EXPLICIT BOUNDS ON GAPS BETWEEN CONSECUTIVE TERMS IN AN ADDITION CHAIN

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ABSTRACT. We develop explicit bounds for the gap between consecutive terms in an addition chain leading to a fixed target $n \in [2^m, 2^{m+1})$.

1. Introduction and motivation

A classical problem asks whether the Ulam sequences have zero asymptotic density. In our earlier work, we showed the following equivalence:

Theorem 1.1 (The second Ulam density criterion)**.** *Let $\{U_m\}_{m\geq 1}$ denote the sequence of all Ulam numbers. Then the following assertions hold:*

(i) *If $\lim_{m\longrightarrow\infty} \inf(U_{m+1} - U_m) = \infty$, then the sequence $\{U_m\}_{m\geq 1}$ have zero density.*

(ii) *If the sequence $\{U_m\}_{m\geq 1}$ has zero density, then $\lim_{m\longrightarrow\infty} \sup(U_{m+1} - U_m) = \infty$.*

Since any finite truncation of the Ulam sequence can be viewed as an addition chain leading to that Ulam number, the explicit run-gap inequalities developed here offer a new toolkit for probing whether these gaps must grow with or without bound.

2. Preliminaries and Setup

Let $l(n)$ be the length of an addition chain leading to n, denoted $E(n)$, of the form

$$E(n) : s_o = 1, s_1 = 2, \ldots, s_{l(n)} = n$$

with $2^m \leq n < 2^{m+1}$ such that $l(n) := \beta(m)$. By adapting the ideas of the paper [1], we partition the steps in an addition chain into the following classes of steps

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$$\mathcal{A} := \{i \ : \ s_i = 2s_{i-1}\} \quad (\textbf{doubling steps})$$

$$\mathcal{B} := \{i \ : \ \gamma s_{i-1} \leq s_i < 2s_{i-1}\} \quad (\textbf{large steps})$$

where $\gamma := \frac{1+\sqrt{5}}{2}$ is the *golden* ratio

$$\mathcal{C} := \{i \ : \ (1+\delta)s_{i-1} \leq s_i < \gamma s_{i-1}\} \quad (\textbf{medium}-\textbf{sized steps})$$

where $\delta := \delta(m) \longrightarrow 0$ as $m \longrightarrow \infty$. In particular

$$\delta := \delta(m) = \frac{1}{\log m}$$

$$\mathcal{D} := \{i \ : \ s_i < (1+\delta)s_{i-1}\} \quad (\textbf{small steps}).$$

We denote the cardinality of the sets to be

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

We call steps in $\mathcal{B}, \mathcal{C}, \mathcal{D}$ as *non-doubling steps*. We have therefore the relation

$$A + B + C + D = \beta(m).$$

Because each non-doubling step in an addition chain cannot grow faster than a corresponding step in a Fibonacci sequence, we have (by induction) the inequality

$$2^m \leq n \leq 2^A \gamma^{B+C+D} = 2^{\beta(m)} \left(\frac{\gamma}{2}\right)^{B+C+D}$$

and we deduce from this relation an upper control for the total number of non-doubling steps in an addition chain of length $\beta(m)$ to be

Lemma 2.1. *Put*

$$E(n) : s_o = 1, s_1 = 2, \ldots, s_{l(n)} = n$$

be an addition chain with $l(n) := \beta(m)$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be steps in an addition chain of length $\beta(m)$ with cardinality A, B, C, D, respectively. Then we have

$$B + C + D \leq \frac{\beta(m) - m}{1 - \log_2 \gamma}.$$

It turns out that the non-doubling steps in an addition chain have certain structural pattern.

Lemma 2.2. *If $j \in \mathcal{B}$, then $j - 1 \in \mathcal{C} \cup \mathcal{D}$. In particular, each large step in an addition chain must be preceded by either a small step or a medium-sized step.*

Proof. Let $j \in \mathcal{B}$ (large step) then we have by definition

$$\gamma s_{j-1} \leq s_j < 2s_{j-1}$$

where $\gamma := \frac{1+\sqrt{5}}{2}$ is the *golden ratio*. Write $s_j = s_k + s_l$ with $k \geq l$. The inequality $s_j = s_k + s_l \leq s_{j-1} + s_l$ with $s_j < 2s_{j-1}$ implies that

$$\gamma s_{j-1} \leq s_j \leq s_{j-1} + s_{j-2}$$

which further implies

$$(\gamma - 1)s_{j-1} < s_{j-2} \iff s_{j-1} < \gamma s_{j-2}$$

since $\gamma = \frac{1}{\gamma-1}$. This proves $j - 1 \in \mathcal{C} \cup \mathcal{D}$. □

3. The run and gap between terms in an addition chain

We begin this section with the following definition.

