))})^{-1}t_1\mathcal{S}^j_s(v)}\, dt_2\Bigr|_2 \\
&\stackrel{2}{\le} \int_0^1 \Bigl|(d\mathcal{S}^j_s|_{(\mathcal{S}^j_s)^{-1}(t_1t_2\mathcal{S}^j_s(v))})^{-1}\underbrace{t_1}_{\le 1}\mathcal{S}^j_s(v)\Bigr|_2 dt_2 \\
&\stackrel{3}{\le} \int_0^1 \mu\Bigl(\bigl|\mathcal{S}^j_s(v)\bigr|_2 + \bigl|(\mathcal{S}^j_s)^{-1}(t_1t_2\mathcal{S}^j_s(v))\bigr|_2 \underbrace{\bigl|\mathcal{S}^j_s(v)\bigr|_1}_{\le 1/2\mu}\Bigr)\, dt_2 \\
&\le \mu\bigl|\mathcal{S}^j_s(v)\bigr|_2 + \tfrac{1}{2}\int_0^1 \bigl|(\mathcal{S}^j_s)^{-1}(t_1t_2\mathcal{S}^j_s(v))\bigr|_2\, dt_2.
\end{aligned}$$

Step 2 is by monotonicity of the integral. Step 3 is by (4.51).

Insert this latter estimate into the previous estimate to obtain

$$|v|_2 \le \left(1+\tfrac{1}{2}\right)\mu\bigl|\mathcal{S}^j_s(v)\bigr|_2 + \tfrac{1}{4}\int_0^1\int_0^1 \bigl|(\mathcal{S}^j_s)^{-1}(t_1t_2\mathcal{S}^j_s(v))\bigr|_2\, dt_2\, dt_1.$$

Estimating the integrand in the second term inductively with the help of (4.51) we obtain for every $n \in \mathbb{N}$ the following estimate

$$|v|_2 \leq \Big(\sum_{j=0}^{n-1} \frac{1}{2^j}\Big)\mu \left|\mathcal{S}_s^j(v)\right|_2 + \frac{1}{2^n}\int_0^1 \dots \int_0^1 \left|(\mathcal{S}_s^j)^{-1}(t_1 \dots t_n \mathcal{S}_s^j(v))\right|_2 dt_n \dots dt_1.$$

We consider the limit $n \to \infty$. First note that there exists a constant c_V depending on V, but not on n, such that $|(\mathcal{S}_s^j)^{-1}(t\mathcal{S}_s^j(v))|_2 \leq c_V$ for every $t \in [0,1]$. Hence we can estimate the second summand from above by $c_V/2^n$. Therefore in the limit, as $n \to \infty$, the second summand vanishes. Using that $\sum_{j=0}^{n-1} \frac{1}{2^j} = 2$ we get (4.62) with $C = 2\mu$, namely

$$|v|_2 \leq 2\mu \left|\mathcal{S}_s^j(v)\right|_2.$$

General case. Maybe after shrinking V we can assume that $\mathcal{S}_s^j(V)$ is convex and contained in the radius-$\frac{1}{2\mu}$ ball in H_1 around $\mathcal{S}_s^j(v_0)$. For $v \in V \cap H_2$ we estimate by the same techniques as in the special case above

$$\begin{aligned}
|v|_2 &= |v_0 + v - v_0|_2 \\
&\leq |v_0|_2 + |(\mathcal{S}_s^j)^{-1}\mathcal{S}_s^j(v) - (\mathcal{S}_s^j)^{-1}\mathcal{S}_s^j(v_0)|_2 \\
&= |v_0|_2 + \Big|\int_0^1 \overbrace{\underbrace{d(\mathcal{S}_s^j)^{-1}|_{t_1\mathcal{S}_s^j(v)+(1-t_1)\mathcal{S}_s^j(v_0)} \left(\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0)\right)}_{(d\mathcal{S}_s^j|_{(\mathcal{S}_s^j)^{-1}(t_1\mathcal{S}_s^j(v)+(1-t_1)\mathcal{S}_s^j(v_0))})^{-1}(\mathcal{S}_s^j(v)-\mathcal{S}_s^j(v_0))}}^{\frac{d}{dt_1}(\mathcal{S}_s^j)^{-1}\left(t_1\mathcal{S}_s^j(v)+(1-t_1)\mathcal{S}_s^j(v_0)\right)} dt_1\Big|_2 \\
&\leq |v_0|_2 + \int_0^1 \Big|(d\mathcal{S}_s^j|_{(\mathcal{S}_s^j)^{-1}(t_1\mathcal{S}_s^j(v)+(1-t_1)\mathcal{S}_s^j(v_0))})^{-1}(\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0))\Big|_2 dt_1 \\
&\overset{5}{\leq} |v_0|_2 + \int_0^1 \mu(\left|\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0)\right|_2 dt_1 \\
&\quad + \int_0^1 \mu \left|(\mathcal{S}_s^j)^{-1}(t_1\mathcal{S}_s^j(v) + (1-t_1)\mathcal{S}_s^j(v_0))\right|_2 \underbrace{\left|\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0\right|_1}_{\leq 1/2\mu} dt_1 \\
&\leq |v_0|_2 + \mu \left|\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0)\right|_2 + \tfrac{1}{2}\int_0^1 \left|(\mathcal{S}_s^j)^{-1}(t_1\mathcal{S}_s^j(v) + (1-t_1)\mathcal{S}_s^j(v_0))\right|_2 dt_1.
\end{aligned}$$

Step 5 is by (4.51). The integrand in the second summand we estimate further

$$
\begin{aligned}
&\left|(\mathcal{S}_s^j)^{-1}(t_1\mathcal{S}_s^j(v) + (1-t_1)\mathcal{S}_s^j(v_0)) - v_0 + v_0\right|_2 \\
&= \Big|\int_0^1 \overbrace{\underbrace{d(\mathcal{S}_s^j)^{-1}|_{t_1t_2\mathcal{S}_s^j(v)+(1-t_1t_2)\mathcal{S}_s^j(v_0)} t_1\left(\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0)\right)}_{(d\mathcal{S}_s^j|_{(\mathcal{S}_s^j)^{-1}(t_1t_2\mathcal{S}_s^j(v)+(1-t_1t_2)\mathcal{S}_s^j(v_0))})^{-1}t_1\left(\mathcal{S}_s^j(v)-\mathcal{S}_s^j(v_0)\right)}}^{\frac{d}{dt_2}(\mathcal{S}_s^j)^{-1}\left(t_1t_2\mathcal{S}_s^j(v)+(1-t_1t_2)\mathcal{S}_s^j(v_0)\right)}\, dt_2 + v_0\Big|_2 \\
&\overset{2}{\le} |v_0|_2 + \int_0^1 \Big|(d\mathcal{S}_s^j|_{(\mathcal{S}_s^j)^{-1}(t_1t_2\mathcal{S}_s^j(v)+(1-t_1t_2)\mathcal{S}_s^j(v_0))})^{-1} \underbrace{t_1}_{\le 1} \left(\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0)\right)\Big|_2 dt_2 \\
&\overset{3}{\le} |v_0|_2 + \int_0^1 \mu\Big(\left|\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0)\right|_2 \\
&\qquad\qquad + \left|(\mathcal{S}_s^j)^{-1}(t_1t_2\mathcal{S}_s^j(v) + (1-t_1t_2)\mathcal{S}_s^j(v_0))\right|_2 \underbrace{\left|\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0)\right|_1}_{\le 1/2\mu}\Big)\, dt_2 \\
&\le |v_0|_2 + \mu\left|\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0)\right|_2 + \tfrac{1}{2}\int_0^1 \left|(\mathcal{S}_s^j)^{-1}(t_1t_2\mathcal{S}_s^j(v) + (1-t_1t_2)\mathcal{S}_s^j(v_0))\right|_2 dt_2.
\end{aligned}
$$

Step 2 is by the triangle inequality and monotonicity of the integral. Step 3 is by (4.51). Insert this latter estimate into the previous one to obtain

$$
\begin{aligned}
|v|_2 &\le \left(1 + \tfrac{1}{2}\right)|v_0|_2 + \left(1 + \tfrac{1}{2}\right)\mu\left|\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0)\right|_2 \\
&\quad + \tfrac{1}{4}\int_0^1\int_0^1 \left|(\mathcal{S}_s^j)^{-1}(t_1t_2\mathcal{S}_s^j(v) + (1-t_1t_2)\mathcal{S}_s^j(v_0))\right|_2 dt_2\, dt_1.
\end{aligned}
$$

Estimating the integrand in the second term inductively with the help of (4.51) we obtain for every $n \in \mathbb{N}$ the following estimate

$$
\begin{aligned}
|v|_2 &\le \Big(\sum_{j=0}^{n-1}\frac{1}{2^j}\Big)|v_0|_2 + \Big(\sum_{j=0}^{n-1}\frac{1}{2^j}\Big)\mu\left|\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0)\right|_2 \\
&\quad + \frac{1}{2^n}\int_0^1 \dots \int_0^1 \left|(\mathcal{S}_s^j)^{-1}(t_1\dots t_n\mathcal{S}_s^j(v) + (1-t_1\dots t_n)\mathcal{S}_s^j(v_0))\right|_2 dt_n\dots dt_1.
\end{aligned}
$$

We consider the limit $n \to \infty$. First note that there exists a constant C_V depending on V, but not on n, such that $|(\mathcal{S}_s^j)^{-1}(t\mathcal{S}_s^j(v) + (1-t)\mathcal{S}_s^j(v_0))|_2 \le C_V$ for every $t \in [0,1]$. Hence we can estimate the final (third) summand from above by $c_V/2^n$. Therefore in the limit, as $n \to \infty$, the final summand vanishes. Using that $\sum_{j=0}^{n-1}\frac{1}{2^j} = 2$ we estimate

$$
\begin{aligned}
|v|_2 &\le 2|v_0|_2 + 2\mu\left|\mathcal{S}_s^j(v) - \mathcal{S}_s^j(v_0)\right|_2 \\
&\le 2\mu\left|\mathcal{S}_s^j(v)\right|_2 + 2\mu\left|\mathcal{S}_s^j(v_0)\right|_2 + 2|v_0|_2 \\
&\le K\left(|\mathcal{S}_s^j(v)|_2 + 1\right), \quad \boxed{K(s) := 2\max\{\mu, \mu|\mathcal{S}_s^j(v_0)|_2 + |v|_2\}.}
\end{aligned}
\tag{4.63}
$$

This proves the claim (4.62).
Conclusion. We are now in position to prove the tameness estimate (2.3) for the inverse $(\mathcal{S}^j_s)^{-1}\colon H_1 \supset \mathcal{S}^j_s(B_R) \to B_R$. Plugging (4.62) into (4.61) leads to

$$\begin{aligned}
&\left| d^2(\mathcal{S}^j_s)^{-1}|_{\mathcal{S}^j_s(v)}(\xi,\eta)\right|_2 \\
&\le \kappa_*\bigl(|\xi|_1|\eta|_2 + |\xi|_2|\eta|_1 + K\left(|\mathcal{S}^j_s(v)|_2+1\right)|\xi|_1|\eta|_1\bigr) \\
&\overset{2}{\le} \kappa_*\Bigl(|\xi|_1|\eta|_2 + |\xi|_2|\eta|_1 + K|\mathcal{S}^j_s(v)|_2|\xi|_1|\eta|_1 + \tfrac{K}{2}|\xi|_2|\eta|_1 + \tfrac{K}{2}|\xi|_1|\eta|_2\Bigr) \qquad (4.64)\\
&= \kappa_*\left(\tfrac{K}{2}+1\right)|\xi|_1|\eta|_2 + \kappa_*\left(\tfrac{K}{2}+1\right)|\xi|_2|\eta|_1 + \kappa_* K|\mathcal{S}^j_s(v)|_2|\xi|_1|\eta|_1 \\
&\le C_*\bigl(|\xi|_1|\eta|_2 + |\xi|_2|\eta|_1 + |\mathcal{S}^j_s(v)|_2|\xi|_1|\eta|_1\bigr)
\end{aligned}$$

where the final step holds for $C_*(s) := \kappa_* \max\{K_s, \frac{K_s}{2}+1\}$. Step 2 is by (2.2).

This proves tameness of $(\mathcal{S}^j_s)^{-1}$ and completes the proof of Step 3. ☐

Step 4. For any $s \in \mathbb{R}$ the map $\varphi_s\colon \{s\}\times U^s \to H_1$ defined by (4.48) is tame.

Proof. In (4.48) the maps

$$\tilde\psi_{\tilde j}^{-1}\circ\psi_j\colon \psi_j^{-1}(V_j\cap\tilde V_{\tilde j}) \to \tilde\psi_{\tilde j}^{-1}(V_j\cap\tilde V_{\tilde j})$$

are transition maps of a tame atlas and therefore tame. Because by Theorem 2.5 the composition of tame maps is tame, tameness of φ_s follows by (4.48) in view of Step 3. This proves Step 4. ☐

Step 5. The interpolation map, see (4.40) and Figure 5,

$$\mathcal{S}^j\colon [t_j^-, t_j]\times B_R \to H_1,\quad (s,v)\mapsto \mathcal{S}^j_s(v)$$

is parametrized tame.

Proof. We need to check two conditions. 1) The map $\mathcal{S}^j$ is C^2 and its restriction,

$$\mathcal{S}^j|_{[t_j^-,t_j]\times(B_R\cap H_2)}\colon [t_j^-,t_j]\times(B_R\cap H_2)\to H_2$$

is C^2 as well. Using that $\mathcal{S}^j_s$ is in (4.39) defined by

$$\mathcal{S}^j_s := \mathcal{S}^{\varphi_j^{-1}}_{\beta^j_s,\psi_{j+1}^{-1}(x_s)}$$

together with smoothness of the cutoff function β^j_s in the s-variable and the fact that our basic path is a C^2-map $s\mapsto x_s$ to the second manifold level X_2, both C^2-claims follow directly from (B.85).
2) The parametrized tameness estimate (2.8) follows from tameness of ϕ_j^{-1} by noting that ϕ_j^{-1} does not depend on s. This proves 2) and Step 5. ☐

Step 6. The inverted interpolation map family

$$\mathcal{S}^j_-\colon \{(s,v)\in[t_j^-,t_j]\times H_1 \mid v\in\mathcal{S}^j_s(B_R)\}\to B_R,\quad (s,v)\mapsto(\mathcal{S}^j_s)^{-1}(v)$$

is parametrized tame.

Proof. 1) That $\mathcal{S}_-^j$ is C^2 on level 1 and level 2 follows from the fact that this is true for $\mathcal{S}^j$ by Step 5 and a version of the implicit function theorem as explained in Proposition B.5 for $\mathcal{G} = \mathcal{S}_-^j$.
2) The tameness estimate (2.8) follows from the tameness estimate (4.64) by noting that the only s-dependence of the constant C_* comes from the constant $K(s)$ as introduced in (4.63), but this constant can be chosen uniformly in a little neighborhood of $s \in \mathbb{R}$ by using that H_2-norm of $\mathcal{S}_s^j(v_0)$ depends continuously on s. This proves Step 6. □

Step 7. The map $\varphi\colon \mathbb{R} \times H_1 \supset O \to H_1$ defined by (4.48) is asymptotically constant parametrized tame.

Proof. By Steps 5 and 6 the map φ is a composition of parametrized tame maps. The same argument as in the proof of Theorem 2.5 shows that the composition of parametrized tame maps is again parametrized tame. The map φ is asymptotically constant by (4.49) and (4.50). This proves Step 7. □

Step 8. The map $\Phi\colon \mathcal{U}_0 \to \tilde{\mathcal{U}}_0$ defined by (4.44) is C^1.

Proof. This follows from Step 7 and Theorem 3.5. □

Step 9. The inverse $\Phi^{-1}\colon \tilde{\mathcal{U}}_0 \to \mathcal{U}_0$, see (4.45), is C^1.

Proof. In Step 8 interchange the roles of the charts $\mathcal{U}$ and $\tilde{\mathcal{U}}$. □

Step 10. We prove Theorem 4.14.

