v.3

Double Stochastic Quantization:Non perturbative approach to Stochastic Quantization $\lambda\varphi_d^{2n}, n \geq 2, d \geq 4$ model Eucledean Quantum Field Theory

Jaykov Foukzon
Center for Mathematical Sciences, Israel Institution of Technology, Haifa, Israel.
E-mail: jaikovfoukzon@gmail.com
E-mail: jaykovfoukzon@list.ru
https://orcid.org/0000-0002-6837-4004
ID: INSPIRE-00004491

Abstract: In this paper, we show how the finite formulation of quantum field theory based on Langevin equations can be generalized to the case of *nonrenormalizable* theories.The 5th-time stochastic-quantization approach to field theory proposed by Parisi and Wu, is put in a path-integral form in [6]. The procedure of taking the limit $\tau \to \infty$ is analyzed and based on new grounds through the introduction of the vacuum-vacuum generating functional. In this paper non perturbative aprouch related to Parisi and Wu stochastic-quantization of the $\lambda\varphi_d^{2n}, n \geq 2, d \geq 4$ model quantum field theory is considered.

Content.

1.Introduction

Parisi and Wu' proposed the following alternative method to get the quantum averages [1]:

(i) Introduce a 5-th time τ, in addition to the usual four space-time t, and postulate the
following Langevin equation for the dynamics of the field $\varphi(\tau,x)$ in this extra time τ

$$\begin{gathered}\frac{\partial\varphi(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi]}{\delta\varphi(\tau,x)} + \eta(\tau,x),\\ \langle\eta(\tau,x)\rangle_\eta = 0,\\ \langle\eta(\tau,x)\,\eta(\tau',x')\rangle_\eta = 2\delta(\tau-\tau')(x-x'),\end{gathered} \tag{1.1}$$

where the angular brackets denote connected average with respect to the random variable η.

(ii) Evaluate the stochastic average of fields $\phi_\eta(\tau,x)$ satisfying Eq. (1.1), that means

$$\langle\phi_\eta(\tau_1,x_1)\phi_\eta(\tau_2,x_2)\ldots\phi_\eta(\tau_m,x_m)\rangle_\eta. \tag{1.2}$$

(iii) Put $\tau_1 = \tau_2 =\ldots= \tau_m = \tau$ in (1.2) and take the limit

$$\lim_{\tau\to\infty}\langle\phi_\eta(\tau,x_1)\phi_\eta(\tau,x_2)\ldots\phi_\eta(\tau,x_m)\rangle_\eta = G(x_1,x_2,\ldots,x_m) \tag{1.3}$$

It is possible to prove, at least perturbatively, that [6]

$$G(x_1,x_2,\ldots,x_m) = \frac{\int D[\varphi](\varphi(x_1)\varphi(x_2)\ldots\varphi(x_m))\exp\{-S[\varphi]\}}{\int D[\varphi]\exp\{-S[\varphi]\}}. \tag{1.4}$$

To understand this relation see ref.[6].
In this paper in particular we deal with double stochastic relaxation equations of the form:

$$\begin{aligned}\frac{\partial\varphi(\tau,x)}{\partial\tau} &= -\frac{\delta S[\varphi]}{\delta\varphi(\tau,x)} + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x),\\ \langle\eta(\tau,x)\rangle_\eta &= 0, \langle\widetilde{\eta}(\tau,x)\rangle_{\widetilde{\eta}_1} = 0\\ \langle\eta(\tau,x)\eta(\tau',x')\rangle_\eta &= 2\delta(\tau-\tau')(x-x'),\\ \langle\widetilde{\eta}(\tau,x)\widetilde{\eta}(\tau',x')\rangle_{\widetilde{\eta}} &= 2\delta(\tau-\tau')(x-x').\end{aligned} \tag{1.5}$$

Here $\eta(\tau,x) = \eta(\tau,x;\omega)$ is a space-time white noice on probability space $\Sigma = (\Omega,\mathcal{S},P)$ and $\widetilde{\eta}_{1,2}(\tau,x) = \widetilde{\eta}(\tau,x;\varpi)$ are space-time white noices on probability space $\widetilde{\Sigma} = (\widetilde{\Omega},\mathcal{S},P)$.

(ii) Evaluate the stochastic average of fields $\phi_\eta(\tau,x)$ satisfying Eq. (1.5), that means

$$\langle\phi_{\eta,\widetilde{\eta}}(\tau_1,x_1;\epsilon)\phi_{\eta,\widetilde{\eta}}(\tau_2,x_2;\epsilon)\ldots\phi_{\eta,\widetilde{\eta}}(\tau_m,x_m;\epsilon)\rangle_{\eta,\widetilde{\eta}}. \tag{1.6}$$

(iii) Put $\tau_1 = \tau_2 = \ldots = \tau_m = \tau$ in (1.2) and take the limit

$$\lim_{\tau\to\infty}\lim_{\epsilon\to 0}\langle\phi_{\eta,\widetilde{\eta}}(\tau,x_1;\epsilon)\phi_{\eta,\widetilde{\eta}}(\tau,x_2;\epsilon)\ldots\phi_{\eta,\widetilde{\eta}}(\tau,x_m;\epsilon)\rangle_{\eta,\widetilde{\eta}} = G(x_1,x_2,\ldots,x_m;\epsilon) \tag{1.7}$$

To understand this relation we have to introduce the notion of probability (density) $P(\varphi_\eta,\tau)$, that is, the probability (density) of having the system in the configuration $\varphi_\eta(\tau,x)$ at time τ. There exists for $P(\varphi_\eta,\tau)$ an equation that describes its evolution in the time τ. It is called the Stochastic Fokker-Planck (SFP)) equation and it has been derived in [8]:

$$\begin{aligned}\frac{\partial P[\varphi(\tau,x),\tau]}{\partial\tau} &= \int d^4x\frac{\delta}{\delta\varphi(\tau,x)}\left[P[\varphi(\tau,x),\tau]\frac{\delta S^\star[\varphi]}{\delta\varphi(\tau,x)}\right] + \int d^4x\frac{\delta^2 S^\star[\varphi]}{\delta\varphi^2(\tau,x)},\\ S^\star[\varphi] &= S[\varphi] - \int[\varphi(\tau,x)\eta(\tau,x)]d^4x.\end{aligned} \tag{1.8}$$

It is possible to rewrite this equation in a Schrodinger-type form:

$$\begin{aligned}\frac{\partial\Psi[\varphi(\tau,x),\tau]}{\partial\tau} &= -2\mathbf{H}\Psi[\varphi(\tau,x),\tau],\\ \Psi &= P[\varphi(\tau,x),\tau]\exp[S^\star[\varphi(\tau,x)]/2],\end{aligned} \tag{1.9}$$

where

$$\mathbf{H} = -\frac{1}{2}\frac{\delta^2}{\delta\varphi^2} + \frac{1}{8}\left[\frac{\delta S^{\bigstar}[\varphi]}{\delta\varphi}\right]^2 - \frac{1}{4}\frac{\delta^2 S^{\bigstar}[\varphi]}{\delta\varphi^2}. \qquad (1.10)$$

It is a positive semi-definite operator $H\Psi_n = E_n\Psi_n$ whis a ground state E_0 is $\Psi_0[\varphi(\tau,x),\tau] = \exp[S^{\bigstar}[\varphi(\tau,x)]/2]$. The solution of Eq.(1.9) is

$$\Psi[\varphi(\tau,x),\tau] = \sum_{n=0}^{\infty} c_n\Psi_n[\varphi(\tau,x),\tau]\exp(-2E_n\tau), \qquad (1.11)$$

where $\{c_n\}_{n=0}^{\infty}$ are normalizing constants. The probability density $P[\varphi(\tau,x),\tau]$ can be written as

$$P[\varphi(\tau,x),\tau] = \exp[S^{\bigstar}[\varphi(\tau,x)]/2]\sum_{n=0}^{\infty} c_n\Psi_n[\varphi(\tau,x),\tau]\exp(-2E_n\tau). \qquad (1.12)$$

In the limit $\tau \to \infty$ the only term that does not disappear in this expression is $\Psi_0[\varphi(\tau,x),\tau]$, so finally we have

$$\begin{aligned}\lim_{\tau\to\infty} P[\varphi(\tau,x),\tau] &= c_0\exp[-S^{\bigstar}[\varphi(\tau,x)]/2]\exp[-S^{\bigstar}[\varphi(\tau,x)]/2] = \\ &= c_0\exp[-S^{\bigstar}[\varphi(\tau,x)]].\end{aligned} \qquad (1.13)$$

This is the formal reason why Eq.(1.4) holds.

2.Non perturbative approach to Stochastic Quantization $\lambda\varphi^{2n}$ model Quantum Field Theory

2.1.The generating functional

In this paper we deal with a system of the double stochastic relaxation equations of the form:

$$
\begin{aligned}
\frac{\partial \varphi_{1,\epsilon}(\tau,x)}{\partial \tau} &= -\frac{\delta S[\varphi_1,\varphi_2]}{\delta \varphi_1(\tau,x)}\bigg|_{\varphi_1=\varphi_{1,\epsilon},\varphi_2=\varphi_{2,\epsilon}} + \eta(\tau,x) + \epsilon\widetilde{\eta}_1(\tau,x),\\
&\varphi_{1,\epsilon}(0,x) = 0, x \in \mathbb{R}^4, \tau \in \mathbb{R}_+,\\
\frac{\partial \varphi_{2,\epsilon}(\tau,x)}{\partial \tau} &= -\frac{\delta S[\varphi_1,\varphi_2]}{\delta \varphi_2(\tau,x)}\bigg|_{\varphi_1=\varphi_{1,\epsilon},\varphi_2=\varphi_{2,\epsilon}} - \eta(\tau,x) + \epsilon\widetilde{\eta}_2(\tau,x),\\
&\varphi_{2,\epsilon}(0,x) = 0, x \in \mathbb{R}^4, \tau \in \mathbb{R}_+,\\
&S[\varphi_1,\varphi_2] = \\
&\int_{\mathbb{R}^4} d^4x\left[\frac{1}{2}(\partial_\mu\varphi_1(\tau,x)\partial_\mu\varphi_1(\tau,x) + m^2\varphi_1(\tau,x)) + P(\varphi_1(\tau,x))\right] + \\
&\int_{\mathbb{R}^4} d^4x\left[\frac{1}{2}(\partial_\mu\varphi_2(\tau,x)\partial_\mu\varphi_2(\tau,x) + m^2\varphi_2(\tau,x)) + P(\varphi_2(\tau,x))\right] + \\
&+\int_{\mathbb{R}^4} d^4x[\gamma \times \varphi_1(\tau,x)\varphi_2(\tau,x)], \gamma > 0.
\end{aligned}
\tag{2.1.1}
$$

where $0 < \epsilon \ll 1$, $P(\cdot)$ is a polinomial degree $2k, k \geq 2$

$$
P(\varphi_{1,2}) = \int_{\mathbb{R}^4} d^4x[\gamma \times \varphi_1(\tau,x)\varphi_2(\tau,x)], \gamma > 0, \gamma \approx 0. \tag{2.1.2}
$$

and where $\eta(\tau,x;\omega), \widetilde{\eta}_1(\tau,x;\varpi)$,
$\widetilde{\eta}_2(\tau,x;\varpi)$ are Gaussian random variables such that

$$
\begin{aligned}
\langle \eta(\tau,x) \rangle_\eta &= 0,\\
\langle \widetilde{\eta}_1(\tau,x) \rangle_{\widetilde{\eta}_1} = 0, &\langle \widetilde{\eta}_2(\tau,x) \rangle_{\widetilde{\eta}_2} = 0,
\end{aligned}
\tag{2.1.3}
$$

and for the two-point correlation function associated with the random noises fields

$$
\begin{aligned}
\langle \eta(\tau,x)\,\eta(\tau',x') \rangle_\eta &= 2\delta(\tau-\tau')(x-x'),\\
\langle \widetilde{\eta}_1(\tau,x)\,\widetilde{\eta}_1(\tau',x') \rangle_{\widehat{\eta}} &= 2\delta(\tau-\tau')(x-x'),\\
\langle \widetilde{\eta}_2(\tau,x)\,\widetilde{\eta}_2(\tau',x') \rangle_{\widehat{\eta}} &= 2\delta(\tau-\tau')(x-x').
\end{aligned}
\tag{2.1.4}
$$

Remark 2.1.1. Here $\eta(\tau,x) = \eta(\tau,x;\omega)$ is a space-time white noice on probability space $\Sigma = (\Omega, \mathcal{S}, P)$ and $\widetilde{\eta}_{1,2}(\tau,x) = \widetilde{\eta}_{1,2}(\tau,x;\varpi)$ are space-time white noices on probability space $\widetilde{\Sigma} = (\widetilde{\Omega}, \mathcal{S}, P)$. The angular brackets denote connected average with respect to the random variables $\eta, \widetilde{\eta}_{1,2}$.
We want to build a generating functional $Z_\epsilon[J_1,J_2]$ from which the correlations

$$\langle \varphi_{1,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{1,\eta}(\tau_l,x_l;\omega)\varphi_{2,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{2,\eta}(\tau_l,x_r;\omega)\rangle_\eta$$

$$\frac{\partial\varphi_{1,\eta}(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi_1,\varphi_2]}{\delta\varphi_1(\tau,x)}\bigg|_{\varphi_1=\varphi_{1,\eta},\varphi_2=\varphi_{2,\eta}} + \eta(\tau,x),$$

$$\frac{\partial\varphi_{2,\eta}(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi_2,\varphi_2]}{\delta\varphi_2(\tau,x)}\bigg|_{\varphi_1=\varphi_{1,\eta},\varphi_2=\varphi_{2,\eta}} + \eta(\tau,x)$$

can be derived by the following fashion:

$$\begin{gathered}\langle \varphi_{1,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{1,\eta}(\tau_l,x_l;\omega)\varphi_{2,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{2,\eta}(\tau_l,x_r;\omega)\rangle_\eta = \\ \lim_{\epsilon\to 0}\langle \varphi_{1,\eta,\epsilon\tilde{\eta}_1}(\tau_1,x_1;\omega,\varpi)\times\ldots\times\varphi_{1,\eta,\epsilon\tilde{\eta}_1}(\tau_l,x_l;\omega,\varpi)\varphi_{2,\eta,\epsilon\tilde{\eta}_2}(\tau_1,x_1;\omega,\varpi)\times\ldots \\ \ldots\times\varphi_{2,\eta,\epsilon\tilde{\eta}_2}(\tau_l,x_r;\omega,\varpi)\rangle_\eta = \\ =\lim_{\epsilon\to 0}\frac{\delta^l Z_\epsilon[J_1,J_2;\omega]}{\delta J_1(\tau_1,x_1)\ldots\delta J_1(\tau_l,x_l)\delta J_2(\tau_1,x_1)\ldots\delta J_2(\tau_r,x_r)}\bigg|_{J_1=0,J_2=0}\end{gathered} \tag{2.1.5}$$

By canonical definition [6] one obtains

$$\begin{gathered}Z_\epsilon[J_1,J_2;\omega] = \\ N\int D[\eta(\tau',x;\omega)]\Big(\Big\{\int D[\varphi_1(\tau',x;\omega)]D[\varphi_2(\tau',x;\omega)]D[\tilde{\eta}_1(\tau',x;\varpi)]D[\tilde{\eta}_2(\tau',x;\varpi)] \\ \delta(\varphi_1(0,x;\omega))\delta(\varphi_2(0,x;\omega))\delta(\varphi_1(\tau',x;\omega)-\varphi_{1,\eta,\epsilon\tilde{\eta}_1,\epsilon\tilde{\eta}_2})\delta(\varphi_2(\tau',x;\omega)-\varphi_{2,\eta,\epsilon\tilde{\eta}_1,\epsilon\tilde{\eta}_2})\times \\ \times\exp\Big[-\int_0^\tau\int_{\mathbb{R}^4}J_1(\tau',x)\varphi_1(\tau',x;\omega)d^4xd\tau'\Big]\exp\Big[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\tilde{\eta}_1^2(\tau',x;\varpi)d^4xd\tau'\Big]\times \\ \times\exp\Big[-\int_0^\tau\int_{\mathbb{R}^4}J_2(\tau',x)\varphi_2(\tau',x;\omega)d^4xd\tau'\Big]\exp\Big[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\tilde{\eta}_2^2(\tau',x;\varpi)d^4xd\tau'\Big]\Big\}\times \\ \exp\Big[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\Big]\Big),\end{gathered} \tag{2.1.6}$$

where $\varphi_{1,2,\eta,\tilde{\eta}} = \varphi_{1,2}(\tau',x;\omega,\varpi)$ that appears in Eq.(2.1.1)-Eq.(2.1.2) is the solution of the double stochastic Langevin equations (2.1.1), solved with zero initial condidion: $\varphi_{1,2,\eta,\tilde{\eta}}(0,x;\omega,)\equiv 0$ and N is a normalizing constant and

$$D[\varphi_{1,2};\omega] = \lim_{M\to\infty}\prod_{i=0}^{M}D[\varphi_{1,2,\tau_i}(x;\omega)] \tag{2.1.7}$$

where $\varphi_{1,2,\tau_i}(x;\omega)$, are the field configurations at the time τ_i, having sliced the interval 0 to τ in M infinitesimal parts ε with $\tau_i = i\varepsilon$ and $D[\varphi_{1,2,\tau_i}(x;\omega)]$ is Feinman random measure, that is a product of the usual four-dimensional Feinman random measures, see Apendix 4 for rigorous definitions. Formally such random measure defined by

$$D[\varphi_{1,2,\tau_i}(x;\omega)] = \prod_{x\in\mathbb{R}^4} d[\varphi_{1,2,\tau_i}(x;\omega)]. \quad (2.1.7')$$

We sort of define it in the following way

$$\prod_{x\in\mathbb{R}^4} d[\varphi_{1,2,\tau_i}(x;\omega)] \triangleq \lim_{\delta\to 0} \prod_{x_i} d[\varphi_{1,2,\tau_i}(x_i;\omega)], \quad (2.1.7'')$$

where the points x_i belong to a lattice of size δ. Integration over random measure $d[\varphi_{1,2,\tau_i}(x_i;\omega)]$ see Apendix 4. One will first consider a finite volume for this lattice, then take the limit of infinite volume.

Abbreviation 2.1.1. $D[\varphi_{1,2};\omega] \triangleq D[\varphi_1;\omega]D[\varphi_2;\omega], D\left[\tilde{\eta}_{1,2}(\tau',x;\omega)\right] \triangleq D[\tilde{\eta}_1(\tau',x;\omega)] \times D[\tilde{\eta}_2(\tau',x;\omega)], \delta\left(\varphi_{1,2} - \varphi_{1,2,\eta,\epsilon\tilde{\eta}_{1,2}}\right) \triangleq \delta(\varphi_1 - \varphi_{1,\eta,\epsilon\tilde{\eta}_1})\delta(\varphi_2 - \varphi_{2,\eta,\epsilon\tilde{\eta}_2})$,
$J_{1,2}(\tau',x)\varphi_{1,2}(\tau',x;\omega) = J_1(\tau',x)\varphi_1(\tau',x;\omega) + J_2(\tau',x)\varphi_2(\tau',x;\omega)$, etc.

The delta function $\delta(\varphi_{1,2} - \varphi_{1,2,\eta,\tilde{\eta}})$ in Eq.(2.1.6) we can write as

$$\delta(\varphi_{1,2} - \varphi_{1,2,\eta,\tilde{\eta}}) = \delta\left[\frac{\partial\varphi_{1,2,\epsilon}}{\partial\tau} + \frac{\delta\tilde{S}}{\delta\varphi_{1,2}}\Big|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} - \epsilon\tilde{\eta}_{1,2}\right]\left\|\frac{\delta(\epsilon\tilde{\eta})_{1,2}}{\delta\varphi_{1,2,\epsilon}}\right\|, \quad (2.1.8)$$

where $\tilde{S} = S - \int_{\mathbb{R}^4}(\varphi_{1,2}\times\eta)d^4x$ and where $\left\|\delta\tilde{\eta}_{1,2}/\delta\varphi_{1,2,\epsilon}\right\|$ is the Jacobian matrix of the transformation $\tilde{\eta}_{1,2} \to \varphi_{1,2,\epsilon}$, that is

$$\left\|\frac{\delta\left(\epsilon\tilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}}\right\| = N_\epsilon\left\|\frac{\delta\left(\tilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}}\right\|,$$

$$\det\left\|\frac{\delta\left(\tilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}}\right\| = N_\epsilon^{-1}\det\left[\left[\partial_\tau + \frac{\delta^2\tilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\Big|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\delta(\tau-\tau')\right], \quad (2.1.9)$$

$$N_\epsilon = \prod_{i=1}^{\infty} x_i, x_i = \epsilon, i = 1,2,\ldots$$

From Eq.(2.1.9) by canonical calculation we get

$$\det\left\|\frac{\delta\left(\tilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}}\right\| =$$

$$= N_\epsilon^{-1}\exp\left[\mathbf{tr}\ln\left[\left[\partial_\tau + \frac{\partial^2\tilde{S}}{\partial\varphi_{1,2}(\tau)\partial\varphi_{1,2}(\tau')}\Big|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\delta(\tau-\tau')\right]\right] = \quad (2.1.10)$$

$$= N_\epsilon^{-1}\exp\left[\mathbf{tr}\ln\partial_\tau\left[\delta(\tau-\tau') + \partial_\tau^{-1}\left(\frac{\partial^2\tilde{S}}{\partial\varphi_{1,2}(\tau)\partial\varphi_{1,2}(\tau')}\Big|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right)\right]\right],$$

where ∂_τ^{-1} indicate the Geen's function $G(\tau-\tau')$ that satisfies

$$\partial_\tau G(\tau-\tau') = \delta(\tau-\tau'). \quad (2.1.11)$$

The solutions of the Eq.(2.1.11) are: (i) if we choose propagation forward in time

$$G(\tau-\tau') = \theta(\tau-\tau') \tag{2.1.12}$$

(ii) if we choose propagation backward in time

$$G(\tau-\tau') = -\theta(\tau-\tau') \tag{2.1.13}$$

In case, propagation forward in time, we get

$$\det\left\| \frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| = N_\epsilon^{-1}\exp\left\{\mathbf{tr}\left[\ln\partial_\tau + \ln\left[\delta(\tau-\tau') + \theta(\tau-\tau')\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\right]\right\}$$
$$= N_\epsilon^{-1}\exp(\mathbf{tr}\ln\partial_\tau)\exp\left[\mathbf{tr}\ln\left[\delta(\tau-\tau') + \theta(\tau-\tau')\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\right]. \tag{2.1.14}$$

The term $\exp(\mathbf{tr}\ln\partial_\tau)$ can be dropped, as it cancels with the same term in the denominator of (2.1.6), once we normalize $\widetilde{Z}_\epsilon[J_{1,2};\omega] = Z_\epsilon[J_{1,2};\omega]/Z_\epsilon[0,0;\omega]$.

Abbreviation 2.1.2.$Z_\epsilon[J_{1,2};\omega] \triangleq Z_\epsilon[J_1,J_2;\omega]$

So in Eq.(2.1.14) we are left with

$$\det\left\| \frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| = N_\epsilon^{-1}\exp\left[\mathbf{tr}\ln\left[\delta(\tau-\tau') + \theta(\tau-\tau')\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\right]. \tag{2.1.15}$$

By using the canonical expansion for the $\ln$, we obtain

$$\det\left\| \frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| = N_\epsilon^{-1}\exp\left[\mathbf{tr}\left[\theta(\tau-\tau')\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} + \theta(\tau-\tau')\theta(\tau'-\tau)\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau')\delta\varphi_{1,2}(\tau)}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} + \ldots\right]\right] =$$
$$= N_\epsilon^{-1}\exp\left[\int d\tau\int_{\mathbb{R}^4} d^4x\theta(0)\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}^2(\tau)}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} + \int d\tau'\int_{\mathbb{R}^4} d^4x\theta(\tau-\tau')\theta(\tau'-\tau)\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau')\delta\varphi_{1,2}(\tau)}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} + \ldots\right] \tag{2.1.16}$$

The second term in this expression is zero because $\theta(\tau-\tau')\theta(\tau'-\tau)=0$ and the

same for all the subsequent terins. The only one left is the first term and choosing $\theta(0)=1/2$ we get

$$\det\left\|\frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}}\right\| = N_{\epsilon}^{-1}\exp\left[\frac{1}{2}\int d\tau'\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2,\epsilon}^2}\right]. \quad (2.1.17)$$

Inserting Eq.(2.1.17) and Eq.(2.1.8) into Eq.(2.1.6) and performing the $\widetilde{\eta}$ integration, we get

$$\begin{aligned} Z_{\epsilon}[J_1,J_2;\omega] &= N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\times \\ &\times\exp\left\{-\int_0^{\tau}\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau}+\frac{\delta\widetilde{S}}{\delta\varphi_1(\tau',x;\omega)}\right]^2-\frac{1}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_1^2(\tau',x;\omega)}\right]d^4xd\tau'\right\} \\ &\times\exp\left\{-\int_0^{\tau}\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau}+\frac{\delta\widetilde{S}}{\delta\varphi_2(\tau',x;\omega)}\right]^2-\frac{1}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_2^2(\tau',x;\omega)}\right]d^4xd\tau'\right\} \\ &\times\exp\left\{-\int_0^{\tau}\int_{\mathbb{R}^4}J_{1,2}(\tau',x)\varphi_{1,2}(\tau',x;\omega)d^4xd\tau'\right\}\exp\left[-\frac{1}{4}\int_0^{\tau}\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right]. \end{aligned} \quad (2.1.18)$$

From Eq.(2.1.18) finally we obtain

$$\begin{aligned} Z_{\epsilon}[J_{1,2};\omega] &= N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\times \\ &\times\exp\left\{-\int_0^{\tau}\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta\right]^2-\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_1^2}\right]d^4xd\tau'\right. \\ &-\int_0^{\tau}\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta\right]^2-\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_2^2}\right]d^4xd\tau'- \\ &\left.-\int_0^{\tau}\int_{\mathbb{R}^4}J_{1,2}(\tau',x)\varphi_{1,2}(\tau',x;\omega)d^4xd\tau'\right\}\times\exp\left[-\frac{1}{4}\int_0^{\tau}\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right]. \end{aligned} \quad (2.1.19)$$

If we want also to specify that we are interested only in the correlations at the same 5-th time τ_1, we have just to choose $J_{1,2}(x,\tau')$ of the form $J_{1,2}(x,\tau')=\bar{J}(x)\delta(\tau'-\tau_1),\ \tau_1<\tau$ and Eq.(2.1.19) then becomes

$$\begin{aligned} Z_{\epsilon}[J_{1,2};\omega] &= N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\times \\ &\times\exp\left\{-\int_0^{\tau}\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta\widetilde{S}}{\delta\varphi_1(\tau',x;\omega)}\right]^2-\frac{1}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_1^2(\tau',x;\omega)}\right]d^4xd\tau'\right. \\ &-\int_0^{\tau}\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta\widetilde{S}}{\delta\varphi_2(\tau',x;\omega)}\right]^2-\frac{1}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_2^2(\tau',x;\omega)}\right]d^4xd\tau'- \\ &\left.-\int_{\mathbb{R}^4}J_{1,2}(\tau_1,x)\varphi_{1,2}(\tau_1,x;\omega)d^4x\right\}\times\exp\left[-\frac{1}{4}\int_0^{\tau}\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right]. \end{aligned} \quad (2.1.20)$$

Remark 2.1.2. In all this we have to remember, of course, that once we set $\tau_1 \to \infty$ we

have also to extend the interval of integration from $[0,\tau]$ to $[0,\infty]$.

