**Gravitational Time Dilation Sheds New Light on the Metric of a Static Point Mass**

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**Abstract** We show that gravitational time dilation constrains the 00 components of metrics of static gravitational fields of finite extent to be nonnegative. Since this constraint is violated by the textbook "Schwarzschild metric" of a static point mass (which is due to Droste and Hilbert, not Schwarzschild), that metric must be reconsidered. It in fact is only one member of a one-parameter family of similar metrics which all meet the requirements set out in textbooks. Although those requirements don't include a boundary condition at the location of the static point mass, gravitational time dilation causes the 00 component of the metric to vanish there. The consequent unique metric of a static point mass turns out to be precisely the metric given in Karl Schwarzschild's January 13, 1916 paper. Schwarzschild's paper, however, doesn't take gravitational time dilation into account; it instead improperly attaches physical significance to a singularity in a factor of a metric component which itself is nonsingular.

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## 1. Gravitational time dilation constrains static metrics to have nonnegative 00 components

The tick rates of clocks at two locations in a static gravitational field of finite extent are related by, [1]

$$[\text{the tick rate of a clock at } \mathbf{r}_2]\big/[\text{the tick rate of a clock at } \mathbf{r}_1] = \sqrt{g_{00}(\mathbf{r}_2)/g_{00}(\mathbf{r}_1)}. \tag{1.1a}$$

Furthermore, as $|\mathbf{r}_1| \to \infty$, $g_{00}(\mathbf{r}_1) \to \eta_{00} = 1$. Together with Eq. (1.1a) this implies that,

$$[\text{the tick rate of a clock at } \mathbf{r}]\big/[\text{the tick rate of a clock at infinity}] = \sqrt{g_{00}(\mathbf{r})}. \tag{1.1b}$$

Since the quotient of clock tick rates is a real number, Eq. (1.1b) implies that,

$$g_{00}(\mathbf{r}) \geq 0 \text{ for all } \mathbf{r}. \tag{1.1c}$$

The little-known Eq. (1.1c) *constraint* on the 00 component of static gravitational metrics is somewhat similar to the fact that the Lorentz transformation with velocity $\mathbf{v} = (v, 0, 0)$ of $t$ and $\mathbf{r} = (x, y, z)$, i.e.,

$$t' = (t - (vx/c^2))\big/\sqrt{1 - (v/c)^2}, \quad x' = (x - vt)\big/\sqrt{1 - (v/c)^2}, \quad y' = y, \quad z' = z, \tag{1.2}$$

*implies that physical speeds* $|v|$ *are constrained to satisfy* $|v| \leq c$, where $c$ is the speed of light in a vacuum.

The misnamed "Schwarzschild metric" of a static point mass $M$ at $\mathbf{r} = \mathbf{0}$ in gravity textbooks, i.e.,

$$(c\,d\tau)^2 = (1 - (r_s/R))(c\,dt)^2 - (1/(1 - (r_s/R)))(dR)^2 - R^2((d\theta)^2 + (\sin\theta\,d\phi)^2), \tag{1.3}$$

where $R = |\mathbf{r}|$ and $r_s \stackrel{\text{def}}{=} (2GM/c^2)$, which was first broached by Johannes Droste on May 27, 1916 [2] and later strongly promoted by David Hilbert, *violates the* Eq. (1.1c) *constraint,* however, because $g_{00}(R) = (1 - (r_s/R)) < 0$ when $0 \leq R < r_s$, so it needs to be reconsidered.

Although the Eq. (1.3) Droste-Hilbert textbook mertric isn't satisfactory, *it does seem plausible* since it is spherically symmetric, its Ricci tensor vanishes at all $R > 0$, it satisfies Einstein's November 18, 1915 coordinate condition that its determinant always equals $-1$ [3], it approaches the Minkowski metric as $R \to \infty$ and $g_{00}(R)$ is asymptotic to $1 + (2\phi(R)/c^2)$ as $R \to \infty$, where $\phi(R) = -(GM/R)$, the Newtonian-gravity potential for the static point mass $M$ at $R = 0$.

We next show, however, that there exists a one-parameter *family* of metrics *which all satisfy the above-listed requirements,* but that *only one* of those metrics *satisfies an additional boundary condition at* $\mathbf{r} = \mathbf{0}$ *which arises from the presence at* $\mathbf{r} = \mathbf{0}$ *of the static point mass* $M$.

## 2. Beyond the Droste-Hilbert textbook metric of a static point mass

Transformations of the radial parameter $R$ of the Eq. (1.3) unsatisfactory Droste-Hilbert textbook metric preserve its spherical symmetry and the appropriate vanishing of its Ricci tensor. We now obtain transformations $R(r)$ which as well preserve the $-1$ value of its determinant. The transformed metric is,

$$(c\,d\tau)^2 = (1 - (r_s/R(r)))(c\,dt)^2 - (1/(1 - (r_s/R(r))))(dR(r)/dr)^2(dr)^2 -$$
$$(R(r)/r)^2\,r^2((d\theta)^2 + (\sin\theta\,d\phi)^2), \tag{2.1}$$

whose determinant is $-(dR(r)/dr)^2(R(r)/r)^4$, which we set to $-1$ and then extract the equation,

$$R^2 dR = r^2 dr, \tag{2.2a}$$

which when integrated yields,

$$(R(r))^3 = r^3 + \rho, \tag{2.2b}$$

where $\rho$ is an arbitrary integration constant with the dimension of length cubed. The upshot is a one-parameter family of transformations,

$$R(r) = (r^3 + \rho)^{\frac{1}{3}}, \tag{2.2c}$$

which inserted into Eq. (2.1) along with $dR(r)/dr = (r/(r^3 + \rho)^{\frac{1}{3}})^2$, yields the one-parameter metric family,

$$(c\,d\tau)^2 = (1 - (r_s/(r^3 + \rho)^{\frac{1}{3}}))(c\,dt)^2 - (1/(1 - (r_s/(r^3 + \rho)^{\frac{1}{3}})))(r/(r^3 + \rho)^{\frac{1}{3}})^4(dr)^2 -$$
$$((r^3 + \rho)^{\frac{1}{3}}/r)^2\,r^2((d\theta)^2 + (\sin\theta\,d\phi)^2). \tag{2.3}$$