Definition 3.1. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be steps in an addition chain as in the setup. We call a **maximal** consecutive sequence of steps of a given type

$$j_1 < j_2 < \cdots < j_d$$

such that $j_1 - 1, j_d + 1$ cannot be a step of the given type a **run** of the step type. We call the number of terms in the *run* the **run length**.

Lemma 2.2 hints at the core idea that a **run** of step type $\mathcal{C} \cup \mathcal{B}$ or $\mathcal{D} \cup \mathcal{B}$ will always appear among the non-doubling steps in any addition chain, whether or not optimal. More likely it is for chains that are not optimal to have many **run** of types $\mathcal{C} \cup \mathcal{B}$. We begin with the following Lemmas

Lemma 3.2. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. *Denote by* $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *the four step types as in the setup, with*

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

be a ***run*** *os a step of type* $\mathcal{B}$, *then*

$$\frac{1}{2}\gamma^{d-2} < s_{j_d} - s_{j_d-1} < 2^{d-2+\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$.

Proof. Consider a **run** of type $\mathcal{B}$ of the form

$$j_1 < \cdots < j_d.$$

Then $\gamma s_{j_d-1} \leq s_{j_d} < 2s_{j_d-1} \iff (\gamma-1)s_{j_d-1} \leq s_{j_d} - s_{j_d-1} < s_{j_d-1}$. We obtain (by induction)

$$(\gamma-1)\gamma^{d-2} < s_{j_d} - s_{j_d-1} < 2^{d-2+\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$. The inequality follows immediately since $\gamma - 1 > \frac{1}{2}$. □

Lemma 3.3. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$*. Denote by* $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ *the four step types as in the setup, with*

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

be a ***run*** *of a step of type* $\mathcal{C}$*, then*

$$\delta(1+\delta)^{d-2} < s_{j_d} - s_{j_d-1} < \gamma^{d-1}2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$*.*

Proof. Consider a **run** of type $\mathcal{C}$ of the form

$$j_1 < \cdots < j_d.$$

Then $(1+\delta)s_{j_d-1} \leq s_{j_d} < \gamma s_{j_d-1} \iff \delta s_{j_d-1} \leq s_{j_d} - s_{j_d-1} < (\gamma - 1)s_{j_d-1}$. By induction, we deduce

$$\delta(1+\delta)^{d-2} \leq s_{j_d} - s_{j_d-1} < \gamma^{d-1}2^{\alpha_o}$$

for some $\alpha := \alpha_o(d) > 0$. □

Lemma 3.4. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$*. Denote by* $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ *the four step types as in the setup, with*

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

be a ***run*** *of a step of type* $\mathcal{D}$*, then*

$$s_{j_d} - s_{j_d - 1} < \delta(1+\delta)^{d-2} 2^{\alpha_o}$$

for some $\alpha_o = \alpha_o(d) > 0$*.*

Proof. Consider a **run** of type $\mathcal{D}$ of the form

$$j_1 < \cdots < j_d.$$

Then $s_{j_d} < (1+\delta)s_{j_d-1} \iff s_{j_d} - s_{j_d-1} < \delta s_{j_d-1}$. By induction, we deduce

$$s_{j_d} - s_{j_d - 1} < \delta(1+\delta)^{d-2} 2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$. □

Lemma 3.5. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$*. Denote by* $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *the four step types as in the setup, with*

