Should Einstein's 1915 Gravitational Coordinates Replace Friedmann's in Cosmology?

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Abstract The DESI collaboration has found that the acceleration of the universe's expansion weakens with time, and JWST has found a large population of galaxies with z $>$ 10. The Friedmann coordinates of the Robertson-Walker metric imply a Big Bang birth of the universe with unbounded expansion speed and perpetual gravitational deceleration of that expansion. The 1998 discovery that the universe's expansion instead accelerates led to trying a cosmological constant in the Einstein equation, but this doesn't accommodate the acceleration's weakening with time found by DESI. Also, the large population of galaxies with z $>$ 10 doesn't jibe with the universe's unbounded initial expansion speed. Einstein's observationally-tested 1915 coordinate condition, however, is Lorentz covariant, which bounds all speeds by c. It also implies refractive gravitational slowing of incoming light, so outgoing light is accelerated. Outgoing galaxies whose z $>$ 0.94 are similarly accelerated, so the universe's expansion is accelerated, a gravitational effect that inherently weakens as the universe expands. We illustrate the above effects by plotting the time evolution of the simplest expanding-dust-sphere model universe in both Friedmann and Einstein coordinates.

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Introduction: Is gravity's acceleration of light exactly the same as that of a body at rest?

In Newtonian gravity theory a test body's gravitational acceleration $d^2\mathbf{r}/dt^2$ depends *exclusively* on its instantaneous location $\mathbf{r}(t)$ in the gravitational field $-\nabla_{\mathbf{r}}\,\phi(\mathbf{r})$, where $\phi(\mathbf{r})$ is the gravitational potential, i.e.,

$$d^2\mathbf{r}/dt^2 = -\nabla_{\mathbf{r}}\,\phi(\mathbf{r}), \tag{I.1}$$

so in the Newtonian framework a test body's gravitational acceleration is independent of its mass and of its velocity. The gravitational principle of equivalence *rests on these twin pillars.* When the gravitational source is a static point mass M located at $\mathbf{r} = \mathbf{0}$, Newton's Law of Gravity implies that,

$$\phi(\mathbf{r}) = -GM/r, \;\text{ so, }\; -\nabla_{\mathbf{r}}\,\phi(\mathbf{r}) = -GM\mathbf{r}/r^3, \;\text{ where }\; r \stackrel{\text{def}}{=} |\mathbf{r}|. \tag{I.2}$$

For *purely radial motion of a test body with respect to the static point mass M*, Eq. (I.1) thus becomes,

$$d^2r/dt^2 = -GM/r^2. \tag{I.3}$$

If Eq. (I.3) *applies to a radially-incoming wave packet of light exactly as it applies to a test body at rest with respect to the static point mass M*, then the radially-incoming wave packet of light *is immediately accelerated to a speed which exceeds c!* The *pillar* of the gravitational principle of equivalence *that a test body's gravitational acceleration is independent of its velocity* thus *conflicts* with the Lorentzian-relativity precept that physical entities such as wave packets of light and test bodies *cannot breach the speed limit c.*

In 1911 Einstein calculated the value of the deflection of starlight by the sun's gravity which follows *directly* from the gravitational principle of equivalence[1], but in his landmark November 18, 1915 paper [2], Einstein calculated both the deflection of starlight by the sun's gravity and Mercury's remnant perihelion shift *in a different way* which is *consistent* with Lorentzian relativity and *inconsistent* with the gravitational principle of equivalence. Einstein's November 18, 1915 approach subjects the Einstein equation for the metric tensor field $g_{\mu\nu}(x)$ to the coordinate condition $\det(g_{\mu\nu}(x)) = -1$ for all values of x, which is manifestly consistent with $g_{\mu\nu}(x)$ being Lorentz covariant and also with its having a well-defined matrix inverse $g^{\alpha\beta}(x)$. Einstein's November 18, 1915 approach furthermore *effectively replaces the Lagrangian,*

$$L(d\mathbf{r}/dt, \phi(\mathbf{r})) = (m/2)|d\mathbf{r}/dt|^2 - m\,\phi(\mathbf{r}), \tag{I.4}$$

for the Eq. (I.1) Newtonian-gravity equation of test-body motion *with a Lorentz-invariant extension* that has a dynamic symmetric second-rank tensor gravitational potential $\phi_{\mu\nu}(x) = c^2(g_{\mu\nu}(x) - \eta_{\mu\nu})/2$ instead of the static scalar potential $\phi(\mathbf{r})$ of Eq. (I.4). The test-body's resulting gravitational acceleration, however, *depends on its velocity,* as is expected of a Lorentz-covariant theory of gravity, and therefore is automatically *inconsistent with a pillar of the gravitational principle of equivalence.* Thus it isn't surprising that Einstein's November 18, 1915 result for the deflection of starlight by the sun's gravity *differs substantially* from his 1911 result; *it is twice as great.* The *observed value* of the deflection of starlight by the sun's gravity [3] *definitely supports* Einstein's November 18, 1915 *Lorentz-covariant* theory of gravity *instead of* his 1907–1911 concept that gravity is governed by the gravitational principle of equivalence.

The *additional* fact *that the observed value of Mercury's remnant perihelion shift agrees to within 1%* with the prediction of Einstein's November 18, 1915 theory of gravity [4] *lends further support to gravity's Lorentz covariance.* Indeed, since the validity of the Lorentz transformation *is one of the empirically most accurately established facts of physics* [5], it is virtually inconceivable that gravity *isn't* Lorentz-covariant.

We now sketch how gravity's inherent *cause* combines with Lorentz covariance to produce a Lorentz-invariant extension of the Eq (I.4) Newtonian-gravity Lagrangian for a test body's motion in a gravitational field. Newton very reasonably regarded *mass* to be gravity's inherent *cause,* but we have since learned that chemical, nuclear and particle reactions can all to some degree convert *mass* to kinetic and radiant *energy.* Energy and momentum *are interchanged* under Lorentz transformations just as space and time are, so it is reasonable to assume that the conserved energy-momentum four-vector is the inherent cause of gravity in a manner somewhat analogous to conserved charge being the inherent cause of electromagnetism. A *difference* is that the gravitational field *itself* carries energy-momentum, and thus *is part of its own source,* so the theoretical machinery of *Newtonian* gravity, such as the Eq. (I.4) Lagrangian, *applies only when gravity is static and $\phi(\mathbf{r})$ is sufficiently weak* (i.e., $|\phi(\mathbf{r})| \ll c^2$).

In Lorentz-covariant *electromagnetic* theory *the proximate cause of local electromagnetic forces is the four-vector density-flux $j^\mu(x)$ of conserved charge; the local electromagnetic potentials which $j^\mu(x)$ produces* *correspondingly comprise a four-vector entity* $A_\mu(x)$. In the static-potential limit $A_\mu(x) \to A_\mu(\mathbf{r})$, and the component $A_0(\mathbf{r})$ *is the Coulomb potential.* The nonrelativistic equation of motion of a test body of mass m and charge e in the static Coulomb potential $A_0(\mathbf{r})$ is of course,

$$m\, d^2\mathbf{r}/dt^2 = -e\nabla_{\mathbf{r}} A_0(\mathbf{r}), \tag{I.5a}$$

whose corresponding Lagrangian is,

$$L(d\mathbf{r}/dt, A_0(\mathbf{r})) = (m/2)|d\mathbf{r}/dt|^2 - eA_0(\mathbf{r}), \tag{I.5b}$$

which is readily upgraded to a Lorentz-invariant Lagrangian that embraces all four components of $A_\nu(x)$,

$$L(dx^\mu/d\tau, A_\nu(x)) = (mc^2/2) - (m/2)\eta_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau) - (e/c)A_\nu(x)(dx^\nu/d\tau), \tag{I.5c}$$

where $d\tau \stackrel{\text{def}}{=} dt\sqrt{1 - |(d\mathbf{r}/dt)/c|^2}$ is the Lorentz-invariant differential proper time the test body's clock registers, dt is the differential time the observer's clock registers, $dx^0 = c\,dt$ and $\eta_{\mu\nu}$ is the Minkowski metric,

$$\eta_{00} = 1,\ \eta_{11} = \eta_{22} = \eta_{33} = -1,\ \eta_{\mu\nu} = 0 \text{ if } \mu \neq \nu. \tag{I.5d}$$