From Eq.(2.1.21) with $\epsilon \ll 1$ for two-point correlation function $\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle$ defined by

$$\langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2)\rangle = \lim_{\epsilon\to 0}\left[\frac{\delta^l Z_\epsilon[J_1,J_2;\omega]}{\delta J_1(\tau_1,x_1)\delta J_2(\tau_2,x_2)}\right] \tag{2.1.21'}$$

for mutually two-point correlation function $\langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2)\rangle_\eta$ we get

$$\begin{gathered}\langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2)\rangle_\eta \asymp \langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2);\epsilon\rangle \triangleq\\ \triangleq N_\epsilon^{-1}\int D[\eta(\tau',x;\omega)]\exp\left[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right]\times\\ \left(\int D[\varphi_{1,2}(\tau',x;\omega)](\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega))\times\right.\\ \delta(\varphi_{1,2}(0,x;\omega))\times\\ \times\exp\left\{-\frac{1}{4\epsilon^2}\left[\int_0^\tau\int_{\mathbb{R}^4}\left[\frac{\partial\varphi(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2-\right.\right.\\ \left.\left.\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_1^2(\tau',x;\omega)}\right]d^4xd\tau'\right\}\\ \times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^\tau\int_{\mathbb{R}^4}\left[\frac{\partial\varphi(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2-\right.\\ \left.\left.\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_2^2(\tau',x;\omega)}\right]d^4xd\tau'\right\}\right).\end{gathered} \tag{2.1.22}$$

Performing the $\varphi_{1,2}(\tau',x;\omega)$ integration in Eq.(2.1.22) by using saddle point approximation we get

$$
\begin{aligned}
&\langle \varphi_{1,\eta}(\tau_1,x_1)\varphi_{2,\eta}(\tau_2,x_2)\rangle_\eta \asymp \\
&\det\left\| \frac{\delta(\varphi_1)}{\delta\eta} \right\| \det\left\| \frac{\delta(\varphi_2)}{\delta\eta} \right\| \int D[\eta(\tau',x;\omega)](\varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_2,x_2;\omega)) \times \\
&\times \exp\left\{ -\int_0^\tau \left[-\frac{1}{2} \frac{\delta^2 S}{\delta\varphi_1^2(\tau',x;\omega)} \bigg|_{\varphi_{1,2}=\varphi_{1,2,\eta}} \right] d^4x d\tau' \right\} \times \\
&\exp\left\{ -\int_0^\tau \left[-\frac{1}{2} \frac{\delta^2 S}{\delta\varphi_2^2(\tau',x;\omega)} \bigg|_{\varphi_{1,2}=\varphi_{1,2,\eta}} \right] d^4x d\tau' \right\} \\
&\exp\left[-\frac{1}{4} \int_0^\tau \eta^2(\tau',x;\omega) d^4x d\tau' \right] = \\
&\int D[\eta(\tau',x;\omega)](\varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_2,x_2;\omega)) \exp\left[-\frac{1}{4} \int_0^\tau \eta^2(\tau',x;\omega) d^4x d\tau' \right] \asymp \\
&\asymp \langle \varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_1,x_2;\omega)\rangle_\eta
\end{aligned} \tag{2.1.23}
$$

$\varphi_{1,2,\eta}(\tau_1,x_1;\omega)$ that appears in Eq.(2.1.23) is the solution of the Langevin equation (2.1.26), solved with zero initial condition. From Eq.(2.1.22)-Eq.(2.1.23) we get

$$
\begin{aligned}
&\langle \varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega)\rangle_\eta \simeq \\
&N \int D[\varphi_{1,2}(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega)) \\
&\left(\int D[\eta(\tau',x;\omega)](\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega)) \exp\left[-\frac{1}{4} \int_0^\tau \int_{\mathbb{R}^4} \eta^2(\tau',x;\omega) d^4x d\tau' \right] \right) \times \\
&\times \exp\left\{ -\int_0^\tau \left[\frac{1}{4\epsilon^2} \left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'} + \frac{\delta S}{\delta\varphi_1(\tau',x;\omega)} - \eta(\tau',x;\omega) \right]^2 \right] d^4x d\tau' \right\} \\
&\times \exp\left\{ -\int_0^\tau \left[\frac{1}{4\epsilon^2} \left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'} + \frac{\delta S}{\delta\varphi_2(\tau',x;\omega)} - \eta(\tau',x;\omega) \right]^2 \right] d^4x d\tau' \right\}.
\end{aligned} \tag{2.1.24}
$$

From Eq.(2.1.24) finally we get

$$\langle \varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_2,x_2;\omega)\rangle_\eta \simeq \langle \varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2);\epsilon\rangle \triangleq$$

$$\triangleq N_\epsilon^{-1}\int D[\varphi_{1,2}(\tau',x;\omega)]\langle \varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega)\rangle_\eta \times$$

$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_1}\int_{\mathbb{R}^4}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2 d^4x d\tau'\right\}$$

$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_2}\int_{\mathbb{R}^4}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2 d^4x d\tau'\right\} \quad (2.1.25)$$

where

$$\langle \varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega)\rangle_\eta \triangleq$$

$$\int D[\eta(\tau',x;\omega)](\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega))\exp\left[-\frac{1}{4}\int_0^{\tau}\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4x d\tau'\right]$$

Proposition 2.1.1. It follows from Eq.(2.1.22)-Eq.(2.1.23) that in Eq.(2.1.22) we can interchange integration on variable $\eta(\tau',x;\omega)$ and integration on variables $\varphi_{1,2}(\tau',x;\omega)$ in Eq.(1.24)-Eq.(2.1.25)

Remark 2.1.3.Note that for any fixed values of parameters τ_1,x_1,τ_2,x_2 and $\gamma \simeq 0$ we get

$$\langle[\varphi_1(\tau_1,x_1;\omega)-a][\varphi_2(\tau_2,x_2;\omega)+a];\epsilon\rangle_\eta \simeq \langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega);\epsilon\rangle_\eta - a^2, \quad (2.1.26)$$

where by translation invariance

$$\langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega);\epsilon\rangle_\eta - a^2 \simeq 0 \Rightarrow a \simeq a(\tau_1,\tau_2,x_1-x_2). \quad (2.1.27)$$

From Eq.(2.1.25) by the replacement

$$\begin{aligned}
\varphi_1(\tau,x;\omega)-\theta(\tau)a &= \nu_-(\tau,x;\omega),\\
\varphi_2(\tau,x;\omega)+\theta(\tau)a &= \nu_+(\tau,x;\omega),\\
\varphi_1(\tau,x;\omega) &= \nu_-(\tau,x;\omega)+\theta(\tau)a,\\
\varphi_2(\tau,x;\omega) &= \nu_+(\tau,x;\omega)-\theta(\tau)a,\\
\frac{\partial\varphi_1(\tau,x;\omega)}{\partial\tau} &= \frac{\partial\nu_-(\tau,x;\omega)}{\partial\tau}+\delta(\tau)a,\\
\frac{\partial\varphi_2(\tau,x;\omega)}{\partial\tau} &= \frac{\partial\nu_+(\tau,x;\omega)}{\partial\tau}-\delta(\tau)a,
\end{aligned} \quad (2.1.28)$$

we obtain

$$\Omega(\tau_1,\tau_2,x_1-x_2,a;\epsilon) \triangleq$$

$$N_\epsilon^{-1}\int D[\varphi_{1,2}(\tau',x;\omega)]\langle[\varphi_1(\tau_1,x_1;\omega)-a][\varphi_2(\tau_2,x_2;\omega)+a]\rangle_\eta \times$$

$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_1}\int_{\mathbb{R}^4}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2 d^4x d\tau'\right\}\times$$

$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_2}\int_{\mathbb{R}^4}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2 d^4x d\tau'\right\} =$$

$$= N_\epsilon^{-1}\int D[\nu_-(\tau',x;\omega)]D[\nu_+(\tau',x;\omega)]\langle\nu_-(\tau_1,x_1)\nu_+(\tau_1,x_1)\rangle_\eta \times \tag{2.1.29}$$

$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_1}\int_{\mathbb{R}^4}\left[\frac{\partial\nu_-(\tau',x;\omega)}{\partial\tau'}+\delta(\tau)a+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}\right|_{\varphi_1=\nu_-(\tau',x;\omega)+a_-}\right.$$

$$\left.-\eta(\tau',x;\omega)d^4xd\tau'\right]^2\Big\}\times$$

$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_2}\int_{\mathbb{R}^4}\left[\frac{\partial\nu_+(\tau',x;\omega)}{\partial\tau'}-\delta(\tau)a+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}\right|_{\varphi_2=\nu_+(\tau_1,x_1;\omega)-a}\right.$$

$$\left.-\eta(\tau',x;\omega)d^4xd\tau'\right\}.$$

Remark 2.1.4.Note that

$$\lim_{\epsilon\to0}\Omega(\tau_1,\tau_2,x_1-x_2,a;\epsilon)=0 \Rightarrow \lim_{\epsilon\to0}\langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2);\epsilon\rangle-a^2=0. \tag{2.1.30}$$

Definition 2.1.1. Let $\nu_\mp(\tau,x;\omega)$ be the solution of the Langevin equations (2.1.31)

$$\begin{aligned}
\frac{\partial\nu_-(\tau,x;\omega)}{\partial\tau} &= -\delta(\tau)a-\frac{\delta S[\varphi_{1,2}]}{\delta\varphi_1(\tau,x;\omega)}\Bigg|_{\substack{\varphi_1=\nu_-(\tau',x;\omega)+a_-\\ \varphi_2=\nu_+(\tau_1,x_1;\omega)-a}}+\eta(\tau,x;\omega),\\
\frac{\partial\nu_+(\tau,x;\omega)}{\partial\tau} &= \delta(\tau)a-\frac{\delta S[\varphi_{1,2}]}{\delta\varphi_2(\tau,x;\omega)}\Bigg|_{\substack{\varphi_1=\nu_-(\tau',x;\omega)+a_-\\ \varphi_2=\nu_+(\tau_1,x_1;\omega)-a}}+\eta(\tau,x;\omega),\\
\nu_\mp(0,x;\omega) &= 0,
\end{aligned} \tag{2.1.31}$$

Linear stochastic differental *master equation* corresponding to the Langevin equations
(2.1.31) reads

$$\begin{aligned}
\frac{\partial\nu_-(\tau,x,a;\omega)}{\partial\tau} &= -\delta(\tau)a-\mathcal{L}\left\{\frac{\delta S[\varphi_{1,2}]}{\delta\varphi_1(\tau,x;\omega)}\Bigg|_{\varphi_1=\nu_-(\tau',x;\omega)+a_-}\right\}+\eta(\tau,x;\omega),\\
\frac{\partial\nu_+(\tau,x,a;\omega)}{\partial\tau} &= \delta(\tau)a-\mathcal{L}\left\{\frac{\delta S[\varphi_{1,2}]}{\delta\varphi_2(\tau,x;\omega)}\Bigg|_{\varphi_2=\nu_+(\tau_1,x_1;\omega)-a}\right\}+\eta(\tau,x;\omega),\\
\nu_\mp(0,x,a;\omega) &= 0,
\end{aligned} \tag{2.1.32}$$

where

$$\mathcal{L}\left\{\delta S[\varphi_{1,2}]/\delta\varphi_1(\tau,x;\omega)|_{\substack{\varphi_1=v_-(\tau',x;\omega)+a_-\\ \varphi_2=v_+(\tau_1,x_1;\omega)-a}}\right\} \quad (2.1.33)$$

is a linear part of variational derivative

$$\delta S[\varphi_{1,2}]/\delta\varphi_1(\tau,x;\omega)|_{\substack{\varphi_1=v_-(\tau',x;\omega)+a_-\\ \varphi_2=v_+(\tau_1,x_1;\omega)-a}} \quad (2.1.34)$$

and where

$$\mathcal{L}\left\{\delta S[\varphi_{1,2}]/\delta\varphi_2(\tau,x;\omega)|_{\substack{\varphi_1=v_-(\tau',x;\omega)+a_-\\ \varphi_2=v_+(\tau_1,x_1;\omega)-a}}\right\}. \quad (2.1.35)$$

is a linear part of variational derivative

$$\delta S[\varphi_{1,2}]/\delta\varphi_2(\tau,x;\omega)|_{\substack{\varphi_1=v_-(\tau',x;\omega)+a_-\\ \varphi_2=v_+(\tau_1,x_1;\omega)-a}} \quad (2.1.36)$$

2.2.Transcendental master equation corresponding to two-point Green function $G(x_1,x_2,\lambda)$.

Definition 2.2.1.Let $v_{\mp}(\tau,x,a;\omega)$ be the solution of the *stochastic differental master equations* (2.1.32). Transcendental *master equation* corresponding to the stochastic Langevin equation (2.1.31) reads

$$\langle v_-(\tau_1,x_1,a;\omega)v_+(\tau_2,x_2,a;\omega)\rangle_\eta = 0. \quad (2.2.1)$$

Theorem 2.2.1.Let $a_{1,2}(\bar{\tau},\bar{x})$ be an solution of the equation (2.2.1) at fixed point $(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2) \in (\mathbb{R}_+\times\mathbb{R}^4)\times(\mathbb{R}_+\times\mathbb{R}^4)$ i.e.,

$$\langle v_-(\bar{\tau}_1,\bar{x}_1,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\omega)v_+(\bar{\tau}_2,\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\omega)\rangle_\eta = 0. \quad (2.2.2)$$

Let $\Delta(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2)$ be a set such that

$$\begin{gathered} a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2) \in \Delta(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2) \Leftrightarrow \\ \Leftrightarrow \langle v_-(\bar{\tau}_1,\bar{x}_1,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\omega)v_+(\bar{\tau}_2,\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\omega)\rangle_\eta = 0, \end{gathered} \quad (2.2.3)$$

and let $\widetilde{\Omega}(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2))$ be a set such that

$$\begin{gathered} a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2) \in \widetilde{\Omega}(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2)) \Leftrightarrow \\ \Leftrightarrow \underline{\lim}_{\epsilon\to 0}\Omega(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\epsilon) = 0, \end{gathered} \quad (2.2.4)$$

where the quantity $\Omega(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\epsilon)$ defined by Eq.(2.1.29). Then

$$\widetilde{\Omega}(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2)) \subseteq \Delta(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2). \quad (2.2.5)$$

2.3. Double Stochastic Quantization the Free Scalar

Fields

For a scalar field theory governed by the action in terms of the Euclidean spacetime is
given by

$$S_E = \int d^4x \left[\frac{1}{2}(\partial\varphi(x))^2 + \frac{1}{2}(m\varphi(x))^2 \right] \tag{2.3.1}$$

differential *master equations* corresponding to the Langevin equations (2.1.31) reads

$$\begin{aligned} \frac{\partial v_-(x,\tau;\omega,\varpi)}{\partial\tau} &= (\partial^2 - m^2)v_-(x,\tau;\omega,\varpi) + \eta(x,\tau;\omega), \\ \frac{\partial v_+(x,\tau;\omega,\varpi)}{\partial\tau} &= (\partial^2 - m^2)v_+(x,\tau;\omega,\varpi) + \eta(x,\tau;\omega), \\ v_\mp(x,0;\omega,\varpi) &= 0. \end{aligned} \tag{2.3.2}$$

Fourier transformed stochastic differential equations (2.3.2) in k and τ given as

$$\begin{gathered} \frac{\partial}{\partial\tau}\widehat{v}_-(k,\tau) = \\ -(k^2+m^2)\widehat{v}_-(k,\tau) - (2\pi)^4 a\delta(\tau)\delta^4(k) - (2\pi)^4 m^2 a\delta^4(k) + \widehat{\eta}(k,\tau;\omega), \\ \frac{\partial}{\partial\tau}\widehat{v}_+(k,\tau) = \\ -(k^2+m^2)\widehat{v}_+(k,\tau) - (2\pi)^4 a\delta(\tau)\delta^4(k) + (2\pi)^4 m^2 a\delta^4(k) + \widehat{\eta}(k,\tau;\omega). \end{gathered} \tag{2.3.3}$$

Let us consider ODE

$$\dot{x}(\tau,\lambda) + \lambda x(\tau,\lambda) = g(\tau,\lambda), x(0) = 0. \tag{2.3.4}$$

The corresponding solution $x(\tau,\lambda)$ is

$$x(\tau,\lambda) = e^{-\lambda\tau}\int_0^\tau e^{\lambda\tau_1}g(\tau_1,\lambda)d\tau_1. \tag{2.3.5}$$

From Eq.(2.3.3)-Eq.(2.3.5) one obtains

$$
\begin{aligned}
&\widehat{\nu}_-(k,\tau,a) = \\
&e^{-(k^2+m^2)\tau}\int_0^\tau e^{(k^2+m^2)\tau_1}\left[-(2\pi)^4 a\delta(\tau_1)\delta^4(k)-(2\pi)^4 m^2 a\delta^4(k)+\widehat{\eta}(k,\tau_1;\omega)\right]d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau}-(2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\int_0^\tau e^{(k^2+m^2)\tau_1}d\tau_1 + \\
&+e^{-(k^2+m^2)\tau}\int_0^\tau e^{(k^2+m^2)\tau_1}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau}-(2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\int_0^\tau e^{(k^2+m^2)\tau_1}d\tau_1 + \\
&+\int_0^\tau e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} - \\
&(2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\left[\frac{e^{(k^2+m^2)\tau}}{k^2+m^2}-\frac{1}{k^2+m^2}\right]+ \\
&+\int_0^\tau e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau}-(2\pi)^4\frac{m^2 a\delta^4(k)}{k^2+m^2}\left[1-e^{-(k^2+m^2)\tau}\right]+ \\
&+\int_0^\tau e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m^2)\tau}+\frac{m^2}{k^2+m^2}\left[1-e^{-(k^2+m^2)\tau}\right]\right]+ \\
&+\int_0^\tau e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1.
\end{aligned}
\tag{2.3.6}
$$

and

$$
\begin{aligned}
&\widehat{v}_{+}(k,\tau,a) = \\
&e^{-(k^2+m^2)\tau}\int_0^\tau e^{(k^2+m^2)\tau_1}\left[-(2\pi)^4 a\delta(\tau_1)\delta^4(k) + (2\pi)^4 m^2 a\delta^4(k) + \widehat{\eta}(k,\tau_1;\omega)\right]d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} - (2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\int_0^\tau e^{(k^2+m^2)\tau_1}d\tau_1 + \\
&+e^{-(k^2+m^2)\tau}\int_0^\tau e^{(k^2+m^2)\tau_1}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} + (2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\int_0^\tau e^{(k^2+m^2)\tau_1}d\tau_1 + \\
&+\int_0^\tau e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} - \\
&(2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\left[\frac{e^{(k^2+m^2)\tau}}{k^2+m^2} - \frac{1}{k^2+m^2}\right] + \\
&+\int_0^\tau e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} - (2\pi)^4\frac{m^2 a\delta^4(k)}{k^2+m^2}\left[1-e^{-(k^2+m^2)\tau}\right] + \\
&+\int_0^\tau e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m^2)\tau} + \frac{m^2}{k^2+m^2}\left[1-e^{-(k^2+m^2)\tau}\right]\right] + \\
&+\int_0^\tau e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1.
\end{aligned}
\tag{2.3.7}
$$

From Eq.(2.3.6)-Eq.(2.3.7) one obtains

$$
\begin{gathered}
\widehat{\nu}_{-}(k_1,\tau,a)\widehat{\nu}_{+}(k_2,\tau',a') = \\
\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m^2)\tau}+\frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right]+\right. \\
\left.+\int_0^{\tau} e^{-(k_1^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right\}\times \\
\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m^2)\tau'}+\frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_2^2+m^2)\tau'}\right]\right]+\right. \\
\left.+\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_1)}\widehat{\eta}(k_2,\tau_1;\omega)d\tau_1\right\}
\end{gathered}
\qquad (2.3.8)
$$

From Eq.(3.2.11) one obtains

$$
\begin{gathered}
\langle\widehat{\nu}_{-}(k_1,\tau,a)\widehat{\nu}_{+}(k_2,\tau',a')\rangle_{\eta} = \\
\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m^2)\tau}+\frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right]\right\}\times \\
\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m^2)\tau'}+\frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_2^2+m^2)\tau'}\right]\right]+\right. \\
+\left\langle\left(\int_0^{\tau} e^{-(k_1^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right)\left(\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_2)}\widehat{\eta}(k_2,\tau_2;\omega)d\tau_2\right)\right\rangle_{\eta} = \\
-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau}+\frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right]\times \\
\left[e^{-(k_2^2+m^2)\tau'}+\frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_2^2+m^2)\tau'}\right]\right]+ \\
+\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m^2)(\tau-\tau_1)}\langle\widehat{\eta}(k_1,\tau_1;\omega)\widehat{\eta}(k_2,\tau_2;\omega)\rangle_{\eta}d\tau_1 d\tau_2
\end{gathered}
\qquad (2.3.9)
$$

We set now $\tau' = \tau, a' = a$. Note that

$$-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right]\times$$
$$\left[e^{-(k_2^2+m^2)\tau'} + \frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau'}\right]\right]\Bigg|_{\tau'=\tau} =$$
$$-(2\pi)^8 a^2\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right]\times \tag{2.3.10}$$
$$\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right]$$

and

$$\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m^2)(\tau-\tau_1)}\langle\widehat{\eta}(k_1,\tau_1;\omega)\widehat{\eta}(k_2,\tau_2;\omega)\rangle_\eta d\tau_1 d\tau_2\Bigg|_{\tau'=\tau} =$$
$$\delta(k_1+k_2)\int_0^{\tau} e^{-(k_2^2+m^2)(\tau-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m^2)(\tau-\tau_1)}\delta(\tau_1-\tau_2)d\tau_1 d\tau_2 =$$
$$2(2\pi)^4\delta(k_1+k_2)\int_0^{\tau} e^{-(k_1^2+k_2^2+2m^2)(\tau-\tau_1)}d\tau_1 = \tag{2.3.11}$$
$$2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m^2)\tau}\int_0^{\tau} e^{(k_1^2+k_2^2+2m^2)\tau_1}d\tau_1 =$$
$$2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m^2)\tau}\left[\frac{1}{(k_1^2+k_2^2+2m^2)}e^{(k_1^2+k_2^2+m^2)\tau} - \frac{1}{(k_1^2+k_2^2+2m^2)}\right] =$$
$$= \frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m^2} - 2(2\pi)^4\delta(k_1+k_2)\frac{e^{-(k_1^2+k_2^2+2m^2)\tau}}{(k_1^2+k_2^2+2m^2)}$$

From Eq.(2.3.9)-Eq.(2.3.11) we get

$$\lim_{\tau\to\infty}\langle\widehat{v}(k_1,\tau,a)\widehat{v}(k_2,\tau,a)\rangle_\eta = (2\pi)^8 a^2\delta^4(k_1)\delta^4(k_2)\frac{m^4}{(k_1^2+m^2)(k_2^2+m^2)} - \tag{2.3.12}$$
$$-\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m^2}.$$

Therefore

$$
\begin{aligned}
&\lim_{\tau\to\infty}\langle\widehat{v}(x_1,\tau,a)\widehat{v}(x_2,\tau,a)\rangle_\eta = \\
&\times a^2(2\pi)^{-8}\times\int d^4k_1e^{ik_1x_1}\int d^4k_2e^{ik_2x_2}\delta^4(k_1)\delta^4(k_2)\frac{m^4}{(k_1^2+m^2)\times(k_2^2+m^2)} - \\
&-(2\pi)^{-8}\times\int d^4k_1e^{ik_1x_1}\int d^4k_2e^{ik_2x_2}\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m^2} = \\
&= \left[\frac{a^2m^4}{m^2\times m^2}+(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2}\right] = a^2-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2}
\end{aligned}
\tag{2.3.13}
$$

Two point function $G(x_1,x_2)$ of euclidean QFT corresponding to the action (2.3.1) is

$$
G(x_1,x_2) = \lim_{\tau\to\infty}\langle\varphi(x_1,\tau;\omega)\varphi(x_2,\tau;\omega)\rangle_\eta \tag{2.3.14}
$$

Master equation corresponding to two-point function $G(x_1,x_2)$ reads

$$
\lim_{\tau\to\infty}\langle\varphi(x_1,\tau;\omega)\varphi(x_2,\tau;\omega)\rangle - a^2 = \lim_{\tau\to\infty}\langle\widehat{v}(k_1,\tau,a)\widehat{v}(k_2,\tau,a)\rangle_\eta = 0. \tag{2.3.15}
$$

From (2.3.15) we get

$$
\lim_{\tau\to\infty}\langle\varphi(x_1,\tau;\omega)\varphi(x_2,\tau;\omega)\rangle_\eta = a^2. \tag{2.3.16}
$$

From Eq.(2.3.13)-Eq.(2.3.16) we get

$$
\lim_{\tau\to\infty}\langle\widehat{v}(k_1,\tau,a)\widehat{v}(k_2,\tau,a')\rangle_\eta = 0 \Leftrightarrow a^2-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2} = 0 \tag{2.3.17}
$$

and therefore

$$
a^2 = (2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2}. \tag{2.3.18}
$$

From Eq.(2.3.16) and Eq.(2.3.21) finally we get desired result

$$
\begin{aligned}
&G_F(x_1,x_2) = \lim_{\tau\to\infty}\langle\varphi(x_1,\tau;\omega)\varphi(x_2,\tau;\omega)\rangle_\eta = \\
&(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2} = (2\pi)^{-2}\left(\frac{m}{|x|}\right)K_1(m|x_1-x_2|),
\end{aligned}
\tag{2.3.19}
$$

where K_1 is the modified Bessel functions of the second kind, integer order 1, and where we used formula 6.566.2 of [12].