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

be a ***run*** *of a step of type* $\mathcal{B} \cup \mathcal{C}$ *such that there are* $d^{'}$ *and* $d^{''}$ *steps of type* $\mathcal{B}$ *and* $\mathcal{C}$*, respectively, in this* ***run*** *with* $d^{'} + d^{''} = d$*. If* $j_d \in \mathcal{B}$*, then*

$$\frac{1}{2}(1+\delta)^{d^{''}} \gamma^{d^{'}-1} < s_{j_d} - s_{j_d-1} < 2^{d^{'}-1+\alpha_o} \gamma^{d^{''}}$$

for some $\alpha_o := \alpha_o(d) > 0$*. On the other hand, if* $j_d \in \mathcal{C}$*, then*

$$\delta(1+\delta)^{d^{''}-1} \gamma^{d^{'}} < s_{j_d} - s_{j_d-1} < \gamma^{d^{''}} 2^{d^{'}+\alpha_o}$$

for some $\alpha_o = \alpha_o(d) > 0$*.*

Proof. Consider a **run** of type $\mathcal{B} \cup \mathcal{C}$ of the form

$$j_1 < \cdots < j_d.$$

In the case $j_d \in \mathcal{B}$, then $\gamma s_{j_d-1} \leq s_{j_d} < 2 s_{j_d-1} \iff (\gamma-1)s_{j_d-1} \leq s_{j_d} - s_{j_d-1} < s_{j_d-1}$. By induction, we deduce

$$\frac{1}{2}(1+\delta)^{d^{''}} \gamma^{d^{'}-1} < s_{j_d} - s_{j_d-1} < 2^{d^{'}-1+\alpha_o} \gamma^{d^{''}}$$

for some $\alpha_o := \alpha_o(d) > 0$. On the other hand, if $j_d \in \mathcal{C}$, we have

$$(1+\delta)s_{j_d-1} \leq s_{j_d} < \gamma s_{j_d-1} \iff \delta s_{j_d-1} \leq s_{j_d} - s_{j_d-1} < (\gamma-1)s_{j_d-1}.$$

By induction, we deduce

$$\delta(1+\delta)^{d^{''}-1}\gamma^{d^{'}} < s_{j_d}-s_{j_d-1} < \gamma^{d^{''}}2^{d^{'}+\alpha_o}$$

for some $\alpha_o=\alpha_o(d)>0$. □

Lemma 3.6. *Let*

$$E(n): s_o=1<s_1=2<\cdots<s_{\beta(m)}=n$$

be an addition chain leading to $n\in[2^m,2^{m+1})$. Denote by $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ the four step types as in the setup, with

$$\#\mathcal{A}:=A, \quad \#\mathcal{B}=B, \quad \#\mathcal{C}=C, \quad \#\mathcal{D}=D.$$

Furthermore, let

$$j_1<\cdots<j_d$$

*be a **run** of a step of type $\mathcal{B}\cup\mathcal{D}$ such that there are $d^{'}$ and $d^{''}$ steps of type $\mathcal{B}$ and $\mathcal{C}$, respectively, in this **run** with $d^{'}+d^{''}=d$. If $j_d\in\mathcal{B}$, then*

$$\frac{1}{2}\gamma^{d^{'}-1} < s_{j_d}-s_{j_d-1} < 2^{d^{'}-1+\alpha_o}(1+\delta)^{d^{''}}$$

for some $\alpha_o:=\alpha_o(d)>0$. On the other hand, if $j_d\in\mathcal{D}$, then

$$s_{j_d}-s_{j_d-1} < \delta(1+\delta)^{d^{''}-1}\gamma^{d^{'}}2^{\alpha_o}$$

for some $\alpha_o:=\alpha_o(d)>0$.

Proof. Consider a **run** of type $\mathcal{B}\cup\mathcal{D}$ of the form

$$j_1<\cdots<j_d.$$

In the case $j_d\in\mathcal{B}$, then

$$\gamma s_{j_d-1}\leq s_{j_d}<2s_{j_d-1} \iff (\gamma-1)s_{j_d-1}\leq s_{j_d}<s_{j_d-1}.$$

By induction, we deduce

$$\frac{1}{2}\gamma^{d^{'}-1} < s_{j_d}-s_{j_d-1} < 2^{d^{'}-1+\alpha_o}(1+\delta)^{d^{''}}$$

for some $\alpha_o:=\alpha_o(d)>0$. On the other hand, if $j_d\in\mathcal{D}$, then

$$s_{j_d}<(1+\delta)s_{j_d-1} \iff s_{j_d}-s_{j_d-1}<\delta s_{j_d-1}.$$

By induction, we deduce

$$s_{j_d}-s_{j_d-1} < \delta(1+\delta)^{d^{''}-1}\gamma^{d^{'}}2^{\alpha_o}$$

for some $\alpha_o:=\alpha_o(d)>0$. □

Lemma 3.7. *Let*

$$E(n): s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. *Denote by* $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *the four step types as in the setup, with*