Proof. Theorem 4.14 follows by Step 8 and Step 9. □

The proof Theorem 4.14 is complete. □

Corollary 4.16. *The weak tangent map $T\Phi\colon \mathcal{U}_0 \times L^2_{H_1} \to \tilde{\mathcal{U}}_0 \times L^2_{H_1}$ defined by (3.25) is a C^1 diffeomorphism.*

Proof. By Step 7 in the proof of Theorem 4.14 the map φ is asymptotically constant parametrized tame. So by Theorem 3.4 the weak tangent map $T\Phi$ is C^1. And interchanging the roles of $\mathcal{U}_0$ and $\tilde{\mathcal{U}}_0$ its inverse is C^1 as well. This proves Corollary 4.16. □

4.2 Weak tangent bundles

We are now in position to define precisely what the weak tangent bundle is. For a local parametrization $\Psi \in \mathcal{AP}_{x_-x_+}$, see (4.42), we denote by $\mathcal{U}_\Psi \subset W^{1,2}_{H_1} \cap L^2_{H_2}$ the domain of Ψ. Given two local parametrizations $\Psi, \tilde{\Psi} \in \mathcal{AP}_{x_-x_+}$, the corresponding **weak tangent bundle transition map** is defined by

$$\Phi_{\Psi\tilde{\Psi}} := \tilde{\Psi}^{-1} \circ \Psi|_{\mathcal{U}_{\Psi\tilde{\Psi}}} \colon \mathcal{U}_{\Psi\tilde{\Psi}} \to \mathcal{U}_{\tilde{\Psi}\Psi}$$

where $\mathcal{U}_{\Psi\tilde{\Psi}}$ and $\mathcal{U}_{\tilde{\Psi}\Psi}$ are open subsets of the Hilbert space $\left(W^{1,2}_{H_1} \cap L^2_{H_2}\right) \times L^2_{H_1}$ and, as illustrated by Figure 6, they are defined by

$$\mathcal{U}_{\Psi\tilde{\Psi}} := \Psi^{-1}\left(\Psi(\mathcal{U}_\Psi) \cap \tilde{\Psi}(\mathcal{U}_{\tilde{\Psi}})\right), \qquad \mathcal{U}_{\tilde{\Psi}\Psi} := \tilde{\Psi}^{-1}\left(\Psi(\mathcal{U}_\Psi) \cap \tilde{\Psi}(\mathcal{U}_{\tilde{\Psi}})\right).$$

By definition the total space of the **weak tangent bundle** is the quotient space

$$\mathcal{E} = \mathcal{E}_{x_-x_+} := \left(\bigcup_{\Psi \in \mathcal{AP}_{x_-x_+}} \left(\mathcal{U}_\Psi \times L^2_{H_1}\right)\right) / \sim$$

where two points $(x, v) \in \mathcal{U}_\Psi \times L^2_{H_1}$ and $(\tilde{x}, \tilde{v}) \in \mathcal{U}_{\tilde{\Psi}} \times L^2_{H_1}$ are **equivalent**

$$(x, v) \sim (\tilde{x}, \tilde{v}) \quad :\Leftrightarrow \quad T\Phi_{\Psi\tilde{\Psi}}(x, v) = (\tilde{x}, \tilde{v}).$$

It follows from Corollary 4.16 that the transition maps are C^1. Hence $\mathcal{E}_{x_-x_+}$ is a C^1 manifold.

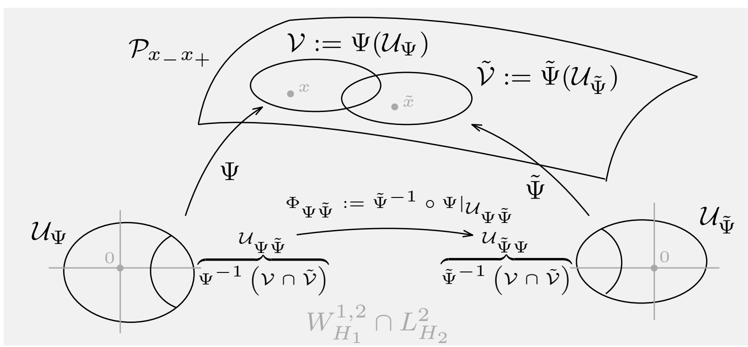

Figure 6: Transition map $\Phi_{\Psi\tilde{\Psi}}$ for path space $\mathcal{P}_{x_-x_+}$ modeled on $W^{1,2}_{H_1} \cap L^2_{H_2}$

Theorem 4.17. *The weak tangent bundle $\mathcal{E}_{x_-x_+}$ is a C^1 manifold modeled on the Hilbert space $\left(W^{1,2}_{H_1} \cap L^2_{H_2}\right) \times L^2_{H_1}$.*

Proof. By Corollary 4.16 the transition maps are C^1. □

A Hilbert space valued Sobolev spaces

In this appendix we collect and detail some results on Hilbert space valued Sobolev spaces, scattered throughout the literature, which we used throughout this paper. As required by the theorem of Pettis we assume that H is a *separable* Hilbert space. We denote the induced norm on H by $|\cdot|$.

A.1 Measurability

Definition A.1. (i) Let X be a topological space and $B(X)$ the **Borel $\boldsymbol{\sigma}$-algebra**, the smallest σ-algebra that contains the open sets; cf. Definition A.5.
(ii) A **measurable space** is a pair $(A, \mathcal{A})$ where A is a set and $\mathcal{A}$ is a σ-algebra in A. The elements of a σ-algebra are called **measurable sets**. A map $f\colon (A, \mathcal{A}) \to (X, \mathcal{B}(X))$ is called a **measurable map** if pre-images of measurable sets are measurable.
(iii) Let $(A, \mathcal{A})$ be a measurable space. A map $s\colon A \to H$ is called **simple** if it is of the form $s = \sum_{k=1}^{N} \chi_{A_k} x_k$ with $A_k \in \mathcal{A}$ and $x_k \in H$. Here χ_{A_k} is the **characteristic function** of the set A_k, i.e. $\chi_{A_k} \equiv 1$ on A_k and $\chi_{A_k} \equiv 0$ else.

Remark A.2. (i) Simple maps $f\colon A \to H$ are measurable. (ii) By the minimality of the Borel σ-algebra, a map $f\colon (A, \mathcal{A}) \to (H, \mathcal{B}(H))$ is measurable iff $f^{-1}(U) \in \mathcal{A}$ for all open sets $U \subset H$; see e.g. [Sal16, Thm. 1.20].

Definition A.3. A map $f\colon A \to H$ is **strongly measurable** if there is a sequence of simple functions $s_k\colon A \to H$ with $\lim_{k\to\infty} s_k = f$ pointwise on A.

Theorem A.4 (Pettis [Pet38])**.** *Let H be a separable Hilbert space and $(A, \mathcal{A})$ a measurable space. For a Hilbert space valued map $f : A \to H$ the following assertions are equivalent.*

(1) Every function $\langle f, x\rangle\colon (A, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $x \in H$, is measurable.

(2) The map $f\colon (A, \mathcal{A}) \to (H, \mathcal{B}(H))$ is measurable.

(3) The map $f\colon (A, \mathcal{A}) \to (H, \mathcal{B}(H))$ is strongly measurable.

Proof. We closely follow the arguments of Neerven [Nee07].
Let $f : A \to H$ be a Hilbert space valued map.
(1)⇒(3): As H is separable, so is the unit sphere, hence we can pick a dense sequence $(e_n)_{n=1}^\infty$ in the unit sphere of H. For any $x \in H$ the function

$$F_x\colon (A, \mathcal{A}) \to ([0, \infty), \mathcal{B}), \quad t \mapsto |f(t) - x| \tag{A.65}$$

is measurable. To prove this we write

$$\begin{aligned} |f(t) - x| = \sup_{|e|=1} |\langle f(t) - x, e\rangle| &= \sup_{n\in\mathbb{N}} |\langle f(t) - x, e_n\rangle| \\ &= \sup_{n\in\mathbb{N}} |\langle f(t), e_n\rangle - \langle x, e_n\rangle|. \end{aligned}$$

The function $t \mapsto \langle f(t), e_n \rangle$ is measurable by (1). Adding to a measurable function a constant term, here $\langle -x, e_n \rangle$, preserves measurability. Absolute value $|\cdot| \colon \mathbb{R} \to [0, \infty)$ is continuous, thus Borel measurable $|\cdot| \colon (\mathbb{R}, \mathcal{B}) \to ([0, \infty), \mathcal{B})$. Measurability is preserved under composition, hence

$$(A, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}) \stackrel{|\cdot|}{\to} ([0, \infty), \mathcal{B}), \quad t \mapsto |\langle f(t), e_n \rangle - \langle x, e_n \rangle|$$

is measurable. Taking the supremum of countably many measurable functions produces a measurable function; see e.g. [Sal16, Thm. 1.24 (ii) and Def. 1.17 (ii)]. This proves that the function (A.65) is measurable.

As H is separable we can pick a dense sequence $(x_n)_{n=1}^\infty$ in H. Define a sequence of functions

$$d_n \colon H \to \{x_1, \dots, x_n\}, \quad y \mapsto d_n(y) := x_{k(n,y)}$$

as follows. For each $y \in H$ let $k(n, y)$ be the least integer $k \in \{1, \dots, n\}$ which minimizes the distance to y, that is

$$|y - x_k| = \min_{j \in \{1,\dots,n\}} |y - x_j|, \qquad \forall \ell < k \colon |y - x_\ell| > |y - x_k|.$$

Observe that

$$\forall y \in H \colon \quad \lim_{n \to \infty} |d_n(y) - y| = 0 \tag{A.66}$$

since $(x_n)_{n=1}^\infty$ is dense in H. Define a sequence of functions

$$\begin{aligned} f_n \colon A &\to \{x_1, \dots, x_n\} \subset H \\ t &\mapsto d_n(f(t)) = x_{k(n,f(t))}. \end{aligned}$$

We show that these functions are a) simple and b) converge pointwise to f. To prove a) observe that for any $k \in \{1, \dots, n\}$ there is the identity

$$\begin{aligned} A_k^n :&= f_n^{-1}(x_k) \\ &= \{t \in A \mid x_{k(n,f(t))} = x_k\} \\ &= \left\{ t \in A \;\middle|\; |f(t) - x_k| = \min_{j \in \{1,\dots,n\}} |f(t) - x_j| \right\} \\ &\quad \bigcap \left\{ t \in A \;\middle|\; \underbrace{|f(t) - x_\ell|}_{=: F_{x_\ell}(t)} > |f(t) - x_k| \text{ for } \ell = 1, \dots, k-1 \right\} \\ &= \left(F_{x_k} - \min_{j \in \{1,\dots,n\}} F_{x_j} \right)^{-1}(0) \quad \cap \bigcap_{\ell=1}^{k-1} \left(F_{x_\ell} - F_{x_k} \right)^{-1}(0, \infty). \end{aligned}$$

Note that the sets on the right hand side are in the Lebesgue σ-algebra, in symbols $A_k^n \in \mathcal{A}$: Indeed functions of the form (A.65) are measurable. Moreover, minima and differences of measurable functions are measurable, see e.g. [Sal16, Thm. 1.24], and the complement of $\{0\}$ as well as $(0, \infty)$ are open sets, thus

Borel measurable sets. This proves that each function f_n is simple. To prove b) note that for each $t \in A$ we have

$$\lim_{n\to\infty} |f_n(t) - f(t)| = \lim_{n\to\infty} |d_n(f(t)) - f(t)| \overset{\text{(A.66)}}{=} 0.$$

This concludes the proof of assertion (3) in the theorem.

(3)⇒(2): To prove that $f^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}(H)$ it suffices to show that $f^{-1}(U) \in \mathcal{A}$ for every open set U in H; see Remark A.2. To this end let $U \subset H$ be open and, by (3), choose a sequence of simple functions $s_n \colon A \to H$ converging pointwise to f. For $r > 0$ define $U_r := \{x \in U \mid \operatorname{dist}(x, U^{\mathrm{c}}) > r\}$, where the closed set $U^{\mathrm{c}} := H \setminus U$ is the complement of U. Then $s_n^{-1}(U_r) \in \mathcal{A}$ for each $n \geq 1$, since U_r is open in H and simple functions are measurable. The equality

$$f^{-1}(U) = \bigcup_{m\geq 1} \bigcup_{n\geq 1} \bigcap_{k\geq n} \underbrace{s_k^{-1}(U_{\frac{1}{m}})}_{\in \mathcal{A}},$$

implies that $f^{-1}(U)$ lies in $\mathcal{A}$ since the right hand side does: Indeed σ-algebras are closed under countable unions and intersections, see e.g. [Sal16, § 1.1]. It remains to prove the equality of sets.

'$\subset$' Let $t \in f^{-1}(U)$. Since U is open and $f(t) \in U$, it holds that $d := \operatorname{dist}(f(t), U^{\mathrm{c}}) > 0$. Since $s_k(t)$ converges to $f(t)$, as $k \to \infty$, there exists $n_t \in \mathbb{N}$ such that $\operatorname{dist}(f(t), s_k(t)) < \frac{d}{2}$ for every $k \geq n_t$. Choose m_t such that $\frac{1}{m_t} < \frac{d}{2}$. We claim that $s_k(t) \in U_{\frac{1}{m_t}}$ for every $k \geq n_t$. To see this pick $x \in U^{\mathrm{c}}$ and let $k \geq n_t$. Then by the triangle inequality

$$d \leq \operatorname{dist}(f(t), x) \leq \operatorname{dist}(f(t), s_k(t)) + \operatorname{dist}(s_k(t), x) < \tfrac{d}{2} + \operatorname{dist}(s_k(t), x).$$

Thus $\operatorname{dist}(s_k(t), x) > \frac{d}{2}$. Since $x \in U^{\mathrm{c}}$ was arbitrary $\operatorname{dist}(s_k(t), U^{\mathrm{c}}) \geq \frac{d}{2} > \frac{1}{m_t}$. Hence $s_k(t) \in U_{\frac{1}{m_t}}$ for every $k \geq n_t$. In other words, the element t lies in

$$t \in \bigcap_{k\geq n_t} s_k^{-1}(U_{\frac{1}{m_t}})$$

for the particular m_t and n_t, therefore it lies in the union over all m and n, in symbols

$$t \in \bigcup_{m\geq 1} \bigcup_{n\geq 1} \bigcap_{k\geq n} s_k^{-1}(U_{\frac{1}{m}})$$

'$\supset$' Let t be element of the right hand side. Then there exist m_t and n_t with

$$t \in \bigcap_{k\geq n_t} s_k^{-1}(U_{\frac{1}{m_t}}).$$

This means that $\operatorname{dist}(s_k(t), U^{\mathrm{c}}) > \frac{1}{m_t}$ whenever $k \geq n_t$. Since $\lim_{k\to\infty} s_k(t) = f(t)$ it follows that $\operatorname{dist}(f(t), U^{\mathrm{c}}) \geq \frac{1}{m_t}$. But this means that $f(t) \in U$.

(2)⇒(1): True by two facts in measure theory: (i) Continuous maps are Borel measurable, here $\langle \cdot, x\rangle : (H, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$. (ii) The composition of a measurable map with a measurable map, here with $f : (A, \mathcal{A}) \to (H, \mathcal{B}(H))$, is measurable.

The proof of Theorem A.4 is complete. □

A.2 Lebesgue measure and really simple functions

Lebesgue measure

Definition A.5. Let $\mathcal{I}$ be the set of all open intervals $(a, b) \subset \mathbb{R}$ with $a < b \in \mathbb{R}$. The smallest σ-algebra containing $\mathcal{I}$ is the **Borel $\boldsymbol{\sigma}$-algebra $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}(\mathbb{R})$**, in symbols $\Sigma_{\mathcal{I}} = \mathcal{B}$; see e.g. [AE09, Thm. IX.1.11]. The Lebesgue measure λ on the Borel σ-algebra $\mathcal{B}$ is the unique translation invariant measure that satisfies $\lambda((a, b)) := b - a$ whenever $a < b$ are real numbers; see e.g. [Sal16, Thm. 2.1].

Definition A.6. A subset A of $\mathbb{R}$ is called **Lebesgue measurable** if there exist $A_- \subset A \subset A_+$ where $A_-, A_+ \in \mathcal{B}$ such that $\lambda(A_+ \setminus A_-) = 0$. For a Lebesgue measurable set A one defines its Lebesgue measure $\lambda(A) := \lambda(A_-) = \lambda(A_+)$.

The family $\mathcal{A}$ which consists of all Lebesgue measurable sets is a σ-algebra on $\mathbb{R}$ which contains the Borel σ-algebra $\mathcal{B}$. It is called the **Lebesgue $\boldsymbol{\sigma}$-algebra**. The map $\lambda \colon \mathcal{A} \to [0, \infty]$ is a measure in the sense that $\lambda(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$ whenever A_i is a sequence of pairwise disjoint Lebesgue measurable sets. One calls λ the **Lebesgue measure on** $\mathbb{R}$.

