Horizons in the tridimensional spherical Natario warp drive using the ADM-MTW-Alcubierre formalism with constant speeds over the x-axis

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Abstract

The Natario warp drive appeared for the first time in 2001.Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.Natario defined a warp drive vector for constant speeds in Polar Coordinates but remember that a real warp drive must be defined in a tridimensional spacetime because a real spaceship is a tridimensional object inserted inside a tridimensional warp bubble that must be defined in real tridimensional coordinates.In this work we present the new warp drive vector in tridimensional $3D$ Spherical Coordinates for constant speeds.One the major drawbacks concerning warp drives is the problem of the Horizons(causally disconnected portions of spacetime) in which an observer in the center of the bubble cannot signal nor control the front part of the bubble.The behavior of a photon sent to the front of the warp bubble in the case of a Natario warp drive with constant velocity and a lapse function is also one of the main purposes of this work.We present the behavior of a photon sent to the front of the bubble in the Natario warp drive in the 1+1 and 3+1 spacetimes with and without the lapse function using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and the ADM(Arnowitt-Dresner-Misner) formalism equations with the approach of MTW(Misner-Thorne-Wheeler) and Alcubierre.

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1 Introduction:

The Natario warp drive appeared for the first time in 2001.([1]).Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.

This propulsion vector nX uses the form $nX = X^i e_i$ where X^i are the shift vectors responsible for the spaceship propulsion or speed and e_i are the Canonical Basis of the Coordinates System where the shift vectors are based or placed.

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).(see Appendix D about Polar Coordinates).The final form of the original Natario warp drive vector is given by $nX = vs * d(r\cos\theta)$.However Polar Coordinates are not real tridimensional $3D$ coordinates since it uses only the two Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

Natario used Polar Coordinates(See pg 4 in [1]) but for a real $3D$ Spherical Coordinates another warp drive vector must be calculated.Remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates.The final form of the Hodge Star for this warp drive vector is calculated no longer over $*d(r\cos\theta)$ but instead over $*d(r\sin\phi\cos\theta)$ since this form uses all the tridimensional $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.(see Appendix E about tridimensional $3D$ Spherical Coordinates).

In this work we present the new Natario warp drive vector in tridimensional $3D$ Spherical Coordinates for constant $nX = vs * d(x)$ speeds.

The warp drive work that predates Natario by 7 years was written by Alcubierre in 1994.(see [16])

Alcubierre([18]) used the so-called $3+1$ original Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([17]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pg 67 in [18],see also eq 1 pg 3 in [16]) and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive spacetime.In this work concerning the ADM formalism we adopt the Alcubierre methodology.

The ADM equation with signature $(-,+,+,+)$ that obeys the original $3+1$ ADM formalism is given below:(see eq (21.40) pg 507 in [17])(see Appendix C).

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{1}$$

In the equation above α is the so-called lapse function,γ_{ij} is the $3D$ diagonalized induced metric and β^i and β^j are the so-called shift vectors.

Combining the eqs (21.40),(21.42) and (21.44) pgs $507, 508$ in [17]
with the eqs (2.2.4),(2.2.5) and (2.2.6) pg 67 in [18] using the signature $(-,+,+,+)$ we get the original matrices of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{2}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{3}$$

The Natario warp drive equation with signature $(-,+,+,+)$ that obeys the original $3+1$ ADM formalism is given below:(see eq 21.40 pg 507 in [17])

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{4}$$

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ making $\alpha = 1$ and inserting the components of the Natario vectors we have in Polar Coordinates:(see Appendix C).

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{5}$$

And in $3D$ Spherical Coordinates:(see also Appendix C).

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{6}$$

The equations above dont have the lapse function.The equivalent equations using the lapse function would then be:

Polar Coordinates:(see Appendix L).

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{7}$$

$3D$ Spherical Coordinates:(see also Appendix L).

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{8}$$

The lapse function is equal to 1 inside and outside the Natario warp bubble while having large values in the Natario warped region.See [10]

In this work we also discuss the Horizon problem for both the Natario warp drive spacetime equations in the $1+1$ and $3+1$ ADM formalisms with and without the lapse function at constant velocities and we arrive at the conclusion that while the equation without the lapse function in the $1+1$ spacetime suffers from the problem of the Horizon and cannot control the warp bubble the new equation in the $1+1$ formalism with the lapse function possessing a value greater than or equal to the bubble velocity in modulus can circumvent the problem of the Horizon because in this case the warp bubble is totally connected due to the presence of the lapse function.

Horizons were deeply covered in the warp drive literature but always for constant velocities and without lapse functions in the $1+1$ spacetime.(see pg 6 in [1],pg 34 in [34],pgs 268 in [35]).The behavior of a photon sent to the front of the warp bubble in the case of a warp drive with constant velocity and a lapse function is one of the main purposes of this work.We present the behavior of a photon sent to the front of the bubble in the Natario warp drive in the $1+1$ and $3+1$ spacetimes with or without the lapse function at constant velocities using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provide here the step by step mathematical calculations in order to outline(or underline or reinforce) the final results found in our work which are the following ones:

- 1)-In the case of the Natario warp drive with fixed velocity and without lapse functions in the $1+1$ spacetime in Polar Coordinates the Horizon exists as expected and in agreement with the current literature.
- 2)-In the case of the Natario warp drive with constant velocity and a lapse function in the $1+1$ spacetime in Polar Coordinates the Horizon do not exists at all.
- 3)-In the case of the Natario warp drive with fixed velocity and without lapse functions in the $3+1$ spacetime in Polar Coordinates the Horizon do not exists at all.
- 4)-In the case of the Natario warp drive with constant velocity and a lapse function in the $3+1$ spacetime in Polar Coordinates the Horizon do not exists at all.
- 5)-In the case of the Natario warp drive with fixed velocity and without lapse functions in the $3+1$ spacetime in $3D$ Spherical Coordinates the Horizon do not exists at all.
- 6)-In the case of the Natario warp drive with constant velocity and a lapse function in the $3+1$ spacetime in $3D$ Spherical Coordinates the Horizon do not exists at all.

In the solutions with the lapse function the whole spacetime geometries are affected by presence of the lapse functions and have different results when compared to the solutions without lapse functions

In the solutions with $3+1$ spacetimes wether the lapse function exists or not the whole spacetime geometries are affected by presence of the $3+1$ spacetimes and have different results when compared to the solutions with only $1+1$ spacetimes.

Remember that we are presenting our results using step by step mathematics in order to better illustrate our point of view. For the solutions of the quadratic forms in $3+1$ spacetimes see Appendices R and S.These solutions are different than the ones obtained only in $1+1$ spacetimes.

We adopt here the Geometrized system of units in which $c = G = 1$ for geometric purposes.

In order to fully understand the main ideas presented in this work:a new Natario warp drive vector in tridimensional $3D$ Spherical Coordinates and the behavior of a photon sent to the front of the bubble in the Natario warp drive in $3+1$ spacetimes with or without the lapse function at constant velocities acquaintance or familiarity with the Natario original warp drive paper is required but we provide all the mathematical demonstration QED(Quod Erad Demonstratum) in the Appendices.

Remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates so a photon sent to the front of the bubble fundamentally moves in a tridimensional spacetime.

This work is organized as follows:

- A)-Section 2 introduces the original Natario warp drive vector in Polar Coordinates $nX = vs * d(x)$ for constant speeds.
- B)-Section 3 introduces the new warp drive vector in tridimensional $3D$ Spherical Coordinates $nX = vs * d(x)$ for constant speeds.
- C)-Section 4 introduces the Horizon problem in Polar Coordinates in a $1+1$ spacetime for constant speeds without the lapse function.
- D)-Section 5 introduces the Horizon problem in Polar Coordinates in a $1+1$ spacetime for constant speeds with the lapse function.
- E)-Section 6 introduces the Horizon problem in Polar Coordinates in a $3+1$ spacetime for constant speeds without the lapse function.
- F)-Section 7 introduces the Horizon problem in Polar Coordinates in a $3+1$ spacetime for constant speeds with the lapse function.
- G)-Section 8 introduces the Horizon problem in $3D$ Spherical Coordinates in a $3+1$ spacetime for constant speeds without the lapse function.
- H)-Section 9 introduces the Horizon problem in $3D$ Spherical Coordinates in a $3+1$ spacetime for constant speeds with the lapse function.

We adopted in this work a pedagogical language and a presentation style that perhaps will be considered as tedious,monotonous, exhaustive or extensive by experienced or seasoned readers and we designated this work for novices,newcomers,beginners or intermediate students providing in our work all the mathematical background needed to understand the process Natario used to generate warp drive vectors.

As a matter of fact if a novice,newcomer,beginner or intermediate student not familiarized with the Natario techniques reads the Natario warp drive paper in first place he(or she) will perhaps feel some difficulties.

We hope our paper is suitable to fill this gap.

Although this work was designed to be independent,self-consistent and self-contained it may be regarded as a companion work to our works in [8],[9],[10],[11][12],[13],[36],[37] and [38].

2 The equation of the original Natario warp drive vector in polar coordinates with a constant speed vs

The equation of the Natario vector nX(pg 2 and 5 in [1]) is given by:

$$nX = X^r e_r + X^\theta e_\theta \tag{9}$$

With the contravariant shift vector components X^{rs} and X^θ given by:(see pg 5 in [1])(see also Appendices A and B for details)

$$X^r = 2v_s f(r) \cos\theta \tag{10}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r)) \sin\theta \tag{11}$$

Considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small r(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G for an explanation about this statement)

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x-axis$ or the equatorial plane r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).

In a $1 + 1$ spacetime the equatorial plane we get¿:

$$nX = X^r e_r \tag{12}$$

The contravariant shift vector component X^r is then:

$$X^r = 2v_s f(r) \tag{13}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f(r) = 0$ resulting in a $X^r = 0$ and outside the bubble $f(r) = \frac{1}{2}$ resulting in a $X^r = vs$ and this illustrates the Natario definition for a warp drive spacetime.See Appendix D

3 The equation of the new Natario warp drive vector in tridimensional $3D$ spherical coordinates with a constant speed vs

The equation of the new Natario warp drive vector in tridimensional $3D$ spherical coordinates with a constant speed vs nX is given by:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{14}$$

With the contravariant shift vector components X^{rs}, X^θ and X^ϕ given by:
(see Appendices J and K for details)

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{15}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{16}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{17}$$

Considering a valid $f(r)$ as a shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small rs(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the warped region:

We must demonstrate that our warp drive vector satisfies the Natario criteria for a warp drive defined by:

any warp drive vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G for an explanation about this statement)

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x - axis$(like Natario did) or the equatorial plane $x - y$ in r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).Also the equatorial plane $x - y$ makes an angle of 90 degrees with the $z - axis$ so $\sin\phi = 1$ and $cos\phi = 0$.

Then the contravariant components reduces to:

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \rightarrow X^r = vs(t)[2f(r)] \rightarrow \sin\phi = 1 \rightarrow\ cos\theta = 1 \tag{18}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] = 0 \rightarrow \sin\phi = 1 \rightarrow \sin\theta = 0 \tag{19}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] = 0 \rightarrow cos\phi = 0 \tag{20}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f(r) = 0$ resulting in a $X^r = 0$ and outside the bubble $f(r) = \frac{1}{2}$ resulting in a $X^r = vs$ and this illustrates the Natario definition for a warp drive spacetime.See Appendix E

4 Horizons in the Natario warp drive with a constant speed vs using the original $1+1$ ADM formalism in Polar Coordinates without the lapse function

The mathematical discussions of this section uses mainly quadratic equations.We choose quadratic equations to outline the problem of the Horizons in the Natario warp drive spacetime because and although quadratic equations are often regarded as being elementary forms of mathematics these quadratic equations can illustrate very well the problem of the Horizons.All the mathematical calculations are presented step by step.

Examining the Natario warp drive equation for constant speed vs in a $1+1$ spacetime:(see Appendix C)

$$ds^2 = (1 - [X^r]^2)dt^2 + 2(X^r dr)dt - dr^2 \tag{21}$$

The contravariant shift vector component X^r is then:

$$X^r = 2v_s f(r) \tag{22}$$

We must analyze what happens in this Natario geometry if an observer in the center of the bubble starts to send photons to the front part of the bubble over the direction of motion.A photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$.Then applying the rule of the null-like geodesics $ds^2 = 0$ to the Natario warp drive equation for constant speed vs in a $1+1$ spacetime we have:

$$0 = (1 - [X^r]^2)dt^2 + 2(X^r dr)dt - dr^2 \tag{23}$$

Dividing both sides by dt^2 we have:

$$0 = (1 - [X^r]^2) + 2(X^r \frac{dr}{dt}) - (\frac{dr}{dt})^2 \tag{24}$$

Making the following algebraic substitution:

$$U = \frac{dr}{dt} \tag{25}$$

We have:

$$0 = (1 - [X^r]^2) + 2(X^r)U - U^2 \tag{26}$$

Multiplying both sides of the equation above by -1 and rearranging the terms of the equation we get the result shown below:

$$U^2 - 2(X^r)U - (1 - [X^r]^2) = 0 \tag{27}$$

The solution of the quadratic equation is then given by:

$$U = \frac{2(X^r) \pm \sqrt{4((X^r)^2) + 4(1 - [X^r]^2)}}{2} \tag{28}$$

$$U = \frac{2(X^r) \pm \sqrt{4((X^r)^2) + 4 - 4([X^r]^2)}}{2} \tag{29}$$

The simplified algebraic expression becomes:

$$U = \frac{2(X^r) \pm \sqrt{4}}{2} \tag{30}$$

Which leads to:

$$U = \frac{2(X^r) \pm 2}{2} \tag{31}$$

And the final result is then given by:

$$U = X^r \pm 1 \tag{32}$$

The above equation have two possible solutions U respectively $U = X^r + 1$ and $U = X^r - 1$ being each solution U a root of the quadratic form.Remember that a photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$ and in our case a photon can be sent to the front or the rear parts of the bubble both parts being encompassed by $ds^2 = 0$ with each part being a root U and a solution of the quadratic form.The solutions U for the front and the rear parts of the bubble are then respectively given by:

$$U_{front} = X^r - 1 \tag{33}$$

$$U_{rear} = X^r + 1 \tag{34}$$

We are interested in the behavior of the photon sent to the front part of the bubble which means:

$$U_{front} = X^r - 1 = 2v_s f(r) - 1 \tag{35}$$

Considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small r(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]) and assuming a continuous behavior for $f(rs)$ from 0 to $\frac{1}{2}$ and in consequence a continuous behavior for $2v_s f(r)$ from 0 to vs we can clearly see that inside the bubble $2v_s f(r) = 0$ because $f(r) = 0$ and outside the bubble $2v_s f(r) = vs$ because $f(r) = \frac{1}{2}$ and assuming also continuous values from 0 to vs with $vs > 1$[1] then somewhere in the Natario warped region where $0 < f(r) < \frac{1}{2}$ we have the situation in which $2v_s f(r) = 1$ because 1 lies in the continuous interval from 0 to vs and in consequence :

$$U_{front} = X^r - 1 = 2v_s f(r) - 1 = 1 - 1 = 0!!! \tag{36}$$

The result is zero !!!.The photon sent to the front of the bubble stops!!!.A Horizon is established!!!.The front part of the bubble is causally disconnected from the observer in the center of the bubble.So photons at light speed cannot be used to send signals to the front of the bubble.The place where the photon stops is the place where $f(r) = \frac{1}{2vs}$ and with $vs > 1$ this place lies well within the Natario warped region.$0 < \frac{1}{2vs} < \frac{1}{2}$

[1]Remember that we are working with Geometrized Units in which $G = c = 1$

Of course this point of view about the Horizons reflects only the geometrical point of view of the Natario warp drive equation for constant speed vs in a $1+1$ spacetime.But we know that in the Natario warp drive the negative energy density covers the entire bubble.(see Appendices M and O).Since the negative energy density have repulsive gravitational behavior(see pg 116 in [19]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.