When the nonrelativistic test-body limit $|(d\mathbf{r}/dt)/c| \to 0$ and the static-potential limit $A_\mu(x) \to A_\mu(\mathbf{r})$ are applied to the Eq. (I.5c) *upgraded Lorentz-invariant Lagrangian* $L(dx^\mu/d\tau, A_\nu(x))$, *it reduces to the* Eq. (I.5b) *Lagrangian* $L(d\mathbf{r}/dt, A_0(\mathbf{r}))$ *for a nonrelativistic test body in the static Coulomb potential* $A_0(\mathbf{r})$. [Note that the constant term $(mc^2/2)$ in the Eq. (I.5c) upgraded Lorentz-invariant Lagrangian $L(dx^\mu/d\tau, A_\nu(x))$ *doesn't contribute to the test body's equation of motion and could be dropped; it is included in* Eq. (I.5c) *to facilitate taking the* $|(d\mathbf{r}/dt)/c| \to 0$ *limit of* $L(dx^\mu/d\tau, A_\nu(x))$.] The Eq. (I.5c) *upgraded Lorentz-invariant Lagrangian* $L(dx^\mu/d\tau, A_\nu(x))$ *furthermore unexpectedly exhibits the symmetry of gauge invariance,* i.e., adding a term of the form $\partial\chi(x)/\partial x^\nu$ to $A_\nu(x)$ *doesn't alter the test body's equation of motion because,*

$$\begin{aligned} L(dx^\mu/d\tau, A_\nu(x) + \partial\chi(x)/\partial x^\nu) &= L(dx^\mu/d\tau, A_\nu(x)) - (e/c)(\partial\chi(x)/\partial x^\nu)(dx^\nu/d\tau) = \\ & L(dx^\mu/d\tau, A_\nu(x)) + d[-(e/c)\chi(x)]/d\tau, \end{aligned} \tag{I.5e}$$

and a Lagrangian term which is a derivative with respect to τ doesn't contribute to the equation of motion. Finally we work out the test-body equation of motion which the Eq. (I.5c) Lorentz-invariant Lagrangian $L(dx^\mu/d\tau, A_\nu(x))$ implies. Lagrangians of the generic form $L(dx^\mu/d\tau, x)$ imply the equation of motion $dp_\mu/d\tau = \partial L/\partial x^\mu$, where the canonical momentum $p_\mu \stackrel{\text{def}}{=} \partial L/\partial(dx^\mu/d\tau)$. For the Eq. (I.5c) Lagrangian, $p_\mu = -m\eta_{\mu\nu}(dx^\nu/d\tau) - (e/c)A_\mu(x)$, $\partial L/\partial x^\mu = -(e/c)(\partial A_\nu(x)/\partial x^\mu)(dx^\nu/d\tau)$ and $dp_\mu/d\tau = -m\eta_{\mu\nu}(d^2x^\nu/d\tau^2) - (e/c)(\partial A_\mu(x)/\partial x^\nu)(dx^\nu/d\tau)$, so the Eq. (I.5c) Lagrangian yields the test-body equation of motion,

$$m\eta_{\mu\nu}(d^2x^\nu/d\tau^2) = (e/c)[(\partial A_\nu(x)/\partial x^\mu) - (\partial A_\mu(x)/\partial x^\nu)](dx^\nu/d\tau), \tag{I.5f}$$

which isn't in the standard form for Lorentzian dynamics because of the Minkowski metric $\eta_{\mu\nu}$ on its left side. We therefore multiply both sides of Eq. (I.5f) by $\eta^{\lambda\mu} = \eta_{\lambda\mu}$, the matrix inverse of the Minkowski metric, which is equal to the Minkowski metric itself, and sum over the index μ to obtain,

$$m(d^2x^\lambda/d\tau^2) = (e/c)\eta^{\lambda\mu}[(\partial A_\nu(x)/\partial x^\mu) - (\partial A_\mu(x)/\partial x^\nu)](dx^\nu/d\tau), \tag{I.5g}$$

electromagnetism's Lorentz Force Law. Note *its gauge invariance* when $A_\nu(x) \to A_\nu(x) + \partial\chi(x)/\partial x^\nu$.

Switching now to Lorentz-covariant *gravity* theory, *the proximate cause of local gravitational forces is the symmetric-second-rank-tensor density-flux* $T^{\mu\nu}(x)$ *of conserved energy-momentum; the local gravitational potentials which* $T^{\mu\nu}(x)$ *produces correspondingly comprise a symmetric-second-rank-tensor entity* $\phi_{\mu\nu}(x)$. In the static-potential limit, $\phi_{\mu\nu}(x) \to \phi_{\mu\nu}(\mathbf{r})$, and the component $\phi_{00}(\mathbf{r})$ *becomes the static Newtonian gravitational potential* $\phi(\mathbf{r})$ *of* Eq. (I.4) at $\mathbf{r}$ *where* $\phi(\mathbf{r})$ *is weak, i.e., where* $|\phi(\mathbf{r})| \ll c^2$. The Lagrangian,

$$L(d\mathbf{r}/dt, \phi(\mathbf{r})) = (m/2)|d\mathbf{r}/dt|^2 - m\,\phi(\mathbf{r}), \tag{I.6a}$$

of Eq. (I.4) *is readily upgraded to a Lorentz-invariant Lagrangian that embraces all ten components of* $\phi_{\mu\nu}(x)$,

$$L(dx^\mu/d\tau, \phi_{\mu\nu}(x)) = (mc^2/2) - (m/2)\eta_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau) - (m/c^2)\phi_{\mu\nu}(x)(dx^\mu/d\tau)(dx^\nu/d\tau). \tag{I.6b}$$

When the nonrelativistic test-body limit $|(d\mathbf{r}/dt)/c| \to 0$ and the static-potential limit $\phi_{\mu\nu}(x) \to \phi_{\mu\nu}(\mathbf{r})$ are applied to the Eq. (I.6b) upgraded Lorentz-invariant Lagrangian $L(dx^\mu/d\tau, \phi_{\mu\nu}(x))$, it reduces to the Newtonian Lagrangian $L(d\mathbf{r}/dt, \phi(\mathbf{r}))$ of Eq. (I.6a) at $\mathbf{r}$ where $|\phi(\mathbf{r})| \ll c^2$, i.e., where and when the Newtonian static weak-field, nonrelativistic-speed approximation can be properly applied. If the $\phi_{\mu\nu}(x)$ *vanish,* the Eq. (I.6b) Lagrangian $L(dx^\mu/d\tau, \phi_{\mu\nu}(x))$ reduces to the Lorentz-invariant *free-body* Lagrangian $L(dx^\mu/d\tau)$,

$$L(dx^\mu/d\tau) = (mc^2/2) - (m/2)\eta_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau). \tag{I.6c}$$

When the following extremely useful definition of the dimensionless gravitational entity $g_{\mu\nu}(x)$,

$$g_{\mu\nu}(x) \stackrel{\text{def}}{=} \eta_{\mu\nu} + (2/c^2)\phi_{\mu\nu}(x), \tag{I.6d}$$

is applied to the Eq. (I.6b) Lorentz-invariant Lagrangian, *it becomes compactly reexpressed as,*

$$L(dx^\mu/d\tau, g_{\mu\nu}(x)) = (mc^2/2) - (m/2)g_{\mu\nu}(x)(dx^\mu/d\tau)(dx^\nu/d\tau). \tag{I.6e}$$

Comparison of the Eq. (I.6c) Lorentz-invariant Lagrangian *for free-body motion* with the Eq. (I.6e) Lorentz-invariant gravitational Lagrangian *for test-body motion in the gravitational entity* $g_{\mu\nu}(x)$ shows, in light of Eq. (I.6d), that gravity can be regarded as *a Lorentz-covariant local distortion* $g_{\mu\nu}(x) = \eta_{\mu\nu} + (2/c^2)\phi_{\mu\nu}(x)$ *of the constant Minkowski metric* $\eta_{\mu\nu}$ *of space-time.* Therefore *the strictly Lorentz-covariant dynamical Lagrangian treatment of gravity leads to a* $g_{\mu\nu}(x)$ *Riemannian-metric interpretation. Moreover, just as the strictly Lorentz-covariant dynamical Lagrangian treatment of electromagnetism leads to its unexpected gauge-invariance symmetry,* we can clearly see that if $g_{\mu\nu}(x)$ transforms as a symmetric second-rank covariant tensor under general coordinate transformations, *then the* Eq. (I.6e) *Lagrangian isn't only Lorentz-invariant, it is generally-invariant as well.* We now work out the test-body equation of gravitational motion for the Eq. (I.6e) Lagrangian $L(dx^\mu/d\tau, g_{\mu\nu}(x)) = (mc^2/2) - (m/2)g_{\mu\nu}(x)(dx^\mu/d\tau)(dx^\nu/d\tau)$. The canonical momentum $p_\mu = -m\, g_{\mu\nu}(x)(dx^\nu/d\tau)$, $\partial L/\partial x^\mu = -(m/2)(\partial g_{\alpha\beta}(x)/\partial x^\mu)(dx^\alpha/d\tau)(dx^\beta/d\tau)$ and $dp_\mu/d\tau = -m\, g_{\mu\nu}(x)(d^2x^\nu/d\tau^2) - m(\partial g_{\mu\alpha}(x)/\partial x^\beta)(dx^\alpha/d\tau)(dx^\beta/d\tau)$, so the Eq. (I.6e) Lagrangian yields the following test-body equation of gravitational motion,

$$m\, g_{\mu\nu}(x)(d^2x^\nu/d\tau^2) = -(m/2)[(\partial g_{\mu\alpha}(x)/\partial x^\beta) + (\partial g_{\mu\beta}(x)/\partial x^\alpha) - (\partial g_{\alpha\beta}(x)/\partial x^\mu)](dx^\alpha/d\tau)(dx^\beta/d\tau), \tag{I.6f}$$

which isn't in the standard form for Lorentzian dynamics because of the metric tensor $g_{\mu\nu}(x)$ on its left side. We therefore *assume* that $g_{\mu\nu}(x)$ has a well-defined matrix inverse $g^{\lambda\mu}(x)$ for all x, which we multiply into both sides of Eq. (I.6f) and then sum over the index μ, with the result,

$$m(d^2x^\lambda/d\tau^2) = -(m/2)g^{\lambda\mu}(x)[(\partial g_{\mu\alpha}(x)/\partial x^\beta) + (\partial g_{\mu\beta}(x)/\partial x^\alpha) - (\partial g_{\alpha\beta}(x)/\partial x^\mu)](dx^\alpha/d\tau)(dx^\beta/d\tau), \tag{I.6g}$$

i.e., gravitation's *geodesic equation,* which is customarily presented in the familiar form,

$$d^2x^\lambda/d\tau^2 + \Gamma^\lambda_{\alpha\beta}(x)(dx^\alpha/d\tau)(dx^\beta/d\tau) = 0, \tag{I.6h}$$

where *the affine connection* $\Gamma^\lambda_{\alpha\beta}(x)$ is defined as,

$$\Gamma^\lambda_{\alpha\beta}(x) \stackrel{\text{def}}{=} \tfrac{1}{2}g^{\lambda\mu}(x)[(\partial g_{\mu\alpha}(x)/\partial x^\beta) + (\partial g_{\mu\beta}(x)/\partial x^\alpha) - (\partial g_{\alpha\beta}(x)/\partial x^\mu)]. \tag{I.6i}$$

We have noted that if $g_{\mu\nu}(x)$ transforms as a symmetric second-rank covariant tensor under general coordinate transformations, then the simple Eq. (I.6e) Lagrangian, which yields the gravitational geodesic equation, is a general invariant, so it isn't surprising that under those circumstances the Eq. (I.6h) gravitational geodesic equation itself transforms as a generally-contravariant vector [6].