Really simple functions

Let H be a separable Hilbert space with induced norm $|\cdot|$. A function $r \colon \mathbb{R} \to H$ is called **really simple** if it is a finite sum of the form

$$r(x) = \sum_{k=1}^N \chi_{I_k} x_k, \qquad I_k = (a_k, b_k), \quad x_k \in H, \quad k = 1, \dots, N.$$

Note that $\nu(I_k) = b_k - a_k$ is finite. Since intervals are Lebesgue measurable r is in particular a simple function with respect to Lebesgue measurable space $(\mathbb{R}, \mathcal{A})$. On the other hand, any simple function $s \colon (a, b) \to H$ (see Definition A.1 for $A = (a, b)$ and $\mathcal{A}_{(a,b)} := \mathcal{A} \cap (a, b)$) can be arbitrarily well approximated in norm by really simple functions.

Theorem A.7. *For every simple function $s \colon (a, b) \to H$ and every $\varepsilon > 0$ there exists a really simple function $r_\varepsilon \colon (a, b) \to H$ such that*

$$\int_a^b |s(t) - r_\varepsilon(t)| \, dt < \varepsilon. \tag{A.67}$$

Proof. See e.g. [LL01, Thm. 1.18] for real valued functions. There simple functions are of the form $\sum_{k=1}^N \chi_{A_k} x_k$ and the sets A_k are required to be of finite measure. We choose as total space $A = (a, b)$, and not $A = \mathbb{R}$, so that this condition is satisfied automatically. Indeed $\lambda(A_k) \le \lambda((a, b)) = b - a < \infty$. Since simple functions take on only finitely many values, the proof for Hilbert valued functions remains the same. ☐

A.3 Bochner integral

In this section H is a separable Hilbert space and $I = (a, b)$ a bounded interval. By $\mathcal{A}$ we denote the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ and $\mathcal{A}_{(a,b)} := \mathcal{A} \cap (a, b)$. By $|\cdot|$ we denote the norm in H induced by the inner product.

Definition A.8. A map $f\colon \mathbb{R} \to H$ is called **Bochner integrable** if it has the following two properties:

(i) $\forall x \in H$ the function $\langle f, x\rangle\colon (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable.

(ii) The integral $\int_{\mathbb{R}} |f(t)|\, dt$ is finite.

Remark A.9. The map $f\colon (\mathbb{R}, \mathcal{A}) \to (H, \mathcal{B}(H))$ is measurable, by property (i) and Theorem A.4. Since $|\cdot|\colon H \to \mathbb{R}$ is continuous, the composition

$$|f(\cdot)|\colon (\mathbb{R}, \mathcal{A}) \xrightarrow{f} (H, \mathcal{B}(H)) \xrightarrow{|\cdot|} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable as well. Therefore the usual scalar Lebesgue integral

$$\int_{-\infty}^{\infty} |f(t)|\, dt \in [0, \infty)$$

is well defined.

In this section we explain how to define the **Bochner integral** $⨍$

$$⨍_I f\, dt \in H, \quad I = (a, b)$$

over bounded intervals ($a < b \in \mathbb{R}$) and show the estimate $|⨍_I f\, dt| \le \int_I |f(t)|\, dt$. The symbol $⨍$ is merely meant to distinguish the Hilbert valued Bochner integral from the usual real valued integral.

Definition A.10 (Bochner integral of simple functions)**.** Consider a simple function $s = \sum_{k=1}^N \chi_{A_k} x_k\colon \mathbb{R} \to H$ where $A_k \in \mathcal{A}$ and $x_k \in H$. Its Bochner integral over a bounded interval $I = (a, b)$ is defined by

$$⨍_I s\, dt := ⨍_{\mathbb{R}} \chi_I s\, dt := \sum_{k=1}^N \lambda(A_k \cap I) x_k \in H. \tag{A.68}$$

Here λ is the Lebesgue measure on $\mathbb{R}$ and $\mu(A_k \cap I) \le \mu(I) = b - a$ is finite. It is routine to show that the Bochner integral of simple functions

a) is independent of the representation of the simple function s;

b) is linear on the real vector space of simple functions;

c) satisfies the inequality $\left|⨍_I s\, dt\right| \le \int_I |s(t)|\, dt$.

Proofs are given e.g. in [AE09, p. 81 Rmk. X.2.1 (b,c,e)].

Definition A.11 (Bochner integral)**.** Let $f\colon \mathbb{R}\to H$ be Bochner integrable. Given a bounded interval $I=(a,b)$, by Theorem A.4 (1)⇔(3) for $A=(a,b)$, there is a sequence of simple functions $s_n\colon (a,b)\to H$ such that $\lim_{n\to\infty} s_n = f$ pointwise. By Proposition A.12 below the limit

$$\fint_a^b f\, dt = \fint_I f\, dt := \lim_{n\to\infty} \fint_I s_n\, dt \in H \tag{A.69}$$

exists in H and is independent of the approximating simple function sequence for f. It is the **Bochner integral of $\boldsymbol{f}$** with respect to Lebesgue measure.

Proposition A.12. *The limit (A.69) exists in H and it is independent of the approximating sequence (s_n) for f.*

Remark A.13. As a consequence of Theorem A.7 the Bochner integral of a Bochner integrable function $f\colon \mathbb{R}\to H$ over bounded intervals I, is given as the limit of Bochner integrals of really simple functions r_n, in symbols

$$\fint_{-T}^T f\, dt = \lim_{n\to\infty} \fint_{-T}^T r_n\, dt \in H. \tag{A.70}$$

The proof of Proposition A.12 is based on the following proposition.

Proposition A.14. *In the presence of the strong measurability property*

(i) *$f\colon (a,b)\to H$ is a map and $s_n\colon (a,b)\to H$ is a sequence of simple functions such that $\lim_{n\to\infty} s_n = f$ pointwise*

the following two properties are equivalent

$$\text{(ii)} \quad \int_a^b |f(t)|\, dt < \infty \qquad \Longleftrightarrow \qquad \text{(iii)} \quad \lim_{n\to\infty} \int_a^b |s_n(t) - f(t)|\, dt = 0$$

and in either case

$$\left|\fint_a^b f\, dt\right| \le \int_a^b |f(t)|\, dt. \tag{A.71}$$

Proof. (i,iii)⇒(ii): Add zero and use the triangle inequality and (iii) to obtain

$$\int_a^b |f(t)|\, dt \le \int_a^b |f(t)-s_n(t)|\, dt + \int_a^b \underbrace{|s_n(t)|}_{\le 1}\, dt \stackrel{n\to\infty}{\longrightarrow} b-a<\infty.$$

(i,ii)⇒(iii): The function defined by

$$\sigma_n := \chi_{\{|s_n|\le 2|f|\}} s_n$$

is simple. Given t, for large n one has $\sigma_n(t) = s_n(t)$. Consequently $\sigma_n\to f$ pointwise. Furthermore, there is the pointwise inequality

$$F_n(t) := |\sigma_n(t)| \le 2|f(t)| =: G(t)$$

between integrable (here (ii) enters) scalar functions. Continuity of the norm yields that $F_n(t) \to F(t) := |f(t)|$ pointwise, as $n \to \infty$. Thus the Lebesgue dominated convergence theorem, see e.g. [AE09, Thm. X.3.12], applies and yields (iii) as well as the following identity to be used right below

$$\lim_{n\to\infty} \int_a^b |\sigma_n(t)|\, dt = \int_a^b |f(t)|\, dt.$$

Final estimate: Suppose (i,ii). Step one in what follows is by (A.69), step two is by continuity of the norm, for the final inequality see Definition A.10 c)

$$\begin{aligned} \left| \fint_a^b f\, dt \right| &= \left| \lim_{n\to\infty} \fint_a^b \sigma_n\, dt \right| \\ &= \lim_{n\to\infty} \left| \fint_a^b \sigma_n\, dt \right| \\ &\overset{3}{\le} \lim_{n\to\infty} \int_a^b |\sigma_n(t)|\, dt. \end{aligned}$$

The previous displayed identity proves the estimate and Proposition A.14. □

Proof of Proposition A.12. The proof has two steps.

Step 1. The elements $y_{s_n} := \fint_a^b s_n\, dt$ form a Cauchy sequence in H.

To see this observe that

$$\begin{aligned} |y_{s_k} - y_{s_\ell}| &= \left| \fint_a^b (s_k - s_\ell)\, dt \right| \\ &\overset{\text{c)}}{\le} \int_a^b |s_k(t) - s_\ell(t)|\, dt \\ &\overset{\Delta}{\le} \int_a^b |s_k(t) - f(t)|\, dt + \int_a^b |f(t) - s_\ell(t)|\, dt \end{aligned}$$

The assertion of Step 1 now follows by part (iii) of Proposition A.14.

Step 2. Let s_n and σ_n be sequences of simple functions both converging pointwise to f. Then both sequences of Bochner integrals y_{s_n} and z_{σ_n} converge in H and have the same limit.

By Step 1 there are elements y and z of H such that $y_{s_n} \to y$ and $z_{\sigma_n} \to z$, as $n \to \infty$. Observe that by continuity of the norm we get inequality one

$$\begin{aligned} |y - z| &\le \lim_{n\to\infty} \left| \fint_a^b s_n\, dt - \fint_a^b \sigma_n\, dt \right| \\ &\overset{\text{c)}}{\le} \lim_{n\to\infty} \int_a^b |s_n(t) - \sigma_n(t)|\, dt \\ &\overset{\Delta}{\le} \lim_{n\to\infty} \int_a^b |s_n(t) - f(t)|\, dt + \lim_{n\to\infty} \int_a^b |f(t) - \sigma_n(t)|\, dt \\ &\overset{\text{(iii)}}{=} 0. \end{aligned}$$

The penultimate step is by the triangle inequality after adding zero and by linearity of the real-valued integral. The ultimate step is by Proposition A.14. This proves Step 2 and Proposition A.12. □

Lemma A.15. *Let $f\colon \mathbb{R} \to H$ be Bochner integrable, $x \in H$, and $a < b \in \mathbb{R}$. Then*

$$\left\langle \fint_a^b f\,dt, x \right\rangle = \int_a^b \langle f(t), x\rangle\,dt.$$

Proof. By linearity of the inner product this is true for simple functions, hence the Lemma follows by approximating f by simple functions in view of the continuity of the inner product. □

A.4 Hilbert space valued Lebesgue and Sobolev spaces

Definition A.16. Let $p \in [1, \infty)$. The space $L^p = L^p(\mathbb{R}, H)$ consists of equivalence classes $[f]$ of functions $f\colon \mathbb{R} \to H$ satisfying the following two conditions.

(i) Every function $\langle f, x\rangle\colon (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$, where $x \in H$, is measurable.

(ii) The $\boldsymbol{L^p}$**-norm** $\|f\|_p := \|f\|_{L^p(\mathbb{R},H)} := \left(\int_{\mathbb{R}} |f(t)|^p dt\right)^{1/p} < \infty$ is finite.

Two functions f and g satisfying (i) and (ii) are called **equivalent** if they coincide outside a set of Lebesgue measure zero. To simplify notation we write the equivalence class $[f]$ still as f. On $L^2(\mathbb{R}, H)$ there is the inner product

$$\langle f, g\rangle_{L^2(\mathbb{R},H)} := \int_{\mathbb{R}} \langle f(t), g(t)\rangle \, dt.$$

The vector space $L^2(\mathbb{R}, H)$ endowed with this inner product becomes a Hilbert space and the $(L^p, \|\cdot\|_p)$ are Banach spaces; see e.g. [FW21, App. A.2].

Lemma A.17. *For $p \in [1, \infty)$ every $f \in L^p(\mathbb{R}, H)$ restricted to a bounded interval $I = (a, b)$ is Bochner integrable along (a, b).*

Proof. By Hölder $\int_{\mathbb{R}} |f(t)\chi_{(a,b)}(t)|\,dt = \int_a^b |f(t)|\,dt \le (b-a)^{1-\frac{1}{p}} \|f\|_p < \infty$. □

Definition A.18 (Sobolev space)**.** Let $W^{1,2}(\mathbb{R}, H)$ be the vector space of all $f \in L^2(\mathbb{R}, H)$ for which there exists an element $v \in L^2(\mathbb{R}, H)$ such that

$$\int_{\mathbb{R}} \langle f(t), \dot\varphi(t)\rangle\,dt = -\int_{\mathbb{R}} \langle v(t), \varphi(t)\rangle\,dt \tag{A.72}$$

for every $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}, H)$. If such a map v exists, then it is unique and called the **weak derivative** of f. We denote v by the symbol $\dot f$ or f'.
The vector space $W^{1,2}(\mathbb{R}, H)$ is endowed with the $\boldsymbol{W^{1,2}}$**-inner product**

$$\langle f, g\rangle_{W^{1,2}(\mathbb{R},H)} := \langle f, g\rangle_{L^2(\mathbb{R},H)} + \langle \dot f, \dot g\rangle_{L^2(\mathbb{R},H)}. \tag{A.73}$$

and the induced $\boldsymbol{W^{1,2}}$**-norm**, notation $\|f\|_{W^{1,2}(\mathbb{R},H)} := (\langle f, g\rangle_{W^{1,2}(\mathbb{R},H)})^{\frac{1}{2}}$.
The vector space $W^{1,2}(\mathbb{R}, H)$ endowed with this inner product is a Hilbert space; see e.g. [FW21, Prop. A.8].

The Banach space $L^\infty(\mathbb{R}, H)$

Definition A.19 (Essentially bounded functions)**.** Let $\mathcal{L}^\infty(\mathbb{R}, H)$ be the set of all functions $f\colon \mathbb{R} \to H$ which have the following two properties:

(a) $\forall x \in H$ the function $\langle f, x\rangle\colon (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable.

(b) There exists $r \in [0, \infty)$ such that $\{\|f\| > r\}$ is of Lebesgue measure zero.

The functions $f \in \mathcal{L}^\infty(\mathbb{R}, H)$ are called **essentially bounded**.

Proposition A.20. *The space $\mathcal{L}^\infty(\mathbb{R}, H)$ is linear and complete with respect to the semi-norm $\|f\|_\infty = \|f\|_{\mathcal{L}^\infty(\mathbb{R},H)}$ defined as the infimum of essential bounds*

$$\|f\|_\infty := \inf\bigl\{r \geq 0 \mid \textit{the set } \{\|f\| > r\} \textit{ has Lebesgue measure zero}\bigr\}.$$

Proof. To see that $\mathcal{L}^\infty(\mathbb{R}, H)$ is a real vector space note that property (a) follows from the fact that the space of measurable functions $\langle f, x\rangle\colon (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is a vector space. Concerning property (b), given $\alpha \in \mathbb{R}$ and $f, g \in \mathcal{L}^\infty(\mathbb{R}, H)$, then $\|\alpha f + g\|_\infty \leq |\alpha|\,\|f\|_\infty + \|g\|_\infty$; see e.g. [AE09, Rmk. X.4.1 (c)]. For a proof that $\mathcal{L}^\infty(\mathbb{R}, H)$ is complete see e.g. [AE09, Thm. X.4.6]. □

On $\mathcal{L}^\infty(\mathbb{R}, H)$ consider the equivalence relation where $f \sim g$ if the two maps are equal outside a set of Lebesgue measure zero. On the quotient space

$$L^\infty(\mathbb{R}, H) := \mathcal{L}^\infty(\mathbb{R}, H)/\sim$$

the semi-norm $\|\cdot\|_\infty$ is a norm. Hence $L^\infty(\mathbb{R}, H)$ is a Banach space by Proposition A.20. To ease notation we still denote the elements of $L^\infty(\mathbb{R}, H)$ by f.

A.5 Differentiable compactly supported approximation

Convolution

Fix a smooth function $\rho\colon \mathbb{R} \to [0, \infty)$ which is supported in $[-1, 1]$, is symmetric $\rho(\cdot) = \rho(-\cdot)$, and satisfies $\int_\mathbb{R} \rho(t)\, dt = 1$; see e.g. [AE09, Ex. X.7.12 b)]. Define

$$\rho_\mu(t) := \tfrac{1}{\mu}\rho(\tfrac{s}{\mu})$$

for $\mu > 0$. This function is supported in $[-\mu, \mu]$ and has the properties that

$$\int_\mathbb{R} \rho_\mu(t)\, dt = \|\rho_\mu\|_{L^1(\mathbb{R})} = \|\rho\|_{L^1(\mathbb{R})} = 1 \tag{A.74}$$

and that $\int_{\mathbb{R}\setminus(-r,r)} \rho_\mu(t)\, dt \to 0$, as $\mu \to 0$, for any given $r > 0$.