The solution that allows contact with the bubble walls was presented in pg 83 in [20].Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg 83 in [20] to be sent to the large warp bubble keeping it in causal contact.Remember that one source of negative energy repels a source of positive energy but attracts another source of negative energy.This idea seems to be endorsed by pg 34 in [34],pg 268 in [35] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

Horizons were deeply covered in the warp drive literature but always for constant velocities and without lapse functions in a $1+1$ spacetime using the Alcubierre 1994 original warp drive paper in [16].(see pg 6 in [1],pg 34 in [34],pg 268 in [35]).In the Appendix R we presented the solutions for the generic quadratic forms in the $3+1$ spacetime without a lapse function.The dimensional reduction of this generic form to a $1+1$ spacetime gives the following result:

$$ds^2 = (1 - X_1 X^1)dt^2 + 2(X_1 dx^1)dt - \gamma_{11}(dx^1)^2 \tag{37}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} = \frac{X_1 + \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{38}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} = \frac{X_1 - \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{39}$$

Examining the Natario warp drive equation for constant speed vs in a $1+1$ spacetime:(see Appendix C)

$$ds^2 = (1 - [X^r]^2)dt^2 + 2(X^r dr)dt - dr^2 \tag{40}$$

But we know that $X^r = X_r = X_1 = 2v_s f(r)$ also $\gamma_{11} = 1$ and $(dx^1)^2 = dr^2$
The solutions are:

$$\frac{dx^1}{dt} = \frac{dr}{dt} = \frac{X_1 + \sqrt{\gamma_{11}}}{\gamma_{11}} = X_1 + 1 = 2v_s f(r) + 1 \tag{41}$$

$$\frac{dx^1}{dt} = \frac{dr}{dt} = \frac{X_1 - \sqrt{\gamma_{11}}}{\gamma_{11}} = X_1 - 1 = 2v_s f(r) - 1 \tag{42}$$

And we recover again the problem of the Horizon.

5 Horizons in the Natario warp drive with a constant speed vs in the original $1+1$ ADM formalism using a lapse function α in Polar Coordinates

The mathematical discussions of this section also uses mainly quadratic equations.We choose quadratic equations to outline the problem of the Horizons in the Natario warp drive spacetime because and although quadratic equations are often regarded as being elementary forms of mathematics these quadratic equations can illustrate very well the problem of the Horizons.All the mathematical calculations are presented step by step.

Examining the Natario warp drive equation for constant speed vs in a $1+1$ spacetime:(see Appendix L)

$$ds^2 = (\alpha^2 - [X^r]^2)dt^2 + 2(X^r dr)dt - dr^2 \tag{43}$$

The contravariant shift vector component X^r is then:

$$X^r = 2v_s f(r) \tag{44}$$

We must analyze what happens in this Natario geometry if an observer in the center of the bubble starts to send photons to the front part of the bubble over the direction of motion.A photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$.Then applying the rule of the null-like geodesics $ds^2 = 0$ to the Natario warp drive equation for constant speed vs in a $1+1$ spacetime we have:

$$0 = (\alpha^2 - [X^r]^2)dt^2 + 2(X^r dr)dt - dr^2 \tag{45}$$

Dividing both sides by dt^2 we have:

$$0 = (\alpha^2 - [X^r]^2) + 2(X^r \frac{dr}{dt}) - (\frac{dr}{dt})^2 \tag{46}$$

Making the following algebraic substitution:

$$U = \frac{dr}{dt} \tag{47}$$

We have:

$$0 = (\alpha^2 - [X^r]^2) + 2(X^r)U - U^2 \tag{48}$$

Multiplying both sides of the equation above by -1 and rearranging the terms of the equation we get the result shown below:

$$U^2 - 2(X^r)U - (\alpha^2 - [X^r]^2) = 0 \tag{49}$$

The solution of the quadratic equation is then given by:

$$U = \frac{2(X^r) \pm \sqrt{4((X^r)^2) + 4(\alpha^2 - [X^r]^2)}}{2} \tag{50}$$

$$U = \frac{2(X^r) \pm \sqrt{4((X^r)^2) + 4\alpha^2 - 4([X^r]^2)}}{2} \tag{51}$$

The simplified algebraic expression becomes:

$$U = \frac{2(X^r) \pm \sqrt{\alpha^2}}{2} \tag{52}$$

Which leads to:

$$U = \frac{2(X^r) \pm \alpha}{2} \tag{53}$$

And the final result is then given by:

$$U = X^r \pm \alpha \tag{54}$$

The above equation have two possible solutions U respectively $U = X^r + \alpha$ and $U = X^r - \alpha$ being each solution U a root of the quadratic form.Remember that a photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$ and in our case a photon can be sent to the front or the rear parts of the bubble both parts being encompassed by $ds^2 = 0$ with each part being a root U and a solution of the quadratic form.The solutions U for the front and the rear parts of the bubble are then respectively given by:

$$U_{front} = X^r - \alpha \tag{55}$$

$$U_{rear} = X^r + \alpha \tag{56}$$

We are interested in the behavior of the photon sent to the front part of the bubble which means:

$$U_{front} = X^r - \alpha = 2v_s f(r) - \alpha \tag{57}$$

Considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small r(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]) and assuming a continuous behavior for $f(r)$ from 0 to $\frac{1}{2}$ and in consequence a continuous behavior for $2v_s f(r)$ from 0 to vs we can clearly see that inside the bubble $2v_s f(r) = 0$ because $f(r) = 0$ and outside the bubble $2v_s f(r) = vs$ because $f(r) = \frac{1}{2}$ and assuming also continuous values from 0 to vs with $vs > 1$[2] then in the Natario warped region where $0 < f(r) < \frac{1}{2}$ we never have the situation in which $2v_s f(r) = \alpha$ because α is greater than $2v_s f(r)$ in modulus and α never lies in the continuous interval from 0 to vs and in consequence :

$$U_{front} = X^r - \alpha = 2v_s f(r) - \alpha \neq 0!!! \tag{58}$$

The result is not zero !!!.The photon sent to the front of the bubble never stops!!!.A Horizon is not established!!!.

[2]Remember that we are working with Geometrized Units in which $G = c = 1$

This is a situation different than the previous case in which:

$$U_{front} = X^r - 1 = 2v_s f(r) - 1 = 1 - 1 = 0!!! \tag{59}$$

A large lapse function may perhaps keeps the Natario warped region causally connected.Like in the previous case this is the geometrical point of view of the Natario warp drive equation for constant speed vs in a $1+1$ spacetime with a lapse function and we know that in the Natario warp drive the negative energy density covers the entire bubble.(see Appendices M and O).Since the negative energy density have repulsive gravitational behavior(see pg 116 in [19]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.[3]

Note that α must be much greater than $2v_s f(r)$ in modulus otherwise if α is less than or equal to $2v_s f(r)$ then the equation $U_{front} = X^r - \alpha = 2v_s f(r) - \alpha$ would have a Horizon when $2v_s f(r) = \alpha$ or when $f(r) = \frac{\alpha}{2v_s}$ assuming a continuous growth from 0 to v_s.Also note that if $\alpha = v_s$ in modulus then $f(r) = \frac{\alpha}{2v_s} = \frac{v_s}{2v_s} = \frac{1}{2}$ and when $f(r) = \frac{1}{2}$ this means the region outside the bubble according to pg 5 in [1].So if the Horizon is established outside the bubble at the end of the Natario warped region this means that the entire Natario warped region in which $0 < f(r) < \frac{1}{2}$ is then totally connected and can be signalized or controlled by an astronaut.

A value of α lesser than v_s in modulus for example $\alpha = \frac{v_s}{2}$ would mean an $f(r) = \frac{\alpha}{2v_s} = \frac{\frac{v_s}{2}}{2v_s} = \frac{1}{4}$ and the Horizon would appear inside the Natario warped region because $0 < \frac{1}{4} < \frac{1}{2}$.

In order to avoid the Horizon and keeps the Natario warped region totally connected the lapse function must have values greater than or equal to the bubble velocity in modulus .

[3]This can perhaps explain the negative value of $2v_s f(r) - \alpha \neq 0$ when $\alpha >> 2v_s f(r)$

In the Appendix S we presented the solutions for the generic quadratic forms in the $3+1$ spacetime with a lapse function.The dimensional reduction of this generic form to a $1+1$ spacetime gives the following result:

$$ds^2 = (\alpha^2 - X_1 X^1)dt^2 + 2(X_1 dx^1)dt - \gamma_{11}(dx^1)^2 \tag{60}$$

$$\frac{dx^1}{dt} = \frac{X_1 + \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} \tag{61}$$

$$\frac{dx^1}{dt} = \frac{X_1 - \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} \tag{62}$$

Examining the Natario warp drive equation for constant speed vs with a lapse function in a $1+1$ spacetime:(see Appendix L)

$$ds^2 = (\alpha^2 - [X^r]^2)dt^2 + 2(X^r dr)dt - dr^2 \tag{63}$$

But we know that $X^r = X_r = X_1 = 2v_s f(r)$ also $\gamma_{11} = 1$ and $(dx^1)^2 = dr^2$
The solutions are:

$$\frac{dx^1}{dt} = \frac{dr}{dt} = \frac{X_1 + \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} = X_1 + \alpha = 2v_s f(r) + \alpha \tag{64}$$

$$\frac{dx^1}{dt} = \frac{dr}{dt} = \frac{X_1 - \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} = X_1 - \alpha = 2v_s f(r) - \alpha \tag{65}$$

And we recover the Horizons solutions with the lapse function presented in this section.

6 Horizons in the Natario warp drive with a constant speed vs in the original $3+1$ ADM formalism without a lapse function α in Polar Coordinates

In the previous sections we used photons sent to the front of the bubble in the $1+1$ spacetime.($1+1$ dimensions).

Now we use photons sent to the front of the bubble in the $2+1$ spacetime.($2+1$ dimensions).

The equation of the Natario warp drive spacetime in Polar Coordinates with a constant speed vs in the original $3+1$ ADM formalism without the lapse function is given by:(see Appendix C)

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{66}$$

Actually Polar Coordinates are given in the $2+1$ spacetime.The generic quadratic form and its solutions for the $2+1$ spacetime are given by:(see Appendix R)

$$ds^2 = (1 - X_1 X^1 - X_2 X^2)dt^2 + 2(X_1 dx^1 + X_2 dx^2)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 \tag{67}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 + \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{68}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 - \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{69}$$

Remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{11} = \gamma_{rr} = 1$ and $\gamma_{22} = \gamma_{\theta\theta} = r^2$. Then the covariant shift vector components $X_1 = X_r$ and $X_2 = X_\theta$ are given by:

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s f(r) \cos\theta = X^r \tag{70}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -r^2 v_s (f(r) + (r)f'(r)) \sin\theta \tag{71}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \sqrt{1+r^2}}{1+r^2} \tag{72}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta - \sqrt{1+r^2}}{1+r^2} \tag{73}$$

Note that now the photon moves in a $2+1$ spacetime and this means motion in r and θ.

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \sqrt{1+r^2}}{1+r^2} = \frac{[2v_s f(r)\cos\theta] + [-r^2 v_s (f(r) + (r)f'(r))\sin\theta] + \sqrt{1+r^2}}{1+r^2} \tag{74}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta - \sqrt{1+r^2}}{1+r^2} = \frac{[2v_s f(r)\cos\theta] + [-r^2 v_s (f(r) + (r)f'(r))\sin\theta] - \sqrt{1+r^2}}{1+r^2} \tag{75}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}=\frac{X_r+X_\theta+\sqrt{1+r^2}}{1+r^2}=\frac{[2v_s f(r)\cos\theta]-[r^2 v_s(f(r)+(r)f'(r))\sin\theta]+\sqrt{1+r^2}}{1+r^2} \tag{76}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}=\frac{X_r+X_\theta-\sqrt{1+r^2}}{1+r^2}=\frac{[2v_s f(r)\cos\theta]-[r^2 v_s(f(r)+(r)f'(r))\sin\theta]-\sqrt{1+r^2}}{1+r^2} \tag{77}$$

In two dimensions the photon moves in a $2+1$ spacetime and this means motion in r and θ the Horizon do not occurs unless $\theta=0$,$\cos\theta=1$,$\sin\theta=0$ and $r^2d\theta^2=0$ and we recover in this case the problem of the Horizon without the lapse function in the $1+1$ spacetime.

Of course this point of view about the Horizons reflects only the geometrical point of view of the Natario warp drive equation for constant speed vs in a $3+1$ spacetime.But we know that in the Natario warp drive the negative energy density covers the entire bubble.(see Appendices M and O).Since the negative energy density have repulsive gravitational behavior(see pg 116 in [19]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.

The solution that allows contact with the bubble walls was presented in pg 83 in [20].Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg 83 in [20] to be sent to the large warp bubble keeping it in causal contact.Remember that one source of negative energy repels a source of positive energy but attracts another source of negative energy.This idea seems to be endorsed by pg 34 in [34],pg 268 in [35] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

7 Horizons in the Natario warp drive with a constant speed vs in the original $3+1$ ADM formalism with a lapse function α in Polar Coordinates

In some of the previous sections we used photons sent to the front of the bubble in the $1+1$ spacetime.($1+1$ dimensions).

Now we use photons sent to the front of the bubble in the $2+1$ spacetime.($2+1$ dimensions).

The equation of the Natario warp drive spacetime in Polar Coordinates with a constant speed vs in the original $3+1$ ADM formalism with the lapse function is given by:(see Appendix L)

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{78}$$

Actually Polar Coordinates are given in the $2+1$ spacetime.The generic quadratic form and its solutions for the $2+1$ spacetime are given by:(see Appendix S)

$$ds^2 = (\alpha^2 - X_1 X^1 - X_2 X^2)dt^2 + 2(X_1 dx^1 + X_2 dx^2)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 \tag{79}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 + \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{80}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 - \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{81}$$

Remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{11} = \gamma_{rr} = 1$ and $\gamma_{22} = \gamma_{\theta\theta} = r^2$. Then the covariant shift vector components $X_1 = X_r$ and $X_2 = X_\theta$ are given by:

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s f(r)\cos\theta = X^r \tag{82}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -r^2 v_s (f(r) + (r)f'(r))\sin\theta \tag{83}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \alpha\sqrt{1 + r^2}}{1 + r^2} \tag{84}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta - \alpha\sqrt{1 + r^2}}{1 + r^2} \tag{85}$$

Note that now the photon moves in a $2+1$ spacetime and this means motion in r and θ.

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \alpha\sqrt{1 + r^2}}{1 + r^2} = \frac{[2v_s f(r)\cos\theta] + [-r^2 v_s (f(r) + (r)f'(r))\sin\theta] + \alpha\sqrt{1 + r^2}}{1 + r^2} \tag{86}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta - \alpha\sqrt{1 + r^2}}{1 + r^2} = \frac{[2v_s f(r)\cos\theta] + [-r^2 v_s (f(r) + (r)f'(r))\sin\theta] - \alpha\sqrt{1 + r^2}}{1 + r^2} \tag{87}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}=\frac{X_r+X_\theta+\alpha\sqrt{1+r^2}}{1+r^2}=\frac{[2v_s f(r)\cos\theta]-[r^2v_s(f(r)+(r)f'(r))\sin\theta]+\alpha\sqrt{1+r^2}}{1+r^2} \tag{88}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}=\frac{X_r+X_\theta-\alpha\sqrt{1+r^2}}{1+r^2}=\frac{[2v_s f(r)\cos\theta]-[r^2v_s(f(r)+(r)f'(r))\sin\theta]-\alpha\sqrt{1+r^2}}{1+r^2} \tag{89}$$

In two dimensions the photon moves in a $2+1$ spacetime and this means motion in r and θ the Horizon do not occurs even if $\theta=0$,$\cos\theta=1$,$\sin\theta=0$ and $r^2d\theta^2=0$ and we recover in this case the problem of the Horizon with the lapse function in the $1+1$ spacetime.

A large lapse function may perhaps keeps the Natario warped region causally connected.Like in the previous case this is the geometrical point of view of the Natario warp drive equation for constant speed vs in a $3+1$ spacetime with a lapse function and we know that in the Natario warp drive the negative energy density covers the entire bubble.(see Appendices M and O).Since the negative energy density have repulsive gravitational behavior(see pg 116 in [19]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.[4]

The lapse function is 1 inside and outside the bubble but with a large value in the Natario warped region.