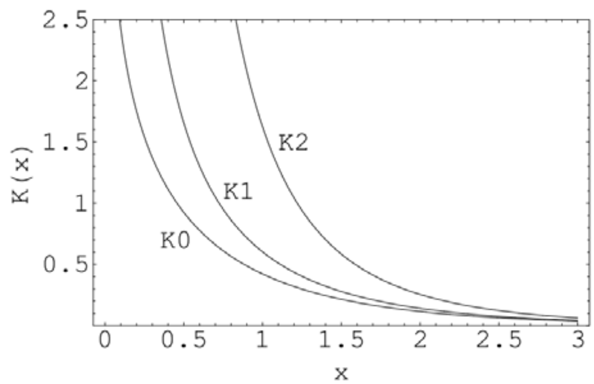


Figure 2.3.1.Plot of the modified Bessel functions of the second kind,integer order 1.

2.4.Double stochastic quantization the $\lambda\phi_d^4$ theory.

In this section we consider a neutral scalar field with a $\frac{\lambda}{4!}\phi_d^4$ $, d \geq 4,$ self-interaction, defined in a d-dimensional Minkowski spacetime. The vacuum persistence functional is the generating functional of all vacuum expectation value of time-ordered products of the theory. The Euclidean field theory can be obtained by analytic continuation to imaginary time supported by the positive energy condition for the relativistic field theory. In the Euclidean field theory, we have the Euclidean counterpart for the vacuum persistence functional, that is, the generating functional of complete Schwinger functions. Actually, the $(\lambda\varphi^4)_d$ Euclidean theory is defined by these Euclidean Green's functions. The Euclidean generating functional $Z[h]$ is formally defined by the following functional integral:

$$Z[h] = \int [d\varphi] \, \exp\Bigl(-S_0 - S_I + \int d^d x \, h(x)\varphi(x)\Bigr), \tag{2.4.1}$$

where the action that usually describes a free scalar field is

$$S_0[\varphi] = \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m_0^2\varphi^2(x)\right), \tag{2.4.2}$$

and the interacting part, defined by the non-Gaussian contribution, is

$$S_I[\varphi] = \int d^d x \, \frac{\lambda}{4!}\,\varphi^4(x). \tag{2.4.3}$$

In Eq.(2.4.1), $[d\varphi]$ is a translational invariant measure, formally given by $[d\varphi] = \prod_{x\in\mathbb{R}^d} d\varphi(x)$. The terms λ and m_0^2 are respectively the bare coupling constant and the squared mass of the model. Finally, $h(x)$ is a smooth function that we introduce to generate the Schwinger functions of the theory by functional derivatives. In the weak-coupling perturbative expansion, which is the conventional procedure, we perform a formal perturbative expansion with respect to the non-Gaussian terms of the action. As a consequence of this formal expansion, all the n-point unrenormalized Schwinger functions are expressed in a powers series of the bare coupling constant λ.

The aim of this section is to discuss the double stochastic quantization of a free scalar field. It can be shown that it is equivalent to the usual path integral quantization. The starting point of the stochastic quantization to obtain the Euclidean field theory is a Markovian Langevin equation. Assume an Euclidean d-dimensional manifold, where we are choosing periodic boundary conditions for a scalar field and also a random noise. In other words, they are defined in a d-torus $\Omega \equiv T^d$. To implement the stochastic quantization we supplement the scalar field $\varphi(x)$ and the random noises $\eta(x)$ and $\widehat{\eta}(\tau,x)$ with an extra coordinate τ, the Markov parameter, such that $\varphi(x) \to \varphi(\tau,x)$ and $\eta(x) \to \eta(\tau,x)$.

Therefore, the fields and the random noises $\eta(\tau,x)$ and $\widetilde{\eta}(\tau,x)$ are defined in a domain: $T^d \times R^{(+)}$. Let us consider that this dynamical system is out of equilibrium, being described by the following equation of evolution:

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -\frac{\delta S_0[\varphi]}{\delta\varphi(x)}\bigg|_{\varphi(x)=\varphi(\tau,x)} + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x), \qquad (2.4.4)$$

where τ is a Markov parameter, $\eta(\tau,x)$ is a random noise field and S_0 is the usual free action defined in Eq.(2.4.2). For a free scalar field, the double stochastic Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -(-\Delta + m_0^2)\varphi(\tau,x) + \eta(\tau,x) + \epsilon\widehat{\eta}(\tau,x), \qquad (2.4.5)$$

where Δ is the d-dimensional Laplace operator. The Eq.(2.4.5) describes a Ornstein-Uhlenbeck process and we are assuming the Einstein relations, that is:

$$\begin{aligned} \langle \eta(\tau,x) \rangle_\eta &= 0, \\ \langle \widetilde{\eta}(\tau,x) \rangle_\eta &= 0, \end{aligned} \qquad (2.4.6)$$

and for the two-point correlation function associated with the random noise fields

$$\begin{aligned} \langle \eta(\tau,x)\,\eta(\tau',x') \rangle_\eta &= 2\delta(\tau-\tau')(x-x'), \\ \langle \widetilde{\eta}(\tau,x)\,\widetilde{\eta}(\tau',x') \rangle_{\widehat{\eta}} &= 2\delta(\tau-\tau')(x-x'), \end{aligned} \qquad (2.4.7)$$

where $\langle \ldots \rangle_\eta$ means stochastic averages. In a generic way, the stochastic average for

any functional of φ given by $F[\varphi]$ is defined by

$$\langle F[\varphi] \rangle_{\eta,\hat{\eta}} = \frac{\int D[\eta]D[\hat{\eta}]F[\varphi]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\eta^2(\tau,x,\omega)\Big]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\widetilde{\eta}^2(\tau,x,\widetilde{\omega})\Big]}{\Big(\int D[\eta]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\eta^2(\tau,x,\omega)\Big]\Big)\Big(\int D[\widetilde{\eta}]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\widetilde{\eta}^2(\tau,x,\widetilde{\omega})\Big]\Big)}. \quad (2.4.8)$$

Let us define the retarded Green function for the diffusion problem that we call $G(\tau-\tau',x-x')$. The retarded Green function satisfies $G(\tau-\tau',x-x') = 0$ if $\tau-\tau' < 0$ and also

$$\Big[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2)\Big]G(\tau-\tau',x-x') = \delta^d(x-x')\delta^d(\tau-\tau'). \quad (2.4.9)$$

Using the retarded Green function and the initial condition $\varphi(\tau,x)|_{\tau=0} = 0$, the solution for Eq.(3.5.5) reads

$$\varphi(\tau,x) = \int_0^\tau d\tau'\int_\Omega d^dx'\,G(\tau-\tau',x-x')[\eta(\tau',x') + \epsilon\hat{\eta}(\tau',x)]. \quad (2.4.10)$$

In the following we are interested in calculating the quantity $\langle\varphi(\tau,x)\varphi(\tau',x')\rangle_{\eta,\hat{\eta}}$. Using Eq.(2.4.6), Eq.(2.4.7) and Eq.(2.4.10), we have

$$\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle_{\eta,\hat{\eta}} = 2\int_0^{\min(\tau_1,\tau_2)} d\tau'\int_\Omega d^dx' G(\tau_1-\tau',x_1-x')\,G(\tau_2-\tau',x_2-x'), \quad (2.4.11)$$

where $\min(\tau_1,\tau_2)$ means the minimum of τ_1 and τ_2. Using a Fourier representation, the two-point correlation function $\langle\varphi(\tau,x)\varphi(\tau',x')\rangle_\eta \equiv D(\tau,x;\tau',x')$ is given by

$$D(\tau,x;\tau',x') = \frac{1}{(2\pi)^d}\int d^dp\,\frac{e^{-ip(x-x')}}{(p^2+m_0^2)}\,e^{-(p^2+m_0^2)(\tau-\tau')}. \quad (2.4.12)$$

It is not difficult to show that Eq.(2.4.12) can be written as:

$$D(\tau,x;\tau',x') = \frac{1}{(2\pi)^d}\sum_{n=0}^\infty\frac{(-1)^n}{\Gamma(1-n)n!}(\tau-\tau')^n\Big(\frac{m_0}{r}\Big)^{\frac{d}{2}+n-1}K_{\frac{d}{2}+n-1}(m_0 r). \quad (2.4.13)$$

where $r = |\ x-x'\ |$ and K_ν is the modified Bessel function of order ν.

We can use the Fourier analysis to show that when the Markov parameters τ and τ' go to infinity we recover the standard Euclidean free field theory. Therefore let us define the Fourier transforms for the field and the noises given by $\varphi(\tau,k)$ and $\eta(\tau,k)$. We have respectively

$$\varphi(\tau,k) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int d^dx\,e^{-ikx}\,\varphi(\tau,x), \quad (2.4.14)$$

and

$$\hat{\eta}(\tau,k) = \frac{1}{(2\pi)^{d/2}} \int e^{-ikx}\, \eta(\tau,x),$$
$$\hat{\tilde{\eta}}(\tau,k) = \frac{1}{(2\pi)^{d/2}} \int e^{-ikx}\, \hat{\tilde{\eta}}(\tau,x) \qquad (2.4.15)$$

Substituting Eq.(2.4.14) in Eq.(2.4.2), the free action for the scalar field in the $(d+1)$-dimensional space writing in terms of the Fourier coefficients reads

$$S_0[\varphi(k)]\,|_{\varphi(k)=\varphi(\tau,k)} = \frac{1}{2} \int d^d k\, \varphi(\tau,k)(k^2+m_0^2)\varphi(\tau,k). \qquad (2.4.16)$$

Substituting Eq.(2.4.14) and Eq.(2.4.15) in Eq.(2.4.5) we have that each Fourier coefficient satisfies a Langevin equation given by

$$\frac{\partial}{\partial\tau}\varphi(\tau,k) = -(k^2+m_0^2)\varphi(\tau,k) + \eta(\tau,k) + \epsilon\hat{\tilde{\eta}}(\tau,k). \qquad (2.4.17)$$

The solution for this equation reads

$$\varphi(\tau,k) =$$
$$\exp(-(k^2+m_0^2)\tau)\varphi(0,k) + \int_0^\tau d\tau' \exp(-(k^2+m_0^2)(\tau-\tau'))[\eta(\tau',k) + \epsilon\hat{\tilde{\eta}}(\tau',k)]. \qquad (2.4.18)$$

Using the Einstein relation, we get that the Fourier coefficients for the random noise satisfies

$$\langle\, \eta(\tau,k)\,\rangle_\eta = 0,$$
$$\langle\, \tilde{\eta}(\tau,k)\,\rangle_\eta = 0 \qquad (2.4.19)$$

and

$$\langle\, \eta(\tau,k)\eta(\tau',k')\,\rangle_\eta = 2(2\pi)^d \delta^d(\tau-\tau')(k+k'),$$
$$\langle\, \hat{\tilde{\eta}}(\tau,k)\hat{\tilde{\eta}}(\tau',k')\,\rangle_\eta = 2(2\pi)^d \delta^d(\tau-\tau')(k+k') \qquad (2.4.20)$$

Before investigate the interacting field theory, let us calculate the Fourier representation for the two-point correlation function, i.e., $\langle\, \varphi(\tau,k)\varphi(\tau',k')\,\rangle_\eta$. Using Eq.(2.4.18), we obtain three contributions to the scalar two-point correlation function. The first one is given by

$$\exp(-(k^2+m_0^2)\,\tau + (k'^2+m_0^2)\,\tau')\varphi(0,k)\varphi(0,k'), \qquad (2.4.21)$$

and decay to zero at long time. Let us assume that $\varphi(\tau,k)|_{\tau=0} = 0$. There are also two crossed terms, each first order in the noise Fourier component given by

$$2\,\varphi(0,k)\exp(-(k^2+m_0^2)\tau) \int_0^{\tau'} ds\, \exp(-(k'^2+m_0^2)(\tau'-s))\Big[\,\eta(s,k') + \epsilon\hat{\tilde{\eta}}(\sigma,k')\,\Big]. \qquad (2.4.22)$$

Since we are assuming the Einstein relations, i.e., $\langle\, \eta(\tau,x)\,\rangle_\eta = 0, \langle\, \tilde{\eta}(\tau,x)\,\rangle_\eta = 0$ on averaging on noise, these cross terms vanish. The final term is second-order in the noise Fourier component. Again, the solution subject to the initial condition $\varphi(\tau,k)|_{\tau=0} = 0$ can be used to give

$$\left\{\int_0^{\tau} ds \exp(-(k^2+m_0^2)(\tau-s))\left[\eta(s,k)+\epsilon\widehat{\widetilde{\eta}}(s,k)\right]\right\}\times \left\{\int_0^{\tau'} d\sigma \exp(-(k'^2+m_0^2)(\tau'-\sigma))\left[\eta(\sigma,k')+\epsilon\widehat{\widetilde{\eta}}(\sigma,k')\right]\right\}. \quad (2.4.23)$$

Again averaging on noises and using the Einstein relation given by Eq.(2.4.20) we have
that this term becomes

$$2\delta^d(k+k_0')\int_0^{\min(\tau,\tau')} ds \exp(-(k^2+m_0)(\tau+\tau'-2s)). \quad (2.4.24)$$

Assuming that $\tau=\tau'$ and using $\langle\varphi(\tau,k)\varphi(\tau',k')\rangle_\eta|_{\tau=\tau'}\equiv D(k,k';\tau,\tau')$ we have

$$D(k;\tau,\tau)=(2\pi)^d\delta^d(k+k')\frac{1}{(k^2+m_0^2)}(1-\exp(-2\tau(k^2+m_0^2))). \quad (2.4.25)$$

In the following, we are redefining the two-point correlation function as
$D(k;\tau,\tau)\to(2\pi)^d D(k;\tau,\tau)$. In the limit when $\tau\to\infty$ we recover the standard two-point
function of the Euclidean free field theory. Before going to the next section, we would
like to mention the existence of more general Markovian Langevin equations. We can
introduce a kernel defined in the d-torus. The kerneled Langevin equation reads:

$$\frac{\partial}{\partial\tau}\varphi(\tau,x)=-\int d^dy\,K(x,y)\frac{\delta S_0}{\delta\varphi(y)}|_{\varphi(y)=\varphi(\tau,y)}+\eta(\tau,x)+\epsilon\widehat{\eta}(\tau,x). \quad (2.4.26)$$

The second moment of the noise fields will be modified to:

$$\begin{aligned}\langle\eta(\tau,x)\,\eta(\tau',x')\rangle_\eta &= 2\delta(\tau-\tau')K(x,x'),\\ \langle\widetilde{\eta}(\tau,x)\,\widetilde{\eta}(\tau',x')\rangle_\eta &= 2\delta(\tau-\tau')K(x,x').\end{aligned} \quad (2.4.27)$$

Choosing an appropriate kernel, it can be shown that all the above conclusions remain
unchanged.

The double stochastic Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi_\epsilon(\tau,x)=(\Delta-m_0^2)\varphi_\epsilon(\tau,x)-\frac{\lambda}{3!}\varphi_\epsilon^3(\tau,x)+\eta(\tau,x)+\epsilon\widetilde{\eta}(\tau,x). \quad (2.4.28)$$

By the replacements

$$\begin{aligned}\varphi_\epsilon(x,\tau;\omega,\varpi) &= v_{\epsilon-}(x,\tau;\omega,\varpi)+\theta(\tau)a,\\ \varphi_\epsilon(x,\tau;\omega,\varpi) &= v_{\epsilon+}(x,\tau;\omega,\varpi)-\theta(\tau)a,\\ \theta(\tau) &= \begin{cases}0 & \text{if } \tau\le 1\\ 1 & \text{if } \tau>1\end{cases}\end{aligned} \quad (2.4.29)$$

we obtain from Eqs.(2.4.28)

$$
\begin{gathered}
\frac{\partial[\nu_{\epsilon-}(x,\tau;\omega,\varpi)+\theta(\tau)a]}{\partial\tau} = \frac{\partial\nu_{\epsilon-}(x,\tau;\omega,\varpi)}{\partial\tau} + a\delta(\tau) = \\
= (\partial^2 - m^2)[\nu_{\epsilon-}(x,\tau;\omega,\varpi)+\theta(\tau)a] - \frac{\lambda}{3!}[\nu_{\epsilon-}(x,\tau;\omega,\varpi)+a]^3 + \\
+\eta(x,\tau;\omega) + \epsilon\tilde{\eta}(x,\tau;\varpi) = \\
(\partial^2 - m^2)[\nu_{\epsilon-}(x,\tau;\omega,\varpi)+\theta(\tau)a] - \frac{\lambda}{3!}(\nu_{\epsilon-}^3 + 3a\nu_{\epsilon-}^2 + 3a^2\nu_{\epsilon-} + a^3) + \\
+\eta(x,\tau;\omega) + \epsilon\tilde{\eta}(x,\tau;\varpi) \qquad (2.4.30) \\
\text{and} \\
\frac{\partial[\nu_{\epsilon+}(x,\tau;\omega,\varpi)-\theta(\tau)a]}{\partial\tau} = \frac{\partial\nu_{\epsilon+}(x,\tau;\omega,\varpi)}{\partial\tau} - a\delta(\tau) = \\
(\partial^2 - m^2)[\nu_{\epsilon+}(x,\tau;\omega,\varpi)-\theta(\tau)a] - \frac{\lambda}{3!}(\nu_{\epsilon+}^3 - 3a\nu_{\epsilon+}^2 + 3a^2\nu_{\epsilon+} - a^3) + \\
+\eta(x,\tau;\omega) + \epsilon\tilde{\eta}(x,\tau;\varpi)
\end{gathered}
$$

Differential master equations corresponding to double stochastic Langevin equations (2.4.30) reads

$$
\begin{gathered}
\frac{\partial\nu_{-}(x,\tau;\omega)}{\partial\tau} = -a\delta(\tau) + [\partial^2 - (m^2 + 0.5\lambda a^2))]\nu_{-}(x,\tau;\omega) - \\
-\frac{\lambda}{6}a^3 - m^2 a + \eta(x,\tau;\omega) = \\
-a\delta(\tau) + [\partial^2 - m_1^2]\nu_{-}(x,\tau;\omega) - \left(\frac{\lambda}{6}a^3 + m^2 a\right) + \eta(x,\tau;\omega) \\
\text{and} \qquad (2.4.31) \\
\frac{\partial\nu_{+}(x,\tau;\omega)}{\partial\tau} = a\delta(\tau) + [\partial^2 - (m^2 + 0.5\lambda a^2))]\nu_{+}(x,\tau;\omega) + \\
+\frac{\lambda}{6}a^3 + m^2 a + \eta(x,\tau;\omega) = \\
a\delta(\tau) + [\partial^2 - m_1^2]\nu_{+}(x,\tau;\omega) + \left(\frac{\lambda}{6}a^3 + m^2 a\right) + \eta(x,\tau;\omega) \\
m_1^2 = m^2 + 0.5\lambda a^2.
\end{gathered}
$$

Consider the Fourier transformed stochastic differential equation (2.4.31) in k and τ given as

$$
\begin{gathered}
\frac{\partial}{\partial \tau}\widehat{v}_{-}(k,\tau) = \\
-(k^2+m_1^2)\widehat{v}_{-}(k,\tau) - (2\pi)^4 a\delta(\tau)\delta^4(k) - (2\pi)^4\left(m^2 a + \frac{\lambda}{6}a^3\right)\delta^4(k) + \widehat{\eta}(k,\tau;\omega) \\
\text{and} \\
\frac{\partial}{\partial \tau}\widehat{v}_{+}(k,\tau) = \\
-(k^2+m_1^2)\widehat{v}_{-}(k,\tau) + (2\pi)^4 a\delta(\tau)\delta^4(k) + (2\pi)^4\left(m^2 a + \frac{\lambda}{6}a^3\right)\delta^4(k) + \widehat{\eta}(k,\tau;\omega)
\end{gathered}
\tag{2.4.32}
$$

Let us consider ODE

$$
\dot{x}\,(\tau,\lambda) + \lambda x(\tau,\lambda) = g(\tau,\lambda), x(0) = 0. \tag{2.4.33}
$$

The corresponding solution $x(t,\lambda)$ reads

$$
x(\tau,\lambda) = e^{-\lambda\tau}\int_0^\tau e^{\lambda\tau_1} g(\tau_1,\lambda)d\tau_1. \tag{2.4.34}
$$

From Eq.(2.4.32)-Eq.(2.4.34) one obtains

$$\widehat{v}_{-}(k,\tau,a) = e^{-(k^2+m_1^2)\tau} \times$$

$$\times\int_0^\tau e^{(k^2+m_1^2)\tau_1}\Big[-(2\pi)^4 a\delta(\tau_1)\delta^4(k) - (2\pi)^4\Big(m^2 a + \frac{\lambda}{6}a^3\Big)\delta^4(k) + \widehat{\eta}(k,\tau_1;\omega)\Big]d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4\Big(m^2 a + \frac{\lambda}{6}a^3\Big)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4 a\Big(m^2 + \frac{\lambda}{6}a^2\Big)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} -$$

$$-(2\pi)^4 a\Big(m^2 + \frac{\lambda}{6}a^2\Big)\delta^4(k)e^{-(k^2+m^2)\tau}\Big[\frac{e^{(k^2+m^2)\tau}}{k^2+m_1^2} - \frac{1}{k^2+m_1^2}\Big] + \qquad (2.4.35)$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4\frac{\Big(m^2 a + \frac{\lambda}{6}a^3\Big)\delta^4(k)}{k^2+m_1^2}\Big[1 - e^{-(k^2+m_1^2)\tau}\Big] +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m_1^2)\tau} + \frac{\Big(m^2 + \frac{\lambda}{6}a^2\Big)}{k^2+m_1^2}\Big[1 - e^{-(k^2+m_1^2)\tau}\Big]\right] +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1,$$

and

$$
\begin{gathered}
\widehat{v}_{+}(k,\tau,a) = e^{-(k^2+m_1^2)\tau}\times \\
\times\int_0^\tau e^{(k^2+m_1^2)\tau_1}\Big[(2\pi)^4 a\delta(\tau_1)\delta^4(k) - (2\pi)^4\Big(-m^2a - \frac{\lambda}{6}a^3\Big)\delta^4(k) \pm \widehat{\eta}(k,\tau_1;\omega)\Big]d\tau_1 = \\
(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4\Big(-m^2a - \frac{\lambda}{6}a^3\Big)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 + \\
+e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
+(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4 a\Big(m^2 + \frac{\lambda}{6}a^3\Big)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 + \\
+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
+(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + \\
+(2\pi)^4 a\Big(m^2 + \frac{\lambda}{6}a^2\Big)\delta^4(k)e^{-(k^2+m^2)\tau}\Big[\frac{e^{(k^2+m^2)\tau}}{k^2+m_1^2} - \frac{1}{k^2+m_1^2}\Big] + \\
+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
+(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4\frac{\Big(m^2 + \frac{\lambda}{6}a^3\Big)\delta^4(k)}{k^2+m_1^2}\Big[1 - e^{-(k^2+m_1^2)\tau}\Big] + \\
+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
+(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m_1^2)\tau} + \frac{\Big(m^2 + \frac{\lambda}{6}a^3\Big)}{k^2+m_1^2}\Big[1 - e^{-(k^2+m_1^2)\tau}\Big]\right] + \\
+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1.
\end{gathered}
\tag{2.4.35'}
$$

From Eq.(2.4.35)-Eq.(2.4.35′) one obtains

$$
\begin{aligned}
&\widehat{v}_{-}(k_1,\tau,a)\widehat{v}_{+}(k_2,\tau',a') = \\
&\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m_1^2)\tau} + \frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]+\right. \\
&\left.+\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right\}\times \\
&\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right]+\right. \\
&\left.+\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_1)}\widehat{\eta}(k_2,\tau_1;\omega)d\tau_1\right\}
\end{aligned}
\tag{2.4.36}
$$

From Eq.(2.4.36) one obtains

$$
\begin{aligned}
&\langle\widehat{v}_{-}(k_1,\tau,a)\widehat{v}_{+}(k_2,\tau',a')\rangle_{\eta} = \\
&\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\right\}\times \\
&\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right]\right\}- \\
&-\left\langle\left(\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right)\left(\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_2)}\widehat{\eta}(k_2,\tau_2;\omega)d\tau_2\right)\right\rangle_{\eta} = \\
&-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m_1^2)\tau} + \frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times \\
&\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right]- \\
&-\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\langle\widehat{\eta}(k_1,\tau_1;\omega)\widehat{\eta}(k_2,\tau_2;\omega)\rangle_{\eta}d\tau_1 d\tau_2
\end{aligned}
\tag{2.4.37}
$$