Since $g^{\lambda\mu}(x)$ is the matrix inverse of the metric tensor $g_{\mu\nu}(x)$, the Eq. (I.6i) affine connection *won't be well-defined for all* x *unless* $\det(g_{\mu\nu}(x)) \neq 0$ *for all* x. The Einstein equation for the metric tensor $g_{\mu\nu}(x) = \eta_{\mu\nu} + (2/c^2)\phi_{\mu\nu}(x)$, namely $G_{\mu\nu}(x) = (8\pi G/c^4)T_{\mu\nu}(x)$, is required to yield Newton's Law of Gravity, $\nabla^2_{\mathbf{r}}\phi(\mathbf{r}) = 4\pi G(T_{00}(\mathbf{r})/c^2)$, in gravity's weak, static limit, and the Einstein tensor $G_{\mu\nu}(x)$ is furthermore required to be a generally-covariant symmetric second-rank tensor whose generally-covariant divergence vanishes. Consequently, $G_{\mu\nu}(x) = \big(R_{\mu\nu}(x) - \tfrac{1}{2}g_{\mu\nu}(x)\big(g^{\alpha\beta}(x)R_{\alpha\beta}(x)\big)\big)$, where $R_{\mu\nu}(x)$ (the Ricci tensor) is given in terms of the affine connection $\Gamma^\lambda_{\alpha\beta}(x)$ by,

$$R_{\mu\nu}(x) = \big(\partial\Gamma^\lambda_{\lambda\mu}(x)/\partial x^\nu\big) - \big(\partial\Gamma^\lambda_{\mu\nu}(x)/\partial x^\lambda\big) + \Gamma^\sigma_{\mu\lambda}(x)\,\Gamma^\lambda_{\nu\sigma}(x) - \Gamma^\sigma_{\mu\nu}(x)\,\Gamma^\lambda_{\lambda\sigma}(x), \tag{I.6j}$$

so the Einstein equation *won't be well-defined unless* $\det(g_{\mu\nu}(x)) \neq 0$ *for all* x. The coordinate condition $\det(g_{\mu\nu}(x)) = -1$ for all x of Einstein's landmark November 18, 1915 paper [2], *is the simplest possible coordinate condition which* (1) ensures that $\det(g_{\mu\nu}(x)) \neq 0$ for all x, (2) is consistent with $g_{\mu\nu}(x)$ being Lorentz covariant and (3) doesn't conflict with the fact that $\det(\eta_{\mu\nu}) = -1$. *Together with the* Eq. (I.6h) *gravitational geodesic equation,* it *also* gets *both* the deflection of starlight by the sun's gravity *and* Mercury's remnant perihelion shift *right, but a pillar of the gravitational principle of equivalence is violated by the dependence of test-body gravitational acceleration on test-body velocity in* Eq. (I.6h).

The *extent* to which the graviational principle of equivalence *can be violated* by this coordinate condition in conjunction with Eq. (I.6h) *is underlined* by the December 8, 1916 thesis of Johannes Droste [7] and the December 23, 1916 paper of David Hilbert [8]. Both treat purely *radial* test-body motion in the static gravitational field of a static point mass M at $r = 0$, and both find that the Eq. (I.3) *Newtonian* test-body acceleration $d^2r/dt^2 = -GM/r^2$ *is replaced in the asymptotic* $r \to \infty$ *limit by,*

$$d^2r/dt^2 \asymp -(GM/r^2)(1 - 3(v/c)^2) \text{ as } r \to \infty, \tag{I.7}$$

where $v \geq 0$ is the test body's asymptotic speed as $r \to \infty$. If $v \ll c$, this modification of Eq. (I.3) *is negligible.* But if a distant galaxy's recession speed exceeds $c/\sqrt{3}$ (i.e, its redshift z exceeds 0.932) the mass within the sphere of matter which isn't as distant, *gravitationally repels this galaxy instead of gravitationally attracting it.* Since galactic redshifts of $z \approx 14$ have been confirmed, *the expansion of the universe itself is gravitationally accelerated instead of gravitationally decelerated.* Eq. (I.7) *also* makes it clear that the ongoing expansion of the universe will cause this gravitational acceleration of its expansion *to slowly wane.* The lamentable fact that the December, 1916 results of Droste and Hilbert *cannot be found in any textbook on gravity* [8] caused the 1998 discovery of the accelerating expansion of the universe *to be greeted with perplexity instead of delight that a long-awaited gravitational phenomenon was observed.* Einstein's *discarded* cosmological constant, *albeit with the opposite sign,* was *instead* revived, *although it prevents us from recovering Newtonian gravity in gravity's weak, static limit.* The cosmological constant *also* disagrees with DESI-collaboration indications that the acceleration of the universe's expansion slowly wanes [9].

Eq. (I.7) reads $d^2r/dt^2 \asymp +2(GM/r^2)$ for a distant radially incoming light packet, so a spherically-symmetric gravitational field *repulsively decelerates* such a light packet *twice as strongly as it attractively accelerates* an equally distant body at rest. This *refutation* of the gravitational principle of equivalence *isn't inherently surprising to astronomers,* who make use of gravity's *refractive slowing of incoming light that causes a foreground galaxy to act as a lens which magnifies the image of a more distant galaxy.*

Owing to historical circumstances, the observationally successful conjunction of Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ for all x with the Eq. (I.6h) gravitational geodesic equation has never, as far as we know, been applied to cosmological models, despite Droste's and Hilbert's breaking of ground for this with their December, 1916 results. Contemporary cosmology's signature Robertson-Walker metric form,

$$(c\,d\tau)^2 = (c\,dt)^2 - (R(t))^2\big[(1/(1 - kr^2))(dr)^2 + r^2\big((d\theta)^2 + (\sin\theta\, d\phi)^2\big)\big], \tag{I.8}$$

which has been its mainstay since 1935, manifestly incorporates Alexandre Friedmann's 1922 coordinate condition $g_{00}(x) = 1$ for all x, *which is incompatible with Lorentz covariance of* $g_{\mu\nu}(x)$*, doesn't guarantee that* $\det(g_{\mu\nu}(x)) \neq 0$ *and completely eliminates gravitational time dilation since that effect is given by,* [10]

$$[(\text{the tick rate of a clock at } x_2)/(\text{the tick rate of a clock at } x_1)] = \sqrt{g_{00}(x_2)/g_{00}(x_1)}. \tag{I.9}$$

As might be expected from the complete elimination of gravitational time dilation by Friedmann's coordinate condition, the chief results of applying the Eq. (I.8) Robertson-Walker metric form to cosmological models *are the same as those of applying Newtonian gravitational dynamics to these models* [11]. Furthemore, the fact that the universe is expanding, together with its description by the Eq. (I.8) Robertson-Walker metric form, implies that the Eq. (I.8) metric function $R(t)$ vanished at a finite past time t_i [12]. This implies that the determinant of the Eq. (I.8) metric vanished at that finite past time t_i, so the affine connection, the Einstein tensor and the Einstein equation *were all undefined at that finite past time* t_i.

Moreover, for equations of state which are plausible, the *density* of the universe increased without bound as $t \to t_i+$ and $R(t) \to 0$ [13], so its *radius* tended toward zero as $t \to t_i+$. Taking into account a key consequence of the Einstein equation for this situation [14] *furthermore yields that the radial expansion speed of the universe increased without bound as* $t \to t_i+$. Thus the Friedmann-coordinate Robertson-Walker metric form implies a Newtonian-gravitational universe which had a Big-Bang commencement at time t_i.

JWST, however, has found *a great many bright compact galaxies with redshifts* $z > 10$ *and elements as heavy as oxygen* [15]. Their *abundance* suggests that the early universe may *not* have expanded as rapidly as Big-Bang cosmology implies. Moreover, *the unbounded expansion speed* of a Big-Bang universe *violates the speed limit* c of the Lorentz transformation, whose validity has been checked to very high accuracy [5].

1. Birkhoff-theorem development of the simplest cosmology model in Friedmann coordinates

The *simplest* cosmological model for the universe is a uniform-density zero-pressure radially-expanding sphere of "dust", whose constituent "dust particles" *interact only gravitationally with each other.* The *Newtonian* gravitational dynamics [11] of the *radius* $r(t)$ of such a radially expanding sphere of uniform-density zero-pressure "dust" is governed, *in light of the Birkhoff theorem*, by the following *two versions* of the *same* familiar Newtonian equation of a test body's purely-radial outward-directed motion in the Newtonian gravitational field of a static point mass M at the origin of coordinates,

$$d^2r/dt^2 = -GM/r^2 \text{ and } dr/dt = \sqrt{(2GM/r) + v^2}, \tag{1.1}$$

where M *is the total conserved energy divided by* c^2 *of the radially-expanding sphere of uniform-density "dust"*, and $v > 0$ is the $r \to \infty$ asymptotic outward-directed velocity of its radius.