[4]This can perhaps explain the negative value of $2v_sf(r)-\alpha\neq 0$ when $\alpha >> 2v_sf(r)$

8 Horizons in the Natario warp drive with a constant speed vs in the original $3+1$ ADM formalism without a lapse function α in $3D$ Spherical Coordinates

In some of the previous sections we used photons sent to the front of the bubble in the $1+1$ spacetime.($1+1$ dimensions).

Now we use photons sent to the front of the bubble in the $3+1$ spacetime.($3+1$ dimensions).

The equation of the Natario warp drive spacetime in $3D$ Spherical Coordinates with a constant speed vs in the original $3+1$ ADM formalism without the lapse function is given by:(see Appendix C)

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (90)$$

The generic quadratic form and its solutions for the $3+1$ spacetime are given by:(see Appendix R)

$$ds^2 = (1-X_1X^1-X_2X^2-X_3X^3)dt^2+2(X_1dx^1+X_2dx^2+X_3dx^3)dt-\gamma_{11}(dx^1)^2-\gamma_{22}(dx^2)^2-\gamma_{33}(dx^3)^2 \quad (91)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 + \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \quad (92)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 - \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \quad (93)$$

Remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ with $\gamma_{11} = \gamma_{rr} = 1$, $\gamma_{22} = \gamma_{\theta\theta} = r^2$ and $\gamma_{33} = \gamma_{\phi\phi} = r^2 \sin^2\theta$. Then the covariant shift vector components $X_1 = X_r$,$X_2 = X_\theta$ and $X_3 = X_\phi$ are given by:

$$X_r = \gamma_{rr}X^r = X_r = \gamma_{rr}X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] = X^r \quad (94)$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = r^2 X^\theta = -r^2 vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \quad (95)$$

$$X_\phi = \gamma_{\phi\phi}X^\phi = r^2 sin^2\theta X^\phi = r^2 sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \quad (96)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi + \sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \quad (97)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi - \sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \quad (98)$$

Note that now the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ.

Note that now the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ.

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{X_r+X_\theta+X_\phi+\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{99}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{X_r+X_\theta+X_\phi-\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{100}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{U+\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{101}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{U-\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{102}$$

$$U=X_r+X_\theta+X_\phi \tag{103}$$

$$U=vs(t)[\sin\phi][2f(r)\ cos\theta]+(-r^2vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta])+r^2sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r))+(rf'(r))] \tag{104}$$

$$U=vs(t)[\sin\phi][2f(r)\ cos\theta]-r^2vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta]+r^2sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r))+(rf'(r))] \tag{105}$$

In three dimensions the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ the Horizon do not occurs unless $\theta=0$,$\cos\theta=1$,$\sin\theta=0$,$\phi=90$,$\sin\phi=1$,$\cos\phi=0$,$r^2d\theta^2=0$ and $r^2\sin^2 d\phi^2=0$ and we recover in this case the problem of the Horizon without the lapse function in the $1+1$ spacetime.

Of course this point of view about the Horizons reflects only the geometrical point of view of the Natario warp drive equation for constant speed vs in a $3+1$ spacetime.But we know that in the Natario warp drive in Polar Coordinates the negative energy density covers the entire bubble.(see Appendices M and O).Since the negative energy density have repulsive gravitational behavior(see pg 116 in [19]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.

But now we are in $3D$ Spherical Coordinates and from Appendix E we know that the Natario warp drive vector in this case can be reduced to a Natario warp drive vector in Polar Coordinates. So it is reasonable to suppose that the negative energy in this case may perhaps cover the entire bubble although we dont have the distribution of energy in the $3D$ Spherical Coordinates.(see bottom of Appendix O).

The solution that allows contact with the bubble walls was presented in pg 83 in [20].Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg 83 in [20] to be sent to the large warp bubble keeping it in causal contact.Remember that one source of negative energy repels a source of positive energy but attracts another source of negative energy.This idea seems to be endorsed by pg 34 in [34],pg 268 in [35] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

9 Horizons in the Natario warp drive with a constant speed vs in the original $3+1$ ADM formalism with a lapse function α in $3D$ Spherical Coordinates

In some of the previous sections we used photons sent to the front of the bubble in the $1+1$ spacetime.($1+1$ dimensions).

Now we use photons sent to the front of the bubble in the $3+1$ spacetime.($3+1$ dimensions).

The equation of the Natario warp drive spacetime in $3D$ Spherical Coordinates with a constant speed vs in the original $3+1$ ADM formalism with a lapse function is given by:(see Appendix L)

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (106)$$

The generic quadratic form and its solutions for the $3+1$ spacetime are given by:(see Appendix S)

$$ds^2 = (\alpha^2 - X_1 X^1 - X_2 X^2 - X_3 X^3)dt^2 + 2(X_1 dx^1 + X_2 dx^2 + X_3 dx^3)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 - \gamma_{33}(dx^3)^2 \quad (107)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 + \alpha\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \quad (108)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 - \alpha\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \quad (109)$$

Remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ with $\gamma_{11} = \gamma_{rr} = 1$, $\gamma_{22} = \gamma_{\theta\theta} = r^2$ and $\gamma_{33} = \gamma_{\phi\phi} = r^2 \sin^2\theta$. Then the covariant shift vector components $X_1 = X_r$,$X_2 = X_\theta$ and $X_3 = X_\phi$ are given by:

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] = X^r \quad (110)$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -r^2 vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \quad (111)$$

$$X_\phi = \gamma_{\phi\phi} X^\phi = r^2 sin^2\theta X^\phi = r^2 sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \quad (112)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi + \alpha\sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \quad (113)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi - \alpha\sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \quad (114)$$

Note that now the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ.

Note that now the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ.

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{X_r+X_\theta+X_\phi+\alpha\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{115}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{X_r+X_\theta+X_\phi-\alpha\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{116}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{U+\alpha\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{117}$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}+\frac{d\phi}{dt}=\frac{U-\alpha\sqrt{1+r^2+r^2sin^2\theta}}{1+r^2+r^2sin^2\theta} \tag{118}$$

$$U=X_r+X_\theta+X_\phi \tag{119}$$

$$U=vs(t)[\sin\phi][2f(r)\ cos\theta]+(-r^2vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta])+r^2sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r))+(rf'(r))] \tag{120}$$

$$U=vs(t)[\sin\phi][2f(r)\ cos\theta]-r^2vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta]+r^2sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r))+(rf'(r))] \tag{121}$$

In three dimensions the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ the Horizon do not occurs unless $\theta=0$,$\cos\theta=1$,$\sin\theta=0$,$\phi=90$,$\sin\phi=1$,$\cos\phi=0$,$r^2d\theta^2=0$ and $r^2\sin^2 d\phi^2=0$ and we recover in this case the problem of the Horizon with the lapse function in the $1+1$ spacetime.

Of course this point of view about the Horizons reflects only the geometrical point of view of the Natario warp drive equation for constant speed vs in a $3+1$ spacetime.But we know that in the Natario warp drive in Polar Coordinates the negative energy density covers the entire bubble.(see Appendices M and O).Since the negative energy density have repulsive gravitational behavior(see pg 116 in [19]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.

But now we are in $3D$ Spherical Coordinates and from Appendix E we know that the Natario warp drive vector in this case can be reduced to a Natario warp drive vector in Polar Coordinates. So it is reasonable to suppose that the negative energy in this case may perhaps cover the entire bubble although we dont have the distribution of energy in the $3D$ Spherical Coordinates.(see bottom of Appendix O).

The lapse function is 1 inside and outside the bubble but with a large value in the Natario warped region.

10 Conclusion

In this work we introduced a new tridimensional $3D$ spherical coordinates warp drive vector using the Natario mathematical techniques.We focused ourselves in the application of the Hodge Star in $3D$ spherical coordinates for constant speeds.

Our focus was concentrated in the Natario methods to obtain a warp drive vector.We know that we used a language and a presentation method or style that may be regarded as exhaustive tedious and monotonous for experienced or seasoned readers but we are concerned about beginners,newcomers,novices or intermediate students not familiarized with the techniques Natario used to develop warp drive vectors so our extensive mathematical demonstrations QED Quod Erad Demonstratum will benefit this audience at least we hope.We gave our best efforts trying to accomplish this goal but only this audience will tell in the future if we succeeded (or not).

Remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates.The final form of the Hodge Star for this warp drive vector was calculated no longer over $*d(r\cos\theta)$ as Natario did but instead over $*d(r\sin\phi\cos\theta)$ since this form uses all the tridimensional $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.

One the major drawbacks concerning warp drives is the problem of the Horizons(causally disconnected portions of spacetime) in which an observer in the center of the bubble cannot signal nor control the front part of the bubble. The behavior of a photon sent to the front of the warp bubble in the case of a Natario warp drive with constant velocity and a lapse function was also one of the main purposes of this work.We presented the behavior of a photon sent to the front of the bubble in the Natario warp drive in the $1+1$ and $3+1$ spacetimes with and without the lapse function using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provided here the step by step mathematical calculations in order to outline the final results found in our work which are the following ones:

For the case of the lapse function the Horizon do not exists at all.Due to the extra terms in the lapse function that affects the whole spacetime geometry this solution allows to circumvent the problem of the Horizon.

For the case of the tridimensional spacetime the Horizon do not exists at all.Due to the presence of the $3D$ dimensions that affects the whole spacetime geometry this solution allows to circumvent the problem of the Horizon.

In this work we developed Horizons but for constant velocities.

The application of the Horizons in the tridimensional $3D$ spherical coordinates warp drive vector using variable speeds and the ADM(Arnowitt-Dresner-Misner) formalism equations in General Relativity with the approach of MTW(Misner-Thorne-Wheeler) and Alcubierre resembling the works [10],[11][12] and [13] will appear in a future work.

The Natario warp drive is possibly the best candidate for interstellar space travel.(see Appendices M,N,O,P and Q)

The warp drive as an artificial superluminal geometric tool that allows to travel faster than light may well have an equivalent in the Nature.According to the modern Astronomy the Universe is expanding and as farther a galaxy is from us as faster the same galaxy recedes from us.The expansion of the Universe is accelerating and if the distance between us and a galaxy far and far away is extremely large the speed of the recession may well exceed the light speed limit.(see pg 98 in [39] and pg 377 in [40]).

For the experimental verification of the acceleration of the Universe see for example the bottom of pg 355 and top of pg 356 eq 8.155 in [42].

11 Appendix A:the mathematical demonstration of the Natario vectors $nX = -vs * dx$ and $nX = vs * dx$ for a constant speed vs in a R^3 space basis-Polar Coordinates

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{122}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta (d\varphi \wedge dr) \tag{123}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{124}$$

From above we get the following results

$$dr \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{125}$$

$$rd\theta \sim r\sin\theta (d\varphi \wedge dr) \tag{126}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{127}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see eq 3.72 pg 69$(a)(b)$ in [2]):

$$*dr = r^2 \sin\theta (d\theta \wedge d\varphi) \tag{128}$$

$$*rd\theta = r\sin\theta (d\varphi \wedge dr) \tag{129}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{130}$$

Back again to the Natario equivalence between polar and cartezian coordinates(pg 5 in [1]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta \cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) \tag{131}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{132}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{133}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{134}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{135}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{136}$$

We know that the following expression holds true(see eq 3.79 pg $70(a)(b)$ in [2]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{137}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{138}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [1]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{139}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{140}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{141}$$

According to eq 3.90 pg $74(a)(b)$ in [2] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{142}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \quad (143)$$

Because and according to eqs 3.90 and 3.91 pg $74(a)(b)$ in [2],tb 3.2 pg $68(a)(b)$ in [2]:

$$*d(\alpha+\beta) = d\alpha + d\beta \quad (144)$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \quad (145)$$

$$*d(dx) = d(dy) = d(dz) = 0 \quad (146)$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \quad (147)$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \quad (148)$$

And then we derived again the Natario result of pg 5 in [1]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \quad (149)$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [1] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \quad (150)$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \quad (151)$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \quad (152)$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \quad (153)$$

$$2f(r)r^2 sin\theta \cos\theta(d\theta \wedge d\varphi) + 2f(r)r\sin^2\theta(dr \wedge d\varphi) + r^2\sin^2\theta f'(r)(dr \wedge d\varphi) \tag{154}$$

Comparing the above expressions with the Natario definitions of pg 4 in [1]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2\sin\theta(d\theta \wedge d\varphi) \tag{155}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \sim -r\sin\theta(dr \wedge d\varphi) \tag{156}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{157}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta \wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dr \wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dr \wedge d\varphi)] \tag{158}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{159}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{160}$$

Defining the Natario Vector as in pg 5 in [1] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{161}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{162}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [1]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{163}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{164}$$

12 Appendix B:the mathematical demonstration of the Natario vectors $nX = -vs * dx$ and $nX = vs * dx$ for a constant speed vs in a R^4 space basis-Polar Coordinates

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [2],eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{165}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta (dt \wedge d\varphi \wedge dr) \tag{166}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{167}$$

From above we get the following results

$$dr \sim r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{168}$$

$$rd\theta \sim r\sin\theta (dt \wedge d\varphi \wedge dr) \tag{169}$$

$$r\sin\theta d\varphi \sim r(dt \wedge dr \wedge d\theta) \tag{170}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see eq 3.74 pg 69$(a)(b)$ in [2]):

$$*dr = r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{171}$$

$$*rd\theta = r\sin\theta (dt \wedge d\varphi \wedge dr) \tag{172}$$

$$*r\sin\theta d\varphi = r(dt \wedge dr \wedge d\theta) \tag{173}$$

Back again to the Natario equivalence between polar and cartezian coordinates(pg 5 in [1]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta\cos\theta dt \wedge d\theta \wedge d\varphi + r\sin^2\theta dt \wedge dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{174}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{175}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{176}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{177}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(dt\wedge d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(dt\wedge d\varphi\wedge dr)] \tag{178}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)] - [r\sin^2\theta(dt\wedge d\varphi\wedge dr)] \tag{179}$$

We know that the following expression holds true(see eq 3.79 pg 70$(a)(b)$ in [2]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{180}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)] + [r\sin^2\theta(dt\wedge dr\wedge d\varphi)] \tag{181}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [1]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{182}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{183}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{184}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) \tag{185}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) \quad (186)$$

Because and according to eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]::

$$*d(\alpha+\beta) = d\alpha + d\beta \quad (187)$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df\wedge\alpha - f\wedge d\alpha \quad (188)$$

$$*d(dx) = d(dy) = d(dz) = 0 \quad (189)$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = dt\wedge d(\sin^2\theta)\wedge d\varphi - dt\wedge\sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) \quad (190)$$

$$*[d(r^2)d\varphi] = 2rdt\wedge dr\wedge d\varphi - dt\wedge r^2\wedge dd\varphi = 2r(dt\wedge dr\wedge d\varphi) \quad (191)$$

And then we derived again the Natario result of pg 5 in [1]

$$r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + r\sin^2\theta(dt\wedge dr\wedge d\varphi) \quad (192)$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [1] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \quad (193)$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \quad (194)$$

$$f(r)r^2 2sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \quad (195)$$

$$2f(r)r^2 sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dt\wedge dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \quad (196)$$

$$2f(r)r^2 sin\theta \cos\theta(dt \wedge d\theta \wedge d\varphi) + 2f(r)r\sin^2\theta(dt \wedge dr \wedge d\varphi) + r^2\sin^2\theta f'(r)(dt \wedge dr \wedge d\varphi) \tag{197}$$

Comparing the above expressions with the Natario definitions of pg 4 in [1]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2\sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{198}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \sim -r\sin\theta(dt \wedge dr \wedge d\varphi) \tag{199}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{200}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(dt \wedge d\theta \wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dt \wedge dr \wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dt \wedge dr \wedge d\varphi)] \tag{201}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{202}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{203}$$

Defining the Natario Vector as in pg 5 in [1] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{204}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{205}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [1]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{206}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{207}$$

13 Appendix C:mathematical demonstration of the Natario warp drive equation for a constant speed vs in the original $3{+}1$ ADM Formalism according to MTW and Alcubierre

General Relativity describes the gravitational field in a fully covariant way using the geometrical line element of a given generic spacetime metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ where do not exists a clear difference between space and time.This generical form of the equations using tensor algebra is useful for differential geometry where we can handle the spacetime metric tensor $g_{\mu\nu}$ in a way that keeps both space and time integrated in the same mathematical entity (the metric tensor) and all the mathematical operations do not distinguish space from time under the context of tensor algebra handling mathematically space and time exactly in the same way.