Note that

$$-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau}+\frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times$$

$$\left.\left[e^{-(k_2^2+m^2)\tau'}+\frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau'}\right]\right]\right|_{\tau'=\tau,a'=a} =$$

$$-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m_1^2)\tau}+\frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times \tag{2.4.38}$$

$$\times\left[e^{-(k_1^2+m^2)\tau}+\frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]$$

and note that

$$\left.\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m^2)(\tau-\tau_1)}\langle\widehat{\eta}(k_1,\tau_1;\omega)\widehat{\eta}(k_2,\tau_2;\omega)\rangle_\eta d\tau_1 d\tau_2\right|_{\tau'=\tau,a'=a} =$$

$$\delta(k_1+k_2)\int_0^{\tau} e^{-(k_2^2+m_1^2)(\tau-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\delta(\tau_1-\tau_2)d\tau_1 d\tau_2 =$$

$$2(2\pi)^4\delta(k_1+k_2)\int_0^{\tau} e^{-(k_1^2+k_2^2+2m_1^2)(\tau-\tau_1)}d\tau_1 = \tag{2.4.39}$$

$$2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m_1^2)\tau}\int_0^{\tau} e^{(k_1^2+k_2^2+2m_1^2)\tau_1}d\tau_1 =$$

$$2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m_1^2)\tau}\left[\frac{1}{(k_1^2+k_2^2+2m_1^2)}e^{(k_1^2+k_2^2+m_1^2)\tau}-\frac{1}{(k_1^2+k_2^2+2m_1^2)}\right] =$$

$$= \frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m_1^2}-2(2\pi)^4\delta(k_1+k_2)\frac{e^{-(k_1^2+k_2^2+2m_1^2)\tau}}{(k_1^2+k_2^2+2m_1^2)}$$

From Eq.(2.4.38)-Eq.(2.4.39) we get

$$\lim_{\tau\to\infty}\langle\widehat{v}_-(k_1,\tau,a)\widehat{v}_+(k_2,\tau,a)\rangle_\eta = (2\pi)^8 a^2\delta^4(k_1)\delta^4(k_2)\frac{\left(m^2+\frac{\lambda}{6}a^2\right)^2}{(k_1^2+m_1^2)(k_2^2+m_1^2)} -$$

$$-\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m_1^2} \tag{2.4.40}$$

$$m_1^2 = m^2+0.5\lambda a^2$$

Therefore

$$\lim_{\tau\to\infty}\langle\hat{v}_-(x_1,\tau,a)\hat{v}_+(x_2,\tau,a)\rangle_\eta =$$

$$-a^2(2\pi)^{-8}\times\int d^4k_1e^{ik_1x_1}\int d^4k_2e^{ik_2x_2}\delta^4(k_1)\delta^4(k_2)\frac{\left(m^2+\frac{\lambda}{6}a^2\right)^2}{(k_1^2+m_1^2)\times(k_2^2+m_1^2)}+$$

$$(2\pi)^{-8}\times\int d^4k_1e^{ik_1x_1}\int d^4k_2e^{ik_2x_2}\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m_1^2} =$$

$$= \left[\frac{a^2\left(m^2+\frac{\lambda}{6}a^2\right)^2}{m_1^2\times m_1^2}-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m_1^2}\right] = \tag{2.4.41}$$

$$\frac{a^2\left(m^2+\frac{\lambda}{6}a^2\right)^2}{m_1^2\times m_1^2}-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m_1^2}$$

$$m_1^2 = m^2+0.5\lambda a^2$$

Transcendental master equation corresponding to two-point Eucledian Green function

$G_{(4)}(x_1,x_2,m,\lambda)$ in Euclidean space E_4 with $\dim E_4 = 4$ reads

$$\frac{\left[a(x_1-x_2)^2\right]\left(m^2+\frac{\lambda}{6}a(x_1-x_2)^2\right)^2}{\left(m^2+0.5\lambda a(x_1-x_2)^2\right)^2}- \\ -(2\pi)^{-4}\int d^4k\frac{\theta_\delta(|x_1-x_2|)e^{ik(x_1-x_2)}}{k^2+m^2+0.5\lambda a(x_1-x_2)^2} = 0. \tag{2.4.41}$$

Transcendental master equation corresponding to two-point Eucledian Green function

$G_{(D)}(x_1,x_2,m,\lambda)$ in Euclidean space E_D with $\dim E_D = D$ reads

$$\frac{\left[a(x_1-x_2)^2\right]\left(m^2+\frac{\lambda}{6}a(x_1-x_2)^2\right)^2}{\left(m^2+0.5\lambda a(x_1-x_2)^2\right)^2}- \\ -(2\pi)^{-D}w\text{-}\lim_{\eta\to 0_+}\int d^Dk\frac{e^{-\eta k+ik(x_1-x_2)}}{k^2+m^2+0.5\lambda a(x_1-x_2)^2} = 0. \\ \eta > 0, \tag{2.4.42}$$

where weak limit taken in $\mathcal{L}'(\mathbb{R}_x^D)$, see Apendix 1.

Note that

$$\int d^4k \frac{e^{ik(x_1-x_2)}}{\left[k^2+m^2+0.5\lambda a(x_1-x_2)^2\right]^2} =$$

$$(4\pi)^{-2}\left(\frac{m_1}{|x_1-x_2|}\right)K_0(m_1|x_1-x_2|) \tag{2.4.43}$$

$$m_1 = \sqrt{m^2+0.5\lambda a(x_1-x_2)^2},$$

$$K_0(m_1|x_1-x_2|) \asymp -\ln\left(\frac{m_1|x_1-x_2|}{2}\right)-\gamma \text{ as } m_1|x_1-x_2| \to 0$$

see Apendix 1. In order to derive Eq.(2.4.43) we applied Eq.(2.4.44) (see formula 13.6 (2) from [13]).

$$\int_0^\infty \frac{y^{\nu+1}J_\nu(by)dy}{(y^2+p^2)^{\mu+1}} = \frac{b^\mu p^{\nu-\mu}}{2^\mu\Gamma(\mu+1)}K_{\nu-\mu}(bp) \tag{2.4.44}$$

$$-1 < \operatorname{Re}\nu < 2\operatorname{Re}\mu + 1.5.$$

2.5.Double stochastic quantization of the $\lambda\phi_4^6$ theory.

In this section we consider a neutral scalar field with a $\frac{\lambda}{6!}\phi_d^6$, $d \geq 4$, self-interaction, defined in a 4-dimensional Minkowski spacetime. It well known that all these theories is nonrenormalizable [14]. The vacuum persistence functional is the generating functional of all vacuum expectation value of time-ordered products of the theory. Thus we deal now with simple *nonrenormalizable* theory with the Lagrangian

$$\mathcal{L} = S_0[\varphi] + S_I[\varphi], \tag{2.5.1}$$

where the action that usually describes a free scalar field is

$$S_0[\varphi] = \int d^4x \left(\frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi) + \frac{1}{2}m_0^2\varphi^2(x)\right), \tag{2.5.2}$$

and the interacting part, defined by the non-Gaussian contribution, is

$$S_I[\varphi] = \int d^4x \frac{\lambda}{6!}\varphi^6(x). \tag{2.5.3}$$

The double stochastic Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi_\epsilon(\tau,x) = (\Delta - m_0^2)\varphi_\epsilon(\tau,x) - \frac{\lambda}{5!}\varphi_\epsilon^5(\tau,x) + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x). \tag{2.5.4}$$

By the replacements

$$\begin{aligned}\varphi_\epsilon(x,\tau;\omega,\varpi) &= v_{\epsilon-}(x,\tau;\omega,\varpi)+\theta(\tau)a,\\ \varphi_\epsilon(x,\tau;\omega,\varpi) &= v_{\epsilon+}(x,\tau;\omega,\varpi)-\theta(\tau)a,\\ \theta(\tau) &= \begin{cases} 0 & \text{if } \tau\leq 1\\ 1 & \text{if } \tau>1\end{cases}\end{aligned} \tag{2.5.5}$$

we obtain from Eq.(2.5.4)

$$\begin{gathered}\frac{\partial[v_{\epsilon-}(x,\tau;\omega,\varpi)+\theta(\tau)a]}{\partial\tau} = \frac{\partial v_{\epsilon-}(x,\tau;\omega,\varpi)}{\partial\tau}+a\delta(\tau) =\\ =(\partial^2-m^2)[v_{\epsilon-}(x,\tau;\omega,\varpi)+\theta(\tau)a]-\frac{\lambda}{5!}[v_{\epsilon-}(x,\tau;\omega,\varpi)+a]^5+\\ +\eta(x,\tau;\omega)+\epsilon\widetilde{\eta}(x,\tau;\varpi)=\\ (\partial^2-m^2)[v_{\epsilon-}(x,\tau;\omega,\varpi)+\theta(\tau)a]-\\ -\frac{\lambda}{5!}(a^5+5a^4v_{\epsilon-}+10a^3v_{\epsilon-}^2+10a^2v_{\epsilon-}^3+5av_{\epsilon-}^4+v_{\epsilon-}^5)+\\ +\eta(x,\tau;\omega)+\epsilon\widetilde{\eta}(x,\tau;\varpi)\\ \text{and}\\ \frac{\partial[v_{\epsilon+}(x,\tau;\omega,\varpi)-\theta(\tau)a]}{\partial\tau} = \frac{\partial v_{\epsilon+}(x,\tau;\omega,\varpi)}{\partial\tau}-a\delta(\tau) =\\ (\partial^2-m^2)[v_{\epsilon+}(x,\tau;\omega,\varpi)-\theta(\tau)a]-\\ -\frac{\lambda}{5!}(-a^5+5a^4v_{\epsilon+}-10a^3v_{\epsilon+}^2+10a^2v_{\epsilon+}^3-5av_{\epsilon+}^4+v_{\epsilon+}^5)+\\ +\eta(x,\tau;\omega)+\epsilon\widetilde{\eta}(x,\tau;\varpi)\end{gathered} \tag{2.5.6}$$

Differential master equations corresponding to double stochastic Langevin equations (2.5.6) reads

$$\begin{gathered}\frac{\partial v_-(x,\tau;\omega)}{\partial\tau}=-a\delta(\tau)+\left[\partial^2-\left(m^2+\frac{\lambda}{4!}a^4\right))\right]v_-(x,\tau;\omega)-\\ -\frac{\lambda}{5!}a^5-m^2a+\eta(x,\tau;\omega)=\\ -a\delta(\tau)+[\partial^2-m_1^2]v_-(x,\tau;\omega)-\left(\frac{\lambda}{5!}a^5+m^2a\right)+\eta(x,\tau;\omega)\\ \text{and}\\ \frac{\partial v_+(x,\tau;\omega)}{\partial\tau}=a\delta(\tau)+\left[\partial^2-\left(m^2+\frac{\lambda}{5!}a^5\right))\right]v_+(x,\tau;\omega)+\\ +\frac{\lambda}{5!}a^5+m^2a+\eta(x,\tau;\omega)=\\ a\delta(\tau)+[\partial^2-m_1^2]v_+(x,\tau;\omega)+\left(\frac{\lambda}{5!}a^5+m^2a\right)+\eta(x,\tau;\omega)\\ m_1^2=m^2+\frac{\lambda}{4!}a^4.\end{gathered} \tag{2.5.7}$$

Consider the Fourier transformed stochastic differential equation (2.5.7) in k given as

$$\frac{\partial}{\partial\tau}\widehat{v}_-(k,\tau) =$$
$$-(k^2+m_1^2)\widehat{v}_-(k,\tau) - (2\pi)^4 a\delta(\tau)\delta^4(k) - (2\pi)^4\left(m^2 a + \frac{\lambda}{5!}a^5\right)\delta^4(k) + \widehat{\eta}(k,\tau;\omega)$$

and

$$\frac{\partial}{\partial\tau}\widehat{v}_+(k,\tau) =$$
$$-(k^2+m_1^2)\widehat{v}_-(k,\tau) + (2\pi)^4 a\delta(\tau)\delta^4(k) + (2\pi)^4\left(m^2 a + \frac{\lambda}{5!}a^5\right)\delta^4(k) + \widehat{\eta}(k,\tau;\omega) \qquad (2.4.32)$$

Let us consider ODE

$$\dot{x}(\tau,\lambda) + \lambda x(\tau,\lambda) = g(\tau,\lambda), x(0) = 0. \qquad (2.5.8)$$

The corresponding solution $x(t,\lambda)$ reads

$$x(\tau,\lambda) = e^{-\lambda\tau}\int_0^\tau e^{\lambda\tau_1} g(\tau_1,\lambda)d\tau_1. \qquad (2.5.9)$$

From Eq.(2.4.32)-Eq.(2.4.34) one obtains

$$\widehat{v}_{-}(k,\tau,a) = e^{-(k^2+m_1^2)\tau} \times$$

$$\times\int_0^\tau e^{(k^2+m_1^2)\tau_1}\Big[-(2\pi)^4 a\delta(\tau_1)\delta^4(k) - (2\pi)^4\Big(m^2a + \frac{\lambda}{5!}a^5\Big)\delta^4(k) + \widehat{\eta}(k,\tau_1;\omega)\Big]d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4\Big(m^2a + \frac{\lambda}{5!}a^5\Big)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4 a\Big(m^2 + \frac{\lambda}{5!}a^4\Big)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} -$$

$$-(2\pi)^4 a\Big(m^2 + \frac{\lambda}{5!}a^4\Big)\delta^4(k)e^{-(k^2+m^2)\tau}\left[\frac{e^{(k^2+m^2)\tau}}{k^2+m_1^2} - \frac{1}{k^2+m_1^2}\right] + \tag{2.5.10}$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4\frac{a\Big(m^2 + \frac{\lambda}{5!}a^4\Big)\delta^4(k)}{k^2+m_1^2}\Big[1 - e^{-(k^2+m_1^2)\tau}\Big] +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m_1^2)\tau} + \frac{\Big(m^2 + \frac{\lambda}{5!}a^4\Big)}{k^2+m_1^2}\Big[1 - e^{-(k^2+m_1^2)\tau}\Big]\right] +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1,$$

$$m_1^2 = m^2 + \frac{\lambda}{4!}a^4,$$

and

$$\hat{v}_+(k,\tau,a) = e^{-(k^2+m_1^2)\tau} \times$$

$$\times \int_0^{\tau} e^{(k^2+m_1^2)\tau_1} \Big[(2\pi)^4 a\delta(\tau_1)\delta^4(k) - (2\pi)^4 \Big(-m^2 a - \frac{\lambda}{5!}a^5 \Big)\delta^4(k) + \hat{\eta}(k,\tau_1;\omega) \Big] d\tau_1 =$$

$$(2\pi)^4 a\delta^4(k) e^{-(k^2+m_1^2)\tau} - (2\pi)^4 \Big(-m^2 a - \frac{\lambda}{5!}a^5 \Big)\delta^4(k) e^{-(k^2+m_1^2)\tau} \int_0^{\tau} e^{(k^2+m_1^2)\tau_1} d\tau_1 +$$

$$+ e^{-(k^2+m_1^2)\tau} \int_0^{\tau} e^{(k^2+m_1^2)\tau_1} \hat{\eta}(k,\tau_1;\omega) d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k) e^{-(k^2+m_1^2)\tau} + (2\pi)^4 a \Big(m^2 + \frac{\lambda}{5!}a^4 \Big)\delta^4(k) e^{-(k^2+m_1^2)\tau} \int_0^{\tau} e^{(k^2+m_1^2)\tau_1} d\tau_1 +$$

$$+ \int_0^{\tau} e^{-(k^2+m_1^2)(\tau-\tau_1)} \hat{\eta}(k,\tau_1;\omega) d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k) e^{-(k^2+m_1^2)\tau} +$$

$$+(2\pi)^4 a \Big(m^2 + \frac{\lambda}{5!}a^4 \Big)\delta^4(k) e^{-(k^2+m^2)\tau} \Big[\frac{e^{(k^2+m^2)\tau}}{k^2+m_1^2} - \frac{1}{k^2+m_1^2} \Big] + \qquad (2.5.11')$$

$$+ \int_0^{\tau} e^{-(k^2+m_1^2)(\tau-\tau_1)} \hat{\eta}(k,\tau_1;\omega) d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k) e^{-(k^2+m_1^2)\tau} + (2\pi)^4 \frac{a\Big(m^2 + \frac{\lambda}{5!}a^4 \Big)\delta^4(k)}{k^2+m_1^2} \Big[1 - e^{-(k^2+m_1^2)\tau} \Big] +$$

$$+ \int_0^{\tau} e^{-(k^2+m_1^2)(\tau-\tau_1)} \hat{\eta}(k,\tau_1;\omega) d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k) \left[e^{-(k^2+m_1^2)\tau} + \frac{\Big(m^2 + \frac{\lambda}{5!}a^4 \Big)}{k^2+m_1^2} \Big[1 - e^{-(k^2+m_1^2)\tau} \Big] \right] +$$

$$+ \int_0^{\tau} e^{-(k^2+m_1^2)(\tau-\tau_1)} \hat{\eta}(k,\tau_1;\omega) d\tau_1 .$$

$$m_1^2 = m^2 + \frac{\lambda}{4!}a^4$$

From Eq.(2.5.10)-Eq.(2.5.11) one obtains

$$\widehat{v}_{-}(k_1,\tau,a)\widehat{v}_{+}(k_2,\tau',a') =$$

$$\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m_1^2)\tau} + \frac{m^2+\frac{\lambda}{5!}a^4}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]+\right.$$

$$\left.+\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right\}\times \qquad (2.5.12)$$

$$\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{5!}a^4}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right]+\right.$$

$$\left.+\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_1)}\widehat{\eta}(k_2,\tau_1;\omega)d\tau_1\right\}$$

From Eq.(2.5.12) one obtains

$$\langle\widehat{v}_{-}(k_1,\tau,a)\widehat{v}_{+}(k_2,\tau',a')\rangle_{\eta} =$$

$$\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2+\frac{\lambda}{5!}a^4}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\right\}\times$$

$$\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{5!}a'^4}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right]\right\}-$$

$$-\left\langle\left(\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right)\left(\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_2)}\widehat{\eta}(k_2,\tau_2;\omega)d\tau_2\right)\right\rangle_{\eta} = \qquad (2.5.13)$$

$$-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m_1^2)\tau} + \frac{m^2+\frac{\lambda}{5!}a^4}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times$$

$$\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{5!}a'^4}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right]-$$

$$-\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\langle\widehat{\eta}(k_1,\tau_1;\omega)\widehat{\eta}(k_2,\tau_2;\omega)\rangle_{\eta}d\tau_1 d\tau_2$$

Note that

$$-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2 + \frac{\lambda}{5!}a^4}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times$$

$$\left[e^{-(k_2^2+m^2)\tau'} + \frac{m^2 + \frac{\lambda}{5!}a'^4}{k_2^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau'}\right]\right]\Bigg|_{\tau'=\tau, a'=a} =$$

$$-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m_1^2)\tau} + \frac{m^2 + \frac{\lambda}{5!}a^4}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times \tag{2.5.14}$$

$$\times\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2 + \frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]$$

$$m_1^2 = m^2 + \frac{\lambda}{4!}a^4$$

and note that

$$\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\langle\widehat{\eta}(k_1,\tau_1;\omega)\widehat{\eta}(k_2,\tau_2;\omega)\rangle_\eta d\tau_1 d\tau_2\Bigg|_{\tau'=\tau, a'=a} =$$

$$\delta(k_1+k_2)\int_0^{\tau} e^{-(k_2^2+m_1^2)(\tau-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\delta(\tau_1-\tau_2)d\tau_1 d\tau_2 =$$

$$2(2\pi)^4\delta(k_1+k_2)\int_0^{\tau} e^{-(k_1^2+k_2^2+2m_1^2)(\tau-\tau_1)}d\tau_1 =$$

$$2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m_1^2)\tau}\int_0^{\tau} e^{(k_1^2+k_2^2+2m_1^2)\tau_1}d\tau_1 = \tag{2.5.15}$$

$$2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m_1^2)\tau}\left[\frac{1}{(k_1^2+k_2^2+2m_1^2)}e^{(k_1^2+k_2^2+m_1^2)\tau} - \frac{1}{(k_1^2+k_2^2+2m_1^2)}\right] =$$

$$= \frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m_1^2} - 2(2\pi)^4\delta(k_1+k_2)\frac{e^{-(k_1^2+k_2^2+2m_1^2)\tau}}{(k_1^2+k_2^2+2m_1^2)}$$

$$m_1^2 = m^2 + \frac{\lambda}{4!}a^4$$

Transcendental master equation corresponding to two-point Eucledian Green function
$G_{(4)}(x_1,x_2,m,\lambda)$ in Euclidean space E_4 with $\dim E_4 = 4$ reads

$$
\begin{gathered}
\frac{\left[a(x_1-x_2)^2\right]\left(m^2+\frac{\lambda}{5!}a(x_1-x_2)^4\right)^2}{\left(m^2+\frac{\lambda}{4!}a(x_1-x_2)^4\right)^2}- \\
-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2+\frac{\lambda}{4!}a(x_1-x_2)^4}=0.
\end{gathered}
\tag{2.5.16}
$$

Transcendental master equation corresponding to two-point Eucledian Green function

$G_{(D)}(x_1,x_2,m,\lambda)$ in Euclidean space E_D with $\dim E_D = D$ reads

$$
\begin{gathered}
\frac{\left[a(x_1-x_2)^2\right]\left(m^2+\frac{\lambda}{5!}a(x_1-x_2)^4\right)^2}{\left(m^2+\frac{\lambda}{4!}a(x_1-x_2)^4\right)^2}- \\
-(2\pi)^{-D}w\text{-}\lim_{\eta\to 0_+}\int d^Dk\frac{e^{-\eta k+ik(x_1-x_2)}}{k^2+m^2+\frac{\lambda}{4!}a(x_1-x_2)^4}=0. \\
\eta>0,
\end{gathered}
\tag{2.5.17}
$$

where a weak limit taken in $\mathcal{L}'(\mathbb{R}^D_x)$, see Apendix 1.

Note that

$$
\begin{gathered}
\int d^4k\frac{e^{ik(x_1-x_2)}}{\left[k^2+m^2+\frac{\lambda}{4!}a(x_1-x_2)^4\right]^2}= \\
(4\pi)^{-2}\left(\frac{m_1}{|x_1-x_2|}\right)K_0(m_1|x_1-x_2|) \\
m_1=\sqrt{m^2+\frac{\lambda}{4!}a(x_1-x_2)^4}, \\
K_0(m_1|x_1-x_2|)\asymp -\ln\left(\frac{m_1|x_1-x_2|}{2}\right)-\gamma \text{ as } m_1|x_1-x_2|\to 0
\end{gathered}
\tag{2.5.18}
$$

see Apendix 1. In order to derive Eq.(2.5.18) we applied Eq.(2.5.19) (see formula 13.6 (2) from [13]).

$$
\begin{gathered}
\int_0^\infty\frac{y^{\nu+1}J_\nu(by)dy}{(y^2+p^2)^{\mu+1}}=\frac{b^\mu p^{\nu-\mu}}{2^\mu\Gamma(\mu+1)}K_{\nu-\mu}(bp) \\
-1<\operatorname{Re}\nu<2\operatorname{Re}\mu+1.5.
\end{gathered}
\tag{2.5.19}
$$

.

3.Weak coupling. Nonperturbative result.

3.1.Weak coupling.The $\lambda\phi_d^4$ theory

We assume now that

$$\varepsilon = \frac{\lambda\theta_\delta(|x|)a(x_1 - x_2)^2}{m^2} \ll 1, \tag{3.1.1}$$

where $\theta_\delta(|x|) = \theta_\delta(|x_1 - x_2|) = \theta(|x_1 - x_2| - \delta), x = |x_1 - x_2|$.
From Eq.(2.4.41) and Eq.(3.1.1) we get

$$\theta_\delta(|x|)a^2(x)\frac{\left(1 + \frac{\lambda a^2(x)}{6m^2}\right)^2}{\left(1 + \frac{0.5\lambda a^2(x)}{m^2}\right)^2} - (2\pi)^{-4}\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2 + k^2)\left[1 + \frac{0.5\lambda a^2(x)}{m^2 + k^2}\right]}. \tag{3.1.2}$$

and

$$\theta_\delta(|x|)a^2(x) = (2\pi)^{-4}\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2 + k^2)}\left[1 - \frac{0.5\lambda a^2(x)}{m^2 + k^2} + \left(\frac{0.5\lambda a^2(x)}{m^2 + k^2}\right)^2 + \ldots\right] =$$
$$(2\pi)^{-4}\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2 + k^2)} - 0.5\lambda(2\pi)^{-4}a^2(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2 + k^2)^2} + \tag{3.1.3}$$
$$+0.25\lambda^2(2\pi)^{-4}a^4(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2 + k^2)^3} + \ldots$$

Therefore under condition (3.1.41) we get

$$a^2(x)\left[1 + 0.5\lambda(2\pi)^{-4}\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2 + k^2)^2}\right] = (2\pi)^{-4}\int \frac{\theta_\delta(|x|)d^4ke^{ikx}}{(m^2 + k^2)} + O(\lambda^2), \tag{3.1.4}$$

where constant in symbol $O(\lambda^2)$ depend on m^2 and δ.

Remark 3.1.1. Note that for a given values of the parameters m^2 and δ we can choose value of the parameter λ such that the inequality (3.1.1) is satisfied.