Definition A.21. Let $f\colon \mathbb{R} \to H$ be a function which is Bochner integrable. The **convolution** of f by ρ_μ at time $t \in \mathbb{R}$ is the Bochner integral

$$f_\mu(t) := (f * \rho_\mu)(t) := \fint_{-\mu}^{\mu} f(t-s)\rho_\mu(s)\, ds = \fint_{t-\mu}^{t+\mu} f(s)\rho_\mu(t-s)\, ds. \tag{A.75}$$

Lemma A.22. *The Bochner integral (A.75), thus convolution, is well defined.*

Proof. (i) The integrand is measurable. Indeed, given $t \in \mathbb{R}$ and $x \in H$, then the map $(\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$, $s \mapsto \langle \rho_\mu(s) f(t-s), x\rangle = \rho_\mu(s) \langle f(t-s), x\rangle$, is measurable since ρ_μ is continuous, and therefore measurable, the second term is measurable by assumption that f is Bochner integrable, see Definition A.8, and the product of measurable functions is measurable. (ii) By Young's inequality for the convolution of real valued functions the norm $\|\rho_\mu * f\|_{L^1(\mathbb{R},H)} \le \|\rho_\mu\|_{L^1(\mathbb{R},H)} \|f\|_{L^1(\mathbb{R},H)} = \|f\|_{L^1(\mathbb{R},H)}$ is finite. ☐

Lemma A.23. *For $f \in L^2(\mathbb{R}, H)$ the convolution $f_\mu := f * \rho_\mu$ lies in $C^1(\mathbb{R}, H)$ and the derivative is $\frac{d}{dt}(f * \rho_\mu) = f * \frac{d}{dt}\rho_\mu$ whenever $\mu > 0$.*

Proof. To see this let $\mu > 0$. Since ρ, and therefore ρ_μ, is C^1 and compactly supported its derivative $\dot\rho_\mu$ is uniformly continuous, so the following is true: Given $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that if $|\sigma - \tilde\sigma| < \delta_\varepsilon$, then

$$|\dot\rho_\mu(\sigma) - \dot\rho_\mu(\tilde\sigma)| < \varepsilon. \tag{A.76}$$

Choose a nonzero number $h \in (-\delta_\varepsilon, \delta_\varepsilon)$. For $t \in \mathbb{R}$ we calculate

$$\begin{aligned}
\left|\tfrac{\rho_\mu(t+h)-\rho_\mu(t)}{h} - \dot\rho_\mu(t)\right| &= \left|\frac{1}{h}\int_0^1 \underbrace{\tfrac{d}{d\tau}\rho_\mu(t+\tau h)}_{\dot\rho_\mu(t+\tau h)\cdot h}\, d\tau - \dot\rho_\mu(t)\right| \\
&\le \int_0^1 \underbrace{|\dot\rho_\mu(t+\tau h) - \dot\rho_\mu(t)|}_{<\varepsilon \text{ by (A.76)}}\, d\tau \\
&< \varepsilon.
\end{aligned}$$

Use first estimate (A.71) and then the previous estimate to obtain

$$\begin{aligned}
&\left|\tfrac{f_\mu(t+h)-f_\mu(t)}{h} - f * \dot\rho_\mu(t)\right| \\
&= \left|\int_{t-\mu-\delta}^{t+\mu+\delta} f(s)\left(\tfrac{\rho_\mu(t+h-s)-\rho_\mu(t-s)}{h} - \dot\rho_\mu(t-s)\right) ds\right| \\
&\le \int_{t-\mu-\delta}^{t+\mu+\delta} |f(s)| \underbrace{\left|\tfrac{\rho_\mu(t+h-s)-\rho_\mu(t-s)}{h} - \dot\rho_\mu(t-s)\right|}_{<\varepsilon} ds \\
&< \varepsilon\sqrt{2(\mu+\delta)}\,\|f\|_{L^2(\mathbb{R},H)}.
\end{aligned}$$

In the last step we used Hölder's inequality for the product integrand $1 \cdot |f(s)|$. This proves that the derivative of f_μ is given by $f * \dot\rho_\mu$.

It remains to show continuity of the derivative. Given $t \in \mathbb{R}$ and $h \in (-\delta_\varepsilon, \delta_\varepsilon)$

as above, we calculate

$$\begin{aligned}
&\left|\dot f_\mu(t+h)-\dot f_\mu(t)\right|\\
&=\left|\int_{t-\mu-\delta}^{t+\mu+\delta} f(s)\bigl(\dot\rho_\mu(t+h-s)-\dot\rho_\mu(t-s)\bigr)ds\right|\\
&\le \int_{t-\mu-\delta}^{t+\mu+\delta}|f(s)|\underbrace{|\dot\rho_\mu(t+h-s)-\dot\rho_\mu(t-s)|}_{<\varepsilon \text{ by (A.76)}}ds\\
&<\varepsilon\sqrt{2(\mu+\delta)}\|f\|_{L^2(\mathbb{R},H)}.
\end{aligned}$$

This proves continuity of $\dot f_\mu$ and concludes the proof of Claim 1. □

Lebesgue space

Theorem A.24 (C^1_c approximation of L^2)**.** *For any $f\in L^2(\mathbb{R},H)$ it holds that*

$$\lim_{\mu\to 0}\|f_\mu-f\|_{L^2(\mathbb{R},H)}=0,\qquad f_\mu\overset{(A.75)}{:=}f*\rho_\mu\in C^1(\mathbb{R},H). \tag{A.77}$$

More is true, namely $C^1_c(\mathbb{R},H)$ is a dense subset of $L^2(\mathbb{R},H)$.

One can replace $C^1_c(\mathbb{R},H)$ by $C^\infty_c(\mathbb{R},H)$ with some more work, but for our purposes this is not needed.

Proof. The proof has six steps. For $I\subset\mathbb{R}$ we often abbreviate $L^1_I:=L^1(I,H)$.
Step 1 (Reduction to L^1). It suffices to prove approximation (A.77) in L^1. To see this pick $f\in L^2(\mathbb{R},H)$ and $\varepsilon>0$. Then there exists $h=h_\varepsilon>0$ such that $\|f\|_{L^2(\{|f|>h\})}<\frac{\varepsilon}{4}$. For the truncated function $f^h:=\chi_{\|f\|<h}f$ we get

$$\|f^h-f\|_{L^2_{\mathbb{R}}}=\left(\int_{\{|f|>h\}}f^2(t)\,dt\right)^{1/2}<\frac{\varepsilon}{4}.$$

By assumption approximation holds in L^1, so there exists $\mu=\mu_\varepsilon>0$ such that

$$\|(f^h)_\mu-f^h\|_1<\frac{\varepsilon^2}{8h}.$$

Note that $\|f^h\|_\infty\le h$. Now use this estimate to get for every $t\in\mathbb{R}$ the estimate $|(\rho_\mu*f^h)(t)|\le\int_{-\mu}^{\mu}\rho_\mu(s)|f^h(t-s)|\,ds\le h\|\rho_\mu\|_1=h$. With this we get

$$\begin{aligned}
\|(f^h)_\mu-f^h\|_2^2&=\int_{\mathbb{R}}|(f^h)_\mu(t)-f^h(t)|^2\,dt\\
&\le\bigl(\|(f^h)_\mu\|_\infty+\|f^h\|_\infty\bigr)\int_{\mathbb{R}}|(f^h)_\mu(t)-f^h(t)|\,dt\\
&\le 2h\|(f^h)_\mu-f^h\|_1.
\end{aligned}$$

Using the above three displayed inequalities we estimate

$$\begin{aligned}\|f_\mu - f\|_2 &\le \|\rho_\mu * (f - f^h)\|_2 + \|(f^h)_\mu - f^h\|_2 + \|f^h - f\|_2 \\ &\le \|\rho_\mu\|_1 \|f - f^h\|_2 + \sqrt{2h\|(f^h)_\mu - f^h\|_1} + \|f^h - f\|_2 \\ &\le \tfrac{\varepsilon}{2} + \sqrt{\tfrac{\varepsilon^2}{4}} \\ &= \varepsilon.\end{aligned}$$

In step two we used Young's inequality $\|gh\|_r \le \|g\|_p \|h\|_q$ where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ for $r = q = 2$ and $p = 1$. In step three we used (A.74).

Step 2 (C^1 approximation near ∞). Pick $f \in L^2(\mathbb{R}, H)$ and $\varepsilon > 0$. There exists $T = T(\varepsilon) > 1$ such that $\|f_\mu - f\|_{L^1_{\mathbb{R}\setminus[-T,T]}} < \frac{\varepsilon}{2}$ whenever $\mu \in (0, 1]$.

Since $f \in L^1$ there exists $T > 1$ such that the integral near infinity is small

$$\|f\|_{L^1_{\mathbb{R}\setminus[-T+1,T-1]}} < \tfrac{\varepsilon}{4}. \tag{A.78}$$

We estimate

$$\begin{aligned}\|f_\mu\|_{L^1_{\mathbb{R}\setminus[-T,T]}} &= \|\rho_\mu * f\|_{L^1_{\mathbb{R}\setminus[-T,T]}} \\ &= \|\rho_\mu * f\chi_{\mathbb{R}\setminus[-T+1,T-1]}\|_{L^1_{\mathbb{R}\setminus[-T,T]}} \\ &\le \|\rho_\mu * f\chi_{\mathbb{R}\setminus[-T+1,T-1]}\|_{L^1_{\mathbb{R}}} \\ &\le \|\rho_\mu\|_{L^1_{\mathbb{R}}} \|f\chi_{\mathbb{R}\setminus[-T+1,T-1]}\|_{L^1_{\mathbb{R}}} \\ &= \|f\|_{L^1_{\mathbb{R}\setminus[-T+1,T-1]}} \\ &< \tfrac{\varepsilon}{4}.\end{aligned}$$

Step two uses that $\mu \le 1$ and we multiplied by a characteristic function which is identically 1 on that domain. Step three is by monotonicity of the integral. Step four is by Young's inequality. Step five uses that $\|\rho_\mu\|_{L^1(\mathbb{R})} = 1$. The final step six is by (A.78).

To conclude the proof of Step 2 we estimate

$$\|f_\mu - f\|_{L^1_{\mathbb{R}\setminus[-T,T]}} \le \|f_\mu\|_{L^1_{\mathbb{R}\setminus[-T,T]}} + \|f\|_{L^1_{\mathbb{R}\setminus[-T,T]}} < \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} = \tfrac{\varepsilon}{2}.$$

Here we used (A.78) and the subsequent estimate

Step 3 (Approximate f by really simple function r). There is a really simple function $r\colon [-T-1, T+1] \to H$ such that $\|f - r\|_{L^1([-T-1,T+1],H)} < \frac{\varepsilon}{6}$.

By Proposition A.14 (iii) we approximate f by a simple function s which, by (A.67), we approximate by a really simple function r.

Step 4 (C^1 approximation of r). There exists a constant $\mu_\varepsilon \in (0, 1]$ such that $\|r_\mu - r\|_{L^1(\mathbb{R},H)} < \frac{\varepsilon}{6}$ whenever $\mu \in (0, \mu_\varepsilon]$.

A really simple function $r\colon [-T, T] \to H$ is of the form $r = \sum_{k=1}^N \chi_{I_k} x_k$ where each I_k is an interval. Note that $|(\rho_\mu * \chi_{I_k}) - \chi_{I_k}| \le 1$ and that for every t outside of the intervals $(a_k - \mu, a_k + \mu)$ and $(b_k - \mu, b_k + \mu)$ the function $|(\rho_\mu * \chi_{I_k})(t) - \chi_{I_k}(t)| = 0$ vanishes. Consequently $\int_{-\infty}^{\infty} |(\rho_\mu * \chi_{I_k}) - \chi_{I_k}| \, dt \le 4\mu$ and therefore $\|r_\mu - r\|_{L^1(\mathbb{R},H)} \le 4\mu N \kappa$ where $\kappa := \max\{\|x_1\|, \dots, \|x_k\|$. Hence Step 4 follows by choosing $\mu_\varepsilon < \varepsilon/(24N\kappa)$.

Step 5 (C^1 approximation of f). Given $\mu \in (0, \mu_\varepsilon] \subset (0,1]$, then we have $\|f_\mu - f\|_{L^1(\mathbb{R},H)} < \varepsilon$. Equivalently, this proves (A.77).

To prove this we estimate

$$
\begin{aligned}
\|f_\mu - r_\mu\|_{L^1_{[-T,T]}} &= \|\rho_\mu * (f - r)\|_{L^1_{[-T,T]}} \\
&= \|\rho_\mu * (f - r)\chi_{[-T-1,T+1]}\|_{L^1_{[-T,T]}} \\
&\le \|\rho_\mu * (f - r)\chi_{[-T-1,T+1]}\|_{L^1_{\mathbb{R}}} \\
&\le \|\rho_\mu\|_1 \|(f - r)\chi_{[-T-1,T+1]}\|_{L^1_{\mathbb{R}}} \\
&= \|f - r\|_{L^1_{[-T-1,T+1]}} \\
&< \tfrac{\varepsilon}{6}.
\end{aligned}
\tag{A.79}
$$

Step two uses that $\operatorname{supp} \rho_\mu \subset [-\mu, \mu] \subset [-1, 1]$ and we multiplied by a characteristic function which is identically 1 on that domain. Step four is by Young's inequality. Step five uses that $\|\rho_\mu\|_1 = \|\rho\|_1 = 1$. The final step six is by Step 3.

Now we decompose $\mathbb{R} = \mathbb{R} \setminus [-T,T] \cup [-T,T]$ and use Step 2 to obtain

$$
\begin{aligned}
\|f_\mu - f\|_{L^1(\mathbb{R},H)} &= \|f_\mu - f\|_{L^1_{\mathbb{R}\setminus[-T,T]}} + \|f_\mu - f\|_{L^1_{[-T,T]}} \\
&\le \tfrac{\varepsilon}{2} + \|f_\mu - r_\mu\|_{L^1_{[-T,T]}} + \|r_\mu - r\|_{L^1_{[-T,T]}} + \|r - f\|_{L^1_{[-T,T]}} \\
&< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{6} + \tfrac{\varepsilon}{6} + \tfrac{\varepsilon}{6} = \varepsilon.
\end{aligned}
$$

The final inequality is by (A.79) and Steps three and four.

Since $\varepsilon > 0$ was arbitrary we get $\lim_{\mu \to 0} \|f_\mu - f\|_{L^1(\mathbb{R},H)} = 0$. This also proves L^2 convergence by Step 1. The proof of Step 5 is complete.

Step 6 (Compact support). For any $f \in L^{1,2}(\mathbb{R}, H)$ the sequence $F_k \in C^1_0(\mathbb{R}, H)$ defined prior to (A.81) converges to f in L^2.

In (A.81) replace the $W^{1,2}$ norm by the L^2 norm.

This concludes the proof of Theorem A.24. □

Sobolev space

Theorem A.25 (C^1 approximation of $W^{1,2}$)**.** *The set $C^1_{\mathrm{c}}(\mathbb{R}, H)$ of smooth compactly supported maps is a dense subset of the Hilbert space $W^{1,2}(\mathbb{R}, H)$.*

The proof of the theorem uses the following lemma.