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

be a ***run*** *of a step of type* $\mathcal{C} \cup \mathcal{D}$ *such that there are* $d^{'}$ *and* $d^{''}$ *steps of type* $\mathcal{C}$ *and* $\mathcal{D}$*, respectively, in this* ***run*** *with* $d^{'} + d^{''} = d$. *If* $j_d \in \mathcal{C}$*, then*

$$\delta(1+\delta)^{d^{'}} \leq s_{j_d} - s_{j_d - 1} < \gamma^{d^{'}}(1+\delta)^{d^{''}} 2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$. *On the other hand, if* $j_d \in \mathcal{D}$ *then*

$$s_{j_d} - s_{j_d - 1} < \delta(1+\delta)^{d^{''}-1}\gamma^{d^{'}} 2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$.

Proof. Consider a **run** of type $\mathcal{C} \cup \mathcal{D}$ of the form

$$j_1 < \cdots < j_d.$$

If $j_d \in \mathcal{C}$, then

$$(1+\delta)s_{j_d - 1} \leq s_{j_d} < \gamma s_{j_d - 1} \iff \delta s_{j_d - 1} \leq s_{j_d} - s_{j_d - 1} < (\gamma - 1)s_{j_d - 1}.$$

By induction, we deduce

$$\delta(1+\delta)^{d^{'}} \leq s_{j_d} - s_{j_d - 1} < \gamma^{d^{'}}(1+\delta)^{d^{''}} 2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$. On the other hand, if $j_d \in \mathcal{D}$, then

$$s_{j_d} < (1+\delta)s_{j_d - 1} \iff s_{j_d} - s_{j_d - 1} < \delta s_{j_d - 1}.$$

By induction, we deduce

$$s_{j_d} - s_{j_d - 1} < \delta(1+\delta)^{d^{''}-1}\gamma^{d^{'}} 2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$. □

Remark 3.8. From now on, we fix

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D$$

for an addition chain

$$E(n): s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

leading to $n \in [2^m, 2^{m+1})$. Consequently, the numbers $c, c', c'' > 0$ appearing in the theorems are absolute constants, because once the proportion of each step type is fixed, these do not vary with m. We establish explicit bounds for the gap between consecutive terms in a **run** of any type. In contrast, each appearance of α_o in the lemmas depended only on d, while each α_o in the theorem depends only on m.

Theorem 3.9. *Let*

$$E(n): s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the setup and fix

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type $\mathcal{C} \cup \mathcal{D}$. If $j_d \in \mathcal{D}$ then*

$$s_{j_d} - s_{j_d-1} < \exp\left((\frac{c''}{\log m} + c' \log \gamma)\beta(m) + \alpha_o \log 2 - \log\log m \right)$$

for some fixed $c'', c' > 0$ and $\alpha_o := \alpha_o(m) > 0$. On the other hand, if $j_d \in \mathcal{C}$ then

$$\exp\left(\frac{c'}{2\log m}\beta(m) - \log\log m \right) < s_{j_d} - s_{j_d-1} < \exp\left((\frac{c''}{\log m} + c' \log \gamma)\beta(m) + \alpha_o \log 2 \right)$$

for some fixed $c', c'' > 0$ and $\alpha_o := \alpha_o(m) > 0$.

Proof. Consider a **run** of type $\mathcal{C} \cup \mathcal{D}$ of the form

$$j_1 < \cdots < j_d.$$

In the case $j_d \in \mathcal{D}$, then Lemma 3.7 gives

$$s_{j_d} - s_{j_d-1} < \delta(1+\delta)^{d''-1}\gamma^{d'} 2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$. Since $\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D$ are each fixed and each term in this run is distributed in the step type $\mathcal{C} \cup \mathcal{D}$, there exist fixed constants $c', c'' > 0$ such that $d'' = c''\beta(m)$ and $d' = c'\beta(m)$. We can now write

$$\begin{aligned} s_{j_d} - s_{j_d-1} &< \exp(\log\delta + d''\log(1+\delta) + d'\log\gamma + \alpha_o(d)\log 2) \\ &\leq \exp(\log\delta + d''\delta + d'\log\gamma + \alpha_o(d)\log 2) \\ &= \exp\left(\frac{c''}{\log m}\beta(m) + c'(\log\gamma)\beta(m) + \alpha_o(m)\log 2 - \log\log m \right) \end{aligned}$$

with $\delta := \delta(m) = \frac{1}{\log m}$, which furnishes the upper bound in this case. In the case $j_d \in \mathcal{C}$, Lemma 3.7 gives

$$(1+\delta)s_{j_d-1} \leq s_{j_d} < \gamma s_{j_d-1} \iff \delta s_{j_d-1} \leq s_{j_d} - s_{j_d-1} < (\gamma - 1)s_{j_d-1}.$$

