Existence of symmetric equilibrium for symmetric bi-matrix games: A Quadratic Programming Approach

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Abstract

We provide a proof of existence of symmetric equilibrium for symmetric bi-matrix games, a result implied by a more general result that was proved by John Nash. Our proof, unlike the original proof due to Nash, does not appeal to the Brouwer fixed point theorem. We prove that any solution to a certain specific quadratic programming problem, is a symmetric equilibrium for the associated symmetric bi-matrix game.

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1. Introduction: Bi-matrix games (see Chandrasekaran (nd)) are two-person (two-player) games (interactive decision-making problems), in which each player has a finite number of strategies to choose from. A pair of strategies- one for each player- is an equilibrium, if each player's strategy is a "best reply" against the strategy of the other player. Allowing for randomization over strategies, Nash (1951) proved a general result that implies all bi-matrix games have at least one equilibrium. A very "witty" justification for randomized strategies is available in (page 17, in chapter 3 of) Washburn (2014): "It is the key to resolving the '… if he thinks that I think that he thinks that…' impasse- you can't outwit someone who refuses to think".

It has been shown in Chandrasekaran (nd), that an equilibrium for any bi-matrix game can be derived from any "symmetric equilibrium" of an associated "symmetric bi-matrix game". A symmetric bi-matrix game is a bi-matrix game in which if the players exchange their strategies, then it results in them exchanging their pay-offs. However, all existing proofs of existence of equilibrium of bi-matrix games, whether symmetric or not, require the use of a comparatively advanced result in mathematics, known as "Brouwer fixed point theorem". This, is as true for the proof of the general result in Nash (1951), as it is for the various proofs for the existence result for symmetric bi-matrix games that are part of "folklore" and may be found in Chandrasekaran (nd) or Lahiri (2025).

Our purpose here is to prove the existence of symmetric equilibrium for symmetric bi-matrix games, by solving a very simple quadratic programming problem for which a solution is easily seen to exist. It takes no more than continuity properties of simple quadratic functions and the mean value theorem of differential calculus, to show that every solution of the quadratic programming problem solves an associated linear programming problem. The structure of the associated linear programming is such that every solution of the quadratic programming problem, yields a symmetric equilibrium for the symmetric bi-matrix game. This in particular implies a simple proof of the existence of equilibrium for the entire class of bi-matrix games, using results from a first course in calculus, and without requiring the use of any fixed-point theorem or any other result in advanced mathematics.

2. Symmetric bi-matrix games: For a positive integer ℓ, let $\Delta^{\ell-1} = \{x \in \mathbb{R}^{\ell}_{+} | \sum_{j=1}^{\ell} x_j = 1\}$.

For a positive integer n, let A be an n×n matrix.

The pair (A, A^T) is said to be a **symmetric bi-matrix game**.

$x^* \in \Delta^{n-1}$ is said to be a **symmetric equilibrium** of (A, A^T) if $x^{*T}Ax^* \geq x^TAx^*$ for all $x \in \Delta^{n-1}$.

The set $\{x^* | x^*$ is a symmetric equilibrium of $(A, A^T)\}$ is **the set of all symmetric equilibria of** (A, A^T).

3. The main result: We now prove the main result presented here. The implications of the theorem we prove are discussed in Lahiri (2025).

Theorem (Nash (1951)): There exists a symmetric equilibrium for (A, A^T).

Proof: Consider the function $f:\mathbb{R}^n \rightarrow \mathbb{R}$ defined as follows: for all $x \in \mathbb{R}^n$, $f(x) = x^TA^Tx$. f is continuous and twice continuously differentiable continuous on $\mathbb{R}^n$, with $Dg(x) = x^TA^T$ for all $x \in \mathbb{R}^n$. Since Δ^{n-1} is a closed and bounded subset of $\mathbb{R}^n$, and f is continuous, by Weirstrass's theorem (Corollary 5-2.1 on page 95 of Smith and Albrecht (1987)), $\underset{x \in \Delta^{n-1}}{\operatorname{argmax}} f(x) \neq \phi$.

Let $x^* \in \underset{x \in \Delta^{n-1}}{\operatorname{argmax}} f(x)$

Consider the quadratic maximization problem: Maximize f(x), subject to $x \in \Delta^{n-1}$.

Clearly x^* solves the quadratic programming problem.

We need to show that x^* solves: Maximize $x^{*T}A^Tx$, subject to $x \in \Delta^{n-1}$.

Suppose towards a contradiction x^* does not solve the linear programming problem.

Then there exists $x \in \Delta^{n-1}$ such that $Df(x^*)(x-x^*) > 0$.

Note that $x^* + t(x-x^*) \in \Delta^{n-1}$ for all $t \in [0, 1]$.

By the Mean Value Theorem (see theorem 5-6.2 on page 107 of Smith and Albrecht (1987)), for all t in (0, 1), $f(x^* + t(x-x^*)) - f(x^*) = tDf(x^* + \xi(x-x^*))(x-x^*)$ for some $\xi \in (0, t)$, possibly depending on t.

Since $Dg(x^*)(x-x^*) > 0$, by the continuity of the function $y \mapsto Df(y)(x-x^*) = y^TA^T(x-x^*)$ on $\mathbb{R}^n$, there exists $s \in (0, 1)$ such that for all $0 < r \leq s$, $Df(x^* + r(x-x^*)(x -x^*) > 0$.

Let t belong to (0, s). Then, $f(x^* + t(x-x^*)) - f(x^*) = tDf(x^* + \xi(x-x^*))(x-x^*)$ for some $\xi \in (0, t)$, possibly depending on t.

Since $0 < \xi < t < s$, it must be that $Df(x^* + \xi(x-x^*))(x-x^*) > 0$, whence $tDf(x^* + \xi(x-x^*))(x-x^*) > 0$.

Thus, $f(x^* + t(x-x^*)) - f(x^*) > 0$, contradicting x^* solves the quadratic programming problem.

Thus, it must be the case that $Dg(x^*)(x-x^*) \leq 0$ for all $x \in \Delta^{n-1}$.

This establishes the desired implication.

Thus, $x^{*T}A^Tx^* \geq x^{*T}A^Tx$ for all $x \in \Delta^{n-1}$.

Hence, $x^{*T}Ax \geq x^TAx^*$ for all $x \in \Delta^{n-1}$, i.e., x^* is a symmetric equilibrium for (A, A^T). Q.E.D.

The proof of theorem 1 is also a proof of the following result.

Theorem 2: (i) The quadratic maximization problem [Maximize x^TAx, subject to $x \in \Delta^{n-1}$] has a solution. (ii) Every solution of this problem is a symmetric equilibrium for (A, A^T).

Proof: Follows immediately from the fact that $x^TAx = x^TA^Tx$ for all $x \in \mathbb{R}^n$. Q. E. D.

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