One of *the most distant galaxies which has been observed* has a redshift z of 14, which corresponds to a recession speed of $0.99c$. Therefore it is reasonable to put the value of the dust-sphere model universe's asymptotic outward-directed radial velocity v to c in Eq. (1.1) above. Thereupon it is very convenient to *reexpress* Eq. (1.1) in terms of the dust-sphere model universe's *dimensionless scaled "time" variable* $u \stackrel{\text{def}}{=} t/(r_s/c)$, where $r_s \stackrel{\text{def}}{=} (2GM/c^2)$, the model universe's Schwarzschild radius, and *also* in terms of the model universe's corresponding *dimensionless scaled "radius" variable* $q(u) \stackrel{\text{def}}{=} r(t)/r_s$. Eq. (1.1) is thereby simplified to the two corresponding *dimensionless scaled equations*,

$$d^2q/du^2 = -1/(2q^2) \text{ and } dq/du = \sqrt{(1/q) + 1}. \tag{1.2a}$$

The solutions $q(u)$ of the second differential equation $dq/du = \sqrt{(1/q) + 1}$ of Eq. (1.2a) *can't* be expressed in terms of elementary functions, *but their inverse functions* $u(q)$ *can be so expressed*, i.e.,

$$u_{q_0}(q) = \int_{q_0}^{q} dq' \sqrt{q'/(q'+1)} =$$

$$\left[\sqrt{q(q+1)} - \ln\left(\sqrt{q} + \sqrt{q+1}\right)\right] - \left[\sqrt{q_0(q_0+1)} - \ln\left(\sqrt{q_0} + \sqrt{q_0+1}\right)\right]. \tag{1.2b}$$

For *the initial condition* $q(u = 0) = 1$, namely that the expanding dust-sphere model universe *attains its Schwarzschild radius at time zero*, the *corresponding* dimensionless scaled radius solution $q(u)$ of the second differential equation in Eq. (1.2a) *is the inverse of the* Eq. (1.2b) *function* $u_{q_0=1}(q)$ *because* $u_{q_0=1}(q = 1) = 0$. Thus the $q(u)$ *which satisfies* $q(u = 0) = 1$, while *not* directly expressible in terms of elementary functions, *is readily plotted using* Eq. (1.2b). Its plot is displayed as the red curve of Figure 1, and the corresponding plots of its dimensionless scaled radial velocity $dq(u)/du = \sqrt{(1/q(u)) + 1}$ and its dimensionless scaled radial acceleration $d^2q(u)/du^2 = -1/(2(q(u))^2)$ are displayed as the red curves of Figures 2 and 3 respectively. These three red curves depict a Big-Bang dust-sphere model universe whose dimensionless scaled radius $q(u)$ *suddenly begins steeply increasing from the value zero at the finite initial dimensionless scaled time* $u = u_i \stackrel{\text{def}}{=} u_{q_0=1}(q = 0) = -\sqrt{2} + \ln(1 + \sqrt{2}) = -0.53284$ *when the Big Bang occurs; the fact that* $q(u_i) = 0$ implies that at the finite dimensionless scaled time $u = u_i$ when the Big Bang occurs the dust-sphere model universe's dimensionless scaled radial velocity $dq(u)/du|_{u=u_i} = \sqrt{(1/(q(u_i) = 0)) + 1}$ *is infinite* and its dimensionless scaled radial acceleration $d^2q(u)/du^2|_{u=u_i} = -1/(2(q(u_i) = 0)^2)$ *is infinite as well.*

Furthermore, as we see from the red curve of Figure 2, *for all* $u \geq u_i = -\sqrt{2} + \ln(1 + \sqrt{2}) = -0.53284$, $dq(u)/du > 1$, *i.e., from the time of the Big Bang onward, the dust-sphere model universe's expansion velocity exceeds* c, which flagrantly contradicts the validity of the Lorentz transformation, one of the empirically most accurately checked facts of physics [5].

The vanishing of the dimensionless scaled radius function $q(u)$ at the finite dimensionless scaled time u_i of the occurrence of the Big Bang naturally corresponds to the vanishing of the dimensionless Robertson-Walker metric function $R(t)$ at that finite time t_i of the occurrence of the Big Bang [12]. We have pointed out that the vanishing of $R(t_i)$ implies the vanishing of the determinant of the Eq. (I.8) Robertson-Walker metric form at the finite time t_i of the Big Bang, and consequently that the affine connection, the Einstein tensor and the Einstein equation are all undefined at the finite time t_i of the Big Bang.

The preceding two paragraphs make it apparent that Friedmann-coordinate Big-Bang cosmology, which implicitly uses Newtonian gravity, *has significant issues of physical soundness;* these are *in addition* to the issue of its compatibility with the abundant compact galaxies revealed by JWST that have redshifts $z > 10$.

However, *unlike* the Friedmann coordinate condition $g_{00}(x) = 1$ for all x, the Einstein coordinate condition $\det(g_{\mu\nu}(x)) = -1$ for all x (1) is compatible with Lorentz covariance of the metric tensor $g_{\mu\nu}(x)$, (2) *ensures* that *the contravariant metric tensor* $g^{\alpha\beta}(x)$, the affine connection $\Gamma^{\lambda}_{\mu\nu}(x)$, the Einstein tensor $G_{\mu\nu}(x)$ and the Einstein equation *are well-defined for all* x, (3) in conjunction with the gravitational geodesic equation *gives the observed values of* the deflection of starlight by the sun's gravity and Mercury's remnant perihelion shift, (4) is compatible with *gravitational repulsion of both light and distant galaxies whose speed exceeds* $(c/\sqrt{3})$ [8] and (5) is compatible with *the existence of gravitational time dilation.*

The Eq. (I.8) Robertson-Walker metric form is deemed suitable for cosmological modeling because a mathematical theorem asserts that it solves any Einstein equation whose energy-momentum-tensor source is spherically symmetric and homogeneous *once the correct specifics of its unspecified function* $R(t)$ *and unspecified constant* k *have been determined* [16]. Because the Einstein equation is generally covariant, this property of the Robertson-Walker metric form *is also a property of its coordinate transformations,* so *since only Einstein-coordinate solutions of the Einstein equation are likely to be physically correct,* we next *transform* the Eq. (I.8) Robertson-Walker metric form *from Friedmann to Einstein coordinates.*

2. Transformation of the Robertson-Walker metric form to Einstein coordinates

Following *arbitrary* non-interdependent transformations $t'(t)$ and $r'_k(r)$ of its time t and radius r coordinates, the Eq. (I.8) Robertson-Walker metric form becomes,

$$(c\,d\tau)^2 = (dt(t')/dt')^2(c\,dt')^2 -$$
$$(R(t(t')))^2\big[\big(1/\big(1 - k(r(r'_k))^2\big)\big)(dr(r'_k)/dr'_k)^2(dr'_k)^2 + (r(r'_k)/r'_k)^2(r'_k)^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big)\big], \tag{2.1}$$

whose determinant will be -1, which is *required* of metrics expressed in Einstein coordinates, *if both,*

$$(dt(t')/dt')^2(R(t(t')))^6 = 1 \;\; \textit{and} \;\; \big(1/\big(1 - k(r(r'_k))^2\big)\big)(dr(r'_k)/dr'_k)^2(r(r'_k)/r'_k)^4 = 1 \;\; \textit{are satisfied.} \tag{2.2}$$

The *first* Eq. (2.2) requirement implies that,

$$(dt(t')/dt')^2 = (R(t(t')))^{-6}, \tag{2.3}$$

and it furthermore implies the following unique time transformation $t'(t)$ *that satisfies* $t'(t = 0) = 0$,

$$t'(t) = \textstyle\int_0^t |R(w)|^3 dw, \tag{2.4}$$

while the *second* Eq. (2.2) requirement implies that,

$$(1/(1 - k(r(r'_k))^2))(dr(r'_k)/dr'_k)^2 = (r'_k/r(r'_k))^4, \tag{2.5}$$

and it furthermore implies the following unique radius transformation $r'_k(r)$ *that satisfies* $r'_k(r = 0) = 0$,

$$r'_k(r) = \Big(3\textstyle\int_0^r \big(1 - ks^2\big)^{-\frac{1}{2}} s^2\, ds\Big)^{\frac{1}{3}}. \tag{2.6}$$

Inserting the results given by Eqs. (2.3) and (2.5) into Eq. (2.1) yields the following transformation of the Eq. (I.8) Robertson-Walker metric form to Einstein coordinates,

$$(c\,d\tau)^2 = (R(t(t')))^{-6}(c\,dt')^2 - (R(t(t')))^2\big[(r'_k/r(r'_k))^4(dr'_k)^2 + (r(r'_k)/r'_k)^2(r'_k)^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big)\big], \tag{2.7}$$

where the function $t(t')$ in Eq. (2.7) is the *inverse* of the specific time transformation $t'(t)$ which is explicitly given by Eq. (2.4), while the function $r(r'_k)$ in Eq. (2.7) is the *inverse* of the specific radius transformation $r'_k(r)$ which is explicitly given by Eq. (2.6). The Eq. (2.7) transformation of the Eq. (I.8) Robertson-Walker metric form to Einstein coordinates *is readily verified to satisfy the Einstein coordinate condition* $\det(g_{\mu\nu}(x)) = -1$ *for all* x.

However, to work out the motion in Einstein coordinates *of only the radius* $r(t)$ of a uniform-density zero-pressure radially-expanding sphere of "dust", i.e., the motion in Einstein coordinates *of only the radius* of the *simplest* radially-expanding spherical homogeneous cosmological model, *it definitely isn't necessary* to actually insert the metric form described by Eqs. (2.7), (2.4) and (2.6) above into the appropriate Einstein equation. *In light of the Birkhoff theorem,* one can *instead use the gravitational geodesic equation* to work out a test body's purely radial motion *in the Einstein-coordinate static metric for a static point mass* M *at the* *origin of coordinates*, where M is equal to the total conserved energy divided by c^2 of the uniform-density zero-pressure radially-expanding sphere of "dust".