Lemma A.26. *For $f \in W^{1,2}(\mathbb{R}, H)$ it holds $(f * \rho_\mu)' = f' * \rho_\mu$ whenever $\mu > 0$.*

Proof. Pick $\varphi \in C_c^\infty(\mathbb{R}, H)$. Fix $T > 0$ such that $\operatorname{supp}\varphi \subset [-T, T]$. Abbreviate $I := [-T-\mu, T+\mu]$. We compute by definition (A.75) of convolution

$$\begin{aligned}
\int_{\mathbb{R}_t} \langle (f' * \rho_\mu)(t), \varphi(t)\rangle \, dt &\overset{1}{=} \int_{I_t}\int_{I_s} \langle f'(s)\rho_\mu(t-s), \varphi(t)\rangle \, dsdt \\
&\overset{2}{=} \int_{I_s}\int_{I_t} \langle f'(s), \rho_\mu(t-s)\varphi(t)\rangle \, dtds \\
&\overset{3}{=} \int_{I_s} \langle f'(s), (\rho_\mu * \varphi)(s)\rangle \, ds \\
&\overset{4}{=} -\int_{I_s} \left\langle f(s), \underbrace{(\rho_\mu * \varphi)'(s)}_{(\rho_\mu * \dot\varphi)(s)} \right\rangle ds \\
&\overset{5}{=} -\int_{I_s}\int_{I_t} \underbrace{\langle f(s), \rho_\mu(s-t)\dot\varphi(t)\rangle}_{\langle f(s)\rho_\mu(s-t), \dot\varphi(t)\rangle} dtds \\
&\overset{6}{=} -\int_{\mathbb{R}_t} \langle (f * \rho_\mu)(t), \dot\varphi(t)\rangle \, dt.
\end{aligned}$$

Steps 1, 3, 5, and 6 are by Lemma A.15. Step 2 is by the Theorem of Fubini, see e.g. [AE09, Thm. X.6.16], which applies since the integrand is absolutely integrable: Indeed the integral

$$\begin{aligned}
\int_{I_t}\int_{I_s} |\langle f'(s)\rho_\mu(t-s), \varphi(t)\rangle| \, dsdt &\le 2(T+\mu)\tfrac{1}{\mu}\|\varphi\|_\infty \int_{I_s} |f'(s)| \, ds \\
&\le (2T+2\mu)^{\frac{3}{2}}\tfrac{1}{\mu}\|\varphi\|_\infty \|f'\|_2
\end{aligned}$$

is finite. Step 3 also uses that $\rho(t) = \rho(-t)$ has been chosen symmetric. Step 4 is by definition (A.72) of weak derivative. We then used Lemma A.23 for $H = \mathbb{R}$. Since φ was an arbitrary test function this proves Lemma A.26. □

Proof of Theorem A.25. The proof has two steps.

Step 1 (C^1 approximation). $C^1(\mathbb{R}, H) \cap W^{1,2}(\mathbb{R}, H)$ is dense in $W^{1,2}(\mathbb{R}, H)$.

To prove this pick $f \in W^{1,2}(\mathbb{R}, H)$. In particular f and its weak derivative f' are in $L^2(\mathbb{R}, H)$. Hence, applying twice Theorem A.24, we have convergence

$$f_\mu \xrightarrow{L^2} f, \qquad (f')_\mu \xrightarrow{L^2} f', \qquad \text{as } \mu \to 0.$$

Since $(f')_\mu = (f_\mu)'$, by Lemma A.26, this shows that

$$\lim_{\mu\to 0} \|f_\mu - f\|_{W^{1,2}(\mathbb{R},H)} = 0, \qquad f_\mu \overset{\text{(A.75)}}{:=} f * \rho_\mu \in C^1(\mathbb{R}, H). \tag{A.80}$$

Step 2 (C^1 approximation with compact support). For any $f \in W^{1,2}(\mathbb{R}, H)$ there is a sequence $F_k \in C_0^1(\mathbb{R}, H)$ which converges to f in $W^{1,2}$.

To see this pick $\phi \in C^\infty_c(\mathbb{R},[0,1])$ supported in $[-2,2]$ and with $\phi \equiv 1$ on $[-1,1]$. For $k \in \mathbb{N}$ set $\phi_k(t) := \phi(\frac{t}{k})$ and $g_k := \phi_k f$. We claim that $g_k \to f$ in $W^{1,2}(\mathbb{R},H)$. Indeed since $\phi_k \equiv 1$ on $\{|t| \le k\}$ the integral domain reduces to

$$\begin{aligned}\|f-g_k\|_2^2 &= \int_{|t|>k} \underbrace{(1-\phi_k(t))}_{\le 1}(1-\phi_k(t)) \cdot |f(t)|^2 dt\\ &\le \int_{|t|>k} |f(t)|^2 dt \to 0, \text{ as } k \to 0,\end{aligned}$$

and since $\dot g_k(t) = \phi_k(t)\dot f(t) + k\dot\phi(kt) f(t)$ we get

$$\begin{aligned}\|\dot f - \dot g_k\|_2^2 &= \int_{\mathbb{R}} |(1-\phi(\tfrac{t}{k}))\dot f(t) - \tfrac{1}{k}\dot\phi(\tfrac{t}{k}) f(t)|^2 dt\\ &\le 2\int_{|t|>k} |\dot f(t)|^2 dt + \frac{2}{k^2}\|\dot\phi\|_\infty \|f\|_2^2 \to 0, \text{ as } k \to \infty.\end{aligned}$$

The sequence of compactly supported smooth functions $F_k := \phi_k(f * \rho_k)$ approximates f in $W^{1,2}$, indeed

$$\begin{aligned}\|f-F_k\|_{1,2} &= \|f - \phi_k f + \phi_k(f - f*\rho_k)\|_{1,2}\\ &\le \underbrace{\|f-g_k\|_{1,2}}_{\to 0 \text{ shown above}} + \underbrace{\|f - f*\rho_k\|_{1,2}}_{\to 0 \text{ by (A.80)}}.\end{aligned} \tag{A.81}$$

In step 1 we added zero and step 2 uses the triangle inequality and that $\phi_k \le 1$. This proves Step 2 and concludes the proof of Theorem A.25. □

A.6 Sobolev embedding

Theorem A.27. *Any element $v \in W^{1,2}(\mathbb{R},H)$ satisfies the estimate*

$$\|v\|_\infty \le \|v\|_{1,2}. \tag{A.82}$$

Proof. By Theorem A.25 we can assume without loss of generality that $v \in C^1_c(\mathbb{R},H)$. By the fundamental theorem of calculus in combination with compact support at any time $t \in \mathbb{R}$ we estimate

$$\begin{aligned}|v(t)|^2 &= \int_{-\infty}^t \underbrace{\tfrac{d}{d\sigma}|v(\sigma)|^2}_{=2\langle v'(\sigma),v(\sigma)\rangle}\; d\sigma\\ &\stackrel{2}{\le} \int_{-\infty}^t \left(|v'(\sigma)|^2 + |v(\sigma)|^2\right) d\sigma\\ &\stackrel{3}{\le} \int_{-\infty}^\infty \left(|v'(\sigma)|^2 + |v(\sigma)|^2\right) d\sigma\\ &= \|v\|^2_{W^{1,2}(\mathbb{R},H)}.\end{aligned}$$

In step 2 we used Cauchy-Schwarz and then Young's inequality $ab \le a^2/2 + b^2/2$ whenever $a, b \ge 0$. Since $t \in \mathbb{R}$ was arbitrary, estimate (A.27) follows. □

Remark A.28. The proof shows that if $\tau \in \mathbb{R}$ and $v \in W^{1,2}((-\infty,\tau),H)$, then instead of estimate (A.27) we have

$$\|v\|_{L^\infty((-\infty,\tau),H)} \le \|v\|_{W^{1,2}((-\infty,\tau),H)}.$$

To see this in step 3 of the estimate just replace $\int_{-\infty}^{\infty}$ by $\int_{-\infty}^{\tau}$ Similarly, if $v \in W^{1,2}((\tau,\infty),H)$, then we obtain

$$\|v\|_{L^\infty((\tau,\infty),H)} \le \|v\|_{W^{1,2}((\tau,\infty),H)}.$$

B Implicit Function Theorem

B.1 Quantitative

We denote by $B_r(x;X)$ the open ball of radius r centered at x in a Banach space X. We often abbreviate $B_r(x) := B_r(x;X)$ and $B_r := B_r(0;X)$.

Lemma B.1 ([MS04, Le. A.3.2])**.** *Let $\gamma < 1$ and R be positive real numbers. Let X be a Banach space, $x_0 \in X$, and $\varphi\colon B_R(x_0) \to X$ be a continuously differentiable map such that*

$$\|\mathrm{Id} - d\varphi(x)\| \le \gamma$$

for every $x \in B_R(x_0)$. Then the following holds. The map φ is injective and φ maps $B_R(x_0)$ into an open set in X such that

$$B_{R(1-\gamma)}(\varphi(x_0)) \subset \varphi(B_R(x_0)) \subset B_{R(1+\gamma)}(x_0). \tag{B.83}$$

The inverse $\varphi^{-1}\colon \varphi(B_R(x_0)) \to B_R(x_0)$ is continuously differentiable and

$$d(\varphi^{-1})|_y = (d\varphi|_{\varphi^{-1}(y)})^{-1}. \tag{B.84}$$

Corollary B.2 (Higher differentiability C^k)**.** *Under the assumption of Lemma B.1 assume in addition that φ is C^k. Then the inverse is C^k as well.*

Proof. This follows inductively by the chain rule from (B.84). □

Lemma B.3 (Family version)**.** *Let $\gamma < 1$ and R be positive real numbers. Let X be a Banach space and $x_0 \in X$. Assume that there exist $\varepsilon > 0$ and a C^k map $\varphi\colon (-\varepsilon,\varepsilon) \times B_R(x_0) \to X$ such that for every $s \in (-\varepsilon,\varepsilon)$ the map $\varphi_s := \varphi(s,\cdot)\colon B_R(x_0) \to X$ satisfies the estimate*

$$\|\mathrm{Id} - d\varphi_s(x)\| \le \gamma$$

at every point $x \in B_R(x_0)$. Then the following holds. The C^k map defined by

$$\Phi\colon (-\varepsilon,\varepsilon) \times B_R(x_0) \to (-\varepsilon,\varepsilon) \times X, \quad (s,x) \mapsto (s,\varphi_s(x))$$

has an inverse and Φ^{-1} is of class C^k as well.

Proof. For fixed $s \in (-\varepsilon, \varepsilon)$ the map $\varphi_s := \varphi(s, \cdot)$ is invertible by Lemma B.1. The inverse of Φ is then given by

$$\Phi^{-1} \colon \mathrm{Im}(\Phi) \to (-\varepsilon, \varepsilon) \times B_R(x_0), \quad (s, y) \mapsto (s, {\varphi_s}^{-1}(y))$$

and its derivative by

$$d\Phi^{-1}|_{(s,y)} \begin{bmatrix} \hat{s} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\left(d\varphi_s|_{\varphi_s^{-1}(y)}\right)^{-1} \dot{\varphi}_s|_{\varphi_s^{-1}(y)} & d(\varphi_s)^{-1}|_y \end{bmatrix} \begin{bmatrix} \hat{s} \\ \hat{y} \end{bmatrix}$$

where $\dot{\varphi} := \partial_1 \varphi(\cdot, \cdot)$ is the s-derivative. If φ, hence Φ, is C^k, then using inductively the chain rule on the above formula shows that Φ^{-1} is C^k as well. □

Remark B.4 (Corollary to Lemma B.1)**.** Let $\gamma < 1$ and R be positive real numbers. Let X be a Banach space, $x_0 \in X$, and $\varphi \colon B_R(x_0) \to X$ be a C^k map with

$$\|d\varphi|_x - \mathrm{Id}\| \le \gamma$$

for every $x \in B_R(x_0)$. In particular, by Corollary B.2, the map φ is a C^k diffeomorphism onto its image. For $\beta \in [0, 1]$ we define a map

$$\mathcal{S}^{\varphi}_{\beta,x_0} \colon X \supset B_R \to X, \quad \eta \mapsto (1 - \beta)(\varphi(x_0) + \eta) + \beta\varphi(x_0 + \eta). \tag{B.85}$$

The derivative at $\eta \in B_R$ is given by

$$d\mathcal{S}^{\varphi}_{\beta,x_0}|_\eta \colon X \to X, \quad \hat{\eta} \mapsto (1 - \beta)\hat{\eta} + \beta d\varphi|_{x_0+\eta}\hat{\eta}.$$

Hence

$$d\mathcal{S}^{\varphi}_{\beta,x_0}|_\eta - \mathrm{Id} = \beta \left(d\varphi|_{x_0+\eta} - \mathrm{Id}\right) : X \to X. \tag{B.86}$$

Since $\beta \le 1$ we obtain for the operator norm

$$\|d\mathcal{S}^{\varphi}_{\beta,x_0}|_\eta - \mathrm{Id}\| \le \gamma$$

whenever $\eta \in B_R$. Therefore, by Lemma B.1, all maps

$$\mathcal{S}^{\varphi}_{\beta,x_0} \colon X \supset B_R \to X$$

are C^k diffeomorphisms onto the image.

B.2 Qualitative – family inversion

Proposition B.5. *Let H be a Hilbert space. Consider a family of maps $\mathcal{F} \colon \mathbb{R} \times H \to H$ of class C^2 such that for each $s \in \mathbb{R}$ the map*

$$\mathcal{F}_s := \mathcal{F}(s, \cdot) \colon H \to H$$

is a C^2-diffeomorphism. Then the map defined by

$$\mathcal{G} \colon \mathbb{R} \times H \to H, \quad (s, y) \mapsto {\mathcal{F}_s}^{-1}(y) \tag{B.87}$$

is of class C^2 as well.

Proof. The idea is to apply the Implicit Function Theorem, see e.g. the book by Lang [Lan01, Thm. 5.9], to the following map

$$f\colon \mathbb{R} \times H \times H \to H, \quad (s, x, y) \mapsto \mathcal{F}(s, y) - x.$$

This map is of class C^2 by our assumptions. Abbreviate $z = (s, x)$, then in the notation of [Lan01, Thm. 5.9] it holds that

$$D_2 f(z, y) = d\mathcal{F}_s(y)\colon H \to H$$

and since according to our assumptions $\mathcal{F}_s$ is a C^2-diffeomorphism it follows, that $D_2 f(z, y)$ is an isomorphism.

According to the Implicit Function Theorem [Lan01, Thm. 5.9], there then exists a C^2 map $g : \mathbb{R} \times H \to H$ such that

$$f(z, g(z)) = 0 \qquad (1) \tag{B.88}$$

for every $z \in \mathbb{R} \times H$. Substituting now $z = (s, x)$ in the definition of f we obtain

$$f(z, g(z)) = f(s, x, g(s, x)) = F(s, g(s, x)) - x. \tag{B.89}$$

By (B.88) and (B.89) it follows $x = \mathcal{F}(s, g(s, x)) = \mathcal{F}_s(g(s, x))$ and therefore

$$g(s, x) = {\mathcal{F}_s}^{-1}(x).$$

Since g is of class C^2 this proves Proposition B.5. □

Lemma B.6. *The first two derivatives of $\mathcal{G}$ in (B.87) are (B.90) and (B.95).*

Proof. The proof has two steps.

Step 1. The first derivative of $\mathcal{G}$ for $s, \hat{s} \in \mathbb{R}$ and $y, \hat{y} \in \mathbb{R} \times H$ is given by

$$\boxed{\begin{aligned} d\mathcal{G}|_{(s,y)}(\hat{s}, \hat{y}) &= -(d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)})^{-1} \dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)} \hat{s} + (d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)})^{-1} \hat{y} \\ &= \begin{bmatrix} -(d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)})^{-1} \dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)} & (d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)})^{-1} \end{bmatrix} \begin{bmatrix} \hat{s} \\ \hat{y} \end{bmatrix} \end{aligned}} \tag{B.90}$$

in any direction $(\hat{s}, \hat{y}) \in \mathbb{R} \times H$

Observe that

$$\mathcal{G}_s = {\mathcal{F}_s}^{-1}\colon H \to H, \qquad \mathcal{G}_s \circ \mathcal{F}_s = {\mathcal{F}_s}^{-1} \circ \mathcal{F}_s = \mathrm{id}_H. \tag{B.91}$$

Hence we get

$$d\mathcal{G}|_{(s,y)}(0, \hat{y}) = d(\mathcal{F}_s)^{-1}|_y \hat{y} = (d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)})^{-1} \hat{y} \tag{B.92}$$

for every $\hat{y} \in H$. Abbreviating

$$\dot{\mathcal{F}}_s(\cdot) := \partial_s \mathcal{F}(s, \cdot),$$

then at $(s,x) \in \mathbb{R} \times H$ we get

$$d\mathcal{F}|_{(s,x)}(\hat{s},0) = \dot{\mathcal{F}}_s(x)\,\hat{s}$$

for every $\hat{s} \in \mathbb{R}$. We take the s-derivative of $\mathcal{F}_s \circ {\mathcal{F}_s}^{-1}(y) = y$ to obtain

$$\dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)} + d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)}(\dot{\mathcal{F}_s^{-1}})|_y = 0 \tag{B.93}$$

for every $y \in H$ and thus

$$(\dot{\mathcal{F}_s^{-1}})(y) = -(d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)})^{-1}\dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)}. \tag{B.94}$$

Therefore we obtain

$$\begin{aligned} d\mathcal{G}|_{(s,y)}(\hat{s},0) &= \dot{\mathcal{G}}_s|_y\hat{s} \\ &\stackrel{2}{=} (\dot{\mathcal{F}_s^{-1}})|_y\hat{s} \\ &\stackrel{3}{=} -(d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)})^{-1}\dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)}\hat{s} \end{aligned}$$

where step 2 is by (B.91) and step 3 by (B.94). Together with (B.92) this proves (B.90).