We deduce for the lower bound

$$\begin{aligned} s_{j_d}-s_{j_d-1} &\geq \exp(\log\delta+d'\log(1+\delta))\\ &\geq \exp(\frac{d'}{2}\delta+\log\delta)\\ &= \exp\left(\frac{c'}{2\log m}\beta(m)-\log\log m\right). \end{aligned}$$

Similarly, we deduce for the upper bound

$$\begin{aligned} s_{j_d}-s_{j_d-1} &< \exp(d'\log\gamma+d''\log(1+\delta)+\alpha_o(d)\log 2)\\ &\leq \exp(d'\log\gamma+d''\delta+\alpha_o(d)\log 2)\\ &= \exp\left(c'(\log\gamma)\beta(m)+\frac{c''}{\log m}\beta(m)+\alpha_o(m)\log 2\right) \end{aligned}$$

for some fix $c',c''>0$ and $\alpha_o(m)>0$. □

Theorem 3.10. *Let*

$$E(n): s_o=1<s_1=2<\cdots<s_{\beta(m)}=n$$

be an addition chain leading to $n\in[2^m,2^{m+1})$*. Denote by* $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ *the four types of steps as in the setup and fix*

$$\#\mathcal{A}:=A,\quad \#\mathcal{B}=B,\quad \#\mathcal{C}=C,\quad \#\mathcal{D}=D.$$

Furthermore, let

$$j_1<\cdots<j_d$$

be a ***run*** *of a step of type* $\mathcal{B}\cup\mathcal{D}$*. If* $j_d\in\mathcal{B}$ *then*

$$\exp(c'(\log\gamma)\beta(m)-\log 2\gamma)<s_{j_d}-s_{j_d-1}<\exp\left((\frac{c''}{\log m}+c'(\log 2))\beta(m)+\alpha_o(m)\log 2\right)$$

for some $c',c''>0$*. On the other hand, if* $j_d\in\mathcal{D}$ *then*

$$s_{j_d}-s_{j_d-1}<\exp\left((\frac{c''}{\log m}+c'\log\gamma)\beta(m)+\alpha_o(m)\log 2-\log\log m\right)$$

Proof. Consider a **run** of type $\mathcal{B}\cup\mathcal{D}$ of the form

$$j_1<\cdots<j_d.$$

In the case $j_d\in\mathcal{B}$, Lemma 3.6 gives

$$\frac{1}{2}\gamma^{d'-1}<s_{j_d}-s_{j_d-1}<2^{d'-1+\alpha_o}(1+\delta)^{d''}$$

for some $\alpha_o:=\alpha_o(d)>0$. Since $\#\mathcal{A}:=A,\quad \#\mathcal{B}=B,\quad \#\mathcal{C}=C,\quad \#\mathcal{D}=D$ are each fixed and each term in this run is distributed in the step type $\mathcal{B}\cup\mathcal{D}$, there exist fixed constants $c',c''>0$ such that $d''=c''\beta(m)$ and $d'=c'\beta(m)$. We obtain for the lower bound

$$\begin{aligned}s_{j_d}-s_{j_d-1}&\geq \exp(d'\log\gamma-\log 2\gamma)\\&=\exp(c'(\log\gamma)\beta(m)-\log 2\gamma).\end{aligned}$$