We next discuss *obtaining the Einstein-coordinate exact static metric solution for a static point mass M at the origin of coordinates*, following which *we use the gravitational geodesic equation* to work out a test body's *purely radial motion in this exact static metric solution.*

3. The unique Einstein-coordinate static metric for a static point mass at the origin

The most general static, spherically-symmetric metric $g_{\mu\nu}(r)$ which adheres to the Einstein coordinate condition $\det(g_{\mu\nu}(r)) = -1$ for all $r \geq 0$ has the following form in spherical polar coordinates,[17]

$$(c\,d\tau)^2 = g_{00}(r)(c\,dt)^2 - g_{rr}(r)(dr)^2 - (g_{00}(r)g_{rr}(r))^{-\frac{1}{2}}\,r^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big). \tag{3.1a}$$

Eq. (I.2) gives the Newtonian gravitational potential $\phi(r)$ for a static point mass M at $r = 0$ as,

$$\phi(r) = -GM/r. \tag{3.1b}$$

Eq. (I.6d) informs us that,

$$g_{00}(r) = \eta_{00} + (2/c^2)\phi_{00}(r) = 1 + (2/c^2)\phi_{00}(r), \tag{3.1c}$$

and the discussion preceding Eq. (I.6a) informs us that $\phi_{00}(r)$ is asymptotic to the Newtonian gravitational potential $\phi(r) = -GM/r$ as $|\phi(r)| \to 0$. Therefore $\phi_{00}(r)$ is asymptotic to $-GM/r$ as $r \to \infty$, so,

$$g_{00}(r) \asymp 1 - (r_s/r) \text{ as } r \to \infty, \text{ where } r_s \stackrel{\text{def}}{=} 2GM/c^2. \tag{3.1d}$$

Furthermore, the Eq. (3.1a) metric *must approach the Minkowski metric as $r \to \infty$*, which implies that,

$$g_{rr}(r) \to 1 \text{ as } r \to \infty. \tag{3.1e}$$

The $r \to \infty$ boundary conditions of Eqs. (3.1d) and (3.1e) above *apply to every static, spherically-symmetric source of finite extent,* so *the specific static point-mass M source at $r = 0$ imposes a further boundary condition at $r = 0$.* In *Newtonian* gravity, the point-mass M at $r = 0$ *causes gravitational acceleration to be infinite at $r = 0$*, but in *Lorentzian-relativistic* gravity, *gravitational time dilation compels both velocities and accelerations to vanish at $r = 0$.* In particular, the tick rate of a clock at $r = 0$ *vanishes relative to the tick rate of a clock at any nonzero value of r.* Therefore from Eq. (I.9) we have that,

$$\sqrt{g_{00}(r = 0)/g_{00}(r \neq 0)} = 0, \tag{3.1f}$$

which in turn implies that,

$$g_{00}(r = 0) = 0. \tag{3.1g}$$

Eq. (3.1g) *is the boundary condition at $r = 0$ arising from the presence there of the static point mass M.*

Applying the Einstein equation together with the two $r \to \infty$ boundary conditions of Eqs. (3.1d) and (3.1e) above to the metric form of Eq. (3.1a) yields [17],

$$g_{00}(r) = 1 - \big(r_s/(r^3 + \rho)^{\frac{1}{3}}\big) \quad \text{and} \quad g_{rr}(r) = \big(1/g_{00}(r)\big)\big(r/(r^3 + \rho)^{\frac{1}{3}}\big)^4, \tag{3.1h}$$

where ρ is an integration constant that has the dimension of length cubed. Applying the Eq. (3.1g) boundary condition $g_{00}(r = 0) = 0$ to the Eq. (3.1h) result yields $\rho = r_s^3$, so Eq. (3.1h) becomes,

$$g_{00}(r) = 1 - \big(r_s/(r^3 + r_s^3)^{\frac{1}{3}}\big) \quad \text{and} \quad g_{rr}(r) = \big(1/g_{00}(r)\big)\big(r/(r^3 + r_s^3)^{\frac{1}{3}}\big)^4. \tag{3.1i}$$

Inserting the Eq. (3.1i) result into the Eq. (3.1a) metric form yields,

$$\begin{aligned}(c\,d\tau)^2 = {}& \big(1 - \big(r_s/(r^3 + r_s^3)^{\frac{1}{3}}\big)\big)(c\,dt)^2 - \big(1/\big(1 - \big(r_s/(r^3 + r_s^3)^{\frac{1}{3}}\big)\big)\big)\big(r/(r^3 + r_s^3)^{\frac{1}{3}}\big)^4(dr)^2 - \\ & \big((r^3 + r_s^3)^{\frac{1}{3}}/r\big)^2 r^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big).\end{aligned} \tag{3.2}$$

Lamentably, no textbook on gravity that we know of takes into account the elementary fact that the static point mass M at $r = 0$ imposes a boundary condition at $r = 0$ on the Eq. (3.1a) static, spherically-symmetic metric form. Ignoring the boundary condition at $r = 0$, which is simply $g_{00}(r = 0) = 0$, results in a "loose cannon" integration constant in the metric solution whose evaluation is routinely botched. Indeed, textbooks on gravity virtually always set this constant ρ in the Eq. (3.1h) metric solution to zero merely because that produces by far the algebraically-simplest metric result, namely,

$$(c\,d\tau)^2 = (1-(r_s/r))(c\,dt)^2 - (1/(1-(r_s/r)))(dr)^2 - r^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big). \tag{3.3}$$

After arriving at this botched metric, textbooks on gravity leave their readers to puzzle over the "fact" that a static point-mass singularity at $r=0$ miraculously "produces" a static metric singularity at $r=r_s>0$!

We next use the Eq. (3.2) Einstein-coordinate static metric for a static point mass M at $r=0$ together with the gravitational geodesic equation to work out a test body's purely radial motion in that metric.

4. Radial test-body motion in the Einstein-coordinate static metric for a static point mass

To apply the gravitational geodesic equation to radial test-body motion in the Eq. (3.2) metric, it is convenient to reexpress that metric in the compact Eq. (14) representation of reference [17]. Given the abbreviations $R(r) \stackrel{\text{def}}{=} (r^3 + r_s^3)^{\frac{1}{3}}$, $B(R(r)) \stackrel{\text{def}}{=} (1-(r_s/R(r)))$ and $A(R(r)) \stackrel{\text{def}}{=} (1/B(R(r)))$, it is readily shown that $dR(r)/dr = (r/R(r))^2$, and consequently that the Eq. (3.2) metric can be compactly reexpressed as,

$$(c\,d\tau)^2 = B(R(r))(c\,dt)^2 - A(R(r))(dR(r))^2 - (R(r))^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big). \tag{4.1}$$

The Eq. (4.1) metric representation *itself* immediately yields the following *first-order* equation of test-body gravitational motion,

$$c^2 = c^2B(R(r))(dt/d\tau)^2 - A(R(r))(dR(r)/d\tau)^2 - (R(r))^2\big((d\theta/d\tau)^2 + (\sin\theta\,(d\phi/d\tau))^2\big). \tag{4.2}$$

Since the test body we consider *moves exclusively radially*, its angular frequencies $d\theta/d\tau$ and $d\phi/d\tau$ are both equal to zero, which reduces Eq. (4.2) to,

$$c^2 = \big[c^2B(R(r)) - A(R(r))(dR(r)/dt)^2\big](dt/d\tau)^2. \tag{4.3}$$

To turn Eq. (4.3) into an equation of test-body radial motion, i.e., a differential equation for dr/dt, we need to *evaluate* the Eq. (4.3) factor $(dt/d\tau)^2$. Doing so requires *integrating the time component* of the test body's Eq. (I.6h) second-order in τ *four-vector gravitational geodesic equation of motion*,

$$d^2x^\lambda/d\tau^2 + \tfrac{1}{2}g^{\lambda\mu}(x)\big[\big(\partial g_{\mu\alpha}(x)/\partial x^\beta\big) + \big(\partial g_{\mu\beta}(x)/\partial x^\alpha\big) - \big(\partial g_{\alpha\beta}(x)/\partial x^\mu\big)\big]\big(dx^\alpha/d\tau\big)\big(dx^\beta/d\tau\big) = 0. \tag{4.4}$$

For *the particular metric form given by* Eq. (4.1), the *time component* of the Eq. (4.4) test-body gravitational geodesic equation of motion is,[18]

$$\tfrac{d^2t}{d\tau^2} + \tfrac{dt}{d\tau}\tfrac{dB(R(r))/dR(r)}{B(R(r))}\tfrac{dR(r)}{d\tau} = 0, \tag{4.5a}$$

which can be written,

$$\tfrac{1}{dt/d\tau}\tfrac{d(dt/d\tau)}{d\tau} + \tfrac{dB(R(r))/dR(r)}{B(R(r))}\tfrac{dR(r)}{d\tau} = 0, \tag{4.5b}$$

which in turn can be written,

$$d\big(\ln\big(dt/d\tau\big) + \ln\big(B(R(r))\big)\big)/d\tau = 0, \tag{4.5c}$$

which implies that,

$$\ln\big((dt/d\tau)(B(R(r)))\big) = -C, \tag{4.5d}$$

where C is an arbitrary dimensionless constant. Eq. (4.5d) implies that,

$$dt/d\tau = 1/(KB(R(r))), \tag{4.5e}$$

where $K = \exp(C)$ is an arbitrary dimensionless positive constant. Inserting Eq. (4.5e) into Eq. (4.3) yields,

$$\big(A(R(r))/(B(R(r)))^2\big)(dR(r)/dt)^2 - \big(c^2/B(R(r))\big) = -c^2K^2. \tag{4.6a}$$