Thus finally for two-point for ϕ_4^4 theory in the Euclidean QFT we obtain non perturbative result

$$\theta_\delta(|x_1-x_2|)G(x_1-x_2) = \theta_\delta(|x_1-x_2|)a^2(x_1-x_2) =$$
$$(2\pi)^{-4}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)}\times$$
$$\left[1+0.5\lambda(2\pi)^{-4}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)^2}\right]^{-1} =$$
$$= (2\pi)^{-4}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)} - \tag{3.1.5}$$
$$-0.5\lambda(2\pi)^{-8}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)^2}\times$$
$$\left(\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)}\right)+o(\varepsilon).$$

Remark 3.1.2. To first order in λ, and in coordinate space, the two point function $G_{(4)}(x_1-x_2;\delta) = \theta_\delta(|x_1-x_2|)G_{(4)}(x_1-x_2)$ bounded on region $\mathbb{R}^4\backslash[-\delta,\delta]^4$ in Euclidean space with $\dim = 4$ is

$$G_{(4)}(x_1-x_2;\delta) = \int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(2\pi)^4(m^2+k^2)} -$$
$$-\frac{\lambda}{2}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(2\pi)^4(m^2+k^2)^2}\times\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(2\pi)^4(m^2+k^2)} = \tag{3.1.6}$$
$$\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(2\pi)^4(m^2+k^2)}\left\{1-\frac{\lambda}{2}\frac{G_{(4)F}(x_1-x_2;\delta)}{(m^2+k^2)}\right\},$$

where $G_{(4)F}(x_1-x_2;\delta) = \theta_\delta(|x_1-x_2|)G_{(4)F}(x_1-x_2)$. For $\lambda \ll 1$ we get

$$\left\{1-\frac{\lambda}{2}\frac{G_{(4)F}(x_1-x_2;\delta)}{(m^2+k^2)}\right\} \simeq \left\{1+\frac{0.5\lambda G_{(4)F}(x_1-x_2;\delta)}{(m^2+k^2)}\right\}^{-1}$$
$$= \frac{m^2+k^2}{m^2+0.5\lambda G_{(4)F}(x_1-x_2;\delta)+k^2}. \tag{3.1.7}$$

From Eq.(3.1.6) and Eq.(3.1.7) we get

$$G_{(4)}(x_1-x_2;\delta) = \frac{1}{(2\pi)^4}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{m^2+0.5\lambda G_{(4)F}(x_1-x_2;\delta)+k^2}. \tag{3.1.8}$$

From Eq.(3.1.8) finally we get

$$G_{(4)}(x_1-x_2;\delta) \simeq \frac{1}{(2\pi)^4}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{m^2+0.5\lambda G_{(4)F}(\delta;\delta)+k^2}. \tag{3.1.9}$$

This expression leads us to define $\mu^2_{\mathbf{ren}}$ by

$$\mu_{\mathbf{ren}}^2 = m^2 + 0.5\lambda G_{(4)F}(\delta;\delta) = m^2 + \delta m^2. \qquad (3.1.10)$$

Note that in contrast with canonical perturbative calculation (see Apendix 2,eq.2.28) $\delta m^2 = 0.5\lambda G_{(4)F}(\delta;\delta)$ is finite.

Remark 3.1.3. To first order in λ, and in coordinate space, the two point function $G_{(D)}(x_1 - x_2;\delta) \upharpoonright \mathbb{R}^D \backslash [-\delta,\delta]^D = \theta_\delta(|x_1 - x_2|)G_{(4)}(x_1 - x_2)$ bounded on region $\mathbb{R}^D \backslash [-\delta,\delta]^D$

in Euclidean space E_D with $\dim E_D = D$ is

$$G_{(D)}(x_1 - x_2;\delta) = \int \frac{d^D k e^{ik(x_1 - x_2)}\theta_\delta(|x_1 - x_2|)}{(2\pi)^D(m^2 + k^2)} -$$
$$-\frac{\lambda}{2}\int \frac{d^D k e^{ik(x_1 - x_2)}\theta_\delta(|x_1 - x_2|)}{(2\pi)^D(m^2 + k^2)^2} \times \int \frac{d^D k e^{ik(x_1 - x_2)}\theta_\delta(|x_1 - x_2|)}{(2\pi)^D(m^2 + k^2)} = \qquad (3.1.11)$$
$$\int \frac{d^D k e^{ik(x_1 - x_2)}\theta_\delta(|x_1 - x_2|)}{(2\pi)^D(m^2 + k^2)}\left\{1 - \frac{\lambda}{2}\frac{G_{(D)F}(x_1 - x_2;\delta)}{(m^2 + k^2)}\right\},$$

where $G_{(D)F}(x_1 - x_2;\delta) = \theta_\delta(|x_1 - x_2|)G_{(D)F}(x_1 - x_2)$. For $\lambda \ll 1$ we get

$$\left\{1 - \frac{\lambda}{2}\frac{G_{(D)F}(x_1 - x_2;\delta)}{(m^2 + k^2)}\right\} \simeq \left\{1 + \frac{0.5\lambda G_{(D)F}(x_1 - x_2;\delta)}{(m^2 + k^2)}\right\}^{-1}$$
$$= \frac{m^2 + k^2}{m^2 + 0.5\lambda G_{(D)F}(x_1 - x_2;\delta) + k^2}. \qquad (3.1.12)$$

From Eq.(3.1.11) and Eq.(3.1.12) we get

$$G_{(D)}(x_1 - x_2;\delta) = \frac{1}{(2\pi)^D}\int \frac{d^D k e^{ik(x_1 - x_2)}\theta_\delta(|x_1 - x_2|)}{m^2 + 0.5\lambda G_{(D)F}(x_1 - x_2;\delta) + k^2}. \qquad (3.1.13)$$

From Eq.(3.1.13) finally we get

$$G_{(D)}(x_1 - x_2;\delta) \simeq \frac{1}{(2\pi)^D}\int \frac{d^D k e^{ik(x_1 - x_2)}\theta_\delta(|x_1 - x_2|)}{m^2 + 0.5\lambda G_{(D)F}(\delta;\delta) + k^2}. \qquad (3.1.14)$$

This expression leads us to define $\mu_{\mathbf{ren}}^2$ by

$$\mu_{\mathbf{ren}}^2 = m^2 + 0.5\lambda G_{(D)F}(\delta;\delta) = m^2 + \delta m^2. \qquad (3.1.15)$$

Note that in contrast with canonical perturbative calculation (see Apendix 2,eq.2.28) $\delta m^2 = 0.5\lambda G_{(D)F}(\delta;\delta)$ is finite.

3.2.Weak coupling.The $\lambda\phi_d^4$ theory.Exact nonperturbative solution.

Trancendental master equation corresponding to $\lambda\phi_d^4$ theory [see Eq.(3.1.2)] reads

$$\theta_\delta(|x|)a^2(x)\frac{\left(1+\frac{\lambda a^2(x)}{6m^2}\right)^2}{\left(1+\frac{0.5\lambda a^2(x)}{m^2}\right)^2} - (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)\left[1+\frac{0.5\lambda a^2(x)}{m^2+k^2}\right]}. \qquad (3.2.1)$$

Assuming now for simplisity that $\lambda/m^2 \ll 1$, then from Eq.(3.2.1) we obtain

$$\theta_\delta(|x|)a^2(x) \simeq (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)\left[1+\frac{0.5\lambda a^2(x)}{m^2+k^2}\right]}. \qquad (3.2.2)$$

Note that

$$\frac{1}{1+z} = 1-z+z^2-z^3+z^4-\ldots 1+\sum_{n=1}^{\infty}(-1)^n z^n \qquad (3.2.3)$$
$$z>0, z<1.$$

From Eq.(3.2.3) by setting

$$z := \frac{0.5\lambda a^2(x)}{m^2+k^2} \qquad (3.2.4)$$

we obtain

$$\left(1+\frac{0.5\lambda a^2(x)}{m^2+k^2}\right)^{-1} = 1-\frac{0.5\lambda a^2(x)}{m^2+k^2}+\left(\frac{0.5\lambda a^2(x)}{m^2+k^2}\right)^2-\ldots= \qquad (3.2.5)$$
$$1+\sum_{n=1}^{\infty}(-1)^n\frac{0.5\lambda a^2(x)}{m^2+k^2}.$$

From Eq.(3.2.2) using Eq.(3.2.5) we obtain

$$\theta_\delta(|x|)a^2(x) = (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}\left[1-\frac{0.5\lambda a^2(x)}{m^2+k^2}+\left(\frac{0.5\lambda a^2(x)}{m^2+k^2}\right)^2+\ldots\right]=$$
$$(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)} - 0.5\lambda(2\pi)^{-4}a^2(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^2}+$$
$$-0.25\lambda^2(2\pi)^{-4}a^4(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^3}+\ldots= \qquad (3.2.6)$$
$$=(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}+(2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n a^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}.$$

Let $\mathbb{R}^1$ be metric space equipped with distance $\rho[x_1,y_1] = |x_1-y_1|, x_1,y_1 \in \mathbb{R}$.
Define now a map $\mathcal{F}(x,a(x)) : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$
\begin{gathered}
\mathcal{F}(x,a^2(x)) = \\
(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)} + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n a^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} = \\
(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)} - (2\pi)^{-4}\lambda/2\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n2}} + \\
+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n a^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}
\end{gathered} \tag{3.2.7}
$$

We rewrite now Eq.(3.2.7) of the form

$$
\begin{gathered}
\mathcal{F}(x,b(x)) = \\
g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n b^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} = \\
g_0(x) - \lambda g_1(x)b(x) + \\
+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n b^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\},
\end{gathered} \tag{3.2.8}
$$

where we let for a shortness

$$
\begin{gathered}
b(x) = a^2(x), \\
g_0(x) = (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}, \\
g_1(x) = (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{2(m^2+k^2)^2}.
\end{gathered} \tag{3.2.9}
$$

In Eq.(3.2.8) - Eq.(3.2.9) we set for simplisity but without loss of generality $m^2 = 1$.
We define now infinite sequence by

$$
\begin{gathered}
q_n(x) = \mathcal{F}(x,q_{n-1}(x)), n = 1,2,\ldots \\
q_0(x) = g_0(x).
\end{gathered} \tag{3.2.11}
$$

From Eq.(3.2.8) and Eq.(3.2.9) we obtain

$$
\begin{gathered}
q_1(x) = \mathcal{F}(x,g_0(x)) = \\
g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n g_0^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} = \\
g_0(x) - \lambda g_1(x)g_0(x) + \\
+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n g_0^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}.
\end{gathered} \tag{3.2.12}
$$

From Eq.(3.2.12) we obtain

$$
\begin{gathered}
\rho[q_1(x), g_0(x)] = \rho[\mathcal{F}(x, g_0(x)), g_0(x)] = |q_1(x) - q_0(x)| = \\
(2\pi)^{-4}\left|\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n g_0^n(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right| = \\
\left|-\lambda g_1(x)g_0(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n g_0^n(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right| \leq \\
\lambda g_1(x)g_0(x) + (2\pi)^{-4}\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n g_0^n(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right| \leq \\
\lambda g_1(x)g_0(x) + (2\pi)^{-4}\left(\sum_{n=2}^{\infty}(\lambda/2)^n g_0^n(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right) \leq \\
\lambda g_1(x)g_0(x) + (2\pi)^{-4}\left(\sum_{n=2}^{\infty}(\lambda/2)^n g_0^n(x)\theta_\delta(|x|)\int \frac{d^4k}{(m^2+k^2)^{n+1}}\right).
\end{gathered}
\tag{3.2.13}
$$

From Eq.(3.2.8) we obtain

$$
\begin{gathered}
q_2(x) = \mathcal{F}(x, q_1(x)) = \\
g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n q_1^n(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} = \\
= g_0(x) - \lambda g_1(x)q_1(x) + \\
+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_1^n(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}.
\end{gathered}
\tag{3.2.14}
$$

From Eq.(3.2.12) and Eq.(3.2.14) we obtain

$$
\begin{gathered}
\rho[q_2(x),q_1(x)] = \rho[\mathcal{F}(x,q_1(x)),\mathcal{F}(x,q_0(x))] = \\
\rho[\mathcal{F}(x,q_1(x)),q_1(x)] = |\mathcal{F}(x,q_1(x)) - q_1(x)| = \\
\left|\left(g_0(x) - \lambda g_1(x)q_1(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_1^n(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right) - \right. \\
\left. -\left(g_0(x) - \lambda g_1(x)q_0(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_0^n(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right)\right| = \\
|\lambda g_1(x)(q_0(x) - q_1(x)) + \\
+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n[q_1^n(x) - q_0^n(x)]\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} \Bigg| = \\
|\lambda g_1(x)(q_0(x) - q_1(x)) - \\
-(2\pi)^{-4}\lambda(q_0(x) - q_1(x)) \times \\
\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_0^{n-1}(x) + q_0^{n-2}(x)q_1(x) + \ldots + q_0(x)q_1^{n-2}(x) + q_1^{n-1}(x)] \times \right. \\
\left. \times \int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} \Bigg| = \\
\sqrt{\lambda}\,|(q_0(x) - q_1(x))|\Big|\Big[\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda} \times \\
\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_0^{n-1}(x) + q_0^{n-2}(x)q_1(x) + \ldots + q_0(x)q_1^{n-2}(x) + q_1^{n-1}(x)] \times \right. \\
\left. \times \int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} \Big]\Big| \leq \\
\sqrt{\lambda}\,|(q_0(x) - q_1(x))|\left(\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\right)\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}(n-1)q_0^{n-1}(x)\right| \leq \\
\sqrt{\lambda}\,|(q_0(x) - q_1(x))|O(\lambda), O(\lambda) < 1.
\end{gathered}
\tag{3.2.15}
$$

In the last lines we applied formulas

$$
\begin{gathered}
a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}), \\
\int_0^{\infty} \frac{dr^3}{(m^2+r^2)^{n+1}} = \left(\frac{-1}{2(n-1)(m^2+r^2)^{n-1}} + \frac{m^2}{2n(m^2+r^2)^n}\right)\Bigg|_0^{\infty} = \\
\frac{1}{2(n-1)m^{2(n-1)}} - \frac{1}{2nm^{2(n-1)}} = \frac{1}{2m^{2(n-1)}}\left(\frac{1}{(n-1)} - \frac{1}{n}\right),
\end{gathered}
\tag{3.2.16}
$$

and Euler–Maclaurin sum formula

$$\sum_{n=a}^{\infty} f(n) = \int_a^{\infty} f(x)dx + 0.5f(a) - \sum_{i=2}^{k} \frac{b_i}{i!} f^{(i-1)}(a) - \int_a^{\infty} \frac{B_k(\{1-t\})}{k!} dt \qquad (3.2.17)$$

where b_i is Bernoulli's numbers and $\{x\}$ is the fractional part of x. Note that

$$|q_1(x)| = |\mathcal{F}(x, g_0(x))| \leq g_0(x) + \lambda g_1(x) g_0(x) + \frac{(2\pi)^{-4}(\lambda/2)^2 g_0^2(x)}{1-(\lambda/2)g_0(x)}. \qquad (3.2.18)$$

From Eq.(3.2.8) we obtain

$$\begin{gathered}
|\mathcal{F}(x, b(x))| = \\
\left| g_0(x) + (2\pi)^{-4} \left\{ \sum_{n=1}^{\infty} (-1)^n (\lambda/2)^n b^n(x) \int \frac{d^4k e^{ikx} \theta_\delta(|x|)}{(m^2+k^2)^{n+1}} \right\} \right| \leq \\
g_0(x) + \lambda g_1(x) b(x) + \\
+(2\pi)^{-4} \left| \sum_{n=2}^{\infty} (-1)^n (\lambda/2)^n b^n(x) \int \frac{d^4k e^{ikx} \theta_\delta(|x|)}{(m^2+k^2)^{n+1}} \right| \leq \\
g_0(x) + \lambda g_1(x) b(x) + (2\pi)^{-4} \sum_{n=2}^{\infty} (\lambda/2)^n b^n(x) \int \frac{\theta_\delta(|x|) d^4k}{(m^2+k^2)^{n+1}} \leq \qquad (3.2.19) \\
g_0(x) + \lambda g_1(x) b(x) + (2\pi)^{-4} \sum_{n=2}^{\infty} (\lambda/2)^n b^n(x) = \\
g_0(x) + \lambda g_1(x) b(x) + (2\pi)^{-4} (\lambda/2)^2 b^2(x) \sum_{n=0}^{\infty} (\lambda/2)^n b^n(x) = \\
= g_0(x) + \lambda g_1(x) b(x) + \frac{(2\pi)^{-4}(\lambda/2)^2 b^2(x)}{1-(\lambda/2)b(x)}.
\end{gathered}$$

From Eq.(3.2.8) and Eq.(3.2.14) we obtain

$$
\begin{aligned}
&\rho[q_3(x),q_2(x)] = \rho[\mathcal{F}(x,q_2(x)),\mathcal{F}(x,q_1(x))] = \\
&|\mathcal{F}(x,q_2(x)) - \mathcal{F}(x,q_1(x))| = \\
&\left|\left(g_0(x) - \lambda g_1(x)q_2(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_2^n(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right) - \right.\\
&\left.-\left(g_0(x) - \lambda g_1(x)q_1(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_1^n(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right)\right| = \\
&|\lambda g_1(x)(q_2(x) - q_1(x)) + \\
&+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n[q_2^n(x) - q_1^n(x)]\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Bigg| = \\
&|\lambda g_1(x)(q_2(x) - q_1(x)) - \\
&-(2\pi)^{-4}\lambda(q_2(x) - q_1(x)) \times \\
&\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_1^{n-1}(x) + q_1^{n-2}(x)q_2(x) + \ldots + q_1(x)q_2^{n-2}(x) + q_2^{n-1}(x)] \times\right. \\
&\left.\times\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Bigg| = \\
&\sqrt{\lambda}\,|(q_2(x) - q_1(x))|\Big|\Big[\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\ \times \\
&\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_1^{n-1}(x) + q_1^{n-2}(x)q_2(x) + \ldots + q_1(x)q_2^{n-2}(x) + q_2^{n-1}(x)] \times\right. \\
&\left.\times\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Big]\Big| \leq \\
&\sqrt{\lambda}\,|(q_1(x) - q_2(x))|\left(\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\right)\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}(n-1)q_1^{n-1}(x)\right| \leq \\
&\sqrt{\lambda}\,|(q_1(x) - q_2(x))|O(\lambda), O(\lambda) < 1.
\end{aligned}
\tag{3.2.20}
$$

Processing inductively we get

$$
\begin{gathered}
\rho[q_{n+1}(x),q_n(x)] = \rho[\mathcal{F}(x,q_{n+1}(x)),\mathcal{F}(x,q_n(x))] = \\
|\mathcal{F}(x,q_{n+1}(x)) - \mathcal{F}(x,q_n(x))| = \\
\left|\left(g_0(x) - \lambda g_1(x)q_{n+1}(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_{n+1}^n(x)\int\frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right) -\right. \\
\left.-\left(g_0(x) - \lambda g_1(x)q_n(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_n^n(x)\int\frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right)\right| = \\
|\lambda g_1(x)(q_{n+1}(x) - q_n(x)) + \\
\left.+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n[q_{n+1}^n(x) - q_n^n(x)]\int\frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\ \right| = \\
|\lambda g_1(x)(q_{n+1}(x) - q_n(x)) - \\
-(2\pi)^{-4}\lambda(q_{n+1}(x) - q_n(x)) \times \\
\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_n^{n-1}(x) + q_n^{n-2}(x)q_{n+1}(x) + \ldots + q_n(x)q_{n+1}^{n-2}(x) + q_{n+1}^{n-1}(x)] \times\right. \qquad (3.2.21) \\
\left.\times\int\frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Bigg| = \\
\sqrt{\lambda}\,|(q_{n+1}(x) - q_n(x))|\Big|\Big[\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\ \times \\
\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_n^{n-1}(x) + q_n^{n-2}(x)q_{n+1}(x) + \ldots + q_n(x)q_{n+1}^{n-2}(x) + q_{n+1}^{n-1}(x)] \times\right. \\
\left.\times\int\frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Bigg]\Bigg| \leq \\
\sqrt{\lambda}\,|(q_{n+1}(x) - q_n(x))|\left(\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\right)\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}(n-1)q_n^{n-1}(x)\right| \leq \\
\sqrt{\lambda}\,|(q_{n+1}(x) - q_n(x))|O(\lambda), O(\lambda) < 1.
\end{gathered}
$$

Theorem 3.2.1. Infinite sequence defined by

$$
\begin{gathered}
q_n(x) = \mathcal{F}(x,q_{n-1}(x)), n = 1,2,\ldots \\
q_0(x) = g_0(x)
\end{gathered} \qquad (3.2.22)
$$

for any $x, |x| > \delta$ has a limit: $\lim_{n\to\infty} q_n(x) = \bar{b}(x)$ such that $\mathcal{F}(x,\bar{b}(x)) = \bar{b}(x)$, where

$$\begin{gathered}\mathcal{F}(x,b(x)) = \\ g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n b^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} = \\ g_0(x) - \lambda g_1(x)b(x) + \\ +(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n b^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\},\end{gathered} \tag{3.2.23}$$

and where

$$\begin{gathered}b(x) = a^2(x), \\ g_0(x) = (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}, \\ g_1(x) = (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{2(m^2+k^2)^2}, \\ |x| > \delta.\end{gathered} \tag{3.2.24}$$

Proof. Immediate from (3.2.21) by Generalized Banach fixed-point theorem, see Apendix3 Theorem 3.2..

3.3.Weak coupling.The $\lambda\phi_d^{2n}, n > 2$ theory.Exact nonperturbative solution.

Transcendental master equation (2.5.16) corresponding to two-point Eucledian Green

function $G_{(4)}(x_1,x_2,m,\lambda)$ in Euclidean space E_4 with $\dim E_4 = 4$ reads

$$\begin{gathered}\frac{\left[a(x_1-x_2)^2\right]\left(m^2+\frac{\lambda}{5!}a(x_1-x_2)^4\right)^2}{\left(m^2+\frac{\lambda}{4!}a(x_1-x_2)^4\right)^2} - \\ -(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2+\frac{\lambda}{4!}a(x_1-x_2)^4} = 0.\end{gathered} \tag{3.3.1}$$

Assuming now for simplisity that $\lambda/m^2 \ll 1$, then from Eq.(3.3.1) we obtain

$$\theta_\delta(|x|)a^2(x) \simeq (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)\left[1+\frac{\lambda a^4(x)}{4!(m^2+k^2)}\right]}. \tag{3.3.2}$$

Note that

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - \ldots 1 + \sum_{n=1}^{\infty}(-1)^n z^n \qquad (3.3.3)$$
$$z > 0, z < 1.$$

From Eq.(3.3.3) by setting

$$z := \frac{\lambda a^4(x)}{4!(m^2+k^2)} \qquad (3.3.4)$$

we obtain

$$\left(1 + \frac{\lambda a^4(x)}{4!(m^2+k^2)}\right)^{-1} = 1 - \frac{\lambda a^4(x)}{4!(m^2+k^2)} + \left(\frac{\lambda a^4(x)}{4!(m^2+k^2)}\right)^2 - \ldots = \\ = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{\lambda a^4(x)}{4!(m^2+k^2)}. \qquad (3.3.5)$$

From Eq.(3.3.2) using Eq.(3.3.5) we obtain

$$\theta_\delta(|x|)a^2(x) = \\ (2\pi)^{-4}\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}\left[1 - \frac{\lambda a^4(x)}{4!(m^2+k^2)} + \left(\frac{\lambda a^4(x)}{4!(m^2+k^2)}\right)^2 + \ldots\right] = \\ (2\pi)^{-4}\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{4!(m^2+k^2)} - \lambda(2\pi)^{-4}a^4(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{4!(m^2+k^2)^2} + \\ -\lambda^2(2\pi)^{-4}a^8(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(4!)^2(m^2+k^2)^3} + \ldots = \\ = (2\pi)^{-4}\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)} + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/4!)^n a^{4n}(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}. \qquad (3.3.6)$$

Let $\mathbb{R}^1$ be metric space equipped with distance $\rho[x_1,y_1] = |x_1 - y_1|, x_1, y_1 \in \mathbb{R}$.
Define now a map $\mathcal{F}(x,a(x)) : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$\mathcal{F}(x,a^2(x)) = \\ (2\pi)^{-4}\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)} + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/4!)^n a^{4n}(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} = \\ (2\pi)^{-4}\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)} - (2\pi)^{-4}(\lambda/4!)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n2}} + \\ +(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/4!)^n a^{4n}(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} \qquad (3.3.7)$$

We rewrite now Eq.(3.3.7) of the form

$$\mathcal{F}(x,b(x)) =$$
$$g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/4!)^n b^{4n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} =$$
$$g_0(x) - (\lambda/4!)g_1(x)b(x) +$$
$$+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n b^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}, \quad (3.2.8)$$

where we let for a shortness

$$b(x) = a^2(x),$$
$$g_0(x) = (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)},$$
$$g_1(x) = (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{2(m^2+k^2)^2}. \quad (3.3.9)$$

In Eq.(3.3.8) - Eq.(3.3.9) we set for simplisity but without loss of generality $m^2 = 1$.
We define now infinite sequence by

$$q_n(x) = \mathcal{F}(x,q_{n-1}(x)), n = 1,2,\ldots$$
$$q_0(x) = g_0(x). \quad (3.3.11)$$

From Eq.(3.3.8) and Eq.(3.3.9) we obtain

$$q_1(x) = \mathcal{F}(x,g_0(x)) =$$
$$g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/4!)^n g_0^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} =$$
$$g_0(x) - (\lambda/4!)g_1(x)g_0^2(x) +$$
$$+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/4!)^n g_0^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}. \quad (3.3.12)$$

From Eq.(3.3.12) we obtain

$$
\begin{aligned}
&\rho[q_1(x),g_0(x)] = \rho[\mathcal{F}(x,g_0(x)),g_0(x)] = |q_1(x)-q_0(x)| = |q_1(x)-g_0(x)| = \\
&(2\pi)^{-4}\left|\sum_{n=1}^{\infty}(-1)^n(\lambda/4!)^n g_0^{2n}(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right| = \\
&\left|-(\lambda/4!)g_1(x)g_0^2(x)+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/4!)^n g_0^{2n}(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right| \leq \\
&(\lambda/4!)g_1(x)g_0^2(x)+(2\pi)^{-4}\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/4!)^n g_0^{2n}(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right| \leq \\
&(\lambda/4!)g_1(x)g_0^2(x)+(2\pi)^{-4}\left(\sum_{n=2}^{\infty}(\lambda/4!)^n g_0^{2n}(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right) \leq \\
&(\lambda/4!)g_1(x)g_0^2(x)+(2\pi)^{-4}\left(\sum_{n=2}^{\infty}(\lambda/4!)^n g_0^{2n}(x)\theta_\delta(|x|)\int \frac{d^4k}{(m^2+k^2)^{n+1}}\right).
\end{aligned}
\tag{3.3.13}
$$