However there are situations in which we need to recover the difference between space and time as for example the evolution in time of an astrophysical system given its initial conditions.

The $3+1$ ADM formalism allows ourselves to separate from the generic equation $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ of a given spacetime the 3 dimensions of space and the time dimension.(see pg 64 in [18])

Consider a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg 65) in [18].

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg 65 in [18])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg 66 in [18])
(see also fig 21.2 pg 506 in [17] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\,dx^\mu\,dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg 507 in [17])[5]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface

- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.

- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

[5] we adopt the Alcubierre notation here

Combining the eqs (21.40),(21.42) and (21.44) pgs 507 and 508 in [17]
with the eqs (2.2.5) and (2.2.6) pg 67 in [18] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{208}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{209}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{210}$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{211}$$

But since $dl^2 = \gamma_{ij} dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii} dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{212}$$

$$(dx^i + \beta^i dt)^2 = (dx^i)^2 + 2\beta^i dx^i dt + (\beta^i dt)^2 \tag{213}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\gamma_{ii}\beta^i dx^i dt + \gamma_{ii}(\beta^i dt)^2 \tag{214}$$

$$\beta_i = \gamma_{ii}\beta^i \tag{215}$$

$$\gamma_{ii}(\beta^i dt)^2 = \gamma_{ii}\beta^i\beta^i dt^2 = \beta_i\beta^i dt^2 \tag{216}$$

$$(dx^i)^2 = dx^i dx^i \tag{217}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{218}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{219}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{220}$$

Note that the expression above is exactly the eq (2.2.4) pg 67 in [18].It also appears as eq 1 pg 3 in [16].

With the original equations of the $3+1$ ADM formalism given below:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{221}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{222}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ii} - \frac{\beta^i\beta^i}{\alpha^2} \end{pmatrix} \tag{223}$$

and suppressing the lapse function making $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{224}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{225}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -1 & \beta^i \\ \beta^i & \gamma^{ii} - \beta^i\beta^i \end{pmatrix} \tag{226}$$

changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-1 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{227}$$

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{228}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{229}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{230}$$

Remember that the equations given above corresponds to the generic warp drive metric given below:

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{231}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [1])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{232}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{233}$$

Comparing all these equations

$$ds^2 = (1 - \beta_i \beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{234}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i \beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{235}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i \beta^i \end{pmatrix} \tag{236}$$

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{237}$$

With

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{238}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i \beta^i = X_i X^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{239}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{240}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{241}$$

Looking to the equation of the Natario vector nX in Polar Coordinates(pg 2 and 5 in [1]):

$$nX = X^r e_r + X^\theta e_\theta \tag{242}$$

With the contravariant shift vector components X^{rs} and X^θ given by:(see pg 5 in [1])(see also Appendices A and B for details)

$$X^r = 2v_s f(r) \cos\theta \tag{243}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r)) \sin\theta \tag{244}$$

But remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_r and X_θ are given by:

$$X_i = \gamma_{ii} X^i \tag{245}$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s f(r) \cos\theta = X^r \tag{246}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -r^2 v_s(f(r) + (r)f'(r)) \sin\theta \tag{247}$$

The equations of the Natario warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{248}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{249}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{250}$$

Then the equation of the Natario warp drive spacetime in Polar Coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{251}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{252}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{253}$$

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).

And the Natario warp drive equation in Polar Coordinates in the $1+1$ spacetime is given by:

$$ds^2 = (1 - X_r X^r)dt^2 + 2(X_r dr)dt - dr^2 \tag{254}$$

But since:

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s f(r)\cos\theta = X^r \tag{255}$$

The equation is better rewritten as:

$$ds^2 = (1 - [X^r]^2)dt^2 + 2(X^r dr)dt - dr^2 \tag{256}$$

This equation is useful to analyze the Horizon problem in the 1+1 spacetime without the lapse function.

Considering now the new Natario warp drive vector in $3D$ tridimensional Spherical Coordinates with a constant speed vs nX given by::

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{257}$$

With the contravariant shift vector components X^{rs}, X^θ and X^ϕ given by:
(see Appendices J and K for details)

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{258}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{259}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{260}$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$ with $\gamma_{rr} = 1$, $\gamma_{\theta\theta} = r^2$ and $\gamma_{\phi\phi} = r^2\sin^2\theta$. Then the covariant shift vector components X_r,X_θ and X_ϕ are given by:

$$X_i = \gamma_{ii}X^i \tag{261}$$

$$X_r = \gamma_{rr}X^r = X_r = \gamma_{rr}X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] = X^r \tag{262}$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = r^2 X^\theta = -r^2 vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{263}$$

$$X_\phi = \gamma_{\phi\phi}X^\phi = r^2 sin^2\theta X^\phi = r^2 sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{264}$$

Then the equation of the Natario warp drive spacetime in $3D$ Spherical Coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{265}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr dt + X_\theta d\theta dt + X_\phi d\phi dt) - dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{266}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{267}$$

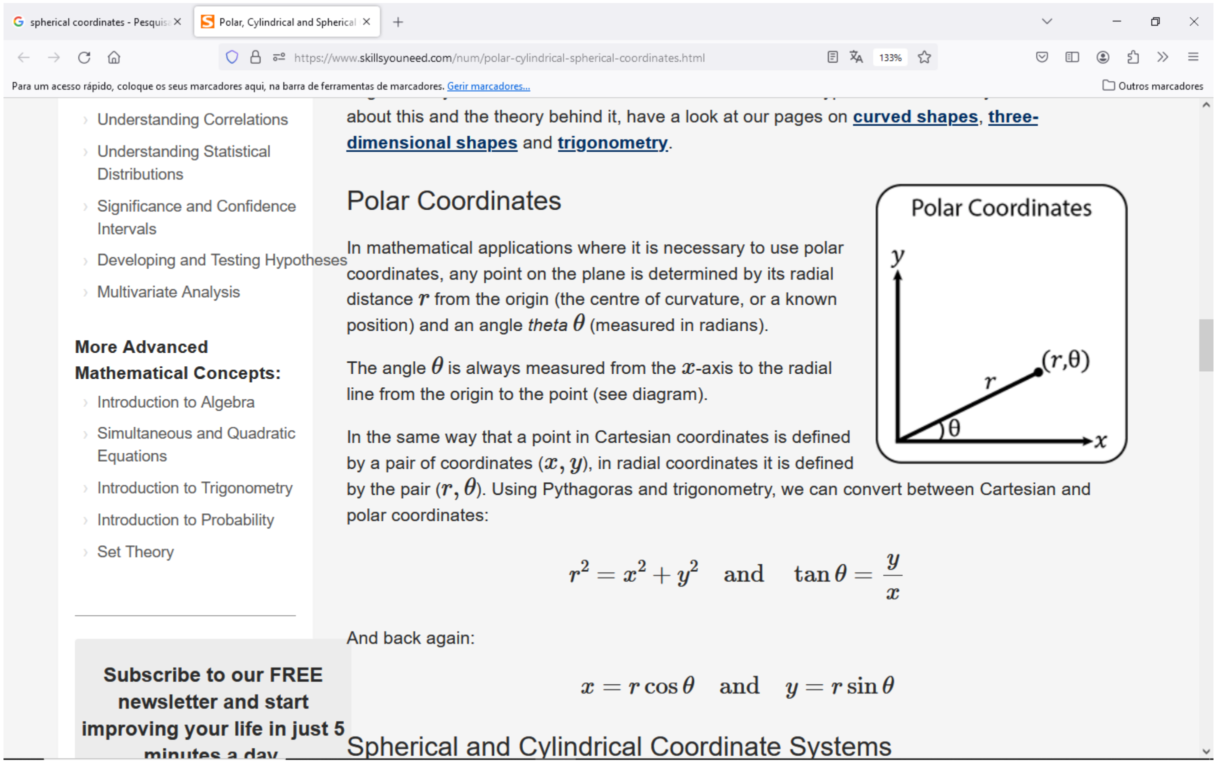

Figure 1: Polar Coordinates.(Source:Internet)

14 Appendix D:Polar Coordinates

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the **constant** speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in **Polar Coordinates**(See pg 4 in [1].(See also Appendices A and B for the detailed calculations).

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim \; \sim r^2\sin\theta\cos\theta d\theta\wedge d\varphi + r\sin^2\theta dr\wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right). \tag{268}$$

Consequently if we set exactly what Natario did in pg 5 in [1]:(we adopted the second expression)

$$\mathbf{X} \sim -v_s(t)d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s f\cos\theta\mathbf{e}_r + v_s(2f + rf')\sin\theta\mathbf{e}_\theta \tag{269}$$

$$\mathbf{X} \sim v_s(t)d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim 2v_s f\cos\theta\mathbf{e}_r - v_s(2f + rf')\sin\theta\mathbf{e}_\theta \tag{270}$$

$$nX = X^r e_r + X^\theta e_\theta \tag{271}$$

$$X^{rs} = 2v_s f\cos\theta \tag{272}$$

$$X^\theta = -v_s(2f + rf')\sin\theta \tag{273}$$

Considering a valid f as a Natario shape function being $f = \frac{1}{2}$ for large r(outside the warp bubble) and $f = 0$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G for an explanation about this statement)

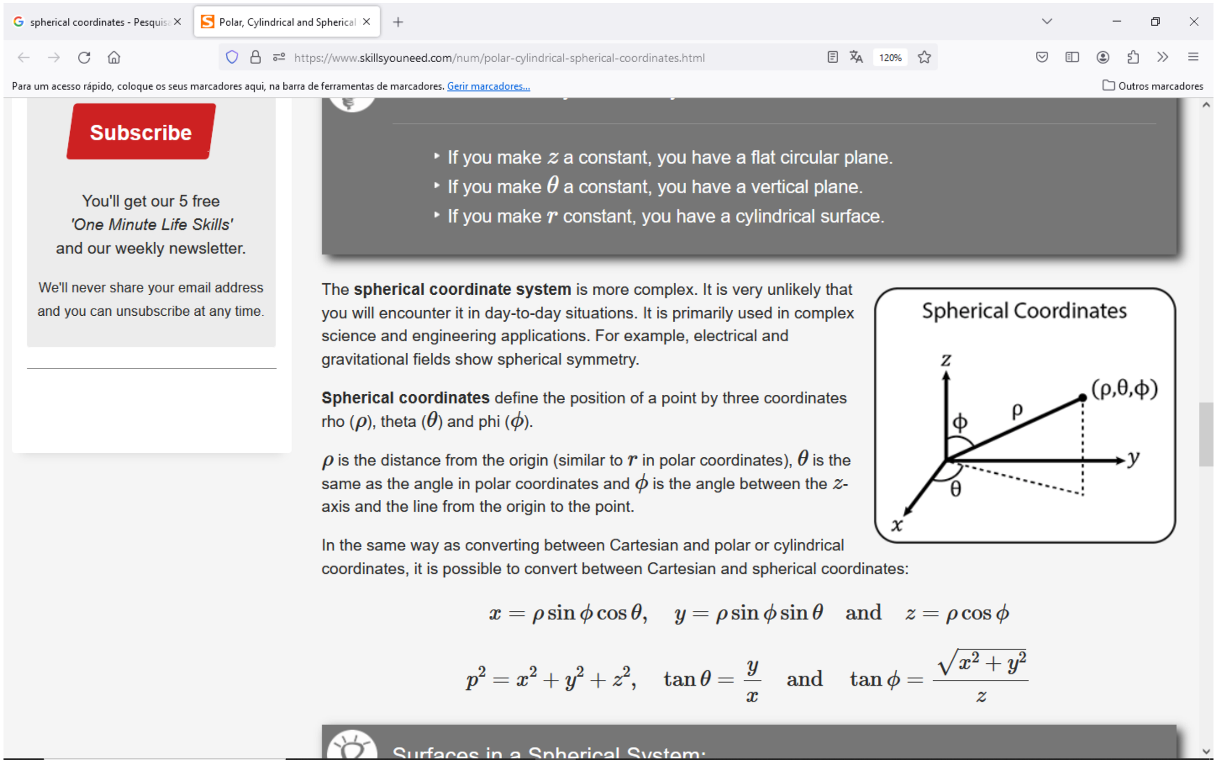

Figure 2: Tridimensional $3D$ Spherical Coordinates.(Source:Internet)

15 Appendix E:Tridimensional $3D$ Spherical Coordinates

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the **constant** speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in **Polar Coordinates**(See pg 4 in [1].(See also Appendix F).

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim \;\sim r^2\sin\theta\cos\theta d\theta\wedge d\varphi + r\sin^2\theta dr\wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right). \tag{274}$$

Note that in this case of Tridimensional $3D$ **Spherical Coordinates** the Hodge Star must be taken no longer over $d(r\cos\theta)$ but instead over $d(\rho\sin\phi\cos\theta)$ and this demands more calculations.Replacing ρ by r we have the following expressions for the Hodge Star:(see Appendices J and K)

$$*dx = *d(r\sin\phi\cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{275}$$

$$\sin\phi[*d[f(r)r^2\sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{276}$$

Our new tridimensional $3D$ **spherical coordinates** warp drive vector in R^3 with **constant speed** or in R^4 with **constant speed** vs $nX = vs * dx$ is given by:

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{277}$$

The corresponding shift vectors are:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{278}$$

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{279}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{280}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{281}$$

The equation of the **polar coordinates** Natario vector nX in **constant speed** vs(pg 2 and 5 in [1]) is given by:

$$nX = X^r e_r + X^\theta e_\theta \tag{282}$$

With the contravariant shift vector components explicitly written:

$$X^r = 2v_s f(r)\cos\theta \tag{283}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{284}$$

Note that Natario in pg 4 in [1] defined the x-axis as the polar axis.if the motion occurs only in the x-axis in **polar coordinates** then the angle between the x-y plane and the z-axis is 90 degrees and in this case $\sin\phi = 1$ and $\cos\phi = 0$ and our new warp drive vector in tridimensional $3D$ **spherical coordinates** reduces to the original Natario warp drive vector in **polar coordinates** both in **constant speed**.

Only in a real tridimensional $3D$ spherical coordinates motion our new warp drive vector accounts for a significant difference

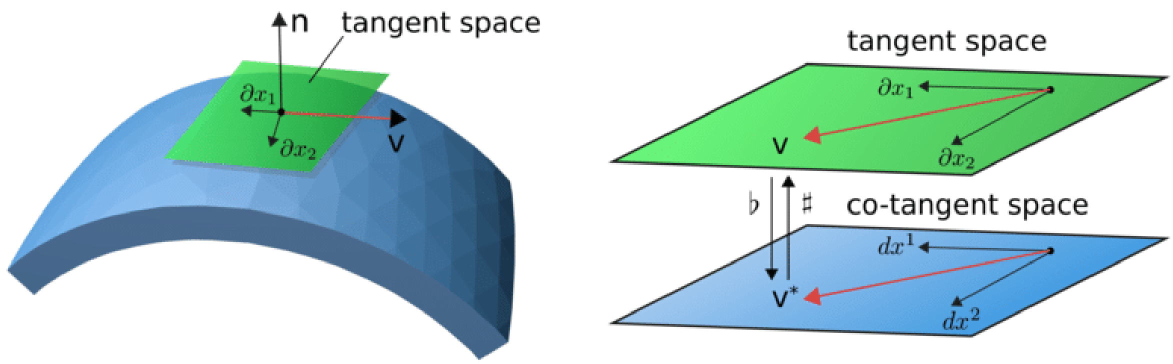

Figure 3: Artistic Presentation of Tangent and Cotangent Spaces I.(Source:Internet)

16 Appendix F:Tangent and Cotangent Spaces I

The Canonical Basis of the Hodge Star * in spherical coordinates in R^3 can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{285}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{286}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{287}$$

The Canonical Basis of the Hodge Star * in spherical coordinates in R^4 can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [2],eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{288}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{289}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{290}$$

In order to study the term $x * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs 363$(a)(b)$ and 364$(a)(b)$ in [2] with the terms $S = u = 1$[6],eq 3.74 pg 69$(a)(b)$ in [2],eqs 11.131 and 11.133 with the term $m = 0$[7] pg 417$(a)(b)$ in [2].)(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{291}$$

As a matter of fact we have for the Canonical Basis and the Hodge Star * in R^4 the following equations (see pg 47 eqs 2.67 to 2.70 in [3]):

$$*e_0 = e_1 \wedge e_2 \wedge e_3 \tag{292}$$

$$*e_1 = e_0 \wedge e_2 \wedge e_3 \tag{293}$$

$$*e_2 = e_0 \wedge e_3 \wedge e_1 \tag{294}$$

$$*e_3 = e_0 \wedge e_1 \wedge e_2 \tag{295}$$

In R^3 the corresponding equations are:(see pg 55 in [5])(see also pg 54 fig 4.2 in [5] for a graphical presentation of the Hodge Star * in R^3)(see pg 18 eq 1.55 in [6]):

$$*e_1 = e_2 \wedge e_3 \tag{296}$$

$$*e_2 = e_3 \wedge e_1 = -e_1 \wedge e_3 \tag{297}$$

$$*e_3 = e_1 \wedge e_2 \tag{298}$$

The Canonical Basis e_i are related to the partial derivatives $\frac{\partial}{\partial x_i}$ or simplifying related to ∂x_i wether in R^3 or R^4 and are graphically represented by the partial derivatives ∂x_i included in the tangent space of the picture given in the beginning of this section.