Step 2. We calculate the second derivative of $\mathcal{G}$ at a point $(s,y) \in \mathbb{R} \times H$. The second derivative of $\mathcal{G}$ at a point $(s,y) \in \mathbb{R} \times H$ is of the form

$$\boxed{d^2\mathcal{G}|_{(s,y)}(\hat{s}_1,\hat{y}_1;\hat{s}_2,\hat{y}_2) = \left(\begin{bmatrix} A & B \\ B & D \end{bmatrix}\begin{bmatrix}\hat{s}_1 \\ \hat{y}_1\end{bmatrix}\right)^t \begin{bmatrix}\hat{s}_2 \\ \hat{y}_2\end{bmatrix}} \tag{B.95}$$

whenever $y_1, \hat{y}_2 \in H$ and $\hat{s}_1, \hat{s}_2 \in \mathbb{R}$. The terms A, B, and D are as follows.

Term A. Take the s-derivative of (B.93) to obtain

$$\begin{aligned} 0 &= \underline{\partial_s\left(\dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)}\right)} + \partial_s\left(d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)}(\dot{\mathcal{F}_s^{-1}})|_y\right) \\ &= \underline{\ddot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)} + d\dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)}(\dot{\mathcal{F}_s^{-1}})|_y} + d\dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)}(\dot{\mathcal{F}_s^{-1}})|_y \\ &\quad + d^2\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)}\left((\dot{\mathcal{F}_s^{-1}})|_y,(\dot{\mathcal{F}_s^{-1}})|_y\right) + d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)}(\ddot{\mathcal{F}_s^{-1}})|_y. \end{aligned}$$

We use this identity in step 3 of the following calculation

$$\begin{aligned} &d^2\mathcal{G}|_{(s,y)}(\hat{s}_1,0;\hat{s}_2,0) \\ &= \ddot{\mathcal{G}}_s|_y\hat{s}_2\hat{s}_2 \\ &\stackrel{2}{=} (\ddot{\mathcal{F}_s^{-1}})|_y\hat{s}_2\hat{s}_2 \\ &\stackrel{3}{=} -\hat{s}_2\hat{s}_2(d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)})^{-1}\Bigg(\ddot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)} + 2d\dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)}\underline{(\dot{\mathcal{F}_s^{-1}})|_y} \\ &\quad + d^2\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)}\left(\underline{(\dot{\mathcal{F}_s^{-1}})|_y},\underline{(\dot{\mathcal{F}_s^{-1}})|_y}\right)\Bigg) \\ &\stackrel{4}{=} -\hat{s}_2\hat{s}_2(d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)})^{-1}\Bigg(\ddot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)} - 2d\dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)}\underline{(d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)})^{-1}\dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)}} \\ &\quad + d^2\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)}\left(\underline{(d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)})^{-1}\dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)}},\underline{(d\mathcal{F}_s|_{{\mathcal{F}_s}^{-1}(y)})^{-1}\dot{\mathcal{F}}_s|_{{\mathcal{F}_s}^{-1}(y)}}\right)\Bigg) \end{aligned}$$

where step 2 is by (B.91) and step 4 by (B.94).

TERM D. Let $s \in \mathbb{R}$ and $y, \hat{y}_1, \hat{y}_2 \in H$. Set $x := {\mathcal{F}_s}^{-1}(y)$. Then we have

$$\begin{aligned} d^2\mathcal{G}|_{(s,y)}(0,\hat{y}_1;0,\hat{y}_2) &\stackrel{1}{=} d^2(\mathcal{F}_s)^{-1}|_y(\hat{y}_1,\hat{y}_2) \\ &\stackrel{2}{=} -(d\mathcal{F}_s|_x)^{-1} d^2\mathcal{F}_s|_x \left((d\mathcal{F}_s|_x)^{-1}\hat{y}_1, (d\mathcal{F}_s|_x)^{-1}\hat{y}_2\right) \end{aligned}$$

Step 1 is by (B.92). Step 2 is analogous to (4.57).

TERM B. Let $s, \hat{s} \in \mathbb{R}$ and $y, \hat{y} \in H$. Set $x := {\mathcal{F}_s}^{-1}(y)$. Then we have

$$\begin{aligned} d^2\mathcal{G}|_{(s,y)}(0,\hat{y};\hat{s},0) &\stackrel{1}{=} \partial_s \left((d\mathcal{F}_s|_x)^{-1}\hat{y}\right)\hat{s} \\ &\stackrel{2}{=} -(d\mathcal{F}_s|_x)^{-1} d\dot{\mathcal{F}}_s|_x (d\mathcal{F}_s|_x)^{-1}\hat{y}\hat{s} \end{aligned}$$

where step 1 is by (B.92) and step 2 by the following consideration. For $x, \xi \in H$ take the s-derivative of $\xi = (d\mathcal{F}_s|_x)^{-1} d\mathcal{F}_s|_x \xi$ to obtain

$$\begin{aligned} 0 &= \left(\partial_s (d\mathcal{F}_s|_x)^{-1}\right) d\mathcal{F}_s|_x \xi + (d\mathcal{F}_s|_x)^{-1} \partial_s (d\mathcal{F}_s|_x) \\ &= \left(\partial_s (d\mathcal{F}_s|_x)^{-1}\right) d\mathcal{F}_s|_x + (d\mathcal{F}_s|_x)^{-1} d\dot{\mathcal{F}}_s|_x \end{aligned}$$

where we used that derivatives commute by the Theorem of Schwarz. Hence

$$\partial_s (d\mathcal{F}_s|_x)^{-1} = -(d\mathcal{F}_s|_x)^{-1} d\dot{\mathcal{F}}_s|_x (d\mathcal{F}_s|_x)^{-1}$$

for all $x, \xi \in H$. This concludes the proof of Step 2 and Lemma B.6. □

C Hilbert manifold structure for the path space of finite dimensional manifolds with the help of the exponential map

Finite dimensional manifolds are automatically tame and therefore Theorem A in particular implies that the space of $W^{1,2}$ paths on a finite dimensional manifold is a C^1 Hilbert manifold. In this appendix we show how this can as well be deduced more traditionally with the help of the exponential map.
If one uses the exponential map on a C^2 manifold, the finite dimensional version of the parametrized version of Theorem B is not quite sufficient since in general one will not have two continuous derivatives in both the parameter s and the space variable. We therefore establish as a technical tool Theorem C.2 which allows us to deal with this complication.

Let M be a C^2 manifold of finite dimension n. Pick two points $x_-, x_+ \in M$. Manifolds of maps $N \to M$ between manifolds have been constructed with the use of an exponential map on the target side. While Elĭasson [Elĭ67] assumes compactness of the domain N but allows infinite dimension of the target, Schwarz [Sch93] deals with maps $\mathbb{R} \to M$ and uses smooth maps $x\colon \mathbb{R} \to M$ which reach $x_\mp$ only asymptotically, i.e. in infinite time $\mp\infty$, as the fundamental paths around which coordinate charts are being built.

As in the construction of Theorem A in §4, but in contrast to Schwarz [Sch93] we build our charts around *basic paths*, i.e. paths which reach $x_\mp$ already in finite time $\mp T$. In Subsection C.2 we recall the elegant construction of coordinate charts of path space using the exponential map associated to the choice of a Riemannian metric on M. Here we already use finite time basic paths.

Definition C.1. Consider a C^2 path $x\colon \mathbb{R} \to M$ with the property that

$$x(s) = \begin{cases} x_- & , s \leq -T, \\ x_+ & , s \geq T, \end{cases}$$

for some $x_-, x_+ \in M$ and some $T > 0$. Such paths are called **basic paths**.

We call two Hilbert spaces **equivalent** if they coincide as vector spaces and their inner products are equivalent. When we refer to an **equivalence class** of Hilbert spaces we mean equivalence with respect to this equivalence relation. For a basic path x we consider the **equivalence class of Hilbert spaces**

$$H^1_x := W^{1,2}(\mathbb{R}, x^*TM). \tag{C.96}$$

In fact, to choose an inner product in H^1_x we have to choose a trivialization $\mathcal{T}^x$ of the bundle $x^*TM \to \mathbb{R}$. As we will explain below, Proposition C.7, choosing different trivializations gives rise to equivalent inner products on H^1_x.

C.1 Technical Tool

The following theorem will be the base to prove Theorem C.5 (transition maps are C^1 diffeomorphisms).

Theorem C.2. *Let $\varphi\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map with the following properties.*

(i) *For each $s \in \mathbb{R}$ the map $\varphi_s := \varphi(s,\cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$ is C^2.*

(ii) *The map $\dot\varphi_s\colon \mathbb{R}^n \to \mathbb{R}^n$ is C^1 where $\dot\varphi := \partial_1\varphi(\cdot,\cdot)$ is the s-derivative.*

(iii) *The map $\mathbb{R}\times\mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n;\mathbb{R}^n)$, $(s,x)\mapsto d^2\varphi_s|_x$, is continuous.*

(iv) *The map $\mathbb{R}\times\mathbb{R}^n \mapsto \mathcal{L}(\mathbb{R}^n)$, $(s,x)\mapsto d\dot\varphi_s|_x$, is continuous.*

(v) *There exist $T>0$ such that $\varphi_s = \varphi_\pm$ whenever $\pm s > T$ and $\varphi_\pm\colon \mathbb{R}^n \to \mathbb{R}^n$ is of class C^2 and maps 0 to 0.*

Then the map

$$\begin{aligned}\Phi\colon W^{1,2}(\mathbb{R},\mathbb{R}^n) &\to W^{1,2}(\mathbb{R},\mathbb{R}^n)\\ \xi &\mapsto [s\mapsto (\Phi(\xi))(s) := \varphi(s,\xi(s))]\end{aligned} \tag{C.97}$$

is well defined and C^1.

Corollary C.3. *Let $\varphi\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map linear in the second variable, i.e. $\varphi_s := \varphi(s,\cdot) \in \mathcal{L}(\mathbb{R}^n)$ for any $s\in\mathbb{R}$, and such that there exists $T>0$ with $\varphi_s = \mathrm{id}_{\mathbb{R}^n}$ whenever $|s|>T$. Then Φ in (C.97) is a bounded linear map.*

Proof. We check that φ satisfies the conditions of Theorem C.2. Since φ_s and $\dot\varphi_s$ are linear maps they are in particular arbitrarily often differentiable. In particular (i) and (ii) hold. Too see that condition (iii) holds consider the map

$$\mathbb{R}\times\mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n;\mathbb{R}^n), \quad (s,x)\mapsto d^2\varphi_s|_x.$$

Observe that $(v,w)\mapsto d^2\varphi_s|_x(v,w) = \varphi_s w$ is independent of v and linear in w. Hence (iii) holds. Observe that $d\dot\varphi_s|_x w = \dot\varphi_s w$ and therefore condition (iv) holds as well. Since $\varphi_s = \mathrm{id}_{\mathbb{R}^n}$ whenever $s>T$ condition (v) is also satisfied. Thus, by Theorem C.2, the map Φ is a well defined bounded linear map on $W^{1,2}$. □

Proof of Theorem C.2. The proof is in three steps. Hypothesis (i) and (ii) serve to formulate (iii) and (iv). We abbreviate $\xi_s := \xi(s)$ and $\dot\xi_s := \frac{d}{ds}\xi_s$.

Step 1. Φ takes values in $W^{1,2}$ (well defined).

Proof. Pick $\xi \in W^{1,2}(\mathbb{R},\mathbb{R}^n)$.
a) We show that $\Phi(\xi)\in L^2(\mathbb{R},\mathbb{R}^n)$. To see this note that since $\xi\in W^{1,2}$ it is continuous and has to decay asymptotically. Therefore there exists a constant c such that $|\xi_s|\le c$ for every $s\in\mathbb{R}$. By continuity of φ, there is a constant $\kappa=\kappa(c)>0$ such that $\|\varphi|_{[-T,T]\times B_c}\|_\infty \le \kappa$ where $B_c\subset\mathbb{R}^n$ is the radius c ball about the origin. Hence $|(\Phi(\xi))(s)|\le\kappa$ for every $s\in[-T,T]$.

Let $\varphi_\pm$ be the maps provided by (v). We next show that there exists a constant $C(c)$ such that

$$\|\varphi_\pm(v)\| \le C|v| \tag{C.98}$$

whenever $|v|\le c$. Indeed, since $d\varphi_\pm$ is C^1, there is a constant C such that

$$\|d\varphi_\pm(x)\| \le C$$

whenever $|x| \leq c$. Now use $\varphi_\pm(0) = 0$ to estimate

$$\begin{aligned}|\varphi_\pm(v)| &= \Big| \int_0^1 \tfrac{d}{dt}\varphi_\pm(tv)\, dt \Big| \\ &\leq \int_0^1 \|d\varphi_\pm(tv)\| \cdot |v|\, dt \\ &\leq C|v|.\end{aligned}$$

This proves (C.98). Using (C.98) in equality 3, we can estimate

$$\begin{aligned}&\|\varphi(\cdot, \xi(\cdot))\|^2_{L^2(\mathbb{R},\mathbb{R}^n)} \\ &= \|\Phi(\xi)\|^2_{L^2(\mathbb{R},\mathbb{R}^n)} \\ &= \|\Phi(\xi)\|^2_{L^2((-\infty,-T),\mathbb{R}^n)} + \|\Phi(\xi)\|^2_{L^2([-T,T],\mathbb{R}^n)} + \|\Phi(\xi)\|^2_{L^2((T,\infty),\mathbb{R}^n)} \\ &\overset{3}{=} C^2\|\xi\|^2_{L^2((-\infty,-T),\mathbb{R}^n)} + \|\Phi(\xi)\|^2_{L^2([-T,T],\mathbb{R}^n)} + C^2\|\xi\|^2_{L^2((T,\infty),\mathbb{R}^n)} \\ &\leq C^2\|\xi\|^2_{W^{1,2}(\mathbb{R},\mathbb{R}^n)} + 2T\kappa^2 < \infty.\end{aligned}$$

b) We show that $\Phi(\xi) \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$. So see this we calculate

$$\partial_s(\Phi(\xi))(s) = \underbrace{\dot\varphi(s, \xi_s)}_{=\,0,\ |s|>T} + \underbrace{d\varphi_s|_{\xi_s}}_{\overset{\text{(v)}}{=}\ d\varphi_\pm|_{\xi_s},\ |s|>T} \dot\xi_s.$$

We show that both summands are in $L^2(\mathbb{R}, \mathbb{R}^n)$.
Summand one: Let the constant c be as in a). Since φ is C^1 it follows that $\dot\varphi$ is continuous. Thus there is a constant $\kappa_2 = \kappa_2(c) > 0$ such that it holds $\|\dot\varphi|_{[-T,T]\times B_c}\|_\infty \leq \kappa_2$. Hence $|\dot\varphi(s, \xi(s))| \leq \kappa_2$ for any $s \in [-T, T]$. We estimate

$$\begin{aligned}\|\dot\varphi(\cdot, \xi(\cdot))\|_{L^2(\mathbb{R},\mathbb{R}^n)} &= \|\dot\varphi(\cdot, \xi(\cdot))\|_{L^2([-T,T],\mathbb{R}^n)} \\ &\leq \kappa_2\sqrt{2T} < \infty\end{aligned}$$

where we used in the identity that $\dot\varphi(s, \cdot) = 0$ whenever $|s| > T$.
Summand two: Since ξ is in $W^{1,2}$ it is in particular continuous and decays asymptotically. So the operator norm of $d\varphi_s|_{\xi_s}$ is uniformly bounded when s lies in compact set $[-T, T]$. Since outside of $[-T, T]$ we have $d\varphi_s|_{\xi_s} = d\varphi_\pm|_{\xi_s}$ by (v), the operator norm is actually uniformly bounded on the whole of $\mathbb{R}$, say by a constant κ_3. Hence

$$\|d\varphi_\cdot|_{\xi(\cdot)}\dot\xi_s\|_{L^2(\mathbb{R},\mathbb{R}^n)} \leq \kappa_3\|\dot\xi_s\|_{L^2(\mathbb{R},\mathbb{R}^n)} < \infty.$$

This proves b) and Step 1. □

Step 2. Φ is differentiable with derivative at $\xi \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ given by

$$\left(d\Phi|_\xi\eta\right)(s) = d\varphi_s|_{\xi_s}\eta_s$$

for every $\eta \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ and every $s \in \mathbb{R}$.