From Eq.(3.3.8) we obtain

$$
\begin{aligned}
&q_2(x) = \mathcal{F}(x,q_1(x)) = \\
&g_0(x)+(2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n q_1^{2n}(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} = \\
&= g_0(x)-\lambda g_1(x)q_1^2(x)+ \\
&+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_1^{2n}(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}.
\end{aligned}
\tag{3.3.14}
$$

From Eq.(3.3.12) and Eq.(3.3.14) we obtain

$$
\begin{aligned}
&\rho[q_2(x),q_1(x)] = \rho[\mathcal{F}(x,q_1(x)),\mathcal{F}(x,q_0(x))] = \\
&= \rho[\mathcal{F}(x,q_1(x)),q_1(x)] = |\mathcal{F}(x,q_1(x)) - q_1(x)| = \\
&\left|\left(g_0(x) - \lambda g_1(x)q_1^2(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_1^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right)-\right. \\
&\left.-\left(g_0(x) - \lambda g_1(x)q_0^2(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_0^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right)\right| = \\
&|\lambda g_1(x)(q_0^2(x) - q_1^2(x)) + \\
&+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n[q_1^{2n}(x) - q_0^{2n}(x)]\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Bigg| = \\
&|\lambda g_1(x)(q_0^2(x) - q_1^2(x)) - \\
&-(2\pi)^{-4}\lambda(q_0(x) - q_1(x)) \times \qquad (3.3.15) \\
&\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_0^{2n-1}(x) + q_0^{2n-2}(x)q_1(x) + \ldots + q_0(x)q_1^{2n-2}(x) + q_1^{2n-1}(x)]\times\right. \\
&\left.\times\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Bigg| = \\
&\sqrt{\lambda}\,|(q_0(x) - q_1(x))||(q_0(x) + q_1(x))|\Big|\Big[\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\ \times \\
&\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_0^{2n-1}(x) + q_0^{2n-2}(x)q_1(x) + \ldots + q_0(x)q_1^{2n-2}(x) + q_1^{2n-1}(x)]\times\right. \\
&\left.\times\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Big]\Big| \leq \sqrt{\lambda}\,|(q_1(x) - q_0(x))| \times c_1, \\
&c_1 \leq 1.
\end{aligned}
$$

In the last lines we applied formula

$$a^{2n} - b^{2n} = (a-b)(a^{2n-1} + a^{2n-2}b + a^{2n-3}b^2 + \ldots\ + ab^{2n-2} + b^{2n-1}) \qquad (3.3.16)$$

and Euler–Maclaurin sum formula

$$\sum_{n=a}^{\infty} f(n) = \int_a^{\infty} f(x)dx + 0.5f(a) - \sum_{i=2}^{k}\frac{b_i}{i!}f^{(i-1)}(a) - \int_a^{\infty}\frac{B_k(\{1-t\})}{k!}dt \qquad (3.3.17)$$

where b_i is Bernoulli's numbers and $\{x\}$ is the fractional part of x. Note that

$$|q_1(x)| = |\mathcal{F}(x,g_0(x))| \leq g_0(x) + \lambda g_1(x)g_0^2(x) + \frac{(2\pi)^{-4}(\lambda/2)^2 g_0^2(x)}{1-(\lambda/2)g_0^2(x)}. \qquad (3.3.18)$$

From Eq.(3.3.8) we obtain

$$
\begin{gathered}
|\mathcal{F}(x,b(x))| = \\
\left| g_0(x) + (2\pi)^{-4}\left\{ \sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n b^{2n}(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}} \right\} \right| \leq \\
g_0(x) + \lambda g_1(x)b^2(x) + \\
+(2\pi)^{-4}\left| \sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n b^{2n}(x)\int \frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}} \right| \leq \\
g_0(x) + \lambda g_1(x)b^2(x) + (2\pi)^{-4}\sum_{n=2}^{\infty}(\lambda/2)^n b^{2n}(x)\int \frac{\theta_\delta(|x|)d^4k}{(m^2+k^2)^{n+1}} \leq \qquad (3.3.19) \\
g_0(x) + \lambda g_1(x)b^2(x) + (2\pi)^{-4}\sum_{n=2}^{\infty}(\lambda/2)^n b^{2n}(x) = \\
g_0(x) + \lambda g_1(x)b(x) + (2\pi)^{-4}(\lambda/2)^2 b^2(x)\sum_{n=0}^{\infty}(\lambda/2)^n b^{2n}(x) = \\
= g_0(x) + \lambda g_1(x)b^2(x) + \frac{(2\pi)^{-4}(\lambda/2)^2 b^2(x)}{1-(\lambda/2)b^2(x)}
\end{gathered}
$$

Processing inductively we get

$$\rho[q_{n+1}(x),q_n(x)] = \rho[\mathcal{F}(x,q_{n+1}(x)),\mathcal{F}(x,q_n(x))] =$$
$$|\mathcal{F}(x,q_{n+1}(x)) - \mathcal{F}(x,q_n(x))| =$$
$$\left|\left(g_0(x) - \lambda g_1(x)q_{n+1}^2(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_{n+1}^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right) -\right.$$
$$\left.-\left(g_0(x) - \lambda g_1(x)q_n^2(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_n^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right)\right| =$$
$$|\lambda g_1(x)(q_{n+1}(x) - q_n(x))(q_{n+1}(x) + q_n(x)) +$$
$$\left.+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n[q_{n+1}^{2n}(x) - q_n^{2n}(x)]\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right| =$$
$$|\lambda g_1(x)(q_{n+1}(x) - q_n(x))(q_{n+1}(x) + q_n(x)) -$$
$$-(2\pi)^{-4}\lambda(q_{n+1}(x) - q_n(x)) \times$$
$$\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_n^{2n-1}(x) + q_n^{2n-2}(x)q_{n+1}(x) + \ldots + q_n(x)q_{n+1}^{2n-2}(x) + q_{n+1}^{2n-1}(x)] \times\right. \tag{3.3.21}$$
$$\left.\left.\times\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right| =$$
$$\sqrt{\lambda}\,|(q_{n+1}(x) - q_n(x))||(q_{n+1}(x) + q_n(x))|\Big|\Big[\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\ \times$$
$$\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_n^{2n-1}(x) + q_n^{2n-2}(x)q_{n+1}(x) + \ldots + q_n(x)q_{n+1}^{2n-2}(x) + q_{n+1}^{2n-1}(x)] \times\right.$$
$$\left.\left.\left.\times\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right]\right| \leq$$
$$\sqrt{\lambda}\,|(q_{n+1}(x) - q_n(x))| \times |(q_{n+1}(x) + q_n(x))| \times$$
$$\left(\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\,\right)\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}(n-1)q_n^{2n-1}(x)\right| \leq$$
$$\sqrt{\lambda}\,|(q_{n+1}(x) - q_n(x))|c_n,$$
$$c_n \leq 1.$$

Theorem 3.3.1. Infinite sequence defined by

$$q_n(x) = \mathcal{F}(x,q_{n-1}(x)), n = 1,2,\ldots$$
$$q_0(x) = g_0(x) \tag{3.3.22}$$

for any $x, |x| > \delta$ has a limit: $\lim_{n\to\infty} q_n(x) = \bar{b}(x)$ such that $\mathcal{F}(x,\bar{b}(x)) = \bar{b}(x)$, where

$$\mathcal{F}(x,b(x)) =$$
$$g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n b^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} =$$
$$g_0(x) - \lambda g_1(x)b^2(x) +$$
$$+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n b^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}, \quad (3.3.23)$$

and where

$$b(x) = a^2(x),$$
$$g_0(x) = (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)},$$
$$g_1(x) = (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{2(m^2+k^2)^2}, \quad (3.3.24)$$
$$|x| > \delta.$$

Proof. Immediate from (3.3.21) by Generalized Banach fixed-point theorem, see Apendix3 Theorem 3.2.

4.1.Double stochastic quantization of Abelian gauge fields.

In this section we consider the simplest case of Abelian gauge fields, for which the double stochastic Langevin equation (which we will call double stochastic Parisi - Wu equations henceforth) reads

$$\frac{\partial A_\mu(x,t)}{\partial t} = (\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A_\nu(x,t) + \eta_\mu(x,t;\omega,\omega') + \epsilon\widetilde{\eta}_\mu(x,t;\varpi,\varpi') \quad (4.1.1)$$

The presence of the $\frac{\partial}{\partial t}$ term for the fields A explicitly breaks the gauge invariance of the theory, meaning that one doesn't need to worry about fixing the gauge at all. In principle, one expects this explicit breaking of the gauge freedom to let us quantize the system without having to worry about fixing the gauge or the problems like Gribov Ambiguity that come with it, but this is not the case as we can see below.

Consider the Langevin equation for the gauge fields in the Fourier space,

$$\frac{\partial A_\mu(k,t)}{\partial t} = (\delta_{\mu\nu}k^2 - k_\mu k_\nu)A_\nu(k,t) + \eta_\mu(k,t;\omega,\omega') + \epsilon\widetilde{\eta}_\mu(k,t;\varpi,\varpi') \quad (4.1.2)$$

We can split the above equation into transverse and longitudinal parts, with

$$
\begin{aligned}
A_\mu^T &= \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) A_\nu = O_{\mu\nu}^T A_\nu \\
A_\mu^L &= \frac{k_\mu k_\nu}{k^2} A_\nu = O_{\mu\nu}^L A_\nu \\
\eta_\mu^T &= \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \eta_\nu = O_{\mu\nu}^T \eta_\nu \\
\eta_\mu^L &= \frac{k_\mu k_\nu}{k^2} \eta_\nu = O_{\mu\nu}^L \eta_\nu
\end{aligned}
\tag{4.1.3}
$$

This splits the Parisi-Wu equation as

$$
\begin{aligned}
\frac{\partial A^T(k,t)}{\partial t} &= -k^2 A^T(k,t) + [\eta^T(k,t) + \epsilon \widetilde{\eta}^T(k,t)] \\
\frac{\partial A^L(k,t)}{\partial t} &= \eta^L(k,t) + \epsilon \widetilde{\eta}^L(k,t).
\end{aligned}
\tag{4.1.4}
$$

This equation has solutions

$$
\begin{aligned}
A^T(k,t) &= \exp(-k^2 t) A_0^T + \exp(-k^2 t) \int_0^t \exp(k^2 \tau)[\eta^T(k,\tau) + \epsilon \widetilde{\eta}^T(k,\tau)] d\tau \\
A^L(k,t) &= A_0^L + \int_0^t \exp(k^2(t-\tau))[\eta^L(k,\tau) + \epsilon \widetilde{\eta}^L(k,\tau)] d\tau.
\end{aligned}
\tag{4.1.5}
$$

From this, we observe that the initial distribution in the transverse modes get dissipated

at infinite time due to the presence of the damping term, while in the case of longitudinal mode, there is no damping term and hence the initial distribution persists

even in equilibrium, i.e. there exists no stationary distribution in the transverse modes.

This is directly the consequence of the gauge invariance. To see how this affects calculations, let us look at the propagator for the gauge fields.

$$
\begin{aligned}
D_{\mu\nu}(k,t|k',t') &= \langle A_\mu(k,t)^L A_\nu(k',t')^L \rangle = \\
&= \delta^4(k+k') \frac{1}{k^2} O_{\mu\nu}^T (\exp(k^2(t - t'^2(t+t')))) + \\
&+2t_< \delta^4(k+k') \frac{k_\mu k_\nu}{k^2} + A_\mu(k,0)^L A_\nu(k',0)^L
\end{aligned}
\tag{4.1.6}
$$

The translational invariance of the propagator requires that the propagator be proportional to $\delta^4(k+k')$, which is broken by the last term in the above-derived propagator. One way to circumvent the problem would be to select the initial configuration of the longitudinal mode to be $A_\mu(k,0) = 0$. But such a distribution is exceptional and therefore we consider a more general distribution that is symmetric around $k = 0$ as

$$A_\mu^L = \frac{k_\mu}{k^2}\phi(k) \tag{4.1.7}$$

and for the propagator, we take another average over the distributions $\phi(k)$. Knowing

$$\langle \phi(k)\phi(k')\rangle_\phi = -\alpha\delta^4(k+k') \tag{4.1.8}$$

where α is the width of $\phi(k)$, we get the limit of the propagator as

$$\lim_{t=t'\to\infty} D_{\mu\nu}(k,t|k',t') = \delta^4(k+k')\left\{\frac{1}{k^2}\left(\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right) + 2t\frac{k_\mu k_\nu}{k^2}\right\} \tag{4.1.9}$$

which is nothing but the usual gauge fixed propagator, with a diverging term proportional to t.Therefore one can see that the stochastic quantization prescription allows one to quantize gauge fields and get the propagator of the theory without having to resort to gauge fixing explicitly, but as seen from the above calculations, the gauge freedom is reflected as a choice in the initial distribution for the longitudinal mode of the gauge fields, the width of the distribution α playing the role of the gauge fixing parameter. Another subtlety worth mentioning is that for gauge-invariant quantities, the term proportional to t in the equilibrium limit goes to zero. But the gauge-variant quantities diverge in the limit $t \to \infty$. This is mainly due to the absence of drift force in the transverse part of Langevin's equation, meaning that the stationary distribution for the transverse part doesn't exist, and a particle undergoing Brownian motion in the configuration space will drift forever in this direction. Speaking in terms of forces, the action provides a force that constrains a particle to a gauge orbit in the configuration space, but the particle is free to move along the gauge orbit indefinitely. It is this free motion along the gauge orbits that causes the gauge-variant quantities to diverge in the equilibrium limit. Therefore, for gauge-invariant quantities, the perturbative stochastic quantization prescription gives the same results as the regular perturbative quantization without having to resort to explicit gauge fixing procedures, but in the case of gauge-invariant observables, the theory again leads to divergences.

We set now

$$\begin{aligned} A_\mu(x,t) - a_\mu\theta(t) &= \underline{A}_\mu(x,t,a_\mu), \\ A_\mu(x,t) &= \underline{A}_\mu(x,t,a_\mu) + a_\mu\theta(t). \end{aligned} \tag{4.1.10}$$

Substitution Eq.(4.1.10)-Eq.(4.3.13) into Eq.(4.1.1) gives

$$\frac{\partial\left(\underline{A}_{\mu}^{T}(x,t,a_{\mu})+a_{\mu}\theta(t)\right)}{\partial t}=$$
$$(\delta_{\mu\nu}\partial^{2}-\partial_{\mu}\partial_{\nu})\underline{A}_{\mu}^{T}(x,t,a_{\mu})+\eta_{\mu}^{T}(x,t;\omega,\omega')+\epsilon\widetilde{\eta}_{\mu}^{T}(x,t;\varpi,\varpi') \tag{4.1.11}$$

and thus

$$\frac{\partial\underline{A}_{\mu}^{T}(x,t,a_{\mu})}{\partial t}=$$
$$-a_{\mu}\delta(t)+(\delta_{\mu\nu}\partial^{2}-\partial_{\mu}\partial_{\nu})\underline{A}_{\mu}^{T}(x,t,a_{\mu})+\eta_{\mu}^{T}(x,t;\omega,\omega')+\epsilon\widetilde{\eta}_{\mu}^{T}(x,t;\varpi,\varpi') \tag{4.1.12}$$

Performing a Fourier transformation to momentum space, we get

$$\frac{\partial\underline{A}_{\mu}^{T}(k,t,a_{\mu})}{\partial t}=$$
$$-(2\pi)^{4}a_{\mu}\delta^{4}(k)\delta(t)-k^{2}T_{\mu\nu}\underline{A}_{\nu}^{T}(k,t,a_{\mu})+\eta_{\mu}^{T}(k,t;\omega,\omega')+\epsilon\widetilde{\eta}_{\mu}^{T}(k,t;\varpi,\varpi'), \tag{4.1.13}$$
$$T_{\mu\nu}=\delta_{\mu\nu}-\frac{k_{\mu}k_{\nu}}{k^{2}},$$

where we introduced the transverse projection operator $T_{\mu\nu}$. In order to solve Eq. (4.1.13) one studies, the Green function $G_{\mu\nu}$ which satisfies Eq. (4.1.14)

$$\left(\delta_{\mu\nu}\frac{\partial}{\partial t}+T_{\mu\nu}k^{2}\right)G_{\mu\nu}(k,t)=\delta_{\mu\nu}\delta(t). \tag{4.1.14}$$

This equation is immediately solved, and solution reads

$$G_{\mu\nu}(k,t)=\theta(t)(\exp(-tk^{2}T))_{\mu\nu}=\theta(t)T_{\mu\nu}\exp(-tk^{2}). \tag{4.1.15}$$

The general solution for $A^{T}(k,t)$ follows:

$$A_{\mu}^{T}(k,t,a_{\mu})=\exp(-k^{2}t)A_{0}^{T}-a_{\mu}\delta^{4}(k)\exp(-k^{2}t)\int_{0}^{t}\delta(\tau)\exp(k^{2}\tau)d\tau+$$
$$\exp(-k^{2}t)\int_{0}^{t}\exp(k^{2}\tau)\left[\eta_{\mu}^{T}(k,\tau,\omega,\omega')+\epsilon\widetilde{\eta}_{\mu}^{T}(k,\tau,\varpi,\varpi)\right]d\tau \tag{4.1.16}$$
$$\eta_{\mu}^{T}(k,\tau,\omega,\omega')=\left(\delta_{\mu\nu}-\frac{k_{\mu}k_{\nu}}{k^{2}}\right)\eta_{\nu}(k,\tau,\omega,\omega').$$

4.2.Double stochastic quantization of massive scalar QED

We now consider scalar QED with the Euclidean action given by

$$S = \int d^4x \Big[(D_\mu\varphi)^*(D_\mu\varphi) - \frac{1}{2}m^2\varphi^{*2} - \frac{1}{2}m^2\varphi^2 + \frac{1}{4}F_{\mu\upsilon}F_{\mu\upsilon} \Big], \qquad (4.2.1)$$

where $F_{\mu\upsilon} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, $D_\mu\varphi = (\partial_\mu - ieA_\mu)\varphi$ is the covariant derivative of the field φ and $D_\mu\varphi^* = (\partial_\mu - ieA_\mu)\varphi^*$ is the covariant derivative of the field φ^*. Associated double stochastic Langevin equations reads

$$\begin{gathered} \frac{\partial}{\partial t}A_\mu = \\ \partial_\nu F_{\mu\upsilon} - ie[\varphi^*\partial_\mu\varphi - (\partial_\mu\varphi^*)\varphi - 2ieA_\mu\varphi^*\varphi] + \eta_\mu(x,t;\omega,,\omega') + \epsilon\widetilde{\eta}_\mu(x,t;\varpi,\varpi') = \\ = (\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A_\nu(x,t) - ie[\varphi^*\partial_\mu\varphi - (\partial_\mu\varphi^*)\varphi - 2ieA_\mu\varphi^*\varphi] + \\ +\eta_\mu(x,t;\omega,\omega') + \epsilon\widetilde{\eta}_\mu(x,t;\varpi,\varpi') \end{gathered} \qquad (4.2.2)$$

$$\begin{gathered} \frac{\partial}{\partial t}\varphi = D_\mu D^\mu\varphi - m^2\varphi + \eta(x,\tau;\omega) + \epsilon\widetilde{\eta}(x,t;\varpi,\varpi') = \\ = \partial^2\varphi - ieA_\mu\partial_\mu\varphi - ie\partial_\mu(A_\mu\varphi) - e^2A_\mu A_\mu\varphi - m^2\varphi + \eta(x,t;\omega,\omega') + \epsilon\widetilde{\eta}(x,t;\varpi,\varpi'), \end{gathered} \qquad (4.2.3)$$

$$\begin{gathered} \frac{\partial}{\partial t}\varphi^* = (D_\mu D^\mu\varphi)^* - m^2\varphi^* + \eta^*(x,t;\omega,\omega') + \epsilon\widetilde{\eta}^*(x,t;\varpi,\varpi') = \\ = \partial^2\varphi^* + ieA_\mu\partial_\mu\varphi^* + ie\partial_\mu(A_\mu\varphi^*) - e^2A_\mu A^\mu\varphi^* - m^2\varphi^* + \eta^*(x,t;\omega,\omega') + \\ +\epsilon\widetilde{\eta}^*(x,t;\varpi,\varpi') \end{gathered} \qquad (4.2.4)$$

with the noise correlations

$$\begin{gathered} \langle\eta_\mu(x,t;\omega,\omega')\rangle = \langle\eta(x,t;\omega,\omega')\rangle = \langle\eta^*(x,t;\omega,\omega')\rangle = 0, \\ \Big\langle\widetilde{\eta}_\mu(x,t;\varpi,\varpi')\Big\rangle = \langle\widetilde{\eta}(x,t;\varpi,\varpi')\rangle = \langle\widetilde{\eta}^*(x,t;\varpi,\varpi')\rangle = 0, \end{gathered} \qquad (4.2.5)$$

$$\begin{gathered} \langle\eta_\mu(x,t)\eta_\mu(x',t')\rangle = 2\delta_{\mu\nu}(x-x')\delta(t-t'), \\ \Big\langle\widetilde{\eta}_\mu(x,t)\widetilde{\eta}_\mu(x',t')\Big\rangle = 2\delta_{\mu\nu}(x-x')\delta(t-t'), \end{gathered} \qquad (4.2.6)$$

$$\begin{gathered} \langle\eta(x,t)\eta^*(x,t)\rangle = 2\delta(x-x')\delta(t-t'), \\ \langle\widetilde{\eta}(x,t)\widetilde{\eta}^*(x,t)\rangle = 2\delta(x-x')\delta(t-t'). \end{gathered} \qquad (4.2.7)$$

We set now

$$A_\mu(x,t) - a_\mu\theta(t) = \underline{A}_\mu(x,t,a_\mu), \qquad (4.2.8)$$

$$\varphi(x,t) - a\theta(t) = \underline{\varphi}(x,t,a), \qquad (4.2.9)$$

$$\varphi^*(x,t) - a^*\theta(t) = \underline{\varphi}^*(x,t,a^*). \qquad (4.2.10)$$

Thus

$$A_\mu(x,t) = \underline{A}_\mu(x,t,a_\mu) + a_\mu\theta(t), \tag{4.2.11}$$

$$\varphi(x,t) = \underline{\varphi}(x,t,a) + a\theta(t), \tag{4.2.12}$$

$$\varphi^*(x,t) = \underline{\varphi}^*(x,t,a^*) + a^*\theta(t). \tag{4.2.13}$$

Substitution Eq.(4.2.11)-Eq.(4.2.13) into Eq.(4.2.2) gives

$$\begin{gathered}\frac{\partial}{\partial t}\left[\underline{A}_\mu(x,t,a_\mu) + a_\mu\theta(t)\right] = \frac{\partial}{\partial t}\underline{A}_\mu(x,t,a_\mu) + a_\mu\delta(t) = \\ (\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)[\underline{A}_\nu(x,t) + a_\mu] - ie\Big[\Big(\underline{\varphi}^* + a^*\Big)\partial_\mu\Big(\underline{\varphi} + a\Big) - \\ \Big(\partial_\mu\Big(\underline{\varphi}^* + a^*\Big)\Big)\Big(\underline{\varphi}(x,t,a) + a\Big) - \\ -2ie\Big(\underline{A}_\mu(x,t,a_\mu) + a_\mu\Big)\Big(\underline{\varphi}^*(x,t,a^*) + a^*\Big)\Big(\underline{\varphi}(x,t,a) + a\Big)\Big] + \\ +\eta_\mu(x,t;\omega) + \epsilon\widetilde{\eta}_\mu(x,t;\varpi)\end{gathered} \tag{4.2.14}$$

From Eq.(4.2.14) we get

$$\begin{gathered}\frac{\partial}{\partial t}\underline{A}_\mu(x,t,a_\mu) + a_\mu\delta(t) = \\ (\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\underline{A}_\nu(x,t) - ie\Big[\Big(\underline{\varphi}^* + a^*\Big)\partial_\mu\underline{\varphi} - \\ \Big(\partial_\mu\underline{\varphi}^*\Big)\Big(\underline{\varphi}(x,t,a) + a\Big) - \\ -2ie\Big(\underline{A}_\mu(x,t,a_\mu) + a_\mu\Big)\Big(\underline{\varphi}^*(x,t,a^*) + a^*\Big)\Big(\underline{\varphi}(x,t,a) + a\Big)\Big] + \\ +\eta_\mu(x,t;\omega) + \epsilon\widetilde{\eta}_\mu(x,t;\varpi) = \\ (\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\underline{A}_\nu(x,t) - ie\Big[\Big(\underline{\varphi}^*\partial_\mu\underline{\varphi} + a^*\partial_\mu\underline{\varphi}\Big) - \\ \Big(\partial_\mu\underline{\varphi}^*\Big)\underline{\varphi}(x,t,a) + a\partial_\mu\underline{\varphi}^*(x,t,a^*) - \\ -2ie\Big(\underline{A}_\mu(x,t,a_\mu)\underline{\varphi}^*(x,t,a^*) + a^*\underline{A}_\mu(x,t,a_\mu) + a_\mu\underline{\varphi}^*(x,t,a^*) + aa_\mu\Big) \times \\ \times\Big(\underline{\varphi}(x,t,a) + a\Big)\Big] + \eta_\mu(x,t;\omega) + \epsilon\widetilde{\eta}_\mu(x,t;\varpi).\end{gathered} \tag{4.2.15}$$