[6]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[7]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

On the other hand in R^4 we also have the following relations for the Hodge Star *:(see pg 92 in [3])

$$*dt = dx \wedge dy \wedge dz \tag{299}$$

$$*dx = dt \wedge dy \wedge dz \tag{300}$$

$$*dy = dt \wedge dz \wedge dx \tag{301}$$

$$*dz = dt \wedge dx \wedge dy \tag{302}$$

Also for R^4 considering the $((w,v)(\epsilon\Lambda_p^3)(R^{1,3}))$ formalism we may have the following relations:(see pg 382 in [4])($x^1 = x$,$x^2 = y$,$x^3 = z$)

$$*dt = dx^1 \wedge dx^2 \wedge dx^3 \tag{303}$$

$$*dx^1 = dt \wedge dx^2 \wedge dx^3 \tag{304}$$

$$*dx^2 = dt \wedge dx^3 \wedge dx^1 \tag{305}$$

$$*dx^3 = dt \wedge dx^1 \wedge dx^2 \tag{306}$$

In R^3 we would have the following relations:(see pg 117 eqs 4.6 and 4.7 in [7])(see pg 298 in [4])

$$*dx = dy \wedge dz \tag{307}$$

$$*dy = dz \wedge dx \tag{308}$$

$$*dz = dx \wedge dy \tag{309}$$

The differentials dx,dy,dz or dx^1, dx^2 and dx^3 are related to the cotangent space differentials included in the picture given in the beginning of this section.

See the graphical presentations of the relations between tangent and cotangent spaces in pg 55 fig 2.28 and pg 70 fig 3.1 in [4].See pg 168 fig 5.19 for a graphical presentation of $dx \wedge dy$,pg 169 fig 5.20 for a graphical presentation of $dy \wedge dz$ and pg 170 fig 5.21 for a graphical presentation of $dz \wedge dx$ all in [4].

Useful relations to deal with the Hodge Star * are given by eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]:See also pg 89 in [3],pg 112 in [4],pg 97 in [5],pg 36 eqs 2.21 and 2.22 in [6],pg 70 eq 3.3 in [7].

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df \wedge \alpha - f \wedge d\alpha \tag{310}$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{311}$$

$$*d(dx) = *d(dy) = *d(dz) = 0 \tag{312}$$

$p = 3$ stands for the R^4 and $p = 2$ stands for the R^3.

See also Appendix I.

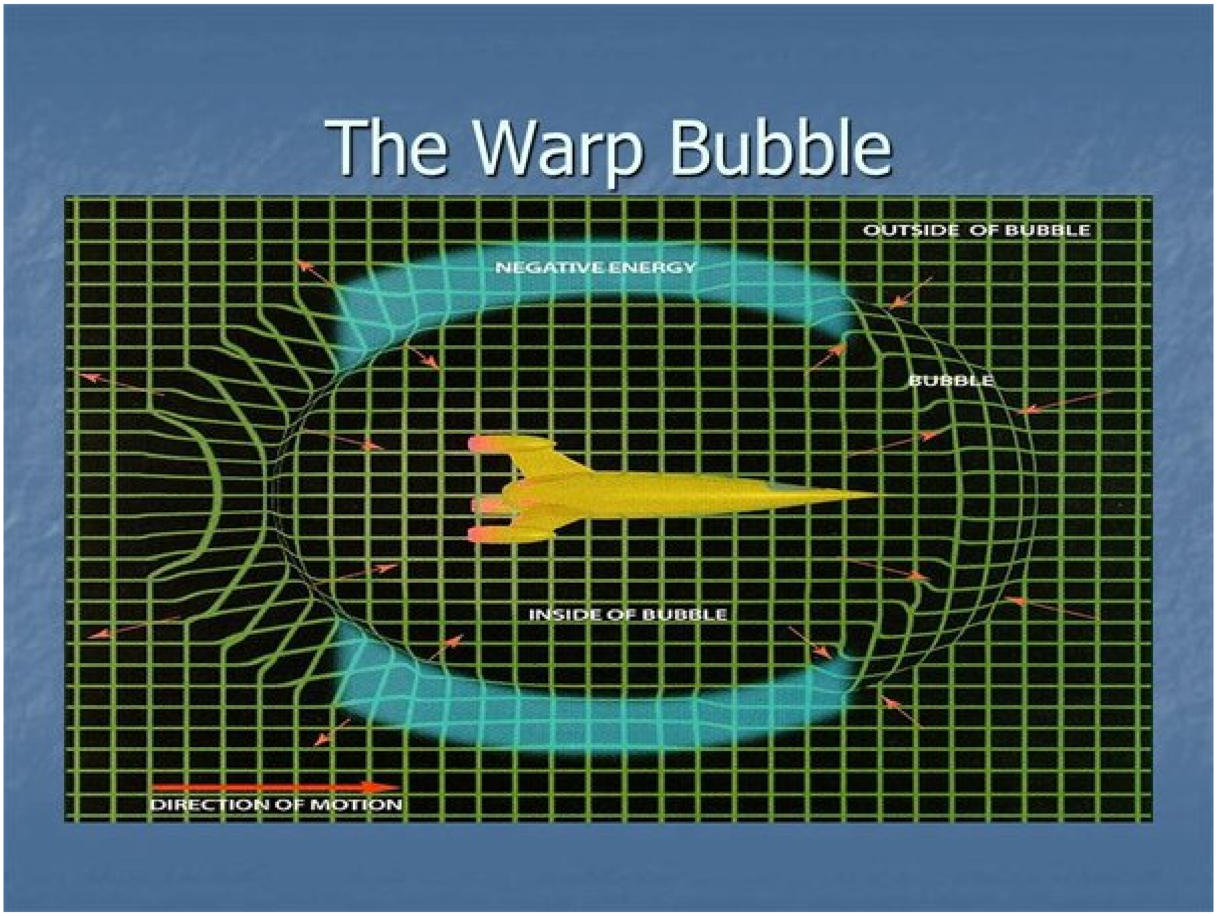

Figure 4: Artistic Presentation of a Warp Bubble.(Source:Internet)

17 Appendix G:Artistic Presentation of a Warp Bubble

In 2001 the Natario warp drive appeared.([1]).This warp drive deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [1]). Imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium.An observer at the rest in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Since the fish is at the rest inside the aquarium the fish would see the observer in the margin passing by him with a large relative speed since for the fish is the margin that moves with a large relative velocity

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

Lets explain better this statement:Natario considered in this case a coordinates reference frame placed inside the bubble where the fish inside the aquarium or the astronaut in a spaceship inside the bubble depicted above are at the rest with respect to their local neighborhoods.Then any Natario vector must be zero inside the bubble or the aquarium or the spaceship.

On the other hand since the fish sees the margin passing by him with a large relative velocity or the astronaut would see a stationary observer in outer space outside the bubble passing by him with a large relative velocity then any Natario vector outside the bubble must have a value equal to the relative velocity seen by both the fish and the astronaut.

Considering a valid f as a Natario shape function being $f = \frac{1}{2}$ for large r(outside the warp bubble) and $f = 0$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):The walls of the bubble the Natario warped region corresponds to the distorted region in the picture depicted in this Appendix.

See also Appendix H.

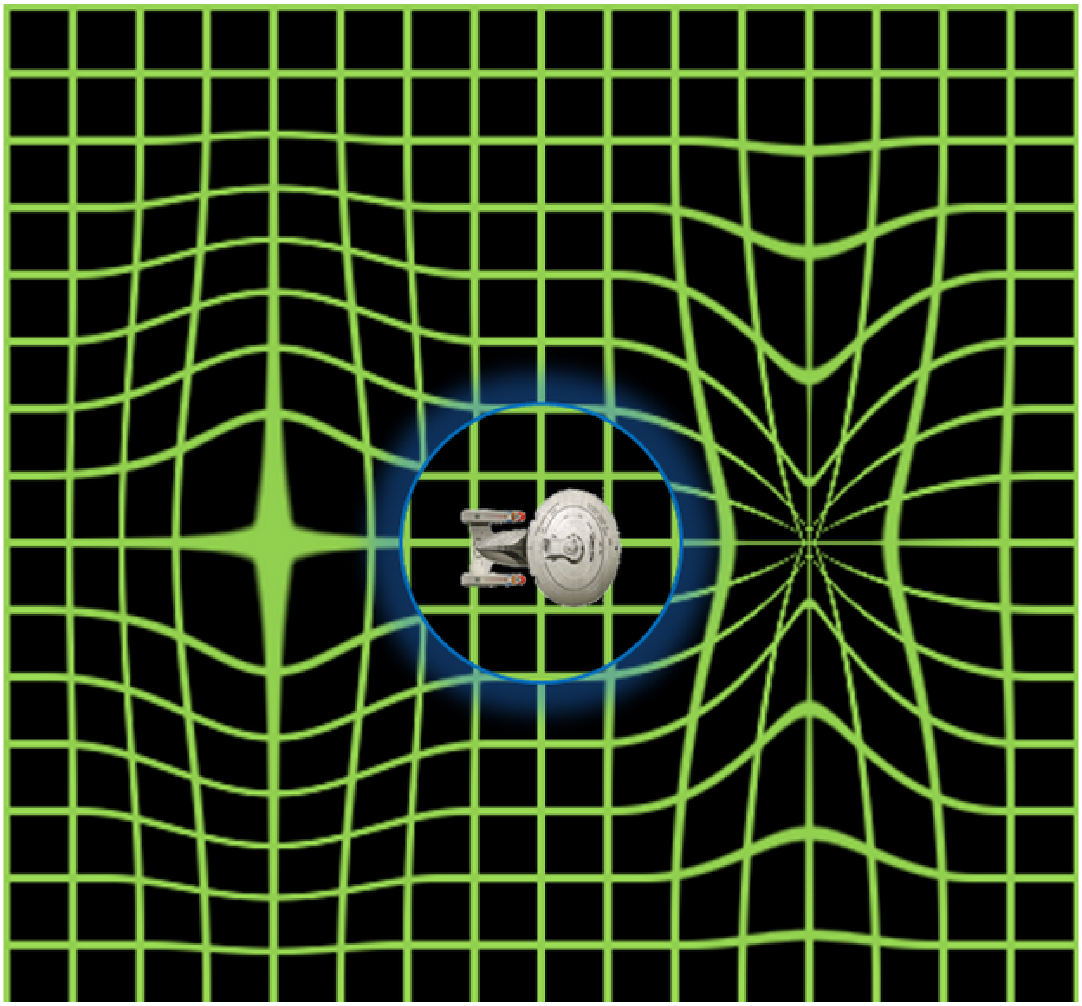

Figure 5: Another Artistic Presentation of a Warp Bubble.(Source:Internet)

18 Appendix H:Another Artistic Presentation of a Warp Bubble

Natario considered a coordinates reference frame placed inside the bubble.Now we must consider a coordinates reference frame placed outside the bubble:In this case the observer at the rest in the margin of the river would see the aquarium passing by him with a large velocity with the fish inside.Also a stationary observer at the rest in outer space would see the spaceship depicted in the picture above passing by him with a large velocity with the astronaut inside.

Now the rules originally defined by Natario are interchanged:

Since the observer in the margin and the observer in outer space are at the rest any Natario vector in this case must be zero outside the bubble.

But since the fish and the spaceship are being seen by the observer at the rest in the margin and the observer at the rest in outer space both fish and spaceship with a large velocity then the Natario vector inside the bubble must have a value equal to the velocity seen by both observers.

Considering a valid f as a Natario shape function being $f = 0$ for large r(outside the warp bubble) and $f = \frac{1}{2}$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region:The walls of the bubble the Natario warped region corresponds to the distorted region the "blue circle" in the picture depicted in this Appendix.

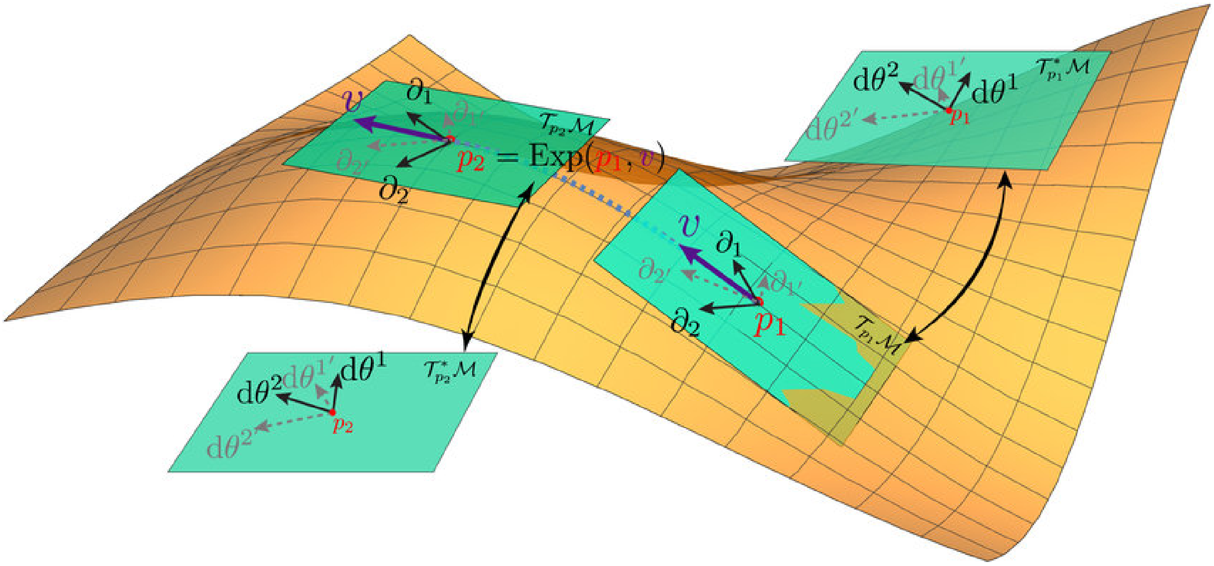

Figure 6: Artistic Presentation of Tangent and Cotangent Spaces II.(Source:Internet)

19 Appendix I:Tangent and Cotangent Spaces II

Consider a curve R in R^4 defined in function of a given set of coordinates u^0,u^1,u^2 and u^3 as being $R = R(u^0, u^1, u^2, u^3)$.