Proof. We need to show that our candidate for the derivative of Φ at ξ in direction η, namely the map $s \mapsto d\varphi_s|_{\xi_s}\eta_s$, lies in $W^{1,2}$. That this map lies in L^2 follows by the argument in the proof of Step 1 b) for the second summand by using continuity of $d\varphi_s$. It remains to show that the s-derivative lies in L^2. This s-derivative is given by

$$\partial_s \left(d\Phi|_\xi \eta\right)(s) = d\dot\varphi_s|_{\xi_s}\eta_s + d^2\varphi_s|_{\xi_s}\left(\dot\xi_s, \eta_s\right) + d\varphi_s|_{\xi_s}\dot\eta_s.$$

That the third summand is in L^2 is the argument in Step 1 b) for summand two. That the first summand is in L^2 follows by the same argument, but now using continuity of $d\dot\varphi_s$, which holds by hypothesis (iv), and since $d\dot\varphi_s = 0$ for $|s| > T$. It remains to discuss the second term. Observe that $d^2\varphi_s(\cdot) = d^2\varphi_\pm(\cdot)$ whenever $|s| > T$. By compactness of $[-T, T]$ and by continuity of the map $d^2\varphi$, by (iii), as well as continuity and asymptotic decay in s of the maps ξ_s and η_s, the operator norm

$$\|v \mapsto d^2\varphi_s|_{\xi_s}(v, \eta_s)\|_{\mathcal{L}(\mathbb{R}^n)} \le c_2\|\eta\|_\infty \le c_3\|\eta\|_{W^{1,2}(\mathbb{R},\mathbb{R}^n)}$$

is bounded uniformly by a constant $c_2 = c_2(T) > 0$ for every $s \in \mathbb{R}$. Here c_3 is the constant of the embedding $W^{1,2} \hookrightarrow C^0$. We obtain

$$\|d^2\varphi_\cdot|_\xi(\dot\xi, \eta)\|_{L^2(\mathbb{R},\mathbb{R}^n)} \le c_3\|\eta\|_{W^{1,2}(\mathbb{R},\mathbb{R}^n)}\|\dot\xi\|_{L^2(\mathbb{R},\mathbb{R}^n)} < \infty.$$

This proves that our derivative candidate $d\varphi_\cdot|_{\xi(\cdot)}\eta(\cdot)$ is element of $\mathcal{L}(W^{1,2})$.

Let us abbreviate $\|\cdot\|_2 := \|\cdot\|_{L^2(\mathbb{R},\mathbb{R}^n)}$ and $\|\cdot\|_{1,2} := \|\cdot\|_{W^{1,2}(\mathbb{R},\mathbb{R}^n)}$. To see that our candidate actually is the derivative we show that the limit

$$\begin{aligned}
&\lim_{h\to 0} \sup_{\|\eta\|_{1,2}\le 1} \frac{\|\Phi(\xi + h\eta) - \Phi(\xi) - h\, d\varphi_\cdot|_\xi \eta\|^2_{W^{1,2}}}{h^2}\\
&= \lim_{h\to 0} \sup_{\|\eta\|_{1,2}\le 1} \frac{\int_{\mathbb{R}} \left|\varphi_s(\xi_s + h\eta_s) - \varphi_s(\xi_s) - h\, d\varphi_s|_{\xi_s}\eta_s\right|^2 ds}{h^2}\\
&\quad + \lim_{h\to 0} \sup_{\|\eta\|_{1,2}\le 1} \frac{\int_{\mathbb{R}} \left|\frac{d}{ds}\left(\varphi_s(\xi_s + h\eta_s) - \varphi_s(\xi_s) - h\, d\varphi_s|_{\xi_s}\eta_s\right)\right|^2 ds}{h^2}
\end{aligned}$$

exists and vanishes. To see that summand one vanishes we use the fundamental theorem of calculus to write it in the form

$$\begin{aligned}
&\lim_{h\to 0} \sup_{\|\eta\|_{1,2}\le 1} \frac{1}{h^2}\int_{\mathbb{R}} \left|\int_0^1 \Big(\underbrace{\tfrac{d}{dt}\varphi_s(\xi_s + th\eta_s)}_{= d\varphi_s|_{\xi_s + th\eta_s} h\eta_s} - h\, d\varphi_s|_{\xi_s}\eta_s\Big) dt\right|^2_{\mathbb{R}^n} ds\\
&= \lim_{h\to 0} \sup_{\|\eta\|_{1,2}\le 1} \underbrace{\int_{\mathbb{R}} \left|\int_0^1 \Big(d\varphi_s|_{\xi_s + th\eta_s} - d\varphi_s|_{\xi_s}\Big)\eta_s dt\right|^2_{\mathbb{R}^n} ds}_{=:F(h,\eta)}.
\end{aligned} \tag{C.99}$$

There exists a constant c_4 such that $|\eta_s| \le c_4$ for all η in the unit $W^{1,2}$ ball and $s \in \mathbb{R}$. Since $d\varphi$ is continuous and $\varphi_s = \varphi_\pm$ is independent of s for $|s| > T$, it is in particular uniformly continuous on compact sets, and therefore for any $\varepsilon > 0$ there exists an $h_0 > 0$ such that for every $h \in [0, h_0]$ there is the estimate

$$\|d\varphi_s|_{\xi_s+th\eta_s} - d\varphi_s|_{\xi_s}\|_{\mathcal{L}(\mathbb{R}^n)} \le \varepsilon$$

whenever $s \in \mathbb{R}$ and $\|\eta\|_{1,2} \le 1$ and $t \in [0,1]$. Hence $F(h,\eta) \le \varepsilon^2 \|\eta\|_2^2 \le \varepsilon^2$ and therefore summand one $\lim_{h\to 0} \sup_{\|\eta\|_{1,2}\le 1} F(h,\eta) = 0$ vanishes.

To see that summand two vanishes as well, we abbreviate and compute

$$\begin{aligned}
G(s) :&= \tfrac{d}{ds}\left(\varphi_s(\xi_s + h\eta_s) - \varphi_s(\xi_s) - h\, d\varphi_s|_{\xi_s}\eta_s\right)\\
&\stackrel{1}{=} \dot\varphi_s|_{\xi_s+h\eta_s} + d\varphi_s|_{\xi_s+h\eta_s}(\dot\xi_s + h\dot\eta_s)\\
&\quad - \dot\varphi_s|_{\xi_s} - d\varphi_s|_{\xi_s}\dot\xi_s\\
&\quad - h\, d\dot\varphi_s|_{\xi_s}\eta_s - h\, d^2\varphi_s|_{\xi_s}(\dot\xi_s,\eta_s) - h\, d\varphi_s|_{\xi_s}\dot\eta_s\\
&\stackrel{2}{=} \underbrace{\dot\varphi_s|_{\xi_s+h\eta_s} - \dot\varphi_s|_{\xi_s}}_{=\int_0^1 \frac{d}{dt}\dot\varphi_s|_{\xi_s+th\eta_s}\,dt} - h\, d\dot\varphi_s|_{\xi_s}\eta_s\\
&\quad + d\varphi_s|_{\xi_s+h\eta_s}\dot\xi_s - d\varphi_s|_{\xi_s}\dot\xi_s - h\, d^2\varphi_s|_{\xi_s}(\dot\xi_s,\eta_s)\\
&\quad + d\varphi_s|_{\xi_s+h\eta_s} h\dot\eta_s - h\, d\varphi_s|_{\xi_s}\dot\eta_s\\
&\stackrel{3}{=} h\int_0^1 \left(d\dot\varphi_s|_{\xi_s+th\eta_s} - d\dot\varphi_s|_{\xi_s}\right)\eta_s\,dt\\
&\quad + h\int_0^1 \left(d^2\varphi_s|_{\xi_s+th\eta_s} - d^2\varphi_s|_{\xi_s}\right)(\dot\xi_s,\eta_s)\,dt\\
&\quad + h\left(d\varphi_s|_{\xi_s+h\eta_s} - d\varphi_s|_{\xi_s}\right)\dot\eta_s
\end{aligned}$$

for every $s \in \mathbb{R}$. Square this identity and integrate to obtain

$$\begin{aligned}
&\sup_{\|\eta\|_{1,2}\le 1} \tfrac{1}{h^2}\int_{\mathbb{R}} |G(s)|^2_{\mathbb{R}^n}\,ds\\
&\le \sup_{\|\eta\|_{1,2}\le 1} \int_{-T}^{T} 3\left|\int_0^1 \left(d\dot\varphi_s|_{\xi_s+th\eta_s} - d\dot\varphi_s|_{\xi_s}\right)\eta_s\,dt\right|^2_{\mathbb{R}^n} ds\\
&\quad + \sup_{\|\eta\|_{1,2}\le 1} \underbrace{\int_{\mathbb{R}} 3\left|\int_0^1 \left(d^2\varphi_s|_{\xi_s+th\eta_s} - d^2\varphi_s|_{\xi_s}\right)(\dot\xi_s,\eta_s)\,dt\right|^2_{\mathbb{R}^n} ds}_{=:F_2(h,\eta,\dot\xi)}\\
&\quad + \sup_{\|\eta\|_{1,2}\le 1} \int_{\mathbb{R}} 3\left|\left(d\varphi_s|_{\xi_s+h\eta_s} - d\varphi_s|_{\xi_s}\right)\dot\eta_s\right|^2_{\mathbb{R}^n} ds.
\end{aligned}$$

Term 1. The limit, as $h \to 0$, of the first of the three terms vanishes by exactly the same argument as in (C.99) by using continuity of $d\dot\varphi$ by (iv).
Term 2. For the second term note that as in (C.99) by continuity (iii) and (v) for every $\varepsilon > 0$ there is $h_0 > 0$ such that for any $h \in [0, h_0]$ there is the estimate

$$\|d^2\varphi_s|_{\xi_s+th\eta_s} - d^2\varphi_s|_{\xi_s}\|_{\mathcal{L}(\mathbb{R}^n)} \le \varepsilon$$

whenever $s \in \mathbb{R}$ and $\|\eta\|_{1,2} \le 1$ and $t \in [0,1]$. There exists a constant $\kappa > 0$ such that $\|\eta\|_\infty \le \kappa_T \|\eta\|_{W^{1,2}}$. So, for $h \in [0, h_0]$, we have

$$\begin{aligned} F_2(h,\eta,\dot\xi) &\le 3\varepsilon^2 \||\dot\xi| \cdot |\eta|\|_2^2 \\ &\le 3\varepsilon^2 \|\eta\|_\infty^2 \|\dot\xi\|_2^2 \\ &\le 3\varepsilon^2\kappa^2 \|\eta\|_{1,2}^2 \|\xi\|_{1,2}^2. \end{aligned} \tag{C.100}$$

Therefore $\sup_{\|\eta\|_{1,2}\le 1} F_2(h,\eta,\dot\xi) \le 3\varepsilon^2\kappa^2\|\xi\|_{1,2}^2$. Consequently the limit vanishes $\lim_{h\to 0} \sup_{\|\eta\|_{1,2}\le 1} F_2(h,\eta,\dot\xi) = 0$.
Term 3. This follows as in (C.99) by noticing that $\|\eta\|_{L^2(\mathbb{R},\mathbb{R}^n)} \le \|\eta\|_{W^{1,2}(\mathbb{R},\mathbb{R}^n)}$.

This concludes the proof of Step 2 (Φ differentiable). □

Step 3. The differential $d\Phi\colon W^{1,2} \to \mathcal{L}(W^{1,2})$ is a continuous map.

Proof. For $\xi, \tilde\xi, \eta \in W^{1,2} = W^{1,2}(\mathbb{R},\mathbb{R}^n)$ we estimate the difference

$$\begin{aligned} &\|d\Phi|_\xi \eta - d\Phi|_{\tilde\xi}\eta\|_{1,2}^2 \\ &= \int_{\mathbb{R}} |(d\varphi_s|_{\xi_s} - d\varphi_s|_{\tilde\xi_s})\eta_s|^2 ds + \int_{\mathbb{R}} |\tfrac{d}{ds}\left[(d\varphi_s|_{\xi_s} - d\varphi_s|_{\tilde\xi_s})\eta_s\right]|^2 ds \\ &\le \int_{\mathbb{R}} |(d\varphi_s|_{\xi_s} - d\varphi_s|_{\tilde\xi_s})\eta_s|^2 ds \\ &\quad + \int_{-T}^{T} |(d\dot\varphi_s|_{\xi_s} - d\dot\varphi_s|_{\tilde\xi_s})\eta_s|^2 ds \\ &\quad + \int_{\mathbb{R}} |(d^2\varphi_s|_{\xi_s}\dot\xi_s - d^2\varphi_s|_{\tilde\xi_s}\dot\xi_s + d^2\varphi_s|_{\tilde\xi_s}\dot\xi_s - d^2\varphi_s|_{\tilde\xi_s}\dot{\tilde\xi}_s)\eta_s|^2 ds \\ &\quad + \int_{\mathbb{R}} |(d\varphi_s|_{\xi_s} - d\varphi_s|_{\tilde\xi_s})\dot\eta_s|^2 ds \\ &\le \int_{\mathbb{R}} |(d\varphi_s|_{\xi_s} - d\varphi_s|_{\tilde\xi_s})\eta_s|^2 ds \\ &\quad + \int_{-T}^{T} |(d\dot\varphi_s|_{\xi_s} - d\dot\varphi_s|_{\tilde\xi_s})\eta_s|^2 ds \\ &\quad + \int_{\mathbb{R}} 2|(d^2\varphi_s|_{\xi_s}\dot\xi_s - d^2\varphi_s|_{\tilde\xi_s}\dot\xi_s)\eta_s|^2 ds \\ &\quad + \int_{\mathbb{R}} 2|d^2\varphi_s|_{\tilde\xi_s}(\dot\xi_s - \dot{\tilde\xi}_s, \eta_s)|^2 ds \\ &\quad + \int_{\mathbb{R}} |(d\varphi_s|_{\xi_s} - d\varphi_s|_{\tilde\xi_s})\dot\eta_s|^2 ds. \end{aligned}$$

Term 1. Since the $W^{1,2}$ norm can be estimated by the C^0 norm, the continuity of $d\varphi$, together with the fact that φ_s is independent of s for $|s| > T$, implies the following: For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|\tilde\xi - \xi\|_{1,2} \le \delta \Rightarrow \|d\varphi_s|_{\xi_s} - d\varphi_s|_{\tilde\xi_s}\|_{\mathcal{L}(\mathbb{R}^n)} \le \varepsilon.$$

In particular, for every $\tilde{\xi}$ in the $W^{1,2}$ δ-ball around ξ it holds that

$$\sup_{\|\eta\|_{1,2}\le 1}\int_{\mathbb{R}}|\big(d\varphi_s|_{\xi_s}-d\varphi_s|_{\tilde{\xi}_s}\big)\eta_s|^2 ds\le\varepsilon^2.$$

Term 2. Maybe after shrinking $\delta>0$ the continuity (iv) of $d\dot{\varphi}$ implies (same argument as for Term1) the estimate

$$\sup_{\|\eta\|_{1,2}\le 1}\int_{-T}^{T}|\big(d\dot{\varphi}_s|_{\xi_s}-d\dot{\varphi}_s|_{\tilde{\xi}_s}\big)\eta_s|^2 ds\le\varepsilon^2.$$

Term 3. Maybe after shrinking $\delta>0$ again, by continuity (iii) of the bi-linear valued map $d^2\varphi$ and by (v), we have the implication

$$\|\tilde{\xi}-\xi\|_{1,2}\le\delta\Rightarrow\|d^2\varphi_s|_{\xi_s}-d^2\varphi_s|_{\tilde{\xi}_s}\|_{\mathcal{L}(R^n,\mathbb{R}^n;\mathbb{R}^n)}\le\varepsilon.$$