From Eq.(4.2.15) finally we get

$$\begin{gathered}\frac{\partial}{\partial t}\underline{A}_\mu(x,t,a_\mu) = \\ -a_\mu\delta(t) + (\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\underline{A}_\nu(x,t,a_\mu) - ie\Big[\Big(\underline{\varphi}^*(x,t,a^*)\partial_\mu\underline{\varphi}(x,t,a) + a^*\partial_\mu\underline{\varphi}(x,t,a)\Big) - \\ -\Big(\partial_\mu\underline{\varphi}^*(x,t,a^*)\Big)\underline{\varphi}(x,t,a) + a\partial_\mu\underline{\varphi}^*(x,t,a^*) - \\ -2ie\Big(\underline{A}_\mu(x,t,a_\mu)\underline{\varphi}^*(x,t,a^*)\underline{\varphi}(x,t,a) + a^*\underline{A}_\mu(x,t,a_\mu)\underline{\varphi}(x,t,a) + a_\mu\underline{\varphi}^*(x,t,a^*)\underline{\varphi}(x,t,a) + \\ +aa_\mu\underline{\varphi}(x,t,a) + a\underline{A}_\mu(x,t,a_\mu)\underline{\varphi}^*(x,t,a^*) + aa^*\underline{A}_\mu(x,t,a_\mu) + aa_\mu\underline{\varphi}^*(x,t,a^*) + a^2a_\mu\Big)\Big] + \\ +\eta_\mu(x,t;\omega) + \epsilon\widetilde{\eta}_\mu(x,t;\varpi).\end{gathered} \tag{4.2.16}$$

From Eq.(4.2.16) we obtain differential master equation corresponding to

$\underline{A}_\mu(x,t,a_\mu)$:

$$\frac{\partial}{\partial t}\underline{A}_\mu(x,t,a_\mu) = \\ -a_\mu\delta(t) + (\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\underline{A}_\nu(x,t,a_\mu) - ie\Big[\Big(a^*\partial_\mu\underline{\varphi}(x,t,a) + a\partial_\mu\underline{\varphi}^*(x,t,a^*)\Big) - \\ -2ie\Big(aa_\mu\underline{\varphi}(x,t,a) + aa^*\underline{A}_\mu(x,t,a_\mu) + aa_\mu\underline{\varphi}^*(x,t,a^*) + a^2a_\mu\Big)\Big] + \eta_\mu(x,t;\omega) \tag{4.2.17}$$

Substitution Eq.(4.2.11)-Eq.(4.2.13) into Eq.(4.2.3) gives

$$\frac{\partial}{\partial t}\Big(\underline{\varphi}(x,t,a) + a\theta(t)\Big) = \\ = \partial^2\Big(\underline{\varphi}(x,t,a) + a\Big) - ie\Big(\underline{A}_\mu(x,t,a_\mu) + a_\mu\Big)\partial_\mu\Big(\underline{\varphi}(x,t,a) + a\Big) - \\ -ie\partial_\mu\Big(\Big(\underline{A}_\mu(x,t,a_\mu) + a_\mu\Big)\Big(\underline{\varphi}(x,t,a) + a\Big)\Big) - e^2\Big(\underline{A}_\mu(x,t,a_\mu) + a_\mu\Big)^2\Big(\underline{\varphi}(x,t,a) + a\Big) \\ -m^2\Big(\underline{\varphi}(x,t,a) + a\Big) + \eta(x,t;\omega) + \epsilon\widetilde{\eta}(x,t;\varpi) = \\ \partial^2\underline{\varphi}(x,t,a) - m^2\underline{\varphi}(x,t,a) - am^2 - ie\Big(\underline{A}_\mu(x,t,a_\mu)\partial_\mu\underline{\varphi}(x,t,a) + a_\mu\partial_\mu\underline{\varphi}(x,t,a)\Big) - \\ -ie\partial_\mu\Big(\underline{A}_\mu(x,t,a_\mu)\underline{\varphi}(x,t,a) + a\underline{A}_\mu(x,t,a_\mu) + a_\mu\underline{\varphi}(x,t,a) + aa_\mu\Big) - \\ -e^2\Big(\underline{A}_\mu(x,t,a_\mu)\underline{A}_\mu(x,t,a_\mu) + 2a_\mu\underline{A}_\mu(x,t,a_\mu) + a_\mu a_\mu\Big)\Big(\underline{\varphi}(x,t,a) + a\Big) + \\ +\eta(x,t;\omega) + \epsilon\widetilde{\eta}(x,t;\varpi) = \\ \partial^2\underline{\varphi}(x,t,a) - m^2\underline{\varphi}(x,t,a) - am^2 - ie\Big(\underline{A}_\mu(x,t,a_\mu)\partial_\mu\underline{\varphi}(x,t,a) + a_\mu\partial_\mu\underline{\varphi}(x,t,a)\Big) - \\ -ie\partial_\mu\Big(\underline{A}_\mu(x,t,a_\mu)\underline{\varphi}(x,t,a)\Big) - iea\partial_\mu\underline{A}_\mu(x,t,a_\mu) - iea_\mu\partial_\mu\underline{\varphi}(x,t,a) - \\ -e^2\Big(\underline{A}_\mu(x,t,a_\mu)\underline{A}_\mu(x,t,a_\mu)\underline{\varphi}(x,t,a) + 2a_\mu\underline{A}_\mu(x,t,a_\mu)\underline{\varphi}(x,t,a) + a_\mu a_\mu\underline{\varphi}(x,t,a)\Big) - \\ -e^2\Big(a\underline{A}_\mu(x,t,a_\mu)\underline{A}_\mu(x,t,a_\mu) + 2aa_\mu\underline{A}_\mu(x,t,a_\mu) + aa_\mu a_\mu\Big) + \eta(x,t;\omega) + \epsilon\widetilde{\eta}(x,t;\varpi). \tag{4.2.18}$$

From Eq.(4.3.18) we obtain differential master equation corresponding to $\underline{\varphi}(x,t,a)$

$$\frac{\partial}{\partial t}\underline{\varphi}(x,t,a) = -a\delta(t) + \partial^2\underline{\varphi}(x,t,a) - m^2\underline{\varphi}(x,t,a) - am^2 - \\ -ie\Big(a_\mu\partial_\mu\underline{\varphi}(x,t,a)\Big) - iea\partial_\mu\underline{A}_\mu(x,t,a_\mu) - iea_\mu\partial_\mu\underline{\varphi}(x,t,a) - \\ -e^2\Big(a_\mu a_\mu\underline{\varphi}(x,t,a) + 2aa_\mu\underline{A}_\mu(x,t,a_\mu) + aa_\mu a_\mu\Big) + \eta(x,t;\omega) + \epsilon\widetilde{\eta}(x,t;\varpi). \tag{4.3.19}$$

Similarly we obtain differential master equation corresponding to $\underline{\varphi}^*(x,t,a)$

$$\frac{\partial}{\partial t}\underline{\varphi}^*(x,t,a^*) = -a^*\delta(t) + \partial^2\underline{\varphi}^*(x,t,a^*) - m^2\underline{\varphi}^*(x,t,a^*) - a^*m^2 - \\ +ie\Big(a_\mu\partial_\mu\underline{\varphi}^*(x,t,a^*)\Big) + iea^*\partial_\mu\underline{A}_\mu(x,t,a_\mu) - iea_\mu\partial_\mu\underline{\varphi}^*(x,t,a^*) - \\ -e^2\Big(a_\mu a_\mu\underline{\varphi}^*(x,t,a^*) + 2a^*a_\mu\underline{A}_\mu(x,t,a_\mu) + a^*a_\mu a_\mu\Big) + \eta^*(x,t;\omega) + \epsilon\widetilde{\eta}^*(x,t;\varpi). \tag{4.2.20}$$

Note that

$$\underline{A}_\mu(k,t,a_\mu) = \int d^4k\exp(ikx)\underline{A}_\mu(x,t,a_\mu), \tag{4.2.21}$$

$$\varphi(k,t,a) = \int d^4k \exp(-ikx)\varphi(x,t,a), \tag{4.2.22}$$

$$\varphi^*(k,t,a^*) = \int d^4k \exp(ikx)\varphi^*(x,t,a). \tag{4.2.23}$$

Performing in differential master equations (4.2.22),(4.2.23) a Fourier transformation (4.2.22)-(4.2.23) to momentum space, we get

$$\begin{gathered}\frac{\partial}{\partial t}\underline{A}_\mu(k,t,a_\mu) = \\ -a_\mu\delta(t)\delta^4(k) - (\delta_{\mu\nu}k^2 + k_\mu k_\nu)\underline{A}_\nu(k,t,a_\mu) - ie\Big[\Big(a^*k_\mu\underline{\varphi}(k,t,a) + ak_\mu\underline{\varphi}^*(k,t,a^*)\Big) \\ -2ie\Big(aa_\mu\underline{\varphi}(x,t,a) + aa^*\underline{A}_\mu(x,t,a_\mu) + aa_\mu\underline{\varphi}^*(x,t,a^*) + a^2a_\mu\delta^4(k)\Big)\Big] + \\ +\eta_\mu(x,t;\omega,\omega').\end{gathered} \tag{4.2.24}$$

$$\begin{gathered}\frac{\partial}{\partial t}\underline{\varphi}(k,t,a) = -a\delta(t)\delta^4(k) - (k^2+m^2)\underline{\varphi}(k,t,a) - am^2\delta^4(k) - \\ -ie\Big(a_\mu k_\mu\underline{\varphi}(k,t,a)\Big) + ieak_\mu\underline{A}_\mu(x,t,a_\mu) - iea_\mu k_\mu\underline{\varphi}(x,t,a) - \\ -e^2\Big(a_\mu a_\mu\underline{\varphi}(k,t,a) + 2aa_\mu\underline{A}_\mu(k,t,a_\mu) + aa_\mu a_\mu\delta^4(k)\Big) + \eta(k,t;\omega,\omega').\end{gathered} \tag{4.2.25}$$

$$\begin{gathered}\frac{\partial}{\partial t}\underline{\varphi}^*(k,t,a^*) = -a^*\delta(t)\delta^4(k) - (k^2+m^2)\underline{\varphi}^*(k,t,a^*) - a^*m^2\delta^4(k) - \\ +ie\Big(a_\mu k_\mu\underline{\varphi}^*(k,t,a^*)\Big) + iea^*k_\mu\underline{A}_\mu(k,t,a_\mu) - iea_\mu k_\mu\underline{\varphi}^*(k,t,a^*) - \\ -e^2\Big(a_\mu a_\mu\underline{\varphi}^*(k,t,a^*) + 2a^*a_\mu\underline{A}_\mu(k,t,a_\mu) + a^*a_\mu a_\mu\delta^4(k)\Big) + \eta^*(k,t;\omega,\omega').\end{gathered} \tag{4.2.26}$$

From (4.2.25),(4.2.26) we get

$$\begin{gathered}\underline{A}_\mu(k,t,a_\mu) = \exp(-k^2t)\int_0^\tau d\tau\exp(k^2t)\{-a_\mu\delta(t)\delta^4(k) \\ -ie\Big[\Big(a^*k_\mu\underline{\varphi}(k,t,a) + ak_\mu\underline{\varphi}^*(k,t,a^*)\Big) \\ -2ie\Big(aa_\mu\underline{\varphi}(x,t,a) + aa^*\underline{A}_\mu(x,t,a_\mu) + aa_\mu\underline{\varphi}^*(x,t,a^*) + a^2a_\mu\delta^4(k)\Big)\Big]\Big\} + \\ +\exp(-k^2t)\int_0^t \exp(k^2\tau)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\eta_\nu(k,\tau;\omega,\omega')d\tau.\end{gathered} \tag{4.2.27}$$

$$\underline{\varphi}(k,t,a) = -a\delta^4(k)\exp[-(k^2+m^2)t]\int_0^t \delta(\tau)\exp[(k^2+m^2)\tau]d\tau +$$
$$+\exp[-(k^2+m^2)t]\int_0^t d\tau\exp[(k^2+m^2)\tau]\{-am^2\delta^4(k) -$$
$$-ie\Big(a_\mu k_\mu \underline{\varphi}(k,t,a)\Big) + ieak_\mu \underline{A}_\mu(x,t,a_\mu) - iea_\mu k_\mu \underline{\varphi}(x,t,a) -$$
$$-e^2\Big(a_\mu a_\mu \underline{\varphi}(k,t,a) + 2aa_\mu \underline{A}_\mu(k,t,a_\mu) + aa_\mu a_\mu \delta^4(k)\Big)\Big\} +$$
$$+\exp[-(k^2+m^2)t]\int_0^t d\tau\exp[(k^2+m^2)\tau]\eta(k,\tau;\omega,\omega') \quad (4.2.28)$$

$$\underline{\varphi}^*(k,t,a) = -(2\pi)^4 a\delta^4(k)\exp[-(k^2+m^2)t]\int_0^t \delta(\tau)\exp[(k^2+m^2)\tau]d\tau +$$
$$+\exp[-(k^2+m^2)t]\int_0^t d\tau\exp[(k^2+m^2)\tau]\Big\{-(2\pi)^4 am^2\delta^4(k) -$$
$$-ie\Big(a_\mu k_\mu \underline{\varphi}^*(k,t,a)\Big) + ieak_\mu \underline{A}_\mu(x,t,a_\mu) - iea_\mu k_\mu \underline{\varphi}^*(x,t,a) -$$
$$-e^2\Big(a_\mu a_\mu \underline{\varphi}^*(k,t,a) + 2aa_\mu \underline{A}_\mu(k,t,a_\mu) + aa_\mu a_\mu (2\pi)^4\delta^4(k)\Big)\Big\} +$$
$$+\exp[-(k^2+m^2)t]\int_0^t d\tau\exp[(k^2+m^2)\tau]\eta^*(k,\tau;\omega,\omega') \quad (4.2.29)$$

Treating the coupling constant e as a small parameter we can solve the equations Eq.(4.2.27)-Eq.(4.2.29) by iteration. We rewrite now Eq.(4.2.27)-Eq.(4.2.29) of the following form

$$\underline{A}_\mu(k,t,a_\mu,e) = \Xi_\mu\Big(\{\underline{A}_\nu(k,t,a_\mu,e)\}_{\nu=1}^4, \underline{\varphi}(k,t,a,e), \underline{\varphi}^*(k,t,a^*,e)\Big)$$
$$\Xi_\mu\Big(\{\underline{A}_\nu(k,t,a_\mu,e)\}_{\nu=1}^4, \underline{\varphi}(k,t,a,e), \underline{\varphi}^*(k,t,a^*,e)\Big) = \quad (4.2.30)$$

$$\underline{\varphi}(k,t,a,e) = \Phi\Big(\{\underline{A}_\nu(k,t,a_\mu,e)\}_{\nu=1}^4, \underline{\varphi}(k,t,a,e), \underline{\varphi}^*(k,t,a^*,e)\Big) \quad (4.2.31)$$

$$\underline{\varphi}^*(k,t,a^*,e) = \Phi^*\Big(\{\underline{A}_\nu(k,t,a_\mu,e)\}_{\nu=1}^4, \underline{\varphi}(k,t,a,e), \underline{\varphi}^*(k,t,a^*,e)\Big) \quad (4.2.32)$$

We define now countable sequences

$$\left\{\underline{A}_\mu^{(n)}(k,t,a_\mu,e)\right\}_{n=0}^{\infty}, \left\{\underline{\varphi}^{(n)}(k,t,a_\mu,e)\right\}_{n=0}^{\infty}, \left\{\underline{\varphi}^{*(n)}(k,t,a_\mu,e)\right\}_{n=0}^{\infty}, \quad (4.2.33)$$

by reqursions

$$\begin{gathered}\underline{A}_\mu^{(n)}(k,t,a_\mu,e) = \Xi_\mu\left(\{\underline{A}_\nu^{(n-1)}(k,t,a_\mu,e)\}_{\nu=1}^{4}, \underline{\varphi}^{(n-1)}(k,t,a,e), \underline{\varphi}^{*(n-1)}(k,t,a^*,e)\right),\\ \underline{A}_\mu^{(0)}(k,t,a_\mu,e) = \underline{A}_\mu^{(0)}(k,t,a_\mu,e=0) = \underline{A}_\mu^{(0)}(k,t);\\ \underline{\varphi}^{(n)}(k,t,a,e) = \Phi\left(\{\underline{A}_\nu^{(n-1)}(k,t,a_\mu,e)\}_{\nu=1}^{4}, \underline{\varphi}^{(n-1)}(k,t,a,e), \underline{\varphi}^{*(n-1)}(k,t,a^*,e)\right),\\ \underline{\varphi}^{(0)}(k,t,a,e) = \underline{\varphi}^{(0)}(k,t,a,e=0) = \underline{\varphi}^{(0)}(k,t);\\ \underline{\varphi}^{*(n)}(k,t,a^*,e) = \Phi^*\left(\{\underline{A}_\nu^{(n-1)}(k,t,a_\mu,e)\}_{\nu=1}^{4}, \underline{\varphi}^{(n-1)}(k,t,a,e), \underline{\varphi}^{*(n-1)}(k,t,a^*,e)\right),\\ \underline{\varphi}^{*(0)}(k,t,a^*,e) = \underline{\varphi}^{*(0)}(k,t,a^*,e=0) = \underline{\varphi}^{*(0)}(k,t).\end{gathered} \quad (4.2.34)$$

In order to obtain $\underline{A}_\mu^{(0)}(k,t), \underline{\varphi}^{(0)}(k,t), \underline{\varphi}^{*(0)}(k,t)$ we set $e = 0$ in Eq.(4.2.27)-Eq.(4.2.29)

$$\frac{\partial}{\partial t}\underline{A}_\mu(k,t) = -k^2 T_{\mu\nu}(k)\underline{A}_\mu + \eta_\mu(x,t;\omega,\omega'), \quad (4.2.\ 35)$$

$$\begin{gathered}\frac{\partial}{\partial t}\underline{\varphi}(k,t,a) = -(2\pi)^4 a\delta(t)\delta^4(k) - p^2\underline{\varphi}(k,t,a) - m^2\underline{\varphi}(k,t,a)\\ -(2\pi)^4 am^2\delta^4(k) + \eta(k,t;\omega,\omega') =\\ -a\delta(t)\delta^4(k) - (p^2+m^2)\underline{\varphi}(k,t,a) - (2\pi)^4 am^2\delta^4(k) + \eta(k,t;\omega,\omega'),\end{gathered} \quad (4.2.36)$$

$$\begin{gathered}\frac{\partial}{\partial t}\underline{\varphi}^*(k,t,a^*) =\\ -(2\pi)^4 a^*\delta(t)\delta^4(k) - p^2\underline{\varphi}(k,t,a^*) - m^2\underline{\varphi}^*(k,t,a^*)\\ -(2\pi)^4 am^2\delta^4(k) + \eta(k,t;\omega,\omega') =\\ -a^*\delta(t)\delta^4(k) - (p^2+m^2)\underline{\varphi}(k,t,a^*) - (2\pi)^4 am^2\delta^4(k) + \eta(k,t;\omega,\omega'),\end{gathered} \quad (4.2.37)$$

This equations is immediately solved, and solutions reads

$$\underline{A}_\mu(k,t,0) = \int_0^t d\tau[T_{\mu\nu}\exp[-k^2(t-\tau)]]\eta_\nu(k,\tau) \quad (4.2.38)$$

$$\underline{\varphi}(k,t,a) = -(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)t}\int_0^t d\tau e^{[-(k^2+m^2)\tau]}\delta(\tau) -$$

$$-(2\pi)^4 am^2\delta^4(k)e^{-(k^2+m^2)t}\int_0^t d\tau e^{[-(k^2+m^2)\tau]} -$$

$$-ieae^{-(k^2+m^2)t}\int_0^t d\tau e^{[-(k^2+m^2)\tau]}\left\{k_\mu \underline{A}_\mu(k,\tau,0) + \eta(k,\tau;\omega)\right\} =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)t} - (2\pi)^4 am^2\delta^4(k)e^{-(k^2+m^2)t}\int_0^t d\tau e^{[-(k^2+m^2)\tau]} -$$

$$-ieae^{-(k^2+m^2)t}\int_0^t d\tau e^{[-(k^2+m^2)\tau]}\left\{k_\mu \underline{A}_\mu(k,\tau,0) + \eta(k,\tau;\omega)\right\} = \quad (4.2.\ )$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)t} - (2\pi)^4 am^2\delta^4(k)e^{-(k^2+m^2)t}\int_0^t d\tau e^{[-(k^2+m^2)\tau]} -$$

$$-ieae^{-(k^2+m^2)t}\int_0^t d\tau e^{[-(k^2+m^2)\tau]}\left\{k_\mu \underline{A}_\mu(k,\tau,0)\right\}$$

$$-ieae^{-(k^2+m^2)t}\int_0^t d\tau e^{[-(k^2+m^2)\tau]}\eta(k,\tau;\omega) =$$

.

4.3. Stochastic quantization of scalar QED by canonical perturbation approach.

We now consider scalar QED with the Euclidean action given by

$$S = \int d^4x\left[(D_\mu\varphi)^*(D_\mu\varphi) - \frac{1}{2}m^2\varphi^{*2} - \frac{1}{2}m^2\varphi^2 + \frac{1}{4}F_{\mu\upsilon}F_{\mu\upsilon}\right], \quad (4.3.1)$$

where $F_{\mu\upsilon} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, $D_\mu\varphi = (\partial_\mu - ieA_\mu)\varphi$ is the covariant derivative of the field φ. Associated Langevin equations reads

$$\frac{\partial}{\partial t}A_\mu =$$
$$\partial_\nu F_{\mu\upsilon} - ie[\varphi^*\partial_\mu\varphi - (\partial_\mu\varphi^*)\varphi - 2ieA_\mu\varphi^*\varphi] + \eta_\mu(x,t;\omega) = \tag{4.3.2}$$
$$(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A_\nu(x,t) - ie[\varphi^*\partial_\mu\varphi - (\partial_\mu\varphi^*)\varphi - 2ieA_\mu\varphi^*\varphi]$$
$$+\eta_\mu(x,t;\omega)$$

$$\frac{\partial}{\partial t}\varphi = D_\mu D^\mu\varphi - m^2\varphi + \eta(x,\tau;\omega) =$$
$$= \partial^2\varphi - ieA_\mu\partial_\mu\varphi - ie\partial_\mu(A_\mu\varphi) - e^2A_\mu A_\mu\varphi - m^2\varphi + \eta(x,t;\omega). \tag{4.3.3}$$

$$\frac{\partial}{\partial t}\varphi^* = (D_\mu D^\mu\varphi)^* - m^2\varphi^* + \eta^*(x,t;\omega) =$$
$$= \partial^2\varphi^* + ieA_\mu\partial_\mu\varphi^* + ie\partial_\mu(A_\mu\varphi^*) - e^2A_\mu A^\mu\varphi^* - m^2\varphi^* + \eta^*(x,t;\omega) \tag{4.3.4}$$

with the noise correlations

$$\begin{gathered}\langle\eta_\mu(x,t)\rangle = 0,\\ \langle\eta_\mu(x,t)\eta_\nu(x,t)\rangle = 2\delta_{\mu\nu}\delta(x-x')\delta(t-t'),\\ \langle\eta(x,t)\rangle = \langle\eta^*(x,t)\rangle = 0,\\ \langle\eta(x,t)\eta^*(x',t')\rangle = 2\delta(x-x')\delta(t-t').\end{gathered} \tag{4.3.5}$$

Note that

$$A_\mu(k,t) = (2\pi)^{-4}\int d^4k\exp(ikx)A_\mu(x,t). \tag{4.3.6}$$

Going to Fourier space, with the additional conventions

$$\varphi(k,t,a) = (2\pi)^{-4}\int d^4k\exp(-ikx)\varphi(x,t,a), \tag{4.3.7}$$

$$\varphi^*(k,t,a^*) = (2\pi)^{-4}\int d^4k\exp(ikx)\varphi^*(x,t), \tag{4.3.8}$$

from Eq.(4.3.2)-Eq.(4.3.8) we obtain the following set of integral equations:

$$A_\mu(k,t) = \int_0^t d\tau[T_{\mu\nu}\exp[-(k^2+m^2)(t-\tau)] + L_{\mu\nu}]\{\eta_\nu(k,\tau) -$$
$$-e\int\frac{d^4pd^4q}{(2\pi)^4}\varphi^*(p,\tau)\varphi(q,\tau)(q_\mu+p_\mu) - \tag{4.3.9}$$
$$-2e^2\int\frac{d^4pd^4qd^4r}{(2\pi)^8}\delta^4(k+p-q+r)A_\nu(p,\tau)\varphi^*(q,\tau)\varphi(r,\tau)$$

$$\varphi(k,t) = \int_0^t d\tau \exp[-(k^2+m^2)(t-\tau)] \times$$

$$\times\left\{[\eta(k,t)] - e\int \frac{d^4pd^4q}{(2\pi)^4}\delta^4(k-p-q)A_\mu(p,t)\right. \tag{4.3.10}$$

$$\left.\times\varphi(q,\tau)(k_\mu+q_\mu) - e^2\int \frac{d^4pd^4qd^4r}{(2\pi)^8}\delta^4(k-p-q-r)A_\mu(p,t)A_\mu(q,t)\varphi(r,\tau)\right\},$$

$$\varphi^*(k,t) = \int_0^t d\tau \exp[-(k^2+m^2)(t-\tau)] \times$$

$$\times\left\{[\eta^*(k,t)] - e\int \frac{d^4pd^4q}{(2\pi)^4}\delta^4(-k-p+q)A_\mu(p,t)\right. \tag{4.3.11}$$

$$\left.\times\varphi^*(q,\tau)(k_\mu+q_\mu) - e^2\int \frac{d^4pd^4qd^4r}{(2\pi)^8}\delta^4(k-p-q+r)A_\mu(p,t)A_\mu(q,t)\varphi^*(r,\tau)\right\},$$

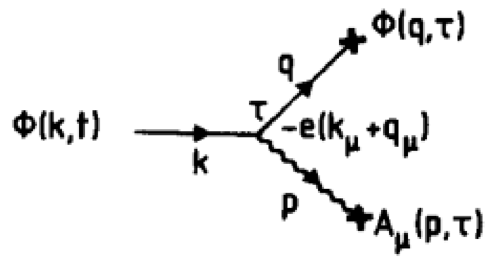
We note that with conventions as in Eq.(4.3.7)-Eq.(4.3.8), the two-point correlations of the scalar fields read

$$\langle \eta(k,t)\eta^*(k',t')\rangle = 2\delta(k-k')\delta(t-t'). \tag{4.3.12}$$

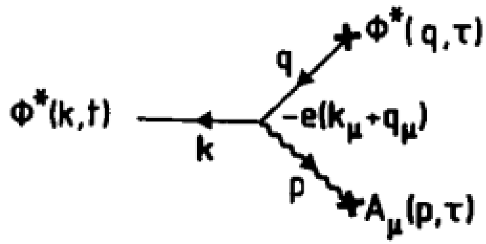
For the assignment of momenta in diagrams, let us explicitly draw the vertices, and indicate the flow of momenta by arrows as in figs. 4.3.1 to 4.3.6.