Similarly for the upper bound, we obtain

$$\begin{aligned}s_{j_d}-s_{j_d-1}&<2^{d'-1+\alpha_o}(1+\delta)^{d''}\\&<\exp\left(d'\log 2+d''\log(1+\delta)+\alpha_o(d)\log 2\right)\\&\leq\exp\left(d'\log 2+d''\delta+\alpha_o(d)\log 2\right)\\&=\exp\left((\frac{c''}{\log m}+c'\log 2)\beta(m)+\alpha_o(m)\log 2\right)\end{aligned}$$

for some fix $c',c''>0$. In the case $j_d\in\mathcal{D}$, Lemma 3.6 gives

$$s_{j_d}-s_{j_d-1}<\delta(1+\delta)^{d''-1}\gamma^{d'}2^{\alpha_o}$$

for some $\alpha_o:=\alpha_o(d)>0$. We deduce

$$\begin{aligned}s_{j_d}-s_{j_d-1}&<\exp\left(\log\delta+d''\log(1+\delta)+d'\log\gamma+\alpha_o(d)\log 2\right)\\&=\exp\left(\log\delta+d''\delta+d'\log\gamma+\alpha_o(d)\log 2\right)\\&=\exp\left((\frac{c''}{\log m}+c'\log\gamma)\beta(m)+\alpha_o(m)\log 2-\log\log m\right)\end{aligned}$$

which establishes the upper bound in this case. □

Theorem 3.11. *Let*

$$E(n):s_o=1<s_1=2<\cdots<s_{\beta(m)}=n$$

be an addition chain leading to $n\in[2^m,2^{m+1})$. *Denote by* $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ *the four types of steps as in the setup and fix*

$$\#\mathcal{A}:=A,\quad \#\mathcal{B}=B,\quad \#\mathcal{C}=C,\quad \#\mathcal{D}=D.$$

Furthermore, let

$$j_1<\cdots<j_d$$

*be a **run** of a step of type* $\mathcal{B}\cup\mathcal{C}$. *If* $j_d\in\mathcal{B}$, *then*

$$s_{j_d}-s_{j_d-1}>\exp\left((\frac{c''}{2\log m}+c'\log\gamma)\beta(m)-\log 2\gamma\right)$$

and

$$s_{j_d}-s_{j_d-1}<\exp\left((c'\log 2+c''\log\gamma)\beta(m)+\alpha_o(m)\log 2\right)$$

for some fixed $c', c'' > 0$. On the other hand, if $j_d \in \mathcal{C}$ then

$$s_{j_d} - s_{j_d-1} > \exp\left((\frac{c''}{2\log m} + c'\log\gamma)\beta(m) - \frac{1}{2\log m} - \log\log m \right)$$

and

$$s_{j_d} - s_{j_d-1} < \exp\left((c''\log\gamma + c'\log 2)\beta(m) + \alpha_o(m)\log 2\right)$$

for some $c', c'' > 0$.

Proof. Consider a **run** of type $\mathcal{B} \cup \mathcal{C}$ of the form

$$j_1 < \cdots < j_d.$$

In the case $j_d \in \mathcal{B}$, Lemma 3.5 gives

$$\frac{1}{2}(1+\delta)^{d''}\gamma^{d'-1} < s_{j_d} - s_{j_d-1} < 2^{d'-1+\alpha_o}\gamma^{d''}$$

for some $\alpha_o := \alpha_o(d) > 0$. Since $\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D$ are each fixed and each term in this run is distributed in the step type $\mathcal{B} \cup \mathcal{D}$, there exist fixed constants $c', c'' > 0$ such that $d'' = c''\beta(m)$ and $d' = c'\beta(m)$. For the lower bound, we can write

$$\begin{aligned} s_{j_d} - s_{j_d-1} &\geq \exp(d''\log(1+\delta) + (d'-1)\log\gamma - \log 2) \\ &\geq \exp(d''\frac{\delta}{2} + d'\log\gamma - \log 2\gamma) \\ &= \exp\left((\frac{c''}{2\log m} + c'\log\gamma)\beta(m) - \log 2\gamma \right) \end{aligned}$$

for a fixed $c', c'' > 0$. We have for the upper bound

$$\begin{aligned} s_{j_d} - s_{j_d-1} &< \exp(d'\log 2 + d''\log\gamma + \alpha_o(d)\log 2) \\ &= \exp\left((c'\log 2 + c''\log\gamma)\beta(m) + \alpha_o(m)\log 2\right) \end{aligned}$$

for fixed $c', c'' > 0$. In the case $j_d \in \mathcal{C}$ Lemma 3.5 gives

$$\delta(1+\delta)^{d''-1}\gamma^{d'} < s_{j_d} - s_{j_d-1} < \gamma^{d''}2^{d'+\alpha_o}$$