The object $dR(r)/dt$ in Eq. (4.6a) is of course equal to $(dR(r)/dr)(dr/dt)$, and we have pointed out above Eq. (4.1) that $dR(r)/dr = (r/R(r))^2$. Inserting this result along with $B(R(r)) = (1-(r_s/R(r)))$ and $A(R(r)) = (1/B(R(r)))$ into Eq. (4.6a) yields,

$$\big((r/R(r))^4/(1-(r_s/R(r)))^3\big)(dr/dt)^2 - \big(c^2/(1-(r_s/R(r)))\big) = -c^2K^2, \tag{4.6b}$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$ and $r_s \stackrel{\text{def}}{=} (2GM/c^2)$. We are now able to write down the Einstein-gravity analog of the Newtonian-gravity equation of motion $(dr/dt)^2 = (2GM/r) + v^2$ of a test body *which moves only radially relative to a static point mass M at $r = 0$*,

$$(dr/dt)^2 = c^2\big((R(r)/r)^2(1 - (r_s/R(r)))\big)^2\big[1 - K^2(1 - (r_s/R(r)))\big], \tag{4.6c}$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$. Defining the dimensionless variable q as $q \stackrel{\text{def}}{=} (r/r_s)$, we note that $(R(r)/r) = \big((q^3+1)^{\frac{1}{3}}/q\big)$ and $(1 - (r_s/R(r))) = \big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)$, so Eq. (4.6c) becomes,

$$(dr/dt)^2 = c^2\big(((q^3+1)^{\frac{1}{3}}/q)^2\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big)^2\big[1 - K^2\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big], \tag{4.6d}$$

where $q \stackrel{\text{def}}{=} (r/r_s)$. In the Newtonian-gravity case where $(dr/dt)^2 = (2GM/r) + v^2$, *$(dr/dt)^2$ grows without bound as $r \to 0$*. Indeed, in the Newtonian-gravity case, *$|dr/dt|$ is asymptotic to $\sqrt{2GM/r}$ as $r \to 0$*.

To work out the asymptotic behavior of $(dr/dt)^2$ as $q \to 0$ in Eq. (4.6d), we note that as $q \to 0$, $(q^3+1)^{\frac{1}{3}}/q \simeq 1/q$ and $\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big) \simeq q^3/3$, so $\big((q^3+1)^{\frac{1}{3}}/q\big)^2\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big) \simeq q/3$, which together with Eq. (4.6d) yields that $((dr/dt)/c)^2 \simeq (q/3)^2$ as $q \to 0$. Thus $|dr/dt| \simeq c(q/3)$ as $q \to 0$, so,

$$\textit{the test body's radial speed } |dr/dt| \text{ is asymptotic to } (c/(3r_s))r \text{ as } r \to 0, \tag{4.6e}$$

which is precisely the opposite of the unbounded speed of the test body as $r \to 0$ in the Newtonian-gravity case. In the Einstein-gravity case, *the gravitational time-dilation effect of very strong gravity reduces speeds.*

We next verify that $(dr/dt)^2 < c^2$. We *first* show that $d\big(((q^3+1)^{\frac{1}{3}}/q)^2\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big)/dq > 0$ when $q > 0$. Since $d\big(((q^3+1)^{\frac{1}{3}}/q)^2\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big)/dq = \big[2 + q^3 - 2(q^3+1)^{\frac{1}{3}}\big]/\big[q^3(q^3+1)^{\frac{2}{3}}\big]$, *we must show that* $2 + q^3 > 2(q^3+1)^{\frac{1}{3}}$ when $q > 0$. We do so by exhibiting a chain of inequalities *which are logically equivalent to $2 + q^3 > 2(q^3+1)^{\frac{1}{3}}$, where the final inequality in the chain is clearly valid when $q > 0$*,

$$\begin{aligned} 2 + q^3 > 2(q^3+1)^{\frac{1}{3}} \iff 1 + (q^3/2) > (1+q^3)^{\frac{1}{3}} \iff 1 + 3(q^3/2) + 3(q^3/2)^2 + (q^3/2)^3 > 1 + q^3 \\ \iff (1/2)q^3 + (3/4)q^6 + (1/8)q^9 > 0 \ \text{ when } \ q > 0. \end{aligned} \tag{4.6f}$$

Therefore $\big(((q^3+1)^{\frac{1}{3}}/q)^2\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big)$ is a strictly increasing function of q when $q > 0$, so when $q > 0$, it is less than its $q \to \infty$ limit, which has the value unity. Consequently, from Eq. (4.6d), $(dr/dt)^2 < c^2\big[1 - K^2\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big] < c^2$ when $q > 0$, because $K^2 > 0$ and $\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big) > 0$ when $q > 0$. Thus, $(dr/dt)^2 < c^2$ when $q > 0$, and, when $q = 0$, Eq. (4.6e) implies that $(dr/dt)^2 = 0$, so $(dr/dt)^2 < c^2$ *under all circumstances; the test body never has a speed as great as c.* This gravitational result *in Einstein coordinates* is physically sensible [5]; *the speeds exceeding* c which the Newtonian gravity equation $(dr/dt)^2 = (2GM/r) + v^2$ permits *in Friedmann coordinates* aren't physically sensible [5].

We next investigate the asymptotic radial speed $|dr/dt|$ of the test body as $r \to \infty$. From Eq. (4.6d) we see that as $q \to \infty$, $(dr/dt)^2 \to c^2\big(1 - K^2\big)$. Therfore,

$$|dr/dt| \to c\sqrt{1 - K^2} \ \text{ as } \ r \to \infty, \tag{4.6g}$$

so $K^2 = \big(1 - (v/c)^2\big)$, where $v \geq 0$ is the test body's asymptotic radial speed. Upon inserting $K^2 = \big(1 - (v/c)^2\big)$ into Eq. (4.6d), it becomes,

$$(dr/dt)^2 = c^2\big(((q^3+1)^{\frac{1}{3}}/q)^2\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big)^2\big[1 - \big(1 - (v/c)^2\big)\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big]. \tag{4.6h}$$

To apply Eq. (4.6h) to the motion of the radius of a uniform-density zero-pressure expanding sphere of dust (the *simplest* model universe) *via the Birkhoff theorem*, we again note that one of most distant known galaxies has a redshift z of 14, whose corresponding recession speed is $0.99c$, so again it is reasonable to put v to c in Eq. (4.6h), which yields,

$$(dr/dt)^2/c^2 = \big(((q^3+1)^{\frac{1}{3}}/q)^2\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big)^2, \tag{4.7a}$$

Since this model universe *is expanding, not contracting*, Eq. (4.7a) becomes.

$$(dr/dt)/c = +\big((q^3+1)^{\frac{1}{3}}/q\big)^2\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big) = (1+q^3)^{\frac{1}{3}}\big((1+q^3)^{\frac{1}{3}} - 1\big)/q^2, \tag{4.7b}$$

and since the dimensionless scaled time u is $t/(r_s/c)$, and $q = r/r_s$ is the dimensionless scaled radius, then in terms of the dimensionless scaled radius q and the dimensionless scaled time u Eq. (4.7b) becomes,

$$dq/du = (1 + q^3)^{\frac{1}{3}} \big((1 + q^3)^{\frac{1}{3}} - 1\big)/q^2. \tag{4.7c}$$

Eq. (4.7c) is the Einstein-coordinate *replacement* of the Friedmann-coordinate *second* differential equation of model-universe radius motion $dq/du = \sqrt{(1/q) + 1}$ of Eq. (1.2a). Here we *again* see that as $q \to 0$, the model-universe dimensionless scaled radial velocity $dq/du \to +\infty$ in Friedmann coordinates, but that as $q \to 0$, the model-universe dimensionless scaled radial velocity $dq/du \to 0$ in Einstein coordinates, which reflects the *absence* of gravitational time dilation in Friedmann coordinates, and the *dominance* of gravitational time dilation in Einstein coordinates when gravity is sufficiently strong.

The Einstein-coordinate *replacement* of the Friedmann-coordinate *first* equation $d^2q/du^2 = -1/(2q^2)$ of Eq. (1.2a) is obtained from the Eq. (4.7c) Einstein-coordinate expression for dq/du as follows,

$$d^2q/du^2 = \{d[(dq/du)]/dq\}\,(dq/du) =$$
$$\big(q^3 - 2\big((1 + q^3)^{\frac{1}{3}} - 1\big)\big)\big((1 + q^3)^{\frac{1}{3}} - 1\big)/\big(q^5(1 + q^3)^{\frac{1}{3}}\big). \tag{4.7d}$$

Analysis of Eq. (4.7d) shows that the radial acceleration d^2q/du^2 of the model universe is *positive* for all $q > 0$ in Einstein coordinates, *in contrast to the negative values* $-1/(2q^2)$ of the radial acceleration d^2q/du^2 of the model universe for all $q > 0$ in Friedmann coordinates. This unexpected *positive acceleration* of the universe's expansion has been observed; it won its discoverers a Nobel prize. Here we see that *positive acceleration of the universe's expansion is an entirely natural gravitational phenomenon in Einstein coordinates*; it *doesn't* require ad hoc insertion of a cosmological-constant term $\lambda g_{\mu\nu}(x)$ into the Einstein equation.

