Two-Person Additively-Separable Sum Games

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Abstract

We consider a sub-class of bi-matrix games which we refer to as two-person (hereafter referred to as two-player) additively-separable sum (TPASS) games, where the sum of the pay-offs of the two players is additively separable. The row player's pay-off at each pair of pure strategies, is the sum of two numbers, the first of which may be dependent on the pure strategy chosen by the column player and the second being independent of the pure strategy chosen by the column player. The column player's pay-off at each pair of pure strategies, is also the sum of two numbers, the first of which may be dependent on the pure strategy chosen by the row player and the second being independent of the pure strategy chosen by the row player. The sum of the inter-dependent components of the pay-offs of the two players is assumed to be zero. We show that a (randomized or mixed) strategy pair is an equilibrium of the game if and only if there exist two other real numbers such that the three together solve a certain linear programming problem.

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1. Consider an ordered triplet (A, π, ρ) where for some positive integers m and n, A is an m×n real valued matrix, π is an m-dimensional real-valued column vector and ρ is an n-dimensional real-valued column vector. For $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, let π_i denote the i^{th} coordinate of π, ρ_j denote the j^{th} coordinate of ρ, and let a_{ij} denote the entry at the intersection of the i^{th} row and j^{th} column of A.

There are two players in this game (i.e., interactive decision-making problem)- the row player and the column player.

For $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, if the row player chooses the i^{th} row and the column player chooses column j, then the payoff to the row player is $a_{ij} + \pi_i$ and the payoff to the column player is $- a_{ij} + \rho_j$.

Note that the sum of the pay-offs to the row player and the column player if the row player chooses the i^{th} row and the column player chooses j^{th} column is $\pi_i + \rho_j$.

We refer to the triplet (A, π, ρ) as a **two-person additively-separable sum** (TPASS) game.

In case all coordinates of π are identical and all coordinates of ρ are identical, a TPASS game reduces to a two-person constant sum (TPCS) game (see chapter 20 of Mote and Madhavan (2016)).

We allow for randomized (mixed) strategies for the row and column players.

For any non-negative integer ℓ, let $\Delta^{\ell} = \{x \in \mathbb{R}_{+}^{\ell+1} | \sum_{k=1}^{\ell+1} x_k = 1\}$. We will interpret points in Δ^{ℓ} as ℓ-dimensional column vectors.

The (randomized or mixed) strategy set for the row player is Δ^{m-1} and the (randomized or mixed) strategy set for the column player is Δ^{n-1}.

A pair $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$ is a (**randomized or mixed**) **strategy pair**.

The pay-off function for the row-player is the function $f^R: \Delta^{m-1} \times \Delta^{n-1} \to \mathbb{R}$ such that for all $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$, $f^R(p, q) = p^T A q + p^T \pi$.

The pay-off function for the column-player is the function $f^C: \Delta^{m-1} \times \Delta^{n-1} \to \mathbb{R}$ such that for all $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$, $f^C(p, q) = -p^T A q + \rho^T q$.

The following concept is available in Nash (1951).

$(p^*, q^*) \in \Delta^{m-1} \times \Delta^{n-1}$ is said to be **an equilibrium** of the TPASS game (A, π) if for all $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$: $f^R(p^*, q^*) \geq f^R(p, q^*)$ and $f^C(p^*, q^*) \geq f^C(p^*, q)$.

Let $R(\pi)$ be the $m \times n$ real matrix, such that for all $i \in \{1, \ldots, m\}$, every entry in the i^{th} row of $R(\pi)$ is π_i.

Let $C(\rho)$ be the $m \times n$ real matrix, such that for all $j \in \{1, \ldots, n\}$, every entry in the j^{th} column of $C(\rho)$ is ρ_j.

Thus, for all $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$, $p^T \pi = p^T R(\pi) q$ and $\rho^T q = p^T C(\rho) q$.

For any positive integer ℓ, let $e^{(\ell)}$ denote the ℓ-dimensional sum column vector, i.e., the ℓ-dimensional vector, all coordinates of which are 1.

2. The following result follows immediately from the "Equivalence Theorem" in section II of Mangasarian and Stone (1964). The Equivalence Theorem of Mangasarian and Stone, uses the result on existence of an equilibrium for a more general class of games, known as bi-matrix games, a proof of the latter being available as the proof of theorem 2 in Chandrasekaran (undated).

Proposition 1: (p^*, q^*) is an equilibrium for the TPASS game (A, π, ρ) <u>if and only if</u> there exist real numbers α^*, β^* such that p^*, q^*, α^*, β^* solve the following "linear programming problem":

Maximize $\pi^T p + \rho^T q - \alpha - \beta$, subject to $Aq + \pi - \alpha e^{(m)} \leq 0$, $-A^T p + \rho - \beta e^{(n)} \leq 0$, $p^T e^{(m)} = 1$, $q^T e^{(n)} = 1$, $p \geq 0$, $q \geq 0$.

Proof: From the Equivalence Theorem in section II of Mangasarian and Stone (1964), (p^*, q^*) is an equilibrium for the TPASS game (A, π, ρ) <u>if and only if</u> there exist real numbers α^*, β^* such that p^*, q^*, α^*, β^* solve the following "<u>bi</u>-linear programming problem":

Maximize $p^T(A+ R(\pi) -A + C(\rho))q - \alpha - \beta$, subject to $(A + R(\pi))q - \alpha e^{(m)} \leq 0$, $(-A+ C(\rho))^T p - \beta e^{(n)} \leq 0$, $p^T e^{(m)} = 1$, $q^T e^{(n)} = 1$, $p \geq 0$, $q \geq 0$.

The above problem is equivalent to:

Maximize $p^T(R(\pi) + C(\rho))q - \alpha - \beta$, subject to $(A + R(\pi))q - \alpha e^{(m)} \leq 0$, $(-A+ C(\rho))^T p - \beta e^{(n)} \leq 0$, $p^T e^{(m)} = 1$, $q^T e^{(n)} = 1$, $p \geq 0$, $q \geq 0$.

Since $R(\pi)q = \pi$ and $p^T C(\rho) = \rho^T$, whence $p^T(R(\pi))q = p^T\pi = \pi^T p$ and $p^T(C(\rho))q = \rho^T q$ for all $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$, we get our desired result. Q.E.D.

Note: Chakrabarti, Gilles and Mallozzi (2024) introduce two classes of non-cooperative games in normal form with the number of players being finite but at least two. The first is the class of "separable games" and the second is the class of "additively separable games". Our TPASS games are "a special case" in the two-person (two-player) context of "additively separable games", the latter being definition 4.1 of Chakrabarti, Gilles and Mallozzi (2024). Since, the two-person version of games considered in Chakrabarti, Gilles and Mallozzi (2024)- including their ""additively separable games"- may not be bi-matrix games, the possibility of extending Proposition 1 in our paper to the class of two-person "additively separable" games is an open question.

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