A total derivative of R is given by:

$$dR = \frac{\partial R}{\partial u^0} du^0 + \frac{\partial R}{\partial u^1} du^1 + \frac{\partial R}{\partial u^2} du^2 + \frac{\partial R}{\partial u^3} du^3 \tag{313}$$

Applying the Einstein summing convention:

$$dR = \frac{\partial R}{\partial u^i} du^i = e_i du^i \tag{314}$$

or

$$dR = \frac{\partial R}{\partial u^j} du^j = e_j du^j \tag{315}$$

With $i, j = 0, 1, 2, 3$ as the coordinates,$\frac{\partial R}{\partial u^i}$ and $\frac{\partial R}{\partial u^j}$ as the directional partial derivatives of R with respect to each coordinate and e_i and e_j are the respective Canonical Basis.

Defining $ds^2 = dR \bigotimes dR$ we have:

$$ds^2 = dR \bigotimes dR = \frac{\partial R}{\partial u^i} du^i \bigotimes \frac{\partial R}{\partial u^j} du^j = e_i du^i \bigotimes e_j du^j \tag{316}$$

$$ds^2 = \frac{\partial R}{\partial u^i} \frac{\partial R}{\partial u^j} du^i du^j = e_i e_j du^i du^j = g_{ij} du^i du^j \tag{317}$$

$$g_{ij} = \frac{\partial R}{\partial u^i} \frac{\partial R}{\partial u^j} = e_i e_j \tag{318}$$

The directional partial derivatives of R and their respective Canonical Basis are related to the ∂_i and ∂_j tangent spaces of the picture depicted in the beginning of this section while the differentials du^i and du^j are related to the respective cotangent spaces.See pg 148 problem 17 in [14],pg 132 eq 10.12 pg 133 eqs 10.14a,10.14b and 10.15 in [15].

$g_{ij} = \frac{\partial R}{\partial u^i}\frac{\partial R}{\partial u^j} = e_i e_j$ is the spacetime metric tensor of General Relativity.

20 Appendix J:the mathematical demonstration of the warp drive vectors $nX = -vs * dx$ and $nX = vs * dx$ for a constant speed vs in a R^3 space basis-$3D$ Spherical Coordinates

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{319}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{320}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{321}$$

Back again to the equivalence between $3D$ spherical and cartezian coordinates $d(\rho\sin\phi\cos\theta)$:(See Appendix E)

We will replace ρ by r and φ by ϕ.Then we have:

$$d(r\sin\phi\cos\theta) = sin\phi[d(r\cos\theta)] + (r\cos\theta)d(\sin\phi) \tag{322}$$

$$d(r\sin\phi\cos\theta) = sin\phi[cos\theta dr + r(d\cos\theta)] + (r\cos\theta)(cos\phi d\phi) \tag{323}$$

$$d(r\sin\phi\cos\theta) = sin\phi[cos\theta(dr) - rsin\theta(d\theta)] + (r\cos\theta)[cos\phi(d\phi)] \tag{324}$$

$$d(r\sin\phi\cos\theta) = sin\phi[cos\theta(dr) - sin\theta(rd\theta)] + cos\phi[(r\cos\theta)(d\phi)] \tag{325}$$

Applying the Hodge Star * to the term $[cos\theta(dr) - sin\theta(rd\theta)]$ we will get the same results already shown in the Appendix A and the first part of the $3D$ spherical warp drive vector is the one of the Appendix A multiplied by $\sin\phi$ .Then we must concern ourselves with the term $cos\phi[(r\cos\theta)(d\phi)]$ and the following Canonical Basis for the Hodge Star * since the other two were covered in the Appendix A.

$$e_\phi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \sim r\sin\theta d\phi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{326}$$

The term $cos\phi[(r\cos\theta)(d\phi)]$ must become compatible with the Canonical Basis for the Hodge Star above and this can be achieved by the following substitution:

$$cos\phi[(r\cos\theta)(d\phi)] = cos\phi[(r\sin\theta\cot\theta)(d\phi)] = cos\phi[\cot\theta(r\sin\theta)(d\phi)] \tag{327}$$

$$cos\phi[\cot\theta * ((r\sin\theta)(d\phi))] = cos\phi[\cot\theta(r(dr \wedge d\theta))] = cos\phi[\cot\theta(e_\phi)] \tag{328}$$

In the Appendix A we used the term $d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)$ and its respective Hodge Star $*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)$ also used by Natario in pg 5 in [1] because this term corresponds to the term $[cos\theta(*dr) - sin\theta(*rd\theta)]$ now being multiplied by $sin\phi$.In the $3D$ spherical warp drive this term also appears multiplied by $sin\phi$ but we must look for a corresponding expression concerning the term $cos\phi[\cot\theta*((r\sin\theta)(d\phi))] = cos\phi[\cot\theta(r(dr\wedge d\theta))]$.

The desired expression is the following one:

$$cos\phi[d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{329}$$

Its respective Hodge Star is:

$$cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{330}$$

Using the relations in the expression above to deal with the Hodge Star * given by eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]:See also pg 89 in [3],pg 112 in [4],pg 97 in [5],pg 36 eqs 2.21 and 2.22 in [6],pg 70 eq 3.3 in [7].

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p=2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{331}$$

$$*d(dx) = *d(dy) = *d(dz) = 0 \tag{332}$$

$p = 2$ stands for the R^3.Then we have:

$$*d[(\frac{1}{2})(r^2)\cot\theta d\theta] = (\frac{1}{2})(\cot\theta)*d(r^2 d\theta) + (\frac{1}{2})(r^2)*d(\cot\theta d\theta) + (\frac{1}{2})(r^2)\cot\theta * d(d\theta) \tag{333}$$

$$*d(r^2 d\theta) = d(r^2)\wedge d\theta + r^2\wedge d(d\theta) = d(r^2)\wedge d\theta = 2rdr\wedge d\theta \tag{334}$$

$$*d(\cot\theta d\theta) = d\cot\theta\wedge d\theta + \cot\theta\wedge d(d\theta) = d\cot\theta\wedge d\theta = -\csc^2\theta d\theta\wedge d\theta = 0 \tag{335}$$

$$*d(d\theta) = 0 \tag{336}$$

$$*d[(\frac{1}{2})(r^2)\cot\theta d\theta] = (\frac{1}{2})(\cot\theta)*d(r^2 d\theta) = (\frac{1}{2})(\cot\theta)(2rdr\wedge d\theta) = (\cot\theta)(rdr\wedge d\theta) \tag{337}$$

And

$$cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] = cos\phi[(\cot\theta)(rdr\wedge d\theta)] = cos\phi[\cot\theta(e_\phi)] \tag{338}$$

Because due to the Canonical Basis of the Hodge Star:

$$e_\phi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \sim r\sin\theta d\phi \sim dr\wedge(rd\theta) \sim r(dr\wedge d\theta) \tag{339}$$

Then in the $3D$ spherical coordinates we have the following Hodge Star:

$$*d(r\sin\phi\cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{340}$$

Also in Appendix A we used the term $*d[f(r)r^2\sin^2\theta d\phi]$ corresponding to the term $*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)$ because Natario also used it in pg 5 in [1].Now this term must be multiplied by $\sin\phi$.

From the Appendix A we have:

$$*d[f(r)r^2\sin^2\theta d\phi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{341}$$

Defining the Natario Vector as in pg 5 in [1] in polar coordinates with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) \tag{342}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) \tag{343}$$

We can get finally the latest expressions for the Natario Vector in polar coordinates nX also shown in pg 5 in [1]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{344}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{345}$$

We choose the polar coordinates Natario vectors $nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right)$ and

$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta$

But in $3D$ spherical coordinates we have:

$$\sin\phi(*d[f(r)r^2\sin^2\theta d\phi]) = \sin\phi(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta) \tag{346}$$

Like the term $*d[f(r)r^2\sin^2\theta d\phi]$ is associated to the term $*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)$ and now these terms must be multiplied by $\sin\phi$ we must find the corresponding term for $cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]]$.

The term we are looking for is the following one:

$$cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{347}$$

Solving the Hodge Star we have:

$$*d[(f(r))(r^2)\cot\theta d\theta] \tag{348}$$

$$(f(r))\cot\theta * d(r^2d\theta) + (f(r))(r^2) * d(\cot\theta d\theta) + (r^2)(\cot\theta) * d(f(r)d\theta)) + ((f(r))(r^2)\cot\theta) * d(d\theta) \tag{349}$$

As already seen before the terms $*d(\cot\theta d\theta) = 0$ and $*d(d\theta) = 0$.Then the Hodge Star becomes:

$$*d[(f(r))(r^2)\cot\theta d\theta] = (f(r))\cot\theta * d(r^2d\theta) + (r^2)(\cot\theta) * d(f(r)d\theta) \tag{350}$$

$$*d(r^2d\theta) = d(r^2)\wedge d\theta + r^2\wedge d(d\theta) = d(r^2)\wedge d\theta = 2rdr\wedge d\theta \tag{351}$$

$$*d(f(r)d\theta) = d(f(r)\wedge d\theta + f(r)\wedge d(d\theta) = d(f(r)\wedge d\theta = f'(r)dr\wedge d\theta \tag{352}$$

Still with the Hodge Star:

$$*d[(f(r))(r^2)\cot\theta d\theta] = (f(r))\cot\theta * d(r^2 d\theta) + (r^2)(\cot\theta) * d(f(r)d\theta) \tag{353}$$

$$*d(r^2 d\theta) = 2rdr \wedge d\theta \tag{354}$$

$$*d(f(r)d\theta) = f'(r)dr \wedge d\theta \tag{355}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = (f(r))\cot\theta(2rdr \wedge d\theta) + (r^2)(\cot\theta)f'(r))(dr \wedge d\theta) \tag{356}$$

The Canonical Basis for the Hodge Star is:

$$e_\phi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \sim r\sin\theta d\phi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{357}$$

Then the Hodge Star now becomes:

$$*d[(f(r))(r^2)\cot\theta d\theta] = 2(f(r))\cot\theta(rdr \wedge d\theta) + (\cot\theta)rf'(r)(rdr \wedge d\theta) \tag{358}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = cot\theta[2(f(r)) + (rf'(r))](rdr \wedge d\theta) \tag{359}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = cot\theta[2(f(r)) + (rf'(r))]e_\phi \tag{360}$$

At last we are ready to present the new tridimensional $3D$ spherical warp drive vector.We already know that in the $3D$ spherical coordinates $d(r\sin\phi\cos\theta)$ we have the following Hodge Star:

$$*d(r\sin\phi\cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{361}$$

But as we already demonstrated in this section the Hodge Star above can be associated to the following one:

$$\sin\phi[*d[f(r)r^2\sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{362}$$

With:

$$*d[f(r)r^2\sin^2\theta d\phi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{363}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = cot\theta[2(f(r)) + (rf'(r))]e_\phi \tag{364}$$

Then our tridimensional $3D$ spherical Hodge Star can be given by:

$$\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi] \tag{365}$$

Natario defined two warp drive vectors in pg 5 in [1] as being:(see Appendix A)

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{366}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{367}$$

$$nX = vs(t) * d\left(f(r)r^2 \sin^2\theta d\phi\right) = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{368}$$

$$nX = -vs(t) * d\left(f(r)r^2 \sin^2\theta d\phi\right) = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{369}$$

We choose this one:$nX = vs(t)*d\left(f(r)r^2 \sin^2\theta d\phi\right) = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta$.Then we have the original Natario warp drive vector in polar coordinates:

$$nX = vs(t) * d\left(f(r)r^2 \sin^2\theta d\phi\right) = vs(t)[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] \tag{370}$$

Now and finally[8] we can present the final form of our new warp drive vector in tridimensional $3D$ spherical coordinates as being:

$$nX = vs(t)[\sin\phi[*d[f(r)r^2 \sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]]] \tag{371}$$

$$nX = vs(t)\sin\phi[*d[f(r)r^2 \sin^2\theta d\phi]] + vs(t)cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{372}$$

$$nX = (\sin\phi)vs(t)[*d[f(r)r^2 \sin^2\theta d\phi]] + (cos\phi)vs(t)[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{373}$$

$$*d[f(r)r^2 \sin^2\theta d\phi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{374}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = cot\theta[2(f(r)) + (rf'(r))]e_\phi \tag{375}$$

$$nX = vs(t)[\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{376}$$

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{377}$$

This is the final form of our new tridimensional $3D$ spherical warp drive vector.Note that Natario in pg 4 in [1] defined the x-axis as the polar axis.if the motion occurs only in the x-axis in polar coordinates then the angle between the x-y plane and the z-axis is 90 degrees and in this case $\sin\phi = 1$ and $\cos\phi = 0$ and our new warp drive vector in tridimensional $3D$ spherical coordinates reduces to the original Natario warp drive vector in polar coordinates.

Only in a real tridimensional $3D$ spherical coordinates motion our new warp drive vector accounts for a significant difference

[8] at last!!!we know that this section is being written in a tedious and monotonous style but we are writing this for beginners or introductory students eagerly needing these mathematical demonstrations QED Quod Erad Demonstratum in order to allow these students to more easily understand the whole process of the obtention of warp drive vectors

For our new tridimensional $3D$ spherical coordinates warp drive vector

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{378}$$

The corresponding shift vectors are:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{379}$$

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{380}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{381}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{382}$$

21 Appendix K:the mathematical demonstration of the warp drive vector $nX = vs * dx$ for a constant speed vs in a R^4 space basis-Tridimensional $3D$ Spherical Coordinates

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [2],eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{383}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{384}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{385}$$

Useful relations to deal with the Hodge Star * are given by eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]:See also pg 89 in [3],pg 112 in [4],pg 97 in [5],pg 36 eqs 2.21 and 2.22 in [6],pg 70 eq 3.3 in [7].

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df \wedge \alpha - f \wedge d\alpha \tag{386}$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{387}$$

$$*d(dx) = *d(dy) = *d(dz) = 0 \tag{388}$$

$p = 3$ stands for the R^4 and $p = 2$ stands for the R^3.

Back again to the equivalence between $3D$ spherical and cartezian coordinates $d(\rho\sin\phi\cos\theta)$:(See Appendix E)

We will replace ρ by r and φ by ϕ.Then we have:

$$d(r\sin\phi\cos\theta) = sin\phi[d(r\cos\theta)] + (r\cos\theta)d(\sin\phi) \tag{389}$$

$$d(r\sin\phi\cos\theta) = sin\phi[cos\theta(dr) - sin\theta(rd\theta)] + cos\phi[(r\cos\theta)(d\phi)] \tag{390}$$

Applying the Hodge Star * to the terms above we will get the same results already shown in the Appendix J.As a matter of fact comparing the Appendices A and B the given final result is the same in both Appendices except for the fact that in Appendix A the Hodge Star is taken over R^3 and in Appendix B the Hodge Star is taken over R^4.

So the expressions for the Hodge Star of the term $d(r\sin\phi\cos\theta)$ covered in the last(and gigantic or enormous) Appendix J taken over R^3 that uses the terms

$$*d(r\sin\phi\cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{391}$$

$$\sin\phi[*d[f(r)r^2\sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{392}$$

Will appear in identical form if we compute the Hodge Star for the same term

$$d(r\sin\phi\cos\theta)$$

in R^4.The only difference is the term in R^4

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df\wedge\alpha - f\wedge d\alpha \tag{393}$$

Different than its counterpart in R^3

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{394}$$

But since the term $f\wedge d\alpha = 0$ wether in R^4 or R^3 the final result of the Hodge Star is the same wether in R^4 or R^3 and we do not need to repeat here the tedious and monotonous piles of calculations shown in the (monster)Appendix J since the results are the same ones.

Our new tridimensional $3D$ spherical coordinates warp drive vector in R^4 with constant speed vs $nX = vs * dx$ is given by:

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r))+(rf'(r))]e_\phi]] \tag{395}$$

The corresponding shift vectors are:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{396}$$

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{397}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{398}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{399}$$

22 Appendix L:mathematical demonstration of the Natario warp drive equation for a constant speed vs in the original $3+1$ ADM Formalism according to MTW and Alcubierre using a lapse function α

This Appendix is a continuation of the Appendix C except for the fact that we do not suppress the lapse function here.Combining the eqs (21.40),(21.42) and (21.44) pgs [507, 508] in [17]
with the eqs (2.2.5) and (2.2.6) pgs [67] in [18] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \quad (400)$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \quad (401)$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \quad (402)$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \quad (403)$$

But since $dl^2 = \gamma_{ij}dx^i dx^j$ is the ADM induced metric and must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \quad (404)$$

Expanding the square term and recombining all the terms we have:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \quad (405)$$

Note that the expression above is exactly the eq (2.2.4) pgs [67] in [18].It also appears as eq 1 pg 3 in [16].

Changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-\alpha^2 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \quad (406)$$

$$ds^2 = (\alpha^2 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \quad (407)$$

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \quad (408)$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i\beta^i = X_i X^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii}dx^i dx^i \quad (409)$$

Then the equation of the Natario warp drive spacetime with a constant speed vs in the original $3+1$ ADM formalism with a lapse function is given by:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{410}$$

Inserting the components of the Natario vector nX(pg 2 and 5 in [1]) in Polar Coordinates:

$$nX = X^r e_r + X^\theta e_\theta \tag{411}$$

We have:

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{412}$$

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{413}$$

The lapse function is equal to 1 inside and outside the Natario warp bubble while having large values in the Natario warped region.See [10]

Reducing to a $1+1$ spacetime as we did in the Appendix C we have:

$$ds^2 = (\alpha^2 - [X^r]^2)dt^2 + 2(X^r dr)dt - dr^2 \tag{414}$$

This equation is useful to compute Horizons.

Considering now the new Natario warp drive vector in $3D$ tridimensional Spherical Coordinates with a constant speed vs nX given by::

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{415}$$

We have:

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr dt + X_\theta d\theta dt + X_\phi d\phi dt) - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{416}$$

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{417}$$

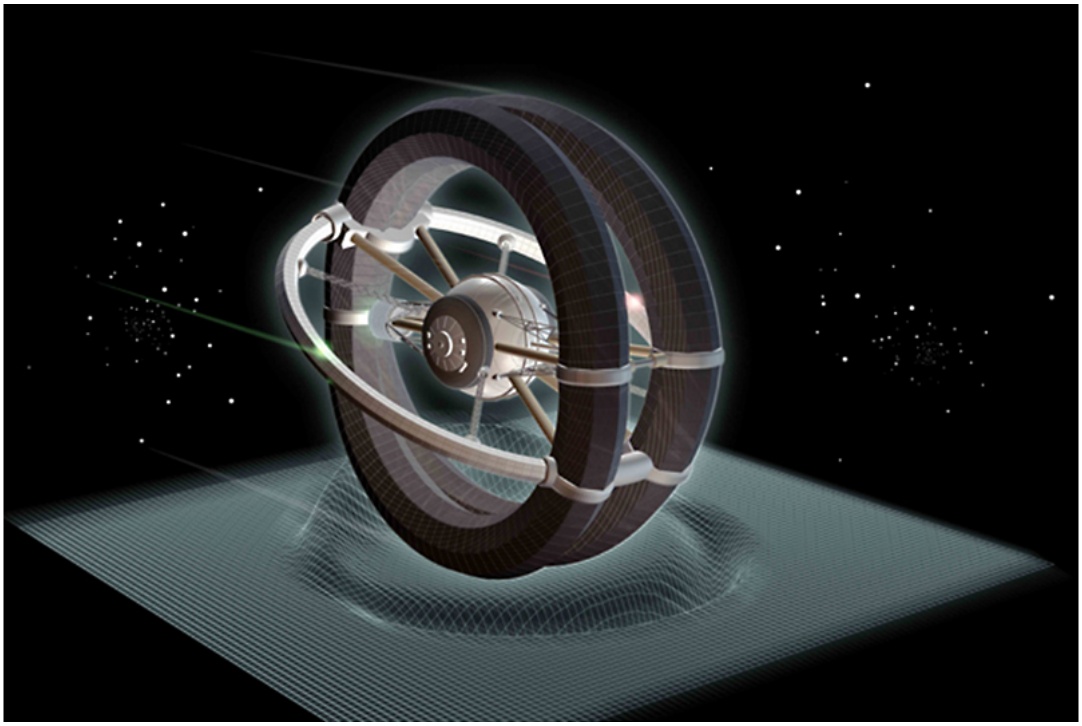

Figure 7: Artistic representation of the Natario warp drive .Note in the bottom of the figure the Alcubierre expansion of the normal volume elements .(Source:Internet)

23 Appendix M:Artistic Presentation of the Natario warp drive-Polar Coordinates

According to the geometry of the Natario warp drive the spacetime contraction in one direction(radial) is balanced by the spacetime expansion in the remaining direction(perpendicular).(pg 5 in [1]).

The expansion of the normal volume elements in the Natario warp drive is given by the following expressions(pg 5 in [1]).

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s f'(r)\cos\theta \tag{418}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = v_s f'(r)\cos\theta; \tag{419}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = v_s f'(r)\cos\theta \tag{420}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{421}$$

If we expand the radial direction the perpendicular direction contracts to keep the expansion of the normal volume elements equal to zero.This figure is a pedagogical example of the graphical presentarion of the Natario warp drive.

The "bars" in the figure were included to illustrate how the expansion in one direction can be counterbalanced by the contraction in the other directions.These "bars" keeps the expansion of the normal volume elements in the Natario warp drive equal to zero.

Note also that the graphical presentation of the Alcubierre warp drive expansion of the normal volume elements according to fig 1 pg 10 in [16] is also included

Note also that the energy density in the Natario warp drive $3+1$ spacetime being given by the following expressions(pg 5 in [1]):

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(f'(r))^2\cos^2\theta + \left(f'(r) + \frac{r}{2}f''(r)\right)^2\sin^2\theta\right]. \tag{422}$$

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{df(r)}{dr})^2\cos^2\theta + \left(\frac{df(r)}{dr} + \frac{r}{2}\frac{d^2f(r)}{dr^2}\right)^2\sin^2\theta\right]. \tag{423}$$

Is being distributed around all the space involving the ship(above the ship $\sin\theta = 1$ and $\cos\theta = 0$ while in front of the ship $\sin\theta = 0$ and $\cos\theta = 1$).The negative energy in front of the ship "deflect" photons or other particles so these will not reach the ship inside the bubble.The illustrated "bars" are the obstacles that deflects photons or incoming particles from outside the bubble never allowing these to reach the interior of the bubble.[9]

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg 116 in [19])

-)-Energy directly above the ship($y - axis$)

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[\left(\frac{df(r)}{dr} + \frac{r}{2}\frac{d^2f(r)}{dr^2}\right)^2\sin^2\theta\right]. \tag{424}$$

-)-Energy directly in front of the ship($x - axis$)

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{df(r)}{dr})^2\cos^2\theta\right]. \tag{425}$$

[9]See also Appendix N

The distribution of energy presented in this Appendix is valid only for the Natario warp drive vector in Polar Coordinates without the lapse function.For the case of the lapse function see Section 3 and Appendix I in [10].

Also the Zero-Expansion behavior is valid only in Polar Coordinates and do not occurs in $3D$ Spherical Coordinates.See [9].

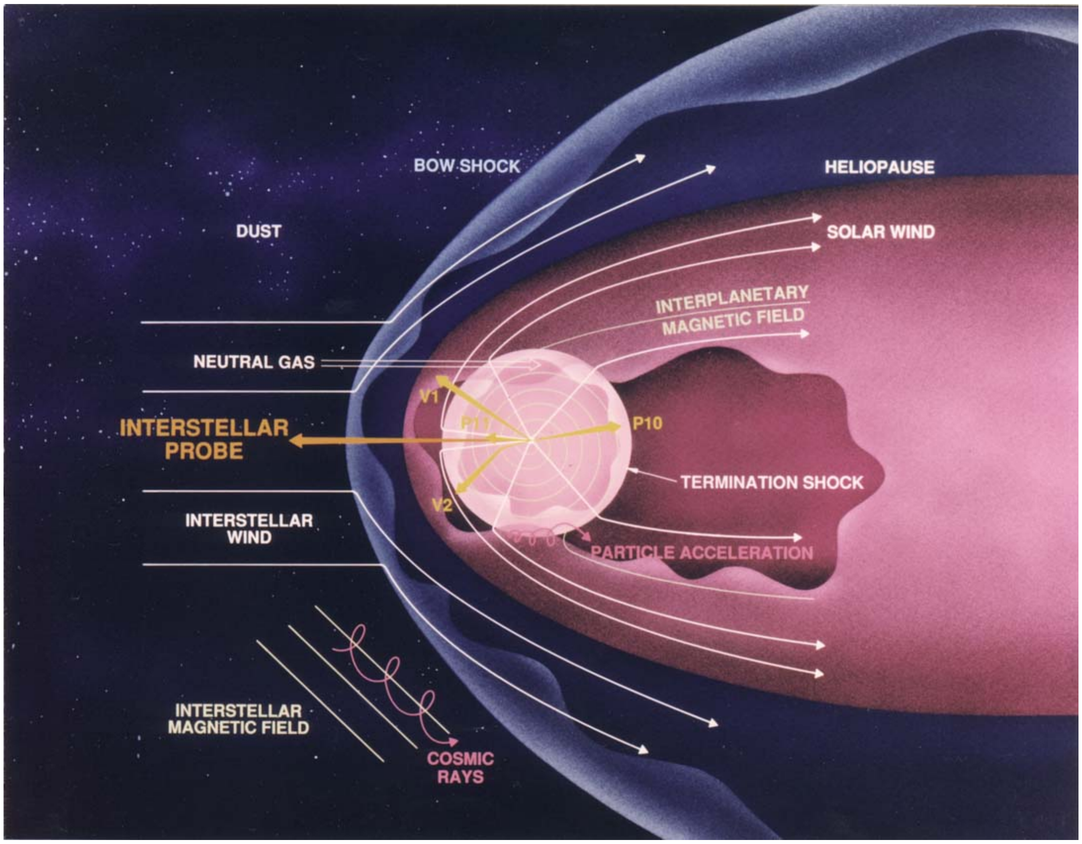

Figure 8: Artistic representation of a Natario warp wrive in a real superluminal space travel .Note the negative energy in front of the ship deflecting incoming hazardous interstellar matter(brown arrows).(Source:Internet)

24 Appendix N:Artistic Presentation of a Natario warp drive in a real faster than light interstellar spaceflight

Above is being presented the artistic presentation of a Natario warp drive in a real interstellar superluminal travel.The "ball" or the spherical shape is the Natario warp bubble with the negative energy surrounding the ship in all directions and mainly protecting the front of the bubble.[10]

The brown arrows in the front of the Natario bubble are a graphical presentation of the negative energy in front of the ship deflecting interstellar dust,neutral gases,hydrogen atoms,interstellar wind photons etc.[11]

The spaceship is at the rest and in complete safety inside the Natario bubble.

[10] See Appendix M

[11] see Appendices P and Q for the composition of the Interstellar Medium IM)

In order to allow to the negative energy density of the Natario warp drive the deflection of incoming hazardous particles from the Interstellar Medium(IM) the Natario warp drive energy density must be heavier or denser when compared to the IM density.

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg 116 in [19])

25 Appendix O:The Natario warp drive negative energy density in Cartezian coordinates

The negative energy density according to Natario in Polar Coordinates is given by(see pg 5 in [1])[12]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(f'(rs))^2\cos^2\theta + \left(f'(rs) + \frac{r}{2}f''(rs)\right)^2\sin^2\theta\right] \tag{426}$$

In the bottom of pg 4 in [1] Natario defined the x-axis as the polar axis.In the top of page 5 we can see that $x = r\cos(\theta)$ implying in $\cos(\theta) = \frac{x}{r}$ and in $\sin(\theta) = \frac{y}{r}$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(f'(rs))^2(\frac{x}{rs})^2 + \left(f'(rs) + \frac{r}{2}f''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{427}$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2 + z^2] = 0$ and $r^2 = [(x - xs)^2]$ and making $xs = 0$ the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $r^2 = x^2$ because in the equatorial plane $y = z = 0$.

Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(f'(rs))^2\right] \tag{428}$$

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg 116 in [19])

The distribution of energy presented in this Appendix is valid only for the Natario warp drive vector in Polar Coordinates without the lapse function.For the case of the lapse function see Section 3 and Appendix I in [10].

But for the case of the warp drive vector in $3D$ Spherical Coordinates equations we can say nothing about the negative energy density at first sight and we need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors.

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pg 342 in [17], pg 276 in [30],pgs $454, 457, 560$ in [31] pg 98 in [32],pg 178 in [33]).

Appendix C pgs $551 - 555$ in [31] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3 + 1$ spacetime metric using $Mathematica$.[13]

[12] $f(r)$ is the Natario shape function.Equation written in the Geometrized System of Units $c = G = 1$

[13] Unfortunately we dont have access to anyone of these programs so we have our hands "tied up"

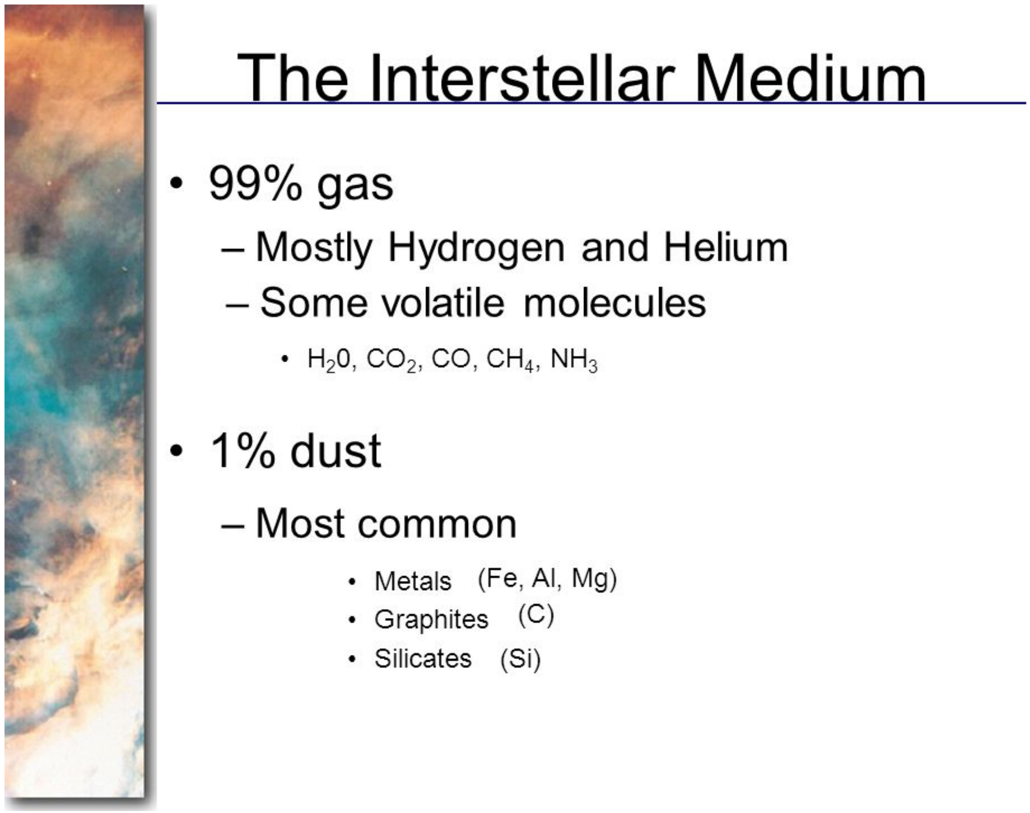

Figure 9: Composition of the Interstellar Medium IM(Source:Internet)

26 Appendix P:Composition of the Interstellar Medium IM

The problem of collisions between a warp drive spaceship moving at superluminal velocity and the potentially dangerous particles from the Interstellar Medium IM is not new.

It was first noticed in 1999 in the work of Chad Clark,Will Hiscock and Shane Larson(see [24]).Later on in 2010 it appeared again in the work of Carlos Barcelo,Stefano Finazzi and Stefano Liberatti(see [25]).In 2012 the same problem of collisions against hazardous IM particles appeared in the work of Brendan McMonigal,Geraint Lewis and Philip O'Byrne(see [21]).

The last work addressing interstellar collisions was the work in ([22]) in 2022.It covers the analysis of Siyu Bian ,Yi Wang,Zun Wang and Mian Zhu.

All these works uses the geometry of the original Alcubierre warp drive 1994 paper in [16] and the results outlined in these works are completely correct.