In particular, for every $\tilde{\xi}$ in the $W^{1,2}$ δ-ball around ξ it holds that

$$\begin{aligned}
&\int_{\mathbb{R}}2|\big(d^2\varphi_s|_{\xi_s}\dot{\xi}_s-d^2\varphi_s|_{\tilde{\xi}_s}\dot{\xi}_s\big)\eta_s|^2 ds\\
&\le 2\varepsilon^2\||\dot{\xi}|\cdot|\eta|\|_2^2\\
&\le 2\varepsilon^2\kappa^2\|\eta\|_{1,2}^2\|\xi\|_{1,2}^2
\end{aligned}$$

where in the last inequality we used estimate (C.100). Therefore we get

$$\sup_{\|\eta\|_{1,2}\le 1}\int_{\mathbb{R}}2|\big(d^2\varphi_s|_{\xi_s}-d^2\varphi_s|_{\tilde{\xi}_s}\big)\big(\dot{\xi}_s,\eta_s\big)|^2 ds\le 2\varepsilon^2\kappa^2\|\xi\|_{1,2}^2.$$

Term 4. By continuity (iii) of $d^2\varphi$ and finite dimension of $\mathbb{R}^n$, and by (v) there exists a constant $c>0$, only depending on ξ but not $\tilde{\xi}$, such that

$$\|\tilde{\xi}-\xi\|_{1,2}\le\delta\Rightarrow\|d^2\varphi_s|_{\tilde{\xi}_s}\|_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n;\mathbb{R}^n)}\le c.$$

Hence we estimate term 4 by

$$\begin{aligned}
&\int_{\mathbb{R}}2|d^2\varphi_s|_{\tilde{\xi}_s}\big(\dot{\xi}_s-\dot{\tilde{\xi}}_s,\eta_s\big)|^2 ds\\
&\le 2c^2\||\dot{\xi}-\dot{\tilde{\xi}}|\cdot|\eta|\|_2^2\\
&\le 2c^2\kappa^2\|\dot{\xi}-\dot{\tilde{\xi}}\|_{1,2}^2\|\eta\|_{1,2}^2
\end{aligned}$$

where in the last inequality we used estimate (C.100). Maybe after shrinking $\delta>0$ we can assume that $\delta<\varepsilon$. Taking the supremum we obtain the estimate

$$\sup_{\|\eta\|_{1,2}\le 1}\int_{\mathbb{R}}2|d^2\varphi_s|_{\tilde{\xi}_s}\big(\dot{\xi}_s-\dot{\tilde{\xi}}_s,\eta_s\big)|^2 ds\le 2c^2\kappa^2\varepsilon^2.$$

Term 5. By the argument from term 1 using that $\|\dot\eta\|_2^2 \le \|\eta\|_{1,2}^2 \le 1$ we get

$$\sup_{\|\eta\|_{1,2}\le 1} \int_{\mathbb{R}} |(d\varphi_s|_{\xi_s} - d\varphi_s|_{\tilde\xi_s})\dot\eta_s|^2 ds \le \varepsilon^2.$$

The term by term analysis above shows that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\begin{aligned}\|d\Phi|_\xi - d\Phi|_{\tilde\xi}\|^2_{\mathcal{L}(W^{1,2})} &= \sup_{\|\eta\|_{1,2}\le 1} \|d\Phi|_\xi\eta - d\Phi|_{\tilde\xi}\eta\|^2_{1,2} \\ &\le \varepsilon^2 \left(3 + 2\kappa^2 \|\xi\|^2_{W^{1,2}(\mathbb{R},\mathbb{R}^n)} + 2c^2\kappa^2\right).\end{aligned}$$

This concludes the proof of Step 3. □

This proves Theorem C.2. □

C.2 Hilbert manifold structure via exponential map

Let M be a C^2 manifold of finite dimension n and let $x_-, x_+ \in M$. We define the space $\mathcal{P}_{x_-x_+}$ of $W^{1,2}$-paths $x\colon \mathbb{R} \to M$ from x_- to x_+ and endow it with the structure of a C^1 Hilbert manifold; see [Elĭ67, Sch93]. To define charts we pick a C^2 Riemannian metric g on M. Let $\iota_{x_0} > 0$ be the injectivity radius at $x_0 \in M$ of the exponential map $\exp\colon T_{x_0}M \to M$.

For convenience of the reader we repeat the construction of coordinate charts of path space using the exponential map, but using already finite time basic paths as coordinate centers. Given a manifold M of finite dimension n, pick a Riemannian metric g on M. The associated exponential map

$$\exp = \exp^g\colon TM \to M \times M, \quad (q, v) \mapsto (q, \exp_q v)$$

when defined on a sufficiently small neighborhood of the zero section is a diffeomorphism onto its image. One defines $\exp_q v := \gamma_v(1)$ where $\gamma_v\colon [0,1] \to M$ is the unique geodesic ($\nabla_t\dot\gamma_v \equiv 0$) with $\gamma_v(0) = q$ and $\dot\gamma_v(0) = v$.

C.2.1 Local parametrizations

Definition C.4 (Local parametrizations)**.** Let x be a basic path connecting two points $x_\mp \in M$. Consider the open neighborhood $\mathcal{U}_x$ of the zero section in the space of $W^{1,2}$ vector fields ξ along x that consists of all ξ such that $|\xi(s)|_g < \iota_{x(s)}$ for every $s \in \mathbb{R}$. We define a **local chart** of $\mathcal{P}_{x_-x_+}$ about x as the inverse of the **local parametrization $\boldsymbol{\Psi_x}$** given by utilizing the exponential map of (M, g) pointwise for every $s \in \mathbb{R}$, in symbols

$$\boxed{\begin{aligned}\Psi_x = \exp_x\colon H^1_x \supset \mathcal{U}_x &\to \exp_x \mathcal{U}_x =: \mathcal{V}_x \subset \mathcal{P}_{x_-x_+} \\ \xi &\mapsto \exp_x \xi := [s \mapsto \exp_{x(s)} \xi(s)].\end{aligned}}$$

These local parametrizations endow $\mathcal{P}_{x_-x_+}$ with the structure of a *topological* Hilbert manifold modelled on $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$.

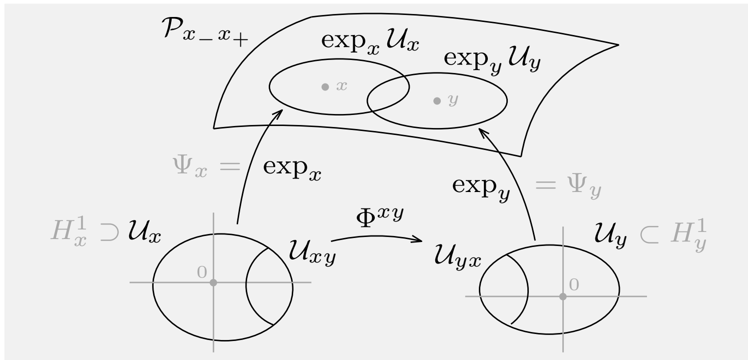

Figure 7: Exponential transition map $\Phi^{xy} \colon \mathcal{U}_{xy} \to \mathcal{U}_{yx}$, $H^1_x := W^{1,2}(\mathbb{S}^1, x^*TM)$

C.2.2 Transition maps and basic trivializations

Theorem C.5 (Exponential parametrization transition maps)**.** *Assume that x and y are two basic paths in M connecting x_- to x_+. Consider the open Hilbert space subsets defined by*

$$\mathcal{U}_{xy} := {\exp_x}^{-1}\left(\exp_x \mathcal{U}_x \cap \exp_y \mathcal{U}_y\right) \subset H^1_x,$$
$$\mathcal{U}_{yx} := {\exp_y}^{-1}\left(\exp_x \mathcal{U}_x \cap \exp_y \mathcal{U}_y\right) \subset H^1_y.$$

Then the chart transition map

$$\boxed{\Phi^{xy} := {\Psi_y}^{-1} \circ \Psi_x|_{\mathcal{U}_{xy}} = {\exp_y}^{-1} \circ \exp_x|_{\mathcal{U}_{xy}} \colon \mathcal{U}_{xy} \to \mathcal{U}_{yx}}$$

is a C^1 diffeomorphism.

Definition C.6 (Basic trivialization)**.** Suppose that x is a basic path such that $x(s) = x_-$ for $s \le -T$ and $x(s) = x_+$ for $s \ge T$. To define an inner product on H^1_x we choose a trivialization

$$\mathcal{T} \colon x^*TM \to \mathbb{R} \times \mathbb{R}^n, \quad (s, v) \mapsto (s, \mathcal{T}_s v)$$

depending continuously differentiable on s, i.e. being of class C^1 in s, and with the property that there are linear isomorphisms

$$\mathcal{T}_\pm \colon T_{x_\pm}M \to \mathbb{R}^n$$

such that

$$\mathcal{T}_s = \mathcal{T}_\pm, \quad \text{whenever } \pm s \ge T.$$

We refer to $\mathcal{T}$ as a **basic trivialization**.

Given a basic trivialization $\mathcal{T}$, we define the inner product of $\xi, \eta \in H^1_x$ by

$$\langle \xi, \eta \rangle := \langle \mathcal{T}(\xi), \mathcal{T}(\eta) \rangle_{W^{1,2}(\mathbb{R},\mathbb{R}^n)}\,.$$

If we choose a different basic trivialization, then we get an equivalent inner product on H^1_x as the following proposition shows.

Proposition C.7. *Assume that $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are two basic trivializations for a basic path x, then*

$$\tilde{\mathcal{T}} \circ \mathcal{T}^{-1} \colon W^{1,2}(\mathbb{R}, \mathbb{R}^n) \to W^{1,2}(\mathbb{R}, \mathbb{R}^n)$$

is a linear isomorphism.

Proof. By Corollary C.3 the map $\tilde{\mathcal{T}} \circ \mathcal{T}^{-1}$ is a well defined linear map from $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ to $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$. By the same corollary the inverse

$$(\tilde{\mathcal{T}} \circ \mathcal{T}^{-1})^{-1} = \mathcal{T} \circ \tilde{\mathcal{T}}^{-1}$$

is as well a well defined linear map from $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ to $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$. Therefore $\tilde{\mathcal{T}} \circ \mathcal{T}^{-1}$ is a linear isomorphism. □

Proof of Theorem C.5. Let x and y be basic paths in the n-dimensional manifold M. In order to apply Theorem C.2 to prove Theorem C.5 we use basic trivializations $\mathcal{T}^x$ and $\mathcal{T}^y$ of the vector bundles $x^*TM \to \mathbb{R}$ and $y^*TM \to \mathbb{R}$, respectively. The **trivialized transition map**

$$\boxed{\Phi := \mathcal{T}^y \circ \Phi^{xy} \circ (\mathcal{T}^x)^{-1} \colon W^{1,2}(\mathbb{R}, \mathbb{R}^n) \to W^{1,2}(\mathbb{R}, \mathbb{R}^n)} \tag{C.101}$$

gives rise to a map φ as in Theorem C.2. Indeed for $s \in \mathbb{R}$ we define

$$\varphi_s := \varphi(s, \cdot) \colon \mathbb{R}^n \overbrace{\xrightarrow{\mathcal{T}_s^{x\,-1}}}^{\text{linear}} \overbrace{T_{x(s)}M \xrightarrow{\exp_{x(s)}} U_{x(s)} \cap U_{y(s)} \xrightarrow{\exp_{y(s)}{}^{-1}} T_{y(s)}M}^{=:\Phi_s^{xy} \in C^2} \overbrace{\xrightarrow{\mathcal{T}_s^y}}^{\text{linear}} \mathbb{R}^n.$$

The map $\varphi \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a composition of the three C^1 maps $\mathcal{T}^x \colon x^*TM \to \mathbb{R} \times \mathbb{R}^n$, $\mathcal{T}^y$, and $\exp \colon TM \to M$ on a neighborhood of the zero section. Therefore φ is C^1. We verify hypotheses (i–v) in Theorem C.2.
(i) As illustrated in the displayed diagram the map $\varphi_s \colon \mathbb{R}^n \to \mathbb{R}^n$ is C^2.
(ii) The s-derivative of $\varphi_s := \varphi(s, \cdot)$ is of the form

$$\dot{\varphi}_s = \partial_s \mathcal{T}_s^{x\,-1} \circ \underbrace{\Phi_s^{xy}}_{\in C^2} \circ \mathcal{T}_s^y + \mathcal{T}_s^{x\,-1} \circ \underbrace{\partial_s \Phi_s^{xy}}_{\in C^1} \circ \mathcal{T}_s^y + \mathcal{T}_s^{x\,-1} \circ \underbrace{\Phi_s^{xy}}_{\in C^2} \circ \partial_s \mathcal{T}_s^y$$

where each summand starts and ends with a linear map. Thus $\dot{\varphi}_s = \partial_s \varphi_s \in C^1$.
(iii) and (iv) follow from the last two displayed diagrams, respectively, since the linear maps $\mathcal{T}_s^{x\,-1}$ and $\mathcal{T}_s^y$ depend continuously on s.
(v) This follows since x and y are basic paths and $\mathcal{T}^x$ and $\mathcal{T}^y$ are basic trivializations and therefore constant whenever $|s|$ is large enough.

It follows from Theorem C.2 that Φ defined by (C.101) is C^1. Interchanging the roles of x and y we get that the inverse map $\Phi^{-1} = \mathcal{T}^x \circ \Phi^{yx} \circ (\mathcal{T}^y)^{-1}$ is C^1 as well. Hence Φ is a diffeomorphism from $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ to $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$.

This proves Theorem C.5 in view of Proposition C.7. □

C.2.3 Independence of choice of Riemannian metric

As a consequence of Theorem C.5 the space $\mathcal{P}_{x_-x_+}$ carries the structure of a C^1 Hilbert manifold. The atlas $\mathcal{A} = \mathcal{A}(g)$ we constructed depends on the choice of a C^2 Riemannian metric g on M. However, the C^1 Hilbert manifold structure of $\mathcal{P}_{x_-x_+}$ does not depend on the choice of C^2 metric as the next theorem shows.

Theorem C.8. *If g and $\tilde{g}$ are two C^2 Riemannian metrics on a C^2 manifold M, then the atlases $\mathcal{A}(g)$ and $\mathcal{A}(\tilde{g})$ are compatible.*

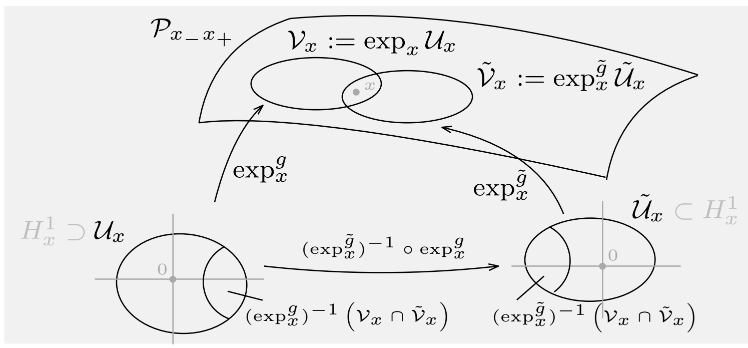

Figure 8: Charts about a basic path x in the atlases $\mathcal{A}(g)$ and $\mathcal{A}(\tilde{g})$

Proof. First note that the notions of basic path and basic trivialization do not depend on the Riemannian metric. Suppose that x is a basic path. Let $\mathcal{U}_x$ and $\tilde{\mathcal{U}}_x$ be the open neighborhoods from Definition C.4 of the zero section with respect to g and $\tilde{g}$, respectively. Define the open image sets in $\mathcal{P}_{x_-x_+}$ by

$$\mathcal{V}_x := \exp_x^g \mathcal{U}_x, \qquad \tilde{\mathcal{V}}_x := \exp_x^{\tilde{g}} \tilde{\mathcal{U}}_x.$$

By the argument in the proof of Theorem C.5 it is a C^1 diffeomorphism the map

$$(\exp_x^{\tilde{g}})^{-1} \circ \exp_x^g \colon (\exp_x^g)^{-1}\left(\mathcal{V}_x \cap \tilde{\mathcal{V}}_x\right) \to (\exp_x^{\tilde{g}})^{-1}\left(\mathcal{V}_x \cap \tilde{\mathcal{V}}_x\right).$$

As the set $C_0^2(\mathbb{R}, \mathbb{R}^n)$ of compactly supported C^2 maps is dense in $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$, basic paths are dense in $\mathcal{P}_{x_-x_+}$ and Theorem C.8 follows. $\square$

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