In Einstein coordinates the evolution in dimensionless scaled time u of the model universe's dimensionless scaled radius q is given by the solution $q(u)$ of the Eq. (4.7c) equation of motion. For the initial condition $q(u = 0) = 1$, i.e., that the model universe attains its Schwarzschild radius at time zero, the numerical solution $q(u)$ of the Eq. (4.7c) equation of motion is displayed as the blue curve of Figure 1. Its corresponding dimensionless scaled radial velocity $dq(u)/du$ and dimensionless scaled radial acceleration $d^2q(u)/du^2$ in Einstein coordinates are displayed as the blue curves of Figures 2 and 3 respectively. These three blue curves show that in Einstein coordinates the model universe *exists at all values of the dimensionless scaled time* u, but the blue curve of Figure 1 shows that at dimensionless scaled times u which are much less than -1 the model universe in Einstein coordinates *is exponentially small relative to its Schwarzschild radius*, and correspondingly has its physical processes and radiation frequencies *so greatly gravitationally time-dilated* that it can aptly be colloquially described as being in a state of "suspended animation". In our simplified model using Einstein coordinates, one might even find an explanation for the excess of particles over antiparticles, because the universe has always existed.

The blue curve of Figure 2 shows that the radial expansion velocity of the model universe in Einstein coordinates *never exceeds* c, in accord with the very thoroughly tested precepts of Lorentzian relativity [5], whereas the red curve of Figure 2 shows that the radial expansion velocity of the model universe in Friedmann coordinates *is unbounded and always exceeds* c, in extreme violation of those precepts [5].

The red curve of Figure 3 shows *unbounded deceleration* of the expansion of the model universe in Friedmann coordinates, whereas the blue curve of Figure 3 *in contrast* shows *perpetual acceleration* of the expansion of the model universe in Einstein coordinates, *with a pronounced peak in that inflation* near $u = 1$.

We now reflect on the reasons why Einstein-coordinate gravity, which *doesn't* suppress gravitational time dilation and *respects* the test-body speed limit c, may cause acceleration *opposite* to that caused by Friedmann-coordinate Newtonian gravity, as evidenced *by the stark contrast* between the blue and red curves of Figure 3. The effect of *gravitational time dilation* is to *decrease* the speed of a test body *which is moving toward* a gravitational center (e.g., a static point mass), and to *increase* the speed of a test body *which is moving away from* that gravitational center. This type of acceleration is indeed *opposite* to that caused by Newtonian gravity; it becomes important for gravitational fields which are *so strong* that this effect of their gravitational time dilation *dominates the opposite effect of their Newtonian gravity.*

Even a *weak* gravitational field, however, turns out, in Einstein coordinates, to *repel* rather than *attract* a test body *which is moving close enough to the speed* c. Consider a light packet which is moving toward a gravitational center, but is distant enough from that center that the gravitational field at the light packet's position is weak. If the light packet *is accelerated toward the gravitational center the same way that a test body at rest at the light packet's position is*, then the light packet's speed *immediately exceeds* c! In fact, in Einstein coordinates such a light packet *is accelerated away from the gravitational center twice as strongly as a test body at rest at the light packet's position is accelerated toward the gravitational center.*

In Einstein coordinates the presence of gravity *always slows light to a speed less than c*, so the effect of gravity on light *is refractive.* Astronomers cleverly exploit *the refractive effect of gravity on light* by using the gravitational field of a foreground galaxy as a *lens* which *magnifies* the image *of a more distant galaxy.*

In Einstein coordinates *it isn't only light* which is accelerated *away* from a gravitational center; a test body which is moving at a speed greater than $c/\sqrt{3} = 0.57735c$ *is also accelerated away from a gravitational center*, although to a lesser extent. Since the speed $c/\sqrt{3}$ corresponds to a redshift z of approximately 1, in Einstein coordinates there exist *many* telescope-visible galaxies which are gravitationally accelerated *away from an observer* by the sphere of cosmic matter whose radius is the observer's distance to such a galaxy.

These nonintuitive aspects of radial gravitational acceleration in Einstein coordinates can be worked out from Eq. (4.6h), which it is convenient to reexpress for that purpose as,

$$(dr/dt)^2 = c^2\big(\chi(q) - \big(1 - (v/c)^2\big)\,\xi(q)\big), \tag{4.8a}$$

where,

$$\chi(q) = \big((q^3+1)^{\frac{1}{3}}/q\big)^4\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)^2 \ \text{ and } \ \xi(q) = \big((q^3+1)^{\frac{1}{3}}/q\big)^4\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)^3. \tag{4.8b}$$

Differentiating both sides of Eq. (4.8a) with respect to t yields,

$$2(dr/dt)(d^2r/dt^2) = c^2\big(d\chi(q)/dq - (1-(v/c)^2)d\xi(q)/dq\big)\,(dq/dr)(dr/dt), \tag{4.8c}$$

where,

$$\begin{aligned} d\chi(q)/dq &= \big(4/q^5\big)\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big(1 + (q^3/2) - (q^3+1)^{\frac{1}{3}}\big) \ \text{ and} \\ d\xi(q)/dq &= \big(4/q^5\big)\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)^2\big(1 + (3q^3/4) - (q^3+1)^{\frac{1}{3}}\big). \end{aligned} \tag{4.8d}$$

Dividing Eq. (4.8c) by $2(dr/dt)$ and noting that $q = (r/r_s)$ and $r_s = (2GM/c^2)$, we rexpress Eq. (4.8c) as,

$$\begin{aligned} d^2r/dt^2 &= (1/2)\,\big(c^2/r_s\big)\,(r_s/r)^2q^2\,\big(d\chi(q)/dq - \big(1-(v/c)^2\big)\,d\xi(q)/dq\big) = \\ &\big(GM/r^2\big)\,\big(\big(q^2\,d\chi(q)/dq\big) - \big(1-(v/c)^2\big)\,\big(q^2\,d\xi(q)/dq\big)\big). \end{aligned} \tag{4.8e}$$

From Eq. (4.8d) we obtain that as $q \to \infty$, $(q^2\,d\chi(q)/dq) \to 2$ and $(q^2\,d\xi(q)/dq) \to 3$, so from Eq, (4.8e),

$$\textit{the test body's radial acceleration } d^2r/dt^2 \textit{ is asymptotic to } -\big(GM/r^2\big)\big(1 - 3(v/c)^2\big) \textit{ as } r \to \infty, \tag{4.8f}$$

which agrees with the Newtonian-gravity acceleration result $d^2r/dt^2 = -(GM/r^2)$ *only when the test body's asymptotic radial velocity* $v \ll c$. On the other hand, when the test body's asymptotic radial velocity $v > c/\sqrt{3} = 0.57735c$, *the test body's asymptotic radial acceleration becomes positive.* Since the speed $c/\sqrt{3}$ corresponds to a redshift z of approximately 1, in Einstein coordinates galaxies whose redshifts exceed 1 are in the process *of increasing their redshifts* (i.e., their acceleration *is in the same direction* as their recession velocity). This is *an entirely natural gravitational phenomenon in Einstein coordinates* which *doesn't* require ad hoc insertion of a cosmological-constant term $\lambda g_{\mu\nu}(x)$ into the Einstein equation. For sufficiently distant galaxies, Eq. (4.8f) becomes $d^2r/dt^2 \approx 2(GM/r^2)$, so their positive radial acceleration *wanes with time* as they move ever further away. This feature of the "tail" of the Figure 3 blue curve is in broad conformity with data released by the Dark Energy Spectroscopic Instrument (DESI) collaboration [9].

For a radially-traveling packet of light, $v = c$, so Eq. (4.8f) tells us that its asymptotic radial acceleration is $+2\,\big(GM/r^2\big)$, *which is opposite in direction and double in magnitude* the radial acceleration $-\big(GM/r^2\big)$ of a test body at rest (i.e., $v = 0$) at the same radius r as that packet of light. It is therefore apparent, as Einstein came to realize in his landmark November 18, 1915 paper [2], that straightforward application of the Principle of Equivalence *fails altogether for light.*

Furthermore, putting the value of v to c in Eq. (4.8a) yields,

$$(dr/dt)^2 = c^2\chi(q), \tag{4.8g}$$

where $\chi(q)$ *increases monotonically from zero at* $q = 0$ *toward unity as* $q \to \infty$, as can be verified by analyzing Eqs. (4.8b) and (4.8d). Therefore in Einstein coordinates a radially-traveling light packet's speed in the gravitational field of a point mass *is less than c*, and the *closer* the light packet is to the point mass, the *slower* its speed is. Thus the effect of gravity on light *is refractive*, as astronomers *are well aware.*

The red and blue $q(u)$ curves of Figure 1 show the growth of the expanding model universe's dimensionless scaled radius q as a function of its dimensionless scaled time u in Friedmann and Einstein coordinates respectively. Although the Einstein-coordinate blue $q(u)$ curve increases exponentially from very slightly positive values when u is much less than -1, *it nevertheless is very quickly overtaken by the Friedmann-coordinate red* $q(u)$ *curve*, which only increases from zero when u is greater than $u_i = -\sqrt{2} + \ln(1 + \sqrt{2}) = -0.53284$. That occurs *because the initial rate of increase of the Friedmann-coordinate red* $q(u)$ *curve is unbounded* (see the $dq(u)/du$ red curve of Figure 2).

The extremely gravitationally time-dilated "suspended animation" state of the model universe in Einstein coordinates *dissipates around* $u = 1$, *after* $q(u)$ passes the value 1 and its inflationary acceleration of expansion *peaks* (see the blue curve of Figure 3), immediately *following* which its radius and radial expansion velocity *are quite appreciably less than those of the model universe in Friedmann coordinates* (compare the blue to the red curves of Figures 1 and 2 during the era of values of u which are roughly between 1 and 5). This makes the model universe in Einstein coordinates more amenable to galaxy formation during that early era of values of u roughly between 1 and 5 than that model universe in Friedmann coordinates is.