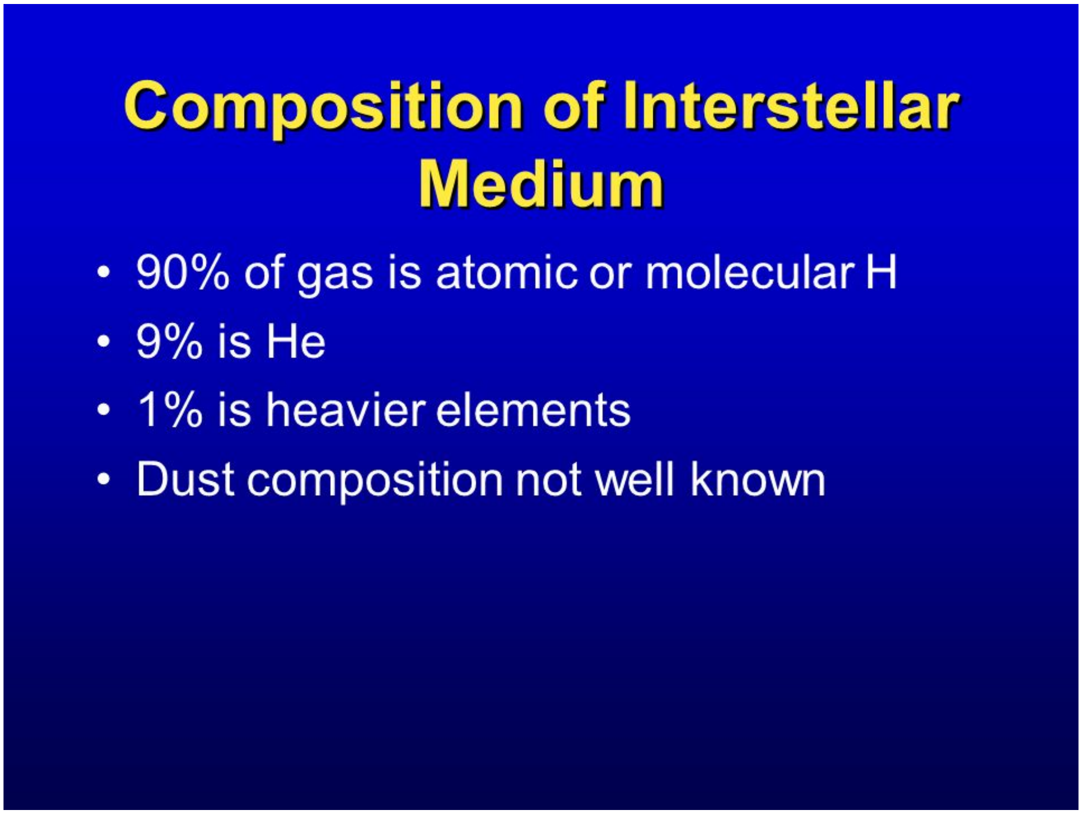

Figure 10: Composition of the Interstellar Medium IM(Source:Internet)

27 Appendix Q:Composition of the Interstellar Medium IM

The Natario warp drive is probably the best candidate(known until now) for an interstellar space travel considering the fact that a spaceship in a real superluminal interstellar spaceflight will encounter(or collide against) hazardous objects(asteroids,comets,interstellar dust and debris etc) and due to a different distribution of the negative energy in front of the ship with repulsive gravitational behavior(see pg 116 in [19]) deflecting all the incoming hazardous particles of the Interstellar Medium(see Appendices M,N and O) the Natario spacetime offers an excellent protection to the crew members as depicted in the works [26],[27] and specially [28],[29] and [23].

28 Appendix R:Generic quadratic forms in the $3+1$ ADM spacetime without the lapse function.

The Natario warp drive equations with signature $(+,-,-,-)$ that obeys the original $3+1$ ADM formalism are given below:

in Polar Coordinates:(see Appendix C).

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{429}$$

in $3D$ Spherical Coordinates:(see also Appendix C).

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{430}$$

Using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity,Horizons can be easily computed for the dimensionally reduced $1+1$ spacetime versions of these equations because only the quadratic form dr^2 exists but in the $3+1$ spacetime we have the presence of 3 quadratic forms respectively dr^2,$r^2 d\theta^2$ and $r^2\sin^2\theta d\phi^2$.Algebraic solutions for the null-like geodesics $ds^2 = 0$ of General Relativity of the $3+1$ equations above are extremely difficult due to the presence of these 3 quadratic forms considering solutions for each quadratic form dr^2 or $r^2 d\theta^2$ or $r^2\sin^2\theta d\phi^2$ isolated.

The best effort to solve the null-like geodesics $ds^2 = 0$ in the case of the $3+1$ spacetime equations given above is to find out a solution that encompasses all the 3 quadratic forms dr^2 and $r^2 d\theta^2$ and $r^2\sin^2\theta d\phi^2$ grouped together.

We will demonstrate all the required mathematics step by step.

Back to the $3+1$ ADM formalism compact generic equation given below:(see Appendix C)

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{431}$$

Expanding the equation above we have:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{432}$$

The null-like geodesics $ds^2 = 0$ is:

$$0 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{433}$$

Dividing by dt^2 we have:

$$0 = (1 - X_i X^i) + 2X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{dx^i dx^i}{dt^2} \tag{434}$$

$$0 = (1 - X_i X^i) + 2X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{(dx^i)^2}{dt^2} \tag{435}$$

$$0 = (1 - X_i X^i) + 2X_i \frac{dx^i}{dt} - \gamma_{ii}(\frac{dx^i}{dt})^2 \tag{436}$$

Introducing the term U^i as being:

$$U^i = \frac{dx^i}{dt} \tag{437}$$

We have now a generic quadratic form in the term U^i:

$$0 = (1 - X_i X^i) + 2X_i U^i - \gamma_{ii}(U^i)^2 \tag{438}$$

Rearranging the terms in this quadratic form we have:

$$\gamma_{ii}(U^i)^2 - 2X_i - (1 - X_i X^i) = 0 \tag{439}$$

$$\gamma_{ii}(U^i)^2 - 2X_i + (X_i X^i - 1) = 0 \tag{440}$$

The solution of this generic quadratic form in the term U^i is given by:

$$U^i = \frac{2X_i \pm \sqrt{[-2X_i]^2 - 4[\gamma_{ii}(X_i X^i - 1)]}}{2\gamma_{ii}} = \frac{2X_i \pm \sqrt{4[X_i]^2 - 4[\gamma_{ii}(X_i X^i] + 4[\gamma_{ii}]}}{2\gamma_{ii}} \tag{441}$$

But since:

$$X_i = \gamma_{ii} X^i \tag{442}$$

We have:

$$U^i = \frac{2X_i \pm \sqrt{4[X_i]^2 - 4[X_i]^2 + 4[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2X_i \pm \sqrt{+4[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2X_i \pm 2\sqrt{[\gamma_{ii}]}}{2\gamma_{ii}} \tag{443}$$

$$U^i = \frac{2X_i \pm 2\sqrt{\gamma_{ii}}}{2\gamma_{ii}} = \frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{444}$$

At last we have the final solution of this generic quadratic form in the term U^i given by:

$$U^i = \frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{445}$$

But this expression actually means:

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{\sum_{i=1}^{3} X_i \pm \sum_{i=1}^{3} \sqrt{\gamma_{ii}}}{\sum_{i=1}^{3} \gamma_{ii}} = \frac{\sum_{i=1}^{3} X_i \pm \sqrt{\sum_{i=1}^{3} \gamma_{ii}}}{\sum_{i=1}^{3} \gamma_{ii}} \tag{446}$$

The subscript γ_{ii} is inside the root $\sqrt{\gamma_{ii}}$ so the sum must be taken also inside the root.(see pg 5,pg 227 section 7.3 and pg 241 section 7.10 in [41]).Then $\sum_{i=1}^{3} \sqrt{\gamma_{ii}}$ actually must be $\sqrt{\sum_{i=1}^{3} \gamma_{ii}}$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 + U^3 = \frac{X_1 + X_2 + X_3 \pm \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{447}$$

The generic quadratic form in the term U^i for the null-like geodesics $ds^2 = 0$ is given by:

$$U^i = \frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{448}$$

Expanding the terms in the expression above we have:

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 + U^3 = \frac{X_1 + X_2 + X_3 \pm \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{449}$$

The line element in the $3+1$ ADM spacetime without the lapse function is:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{450}$$

Expanding the terms in the expression above we have:

$$ds^2 = (1 - X_1X^1 - X_2X^2 - X_3X^3)dt^2 + 2(X_1dx^1 + X_2dx^2 + X_3dx^3)dt - \gamma_{11}dx^1dx^1 - \gamma_{22}dx^2dx^2 - \gamma_{33}dx^3dx^3 \tag{451}$$

$$ds^2 = (1 - X_1X^1 - X_2X^2 - X_3X^3)dt^2 + 2(X_1dx^1 + X_2dx^2 + X_3dx^3)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 - \gamma_{33}(dx^3)^2 \tag{452}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{453}$$

$$U^i = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{454}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 + U^3 = \frac{X_1 + X_2 + X_3 + \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{455}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 + U^3 = \frac{X_1 + X_2 + X_3 - \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{456}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 + \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{457}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 - \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{458}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $3+1$ spacetime equations given above with the solution that encompasses all the 3 quadratic forms $(dx^1)^2$,$(dx^2)^2$ and $(dx^3)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt}$.

The line element in the $2+1$ ADM spacetime without the lapse function is:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{459}$$

Expanding the terms in the expression above we have:

$$ds^2 = (1 - X_1 X^1 - X_2 X^2)dt^2 + 2(X_1 dx^1 + X_2 dx^2)dt - \gamma_{11} dx^1 dx^1 - \gamma_{22} dx^2 dx^2 \tag{460}$$

$$ds^2 = (1 - X_1 X^1 - X_2 X^2)dt^2 + 2(X_1 dx^1 + X_2 dx^2)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 \tag{461}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{462}$$

$$U^i = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{463}$$

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 = \frac{X_1 + X_2 + \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{464}$$

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 = \frac{X_1 + X_2 - \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{465}$$

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 + \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{466}$$

$$\sum_{i=1}^{2}[U^i] = \sum_{i=1}^{2}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 - \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{467}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $2+1$ spacetime equations given above with the solution that encompasses all the 2 quadratic forms $(dx^1)^2$ and $(dx^2)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt}$.

The line element in the $1+1$ ADM spacetime without the lapse function is:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{468}$$

Expanding the terms in the expression above we have:

$$ds^2 = (1 - X_1 X^1)dt^2 + 2(X_1 dx^1)dt - \gamma_{11} dx^1 dx^1 \tag{469}$$

$$ds^2 = (1 - X_1 X^1)dt^2 + 2(X_1 dx^1)dt - \gamma_{11} (dx^1)^2 \tag{470}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{471}$$

$$U^i = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{472}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 = \frac{X_1 + \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{473}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 = \frac{X_1 - \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{474}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} = \frac{X_1 + \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{475}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} = \frac{X_1 - \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{476}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $1+1$ spacetime equations given above with the solution that encompasses the single quadratic forms $(dx^1)^2$.The solution is given in function of $\frac{dx^1}{dt}$.

29 Appendix S:Generic quadratic forms in the $3+1$ ADM spacetime with the lapse function.

This Appendix is a continuation of the Appendix R but this time we consider the lapse function.We provide all the step by step mathematical calculations.

Back to the $3+1$ ADM formalism compact generic equation with the lapse function given below:(see Appendix L)

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{477}$$

Expanding the equation above we have:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{478}$$

The null-like geodesics $ds^2 = 0$ is:

$$0 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{479}$$

Dividing by dt^2 we have:

$$0 = (\alpha^2 - X_i X^i) + 2X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{dx^i dx^i}{dt^2} \tag{480}$$

$$0 = (\alpha^2 - X_i X^i) + 2X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{(dx^i)^2}{dt^2} \tag{481}$$

$$0 = (\alpha^2 - X_i X^i) + 2X_i \frac{dx^i}{dt} - \gamma_{ii} (\frac{dx^i}{dt})^2 \tag{482}$$

Introducing the term U^i as being:

$$U^i = \frac{dx^i}{dt} \tag{483}$$

We have now a generic quadratic form in the term U^i:

$$0 = (\alpha^2 - X_i X^i) + 2X_i U^i - \gamma_{ii}(U^i)^2 \tag{484}$$

Rearranging the terms in this quadratic form we have:

$$\gamma_{ii}(U^i)^2 - 2X_i - (\alpha^2 - X_i X^i) = 0 \tag{485}$$

$$\gamma_{ii}(U^i)^2 - 2X_i + (X_i X^i - \alpha^2) = 0 \tag{486}$$

The solution of this generic quadratic form in the term U^i is given by:

$$U^i = \frac{2X_i \pm \sqrt{[-2X_i]^2 - 4[\gamma_{ii}(X_i X^i - \alpha^2)]}}{2\gamma_{ii}} = \frac{2X_i \pm \sqrt{4[X_i]^2 - 4[\gamma_{ii}(X_i X^i] + 4\alpha^2[\gamma_{ii}]}}{2\gamma_{ii}} \tag{487}$$

But since:

$$X_i = \gamma_{ii} X^i \tag{488}$$

We have:

$$U^i = \frac{2X_i \pm \sqrt{4[X_i]^2 - 4[X_i]^2 + 4\alpha^2[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2X_i \pm \sqrt{+4\alpha^2[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2X_i \pm 2\alpha\sqrt{[\gamma_{ii}]}}{2\gamma_{ii}} \tag{489}$$

$$U^i = \frac{2X_i \pm 2\alpha\sqrt{\gamma_{ii}}}{2\gamma_{ii}} = \frac{X_i \pm \alpha\sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{490}$$

At last we have the final solution of this generic quadratic form for the null-like geodesics $ds^2 = 0$ in the term U^i given by:

$$U^i = \frac{X_i \pm \alpha\sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{491}$$

The solution have two roots:

$$U^i = \frac{X_i + \alpha\sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{492}$$

$$U^i = \frac{X_i - \alpha\sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{493}$$

The subscript γ_{ii} is inside the root $\sqrt{\gamma_{ii}}$ so the sum must be taken also inside the root.(see pg 5,pg 227 section 7.3 and pg 241 section 7.10 in [41]).Then $\sum_{i=1}^{3}\sqrt{\gamma_{ii}}$ actually must be $\sqrt{\sum_{i=1}^{3}\gamma_{ii}}$

Adapting the results from the previous section we have for the equation of the $3+1$ spacetime in the ADM formalism:

$$ds^2 = (\alpha^2 - X_1X^1 - X_2X^2 - X_3X^3)dt^2 + 2(X_1dx^1 + X_2dx^2 + X_3dx^3)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 - \gamma_{33}(dx^3)^2 \tag{494}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 + \alpha\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{495}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 - \alpha\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{496}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $3+1$ spacetime equations given above with the solution that encompasses all the 3 quadratic forms $(dx^1)^2$,$(dx^2)^2$ and $(dx^3)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt}$.

Adapting the results from the previous section we have for the equation of the $2+1$ spacetime in the ADM formalism:

$$ds^2 = (\alpha^2 - X_1X^1 - X_2X^2)dt^2 + 2(X_1dx^1 + X_2dx^2)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 \tag{497}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 + \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{498}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 - \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{499}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $2+1$ spacetime equations given above with the solution that encompasses all the 2 quadratic forms $(dx^1)^2$ and $(dx^2)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt}$.

Adapting the results from the previous section we have for the equation of the $1+1$ spacetime in the ADM formalism:

$$ds^2 = (\alpha^2 - X_1X^1)dt^2 + 2(X_1dx^1)dt - \gamma_{11}(dx^1)^2 \tag{500}$$

$$\frac{dx^1}{dt} = \frac{X_1 + \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} \tag{501}$$

$$\frac{dx^1}{dt} = \frac{X_1 - \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} \tag{502}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $1+1$ spacetime equations given above with the solution that encompasses the single quadratic form $(dx^1)^2$.The solution is given in function of $\frac{dx^1}{dt}$.

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