The Eq. (2.3) one-parameter family of particular radial-coordinate transformations of the Eq. (1.3) Droste-Hilbert textbook metric yields the latter when the parameter $\rho = 0$. Its metrics are spherically symmetric, have determinant $-1$, approach the Minkowski metric as $r \to \infty$ and have $g_{00}(r)$ asymptotic to $1+(2\phi(r)/c^2)$ as $r \to \infty$, where $\phi(r) = -(GM/r)$, the Newtonian-gravity potential for the static point mass $M$ at $r = 0$. We therefore *realize* that *the listed requirements* which the Eq. (1.3) Droste-Hilbert textbook metric satisfies *aren't sufficient to determine it;* the *presence* of the static point mass $M$ at $r = 0$ *obviously imposes an additional boundary condition at* $r = 0$ *which hasn't yet been taken into account.*

In *Newtonian* gravity, the static point mass M at $r = 0$ of course produces *infinite gravitational acceleration at* $r = 0$, but in *Lorentzian* gravity, *the extreme gravitational time dilation at* $r = 0$ *reduces velocities and accelerations at* $r = 0$ *to zero.* In *particular,* the tick rate of a clock at $r = 0$ *will be zero relative to the tick rate of a clock at infinity,* a fact *which together with* Eq. (1.1b) *implies that,*

$$0 = [\text{the tick rate of a clock at } r = 0]/[\text{the tick rate of a clock at infinity}] = \sqrt{g_{00}(r = 0)}, \tag{2.4a}$$

so *the boundary condition at* $r = 0$ which the *presence* of the static point mass at $r = 0$ *imposes* is,

$$g_{00}(r = 0) = 0. \tag{2.4b}$$

Since for the Eq. (2.3) metric family, $g_{00}(r) = 1 - (r_s/(r^3 + \rho)^{\frac{1}{3}})$, we see that *the* Eq. (2.4b) *boundary condition* $g_{00}(r = 0) = 0$ *yields,*

$$\rho = (r_s)^3. \tag{2.5}$$

This *unique value of* $\rho$ inserted into Eq. (2.3) yields the following non-pathological *unique Einstein-coordinate metric produced by a static point mass* $M$ *at* $r = 0$,

$$(c\,d\tau)^2 = (1 - (r_s/(r^3 + (r_s)^3)^{\frac{1}{3}}))(c\,dt)^2 - (1/(1 - (r_s/(r^3 + (r_s)^3)^{\frac{1}{3}})))(r/(r^3 + (r_s)^3)^{\frac{1}{3}})^4(dr)^2 - ((r^3 + (r_s)^3)^{\frac{1}{3}}/r)^2\, r^2((d\theta)^2 + (\sin\theta\, d\phi)^2). \tag{2.6}$$

Harking back to Eqs. (2.2c) and (2.1), *but now replacing* $\rho$ *in* Eq. (2.2c) *by* $(r_s)^3$ produces,

$$R(r) = (r^3 + (r_s)^3)^{\frac{1}{3}}, \tag{2.7a}$$

and produces, from Eq. (2.7a) and Eq. (2.1), *the following compact form of the unique* Eq. (2.6) *metric,*

$$(c\,d\tau)^2 = (1 - (r_s/R(r)))(c\,dt)^2 - (1/(1 - (r_s/R(r))))(dR(r))^2 - (R(r))^2((d\theta)^2 + (\sin\theta\, d\phi)^2). \tag{2.7b}$$

Karl Schwarzschild's January 13, 1916 paper [4] *also* features Eq. (2.3) and Eqs. (2.5)–(2.7b), but instead of taking gravitational time dilation into account, it arrives at Eq. (2.5) *by rendering a chosen factor of the* Eq. (2.3) *metric component* $g_{rr}(r)$ *singular at* $r = 0$ *in analogy with the singularity at* $r = 0$ *of the Newtonian-gravity potential of a static point mass* $M$ *at* $r = 0$. The chosen factor $f_1(r)$ of $g_{rr}(r)$ is $-(g_{rr}(r)/r^4)$, i.e.,

$$f_1(r) = (1/(1 - (r_s/(r^3 + \rho)^{\frac{1}{3}})))(1/(r^3 + \rho)^{\frac{1}{3}})^4,$$

which is singular at $r = 0$ if $\rho = (r_s)^3$, the same equality as in Eq. (2.5). Consequently the equations of Schwarzschild's paper also match Eqs. (2.6)–(2.7b). However the chosen factor $f_1(r) = -(g_{rr}(r)/r^4)$ of the metric component $g_{rr}(r)$ *has no physical significance in itself,* and when $\rho = (r_s)^3$, *the physically significant related metric component* $g_{rr}(r) = -(r^4 f_1(r))$ *in fact vanishes at* $r = 0$ *instead of being singular at* $r = 0$. Thus Schwarzschild's attempt to leverage the singularity of the Newtonian-gravity case is unsuccessful.

REFERENCES

[1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972), Eq. (3.5.3) on p. 80.

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[3] https://arxiv.org/pdf/2111.11238 M. Janssen and J. Renn, "Einstein and the Perihelion Motion of Mercury", Chapter 6, Subsection 6.3.2, pp. 53–61 (2021).

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