Furthermore, the *nature* of the model universe in Einstein coordinates at values of u much less than -1 *is that of an extremely slowly expanding zero-angular-momentum extremely gravitationally time-dilated black hole.* (Note, however, that *physical* black holes, no matter how extremely gravitationally time-dilated, *don't have nonzero-radius event horizons*—see section 3.) It seems *reasonable* that the inflationary-peak *breakdown at values of* u *around* 1 *of such an expanding zero-angular-momentum extremely gravitationally time-dilated black hole produces a great many nonzero-angular-momentum extremely gravitationally time-dilated black holes which are individually stable*, along with a considerably *lesser* amount of matter *not* organized into such nonzero-angular-momentum extremely time-dilated stable black holes. The *existence of such constituents* of a relatively compact, not-too-rapidly-expanding universe would be favorable *to early gravitational condensation of compact galaxies*, and *such compact galactic environments would have promoted the rapid birth of stars, including very short-lived ultraviolet giants, from those galaxies' considerably lesser amount of matter not organized into nonzero-angular-momentum extremely time-dilated stable black holes.* These compact early galaxies would have been bright and hot well into the ultraviolet; their frequency-downshifted black-body radiation may be the cosmic microwave background of the present era.

The *unbounded-radial-velocity explosive Big-Bang birth* of the model universe in Friedmann coordinates *contrariwise is highly unfavorable to early formation of nonzero-angular-momentum extremely gravitationally time-dilated stable black holes or consequent early gravitational condensation of compact galaxies.*

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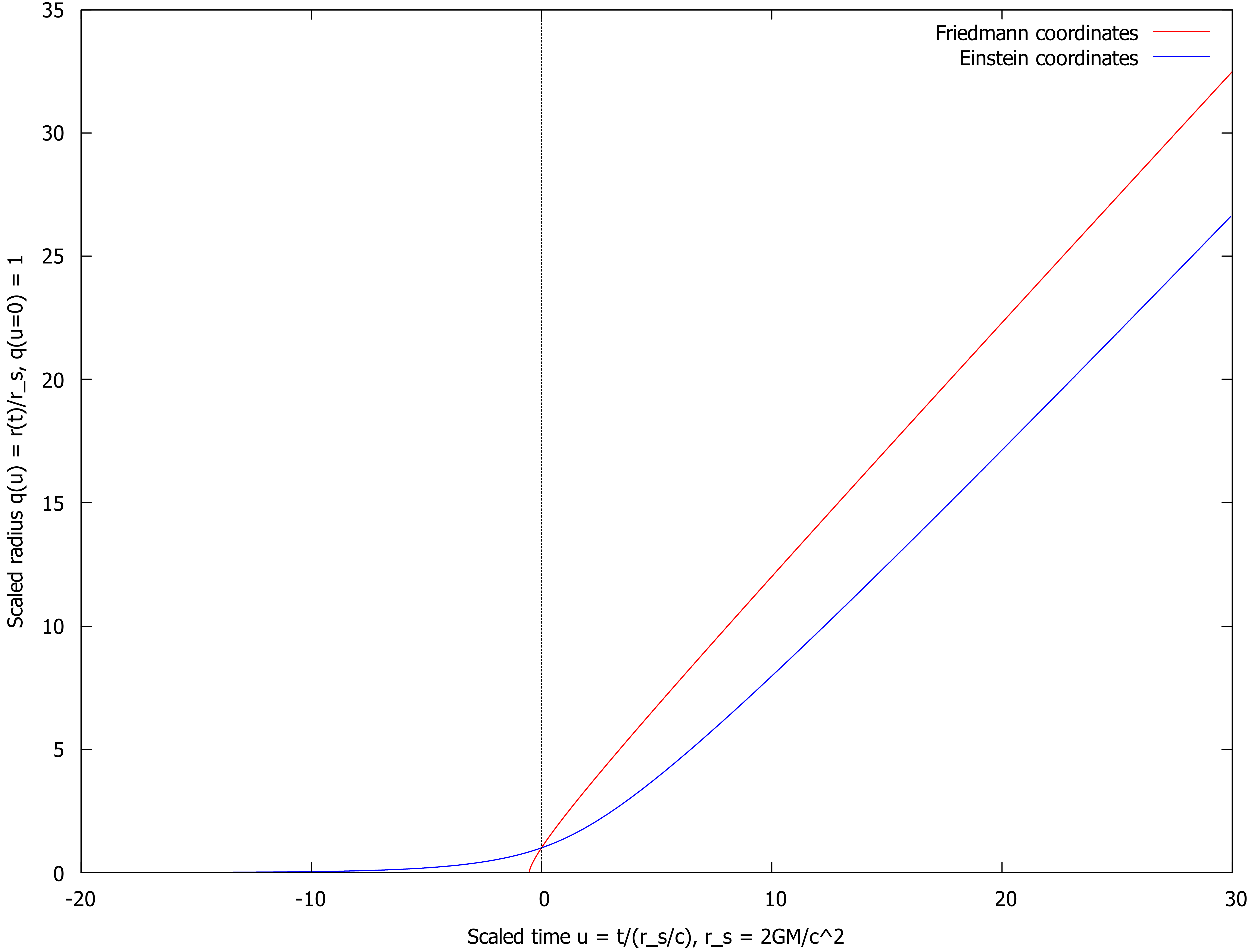
Figure 1 The simplest model universe's scaled radius



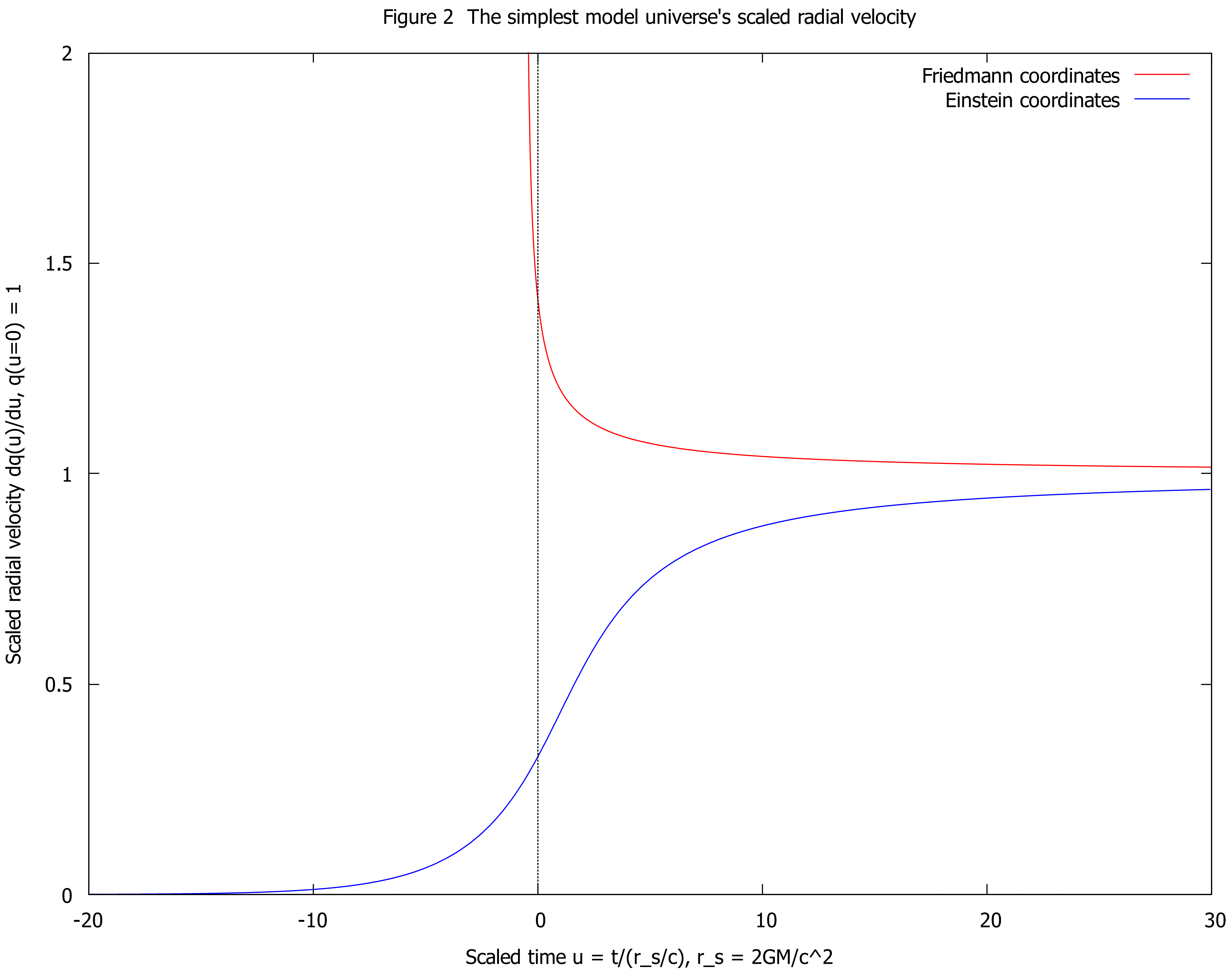


Figure 2 The simplest model universe's scaled radial velocity
Friedmann coordinates
Einstein coordinates
Scaled radial velocity dq(u)/du, q(u=0) = 1
Scaled time u = t/(r_s/c), r_s = 2GM/c^2
2
1.5
1
0.5
0
-20
-10
0
10
20
30

Figure 3 The simplest model universe's scaled radial acceleration