for some $\alpha_o = \alpha_o(d) > 0$. We can write for the lower bound

$$\begin{aligned} s_{j_d} - s_{j_d-1} &> \exp((d''-1)\log(1+\delta) + \log\delta + d'\log\gamma) \\ &\geq \exp(d''\frac{\delta}{2} + \log\delta - \frac{\delta}{2} + d'\log\gamma) \\ &= \exp\left((\frac{c''}{2\log m} + c'\log\gamma)\beta(m) - \frac{1}{2\log m} - \log\log m \right) \end{aligned}$$

for fixed $c', c' > 0$. Similarly, we have for the upper bound

$$\begin{aligned} s_{j_d} - s_{j_d - 1} &< \exp(d'' \log \gamma + d' \log 2 + \alpha_o(d) \log 2) \\ &= \exp\left((c'' \log \gamma + c' \log 2)\beta(m) + \alpha_o(m) \log 2\right) \end{aligned}$$

which establishes the upper in this case. □

Theorem 3.12. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. *Denote by* $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *the four types of steps as in the setup and fix*

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

be a ***run*** *of a step of type* $\mathcal{C}$. *Then*

$$s_{j_d} - s_{j-d-1} > \exp\left(\frac{c}{2\log m}\beta(m) - \frac{1}{\log m} - \log\log m\right)$$

and

$$s_{j_d} - s_{j_d - 1} < \exp(c(\log \gamma)\beta(m) + \alpha_o(m) \log 2)$$

for some fixed $c > 0$.

Proof. Consider a **run** of type $\mathcal{C}$ of the form

$$j_1 < \cdots < j_d.$$

Lemma 3.3 gives

$$\delta(1+\delta)^{d-2} < s_{j_d} - s_{j_d - 1} < \gamma^{d-1} 2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$. Since $\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D$ are each fixed and each term in this run is distributed in the step type $\mathcal{C}$, there exist fixed constants $c > 0$ such that $d = c\beta(m)$. For the lower bound, we can write

$$\begin{aligned} s_{j_d} - s_{j_d - 1} &> \exp(\log \delta + (d-2)\log(1+\delta)) \\ &\geq \exp(\log \delta + (d-2)\frac{\delta}{2}) \\ &= \exp\left((\frac{c}{2\log m}\beta(m) - \frac{1}{\log m} - \log\log m)\right) \end{aligned}$$

for a fixed constant $c > 0$. We have for the upper bound

$$\begin{aligned} s_{j_d} - s_{j_d - 1} &< \exp(d \log \gamma + \alpha_o(d) \log 2) \\ &= \exp(c(\log \gamma)\beta(m) + \alpha_o(m) \log 2) \end{aligned}$$

for a fixed constant $c > 0$. This establishes the inequality. □

Theorem 3.13. *Let*
$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$
be an addition chain leading to $n \in [2^m, 2^{m+1})$. *Denote by* $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *the four types of steps as in the setup and fix*
$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$
Furthermore, let
$$j_1 < \cdots < j_d$$
be a ***run*** *of a step of type* $\mathcal{D}$. *Then*
$$s_{j_d} - s_{j_d - 1} < \exp\left(\frac{c}{\log m}\beta(m) - \log\log m + \alpha_o(m)\log 2\right)$$
for a fixed constant $c > 0$.

Proof. Consider a **run** of type $\mathcal{C}$ of the form
$$j_1 < \cdots < j_d.$$
Lemma 3.4 gives
$$s_{j_d} - s_{j_d - 1} < \delta(1+\delta)^{d-2}2^{\alpha_o}$$
for some $\alpha_o = \alpha_o(d) > 0$. We can now write
$$\begin{aligned} s_{j_d} - s_{j_d - 1} &< \exp(\log\delta + d\log(1+\delta) + \alpha_o(d)\log 2) \\ &= \exp(\log\delta + d\delta + \alpha_o(d)\log 2) \\ &= \exp\left(\frac{c}{\log m}\beta(m) - \log\log m + \alpha_o(m)\log 2\right) \end{aligned}$$
which is the form of the upper bound. □

References

1. J.M. De Koninck, N. Doyon and W. Verreault *On the minimal length of addition chains*, arXiv preprint arXiv:2504.07332, 2025.

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