Let us now calculating the one-loop correction to the scalar propagator $\langle\varphi(k,t)\varphi^*(k',t)\rangle$.

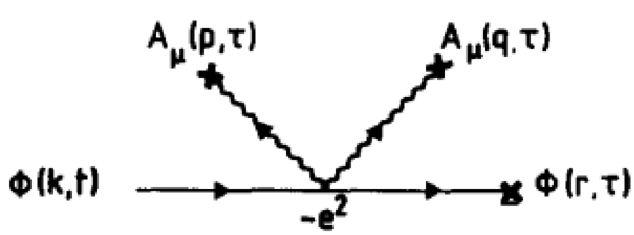
To do so we list first all contributing diagrams in fig. 4.4.1(a)-(g). For the sake of demonstration, we choose this gauge variant quantity on purpose so that we are facing divergences linear in t when performing the equilibrium limit. We would like to split the calculations into two parts, and first obtain the well-converging expression with just transverse gauge field contributions. Subsequently, we will extract the linearly diverging contribution and at the end of this calculation take the limits $\lim_{t\to\infty}\lim_{x\to y}\varphi(x,t)\varphi(y,t)$ upon which the diverging terms disappear explicitly, as generally expected from gauge invariant quantities. Thus we have already acquired all the knowledge to obtain the large t contribution of the diagrams (a) to (e) immediately, avoiding any lengthy calculation. Starting by considering the gauge field as purely transverse, the large t contribution is given by multiplying together vertex factors and projection operators from gauge field lines (at the moment just transverse projection operators); dividing by crossed-line contributions (i.e. the momentum^2 of the corresponding line); multiplying with the fictitious time integration factors (which are identical to the ones discussed in the scalar-field case, as the transverse gauge field has the same exponential fictitious time dependence).



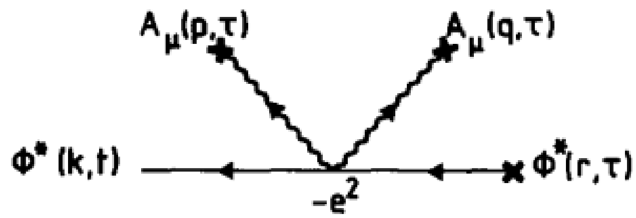
Pic.4.3.1.Vertices in stochastic perturbation theory of scalar QED.



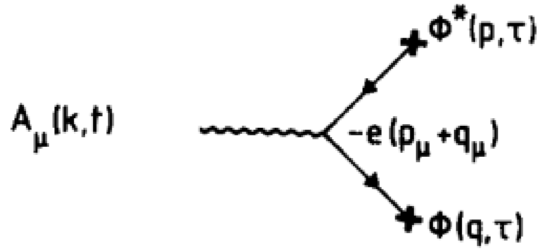
Pic.4.3.2.Vertices in stochastic perturbation theory of scalar QED.



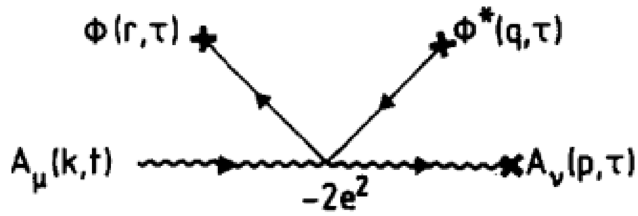
Pic.4.3.3.Vertices in stochastic perturbation theory of scalar QED.



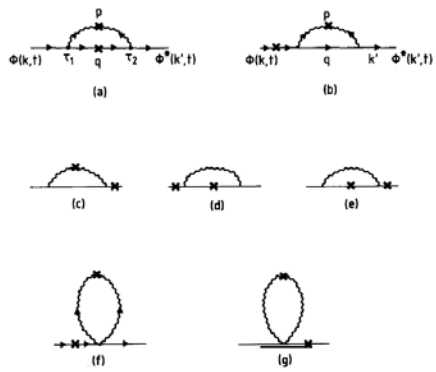
Pic.4.3.4.Vertices in stochastic perturbation theory of scalar QED.



Pic.4.3.5.Vertices in stochastic perturbation theory of scalar QED.



Pic.4.3.6.Vertices in stochastic perturbation theory of scalar QED.



Stochastic diagrams contributing to the one-loop correction to
the $\varphi^*\varphi$ propagator in scalar QED.

Diagram (a) has two different time orderings. The diagram being completely symmetric,
it suffices to take twice the contribution of one of these, say when $\tau_1 < \tau_2$.
Apart from the overall $\int\int d^4pd^4q\delta(k-p-q)\delta(k-k')$ we then get

$$
\begin{aligned}
(a) &= -2e^2\frac{(k_\mu+q_\mu)T_{\mu\nu}(k_\nu+q_\nu)}{p^2q^2}\frac{1}{p^2+q^2+k^2}\frac{1}{2k^2},\\
(a) = (b) &= -e^2\frac{(k_\mu+q_\mu)T_{\mu\nu}(k_\nu+q_\nu)}{p^2q^2}\frac{1}{p^2+q^2+k^2}\frac{1}{2k^2},\\
(d) = (e) &= -e^2\frac{(k_\mu+q_\mu)T_{\mu\nu}(k_\nu+q_\nu)}{p^2q^2}\frac{1}{p^2+q^2+k^2}\frac{1}{2k^2}
\end{aligned}
\tag{4.3.13}
$$

and (not unexpectedly, in fact) the sum gives the conventional result in Landau gauge

$$
-2e^2\int\int d^4pd^4q\delta(k-p-q)\delta(k-k')\frac{(k_\mu+q_\mu)T_{\mu\nu}(k_\nu+q_\nu)}{k^4p^2q^2}. \tag{4.3.14}
$$

The remaining two diagrams can easily be evaluated in a similar way (note that the loop momentum does not contribute in the fictitious time integrations, as its line is connected to just identical fictitious times),

$$
-2e^2\int d^4p\frac{\mathbf{Tr}(T_{\mu\nu})}{k^4p^2} \tag{4.3.15}
$$

Generalizing the results of this little exercise, one concludes that the equivalence proof
for scalar fields carries over directly to scalar QED with transverse gauge bosons, having identical fictitious time dependence. It therefore follows that the sum of stochastic diagrams with transverse gauge field contributions sum up to the corresponding Feynman diagram in Landau gauge.
In scalar QED a general vertex involves always at least one scalar field line which

carries exponential time dependence. Given some fixed time ordering, fictitious time integrations have the general structure

$$\int_0^{\tau_{i+1}} d\tau_i(\tau_i^n \exp(\sum p^2\tau_i^2)) = \quad (4.3.16)$$

5.Double stochastic quantization QFT with massive fermions.

5.1.Basic notions.

Remin that originally developed for bosonic models, it it well known many years that stochastic quantization of fermions received a physically consistent treatment, see ref. [16],[17]. The starting point is a generalization of the original Langevin equation by
means of the introduction of a kernel $k_{ij}(x,y)$

$$\frac{\partial\varphi_i(\tau,x)}{\partial\tau} = -\int d^D y k_{ij}(x,y)\frac{\delta S[\varphi]}{\delta\varphi_i(\tau,y)} + \eta_i(\tau,x), \quad (5.1.1)$$

where $S[\varphi] \triangleq S\left[\{\varphi_i\}_{i=1}^N\right]$ is the Euclidean action, and $\{\eta_i(\tau,x)\}_{i=1}^N$ a distributional noise
field, with correlations

$$\begin{aligned}\langle\eta_i(\tau,x)\rangle &= 0,\\ \langle\eta_i(\tau,x)\eta_j(\tau',x')\rangle &= \delta_{ij}k_{ij}(x,x')\delta(\tau-\tau').\end{aligned} \quad (5.1.2)$$

In the case of free fermions with the classical Euclidean action $S[\overline{\psi}(x)\psi(x)]$, is given by

$$S[\overline{\psi}(x)\psi(x)] = -i\int d^4x\overline{\psi}(x)(\partial\!\!\!/_x+im)\psi(x), \quad (5.1.3)$$

the kernel is given by [1]-[2]:

$$\begin{aligned}k_{ij}(x,y) &= \delta_{ij}(\partial\!\!\!/_x+m)\delta^4(x-y)\\ \partial\!\!\!/ &= \boldsymbol{\gamma}^\mu\partial_\mu, \mu = 0,1,2,3\end{aligned} \quad (5.1.4)$$

and the Langevin equation reads

$$\frac{\partial\psi(\tau,x)}{\partial\tau} = (\partial^2+m^2)\psi(\tau,x) + \vartheta. \quad (5.1.5)$$

The Green function in momentum space reads [1]:

$$G_{\alpha\beta}(k,\tau-\tau') = \delta_{\alpha\beta}e^{-(k^2+m^2)(\tau-\tau')}\widehat{\vartheta}(k,\tau-\tau'). \quad (5.1.6)$$

where

$$\left\langle \widehat{\vartheta}(\mathbf{k},\tau)\widehat{\overline{\vartheta}}(\mathbf{k}',\tau')\right\rangle = (-\not{\mathbf{k}} + m)\delta^4(\mathbf{k}+\mathbf{k}')\delta(\tau-\tau'),$$
$$\left\langle \widehat{\eta}_\mu(\mathbf{k},\tau)\widehat{\eta}_\nu(\mathbf{k}',\tau')\right\rangle =^4 \delta_{\mu\nu}\delta(\mathbf{k}+\mathbf{k}')\delta(\tau-\tau'). \tag{5.1.7}$$

Notice that 2-point function $\langle\psi(x,\tau)\overline{\psi}(x',\tau')\rangle$ corresponding to Euclidean action (5.1.3)
reads

$$\langle\psi(x,\tau)\overline{\psi}(x',\tau')\rangle = \int_0^\infty dt_1 \int_0^\infty dt_2 \int_{\mathbb{R}^4} d^4x_1 \int_{\mathbb{R}^4} d^4x_2 G(x-x_1,\tau-t_1)G(x'-x_2,\tau'-t_2)\times$$
$$\times\langle\vartheta(x_1,t_1)\vartheta(x_2,t_2)\rangle. \tag{5.1.8}$$

Using the stochastic expectation values (4.1.7) and the momentum-space representation (4.1.6) of the Green function, this immediately leads to the momentum-space propagator

$$\langle\psi(\mathbf{k},\tau)\overline{\psi}(\mathbf{k}',\tau')\rangle = -(2\pi)^4\delta^4(\mathbf{k}-\mathbf{k}')\left(\frac{\not{\mathbf{k}}-m}{\mathbf{k}^2+m^2}\right)\times$$
$$\times\exp\{-(\mathbf{k}^2+m^2)(\tau+\tau')\}\times[\exp\{2(\mathbf{k}^2+m^2)\widetilde{\tau}\}-1], \tag{5.1.9}$$

where $\widetilde{\tau} = \min(\tau_1,\tau_2)$. For practical purposes the ‘time ordered’ propagator $\tau<\tau'$ is useful:

$$\Delta(\mathbf{k};\tau,\tau') = -\left(\frac{\not{\mathbf{k}}-m}{\mathbf{k}^2+m^2}\right)\times$$
$$\times\exp\{-(\mathbf{k}^2+m^2)(\tau-\tau')\}\times[1-\exp\{2(\mathbf{k}^2+m^2)\tau'\}]. \tag{5.1.10}$$

Note that for equal times this simply reduces to the standard Euclidean fermion propagator:

$$\lim_{\tau=\tau'\to\infty}\Delta(\mathbf{k};\tau,\tau') = \frac{\not{\mathbf{k}}-m}{\mathbf{k}^2+m^2}. \tag{5.1.11}$$

The four-dimensional action for Euclidean QCD is given by

$$S[A,\psi,\overline{\psi}] = \int d^4x\left[\frac{1}{4}(F^a_{\mu\nu})^2 - \overline{\psi}(x)(\not{\mathbf{D}}_x+im)\psi(x)\right] \tag{5.1.12}$$

where we should use modified covariant kernels

$$k_{\alpha\beta}(x,y) = (\not{\mathbf{D}}_x+im)_{\alpha\beta}\delta^4(x-y) \tag{5.1.13}$$

Thus, the corresponding Langevin equations hold,

$$\frac{\partial}{\partial\tau}\psi_\alpha(x,\tau) = -[(\not{\mathbf{D}}-im)(\not{\mathbf{D}}+im)]_{\alpha\beta}\psi_\beta(x,\tau)+\vartheta_\alpha(x,\tau),$$
$$\frac{\partial}{\partial\tau}\overline{\psi}_\alpha(x,\tau) = -\overline{\psi}_\beta(x,\tau)\left[(\not{\mathbf{D}}'-im)^\top(\not{\mathbf{D}}'-im)^\top\right]_{\alpha\beta}+\overline{\vartheta}_\alpha(x,\tau), \tag{5.1.14}$$

where

$$\mathbf{D}_\mu = \partial_\mu - ieA_\mu,$$
$$\mathbf{D}'_\mu = -\partial_\mu - ieA_\mu, \tag{5.1.15}$$

and for the gauge fields

$$A_\mu = -\partial_\nu F_{\mu\nu} + e\overline{\psi}\gamma_\mu\psi \tag{5.1.16}$$

The Langevin equations for the general non-Abelian case reads

$$\frac{\partial}{\partial\tau}A^a_\lambda(x,\tau) = D_\nu F^a_{\nu\lambda}(x,\tau) + J^a_\lambda(x,\tau) + \eta^a_\lambda(x,\tau)$$
$$J^a_\lambda(x,\tau) = \frac{1}{2}e\overline{\psi}_\alpha(x,\tau)\gamma^\lambda A^a_\lambda(x,\tau)\psi_\alpha(x,\tau) \tag{5.1.17}$$

This corresponds to the Euclidean action

$$S[A,\psi,\overline{\psi}] = \int d^4x\left[\frac{1}{4}(F^a_{\mu\nu})^2 - \overline{\psi}(x)(i\not{D} - m)\psi(x)\right] \tag{5.1.18}$$

with the standard notation

$$F^a_{\lambda\nu}(x,\tau) = \partial_\lambda A^a_\nu(x,\tau) - \partial_\nu A^a_\lambda(x,\tau) - f^{abc}A^b_\lambda(x,\tau)A^c_\nu(x,\tau),$$
$$\not{D} = \not{\partial} - ie\not{A}. \tag{5.1.19}$$

The stochastic expectation values for the $\eta^a_\lambda(x,\tau)$ fields are given by

$$\langle\eta^a_\lambda(x,\tau)\eta^b_\nu(x',\tau')\rangle_\eta = 2\delta^{ab}\delta_{\lambda\nu}\delta^4(x-x')\delta(\tau-\tau'). \tag{5.1.20}$$

5.2. Double stochastic quantization of the Euklidean QED with massive photon.

The Euklidean action for QED with massive photon reads

$$S = \int d^4x\left[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2_{reg}A^\mu A_\mu - i\overline{\psi}(\partial - ieA + im)\psi\right] \tag{5.2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Langevin equations reads

$$\frac{\partial\psi}{\partial t} = \int d^4y K(x,y)[i(\not{D} + im)\psi + \eta_1](y,t) + \eta_2(x,t),$$
$$\frac{\partial\overline{\psi}}{\partial t} = \int d^4y \overline{K}(x,y)^\top[i(\not{D}' + im)\overline{\psi} + \overline{\eta}_2](y,t) + \overline{\eta}_1(x,t),$$
$$\frac{\partial A_\mu}{\partial t} = \partial_\nu F_{\nu\mu} - m^2_{reg}A_\mu + e\overline{\psi}\gamma_\mu\psi + \eta_\mu,$$
$$\mathbf{D}_\mu = \partial_\mu - ieA_\mu, \mathbf{D}'_\mu = -\partial_\mu - ieA_\mu, \tag{5.2.2}$$

where $\top$ means matrix transposition and $\eta_1 = \mathbf{diag}\{\eta_{1,0},\eta_{1,1},\eta_{1,2},\eta_{1,3}\}$, $\eta_2 = \mathbf{diag}\{\eta_{2,0},\eta_{2,1},\eta_{2,2},\eta_{2,3}\}$ are diagonal matrix and where $\overline{\eta}_1,\overline{\eta}_2$ its Dirac adjoint is defined as $\overline{\eta}_1 = \eta_1^\dagger\gamma_0, \overline{\eta}_2 = \eta_2^\dagger\gamma_0$.

$$
\begin{gathered}
\langle\eta_1(x,t)\rangle = \langle\overline{\eta}_2(x,t)\rangle = 0 \\
\langle\eta_i(x,t)\overline{\eta}_k(x',t')\rangle = 2\delta_{ik}\delta^4(x-x')\delta(t-t') \\
i = 1,2; k = 1,2
\end{gathered}
\tag{5.2.3}
$$

From Eq.(5.2.2) one obtains

$$
\begin{aligned}
\frac{\partial\psi(x,t)}{\partial t} &= -(\not{D} - im)(\not{D} + im)\psi(x,t) + i(\not{D} - im)\eta_1(x,t) + \eta_2(x,t) \\
\frac{\partial\overline{\psi}(x,t)}{\partial t} &= -(\not{D}' - im)^{\top}(\not{D}' + im)^{\top}\overline{\psi}(x,t) + i(\not{D}' - im)^{\top}\overline{\eta}_2(x,t) + \overline{\eta}_1(x,t) \\
\frac{\partial A_\mu(x,t)}{\partial t} &= \partial_\nu F_{\nu\mu} - m_{reg}^2 A_\mu(x,t) + e\overline{\psi}(x,t)\gamma_\mu\psi(x,t) + \eta_\mu(x,t)
\end{aligned}
\tag{5.2.4}
$$

The Fourier transform is defined as

$$
\begin{gathered}
\mathcal{F}[f(\mathbf{x})](\mathbf{k}) = \int f(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d^4\mathbf{x} = g(\mathbf{k}), \\
\mathcal{F}^{-1}[g(\mathbf{k})](\mathbf{x}) = (2\pi)^{-4}\int g(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}d^4\mathbf{k}.
\end{gathered}
\tag{5.2.4$'$}
$$

After standard computations, one finds the following set of integral equations, corresponding to (5.2.4) in momentum space, as:

$$
\begin{aligned}
\psi(\mathbf{k},t) = &\int_0^t d\tau e^{-(\mathbf{k}^2+m^2)(t-\tau)}\{(\not{\mathbf{k}} + m)\eta_1(\mathbf{k},\tau) + \eta_2(\mathbf{k},\tau) + \\
&+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}\int d^4\mathbf{q}\delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \times \\
&\times[(\not{\mathbf{p}}\not{\mathbf{A}}(\mathbf{p},\tau) - 2\mathbf{q}_\nu\mathbf{A}_\nu(\mathbf{p},\tau))\psi(\mathbf{q},\tau) + \not{\mathbf{A}}(\mathbf{p},\tau)\eta_1(\mathbf{q},\tau)] - \\
&-\frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}\int d^4\mathbf{r}\delta(\mathbf{k}-\mathbf{p}-\mathbf{q}-\mathbf{r})\mathbf{A}_\nu(\mathbf{p},\tau)\mathbf{A}_\nu(\mathbf{q},\tau)\psi(\mathbf{r},\tau)\Big\} = \\
&= \int_0^t d\tau e^{-(\mathbf{k}^2+m^2)(t-\tau)}\{(\not{\mathbf{k}} + m)\eta_1(\mathbf{k},\tau) + \eta_2(\mathbf{k},\tau) + \\
&+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}\times[(\not{\mathbf{p}}\not{\mathbf{A}}(\mathbf{p},\tau) - 2(\mathbf{k}_\nu - \mathbf{p}_\nu)\mathbf{A}_\nu(\mathbf{p},\tau))\psi(\mathbf{k}-\mathbf{p},\tau) + \not{\mathbf{A}}(\mathbf{p},\tau)\eta_1(\mathbf{k}-\mathbf{p},\tau)] \\
&-\frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}\mathbf{A}_\nu(\mathbf{p},\tau)\mathbf{A}_\nu(\mathbf{q},\tau)\psi(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)\Big\}.
\end{aligned}
\tag{5.2.5}
$$

$$\psi(\mathbf{k},t) = \int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}}+m)\eta_1(\mathbf{k},\tau) + \eta_2(\mathbf{k},\tau) +$$

$$+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}\times \tag{5.2.5'}$$

$$\times[(\not{\mathbf{p}}\not{\mathbf{A}}(\mathbf{p},\tau) - 2(\mathbf{k}_\nu - \mathbf{p}_\nu)\mathbf{A}_\nu(\mathbf{p},\tau))\psi(\mathbf{k}-\mathbf{p},\tau) + \not{\mathbf{A}}(\mathbf{p},\tau)\eta_1(\mathbf{k}-\mathbf{p},\tau)]$$

$$-\frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}\mathbf{A}_\nu(\mathbf{p},\tau)\mathbf{A}_\nu(\mathbf{q},\tau)\psi(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)\Big\}.$$

$$\overline{\psi}(\mathbf{k},t) = \int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}}+m)\eta_2(\mathbf{k},\tau) + \overline{\eta}_1(\mathbf{k},\tau) +$$

$$+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}\int d^4\mathbf{q}\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\times$$

$$\times[\overline{\psi}(\mathbf{q},\tau)(-\not{\mathbf{A}}(\mathbf{p},\tau)\not{\mathbf{p}} - 2q_\nu A_\nu(\mathbf{p},\tau)) + \overline{\eta}_2(\mathbf{k},\tau)\not{\mathbf{A}}(\mathbf{p},\tau)] -$$

$$-\frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}\int d^4\mathbf{r}\delta(-\mathbf{k}-\mathbf{p}-\mathbf{q}+\mathbf{r})A_\nu(\mathbf{p},\tau)A_\nu(\mathbf{q},\tau)\overline{\psi}(\mathbf{r},\tau) = \tag{5.2.6}$$

$$\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}}+m)\eta_2(\mathbf{k},\tau) + \eta_1(\mathbf{k},\tau) +$$

$$+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}[\overline{\psi}(\mathbf{k}-\mathbf{p},\tau)(-\not{\mathbf{A}}(\mathbf{p},\tau)\not{\mathbf{p}} - 2(\mathbf{k}_\nu - \mathbf{p}_\nu)\mathbf{A}_\nu(\mathbf{p},\tau)) + \overline{\eta}_2(\mathbf{k},\tau)\not{\mathbf{A}}(\mathbf{p},\tau)] -$$

$$-\frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}A_\nu(\mathbf{p},\tau)A_\nu(\mathbf{q},\tau)\overline{\psi}(\mathbf{k}+\mathbf{p}+\mathbf{q},\tau)\Big\}.$$

$$\overline{\psi}(\mathbf{k},t) = \int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}}+m)\eta_2(\mathbf{k},\tau) + \overline{\eta}_1(\mathbf{k},\tau) +$$

$$+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}[\overline{\psi}(\mathbf{k}-\mathbf{p},\tau)(-\not{\mathbf{A}}(\mathbf{p},\tau)\not{\mathbf{p}} - 2(\mathbf{k}_\nu - \mathbf{p}_\nu)\mathbf{A}_\nu(\mathbf{p},\tau)) + \overline{\eta}_2(\mathbf{k},\tau)\not{\mathbf{A}}(\mathbf{p},\tau)] - \tag{5.2.6'}$$

$$-\frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}A_\nu(\mathbf{p},\tau)A_\nu(\mathbf{q},\tau)\overline{\psi}(\mathbf{k}+\mathbf{p}+\mathbf{q},\tau)\Big\}.$$

$$A_\mu(\mathbf{k},t) = \int_0^t d\tau\left[e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)}T_{\mu\nu} + L_{\mu\nu}\right]\{\eta_\nu(\mathbf{k},\tau) +$$

$$+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}\int d^4\mathbf{q}\delta(\mathbf{k}+\mathbf{p}-\mathbf{q})\overline{\psi}(\mathbf{p},t)\gamma_\mu\psi(\mathbf{q},t) =$$

$$\int_0^t d\tau\left[e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)}T_{\mu\nu}^{reg} + L_{\mu\nu}^{reg}\right]\left\{\eta_\nu(\mathbf{k},\tau) + \frac{e}{(2\pi)^4}\int d^4\mathbf{p}\overline{\psi}(\mathbf{p},t)\gamma_\mu\psi(\mathbf{k}+\mathbf{p},\tau)\right\}, \tag{5.2.7}$$

$$T_{\mu\nu}^{reg} = \delta_{\mu\nu} - \frac{\mathbf{k}_\mu\mathbf{k}_\nu}{\mathbf{k}^2+m_{reg}^2}, L_{\mu\nu}^{reg} = \frac{\mathbf{k}_\mu\mathbf{k}_\nu}{\mathbf{k}^2+m_{reg}^2}.$$

$$A_\mu(\mathbf{k},t) =$$

$$\int_0^t d\tau\left[e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)}T_{\mu\nu}^{reg} + L_{\mu\nu}^{reg}\right]\left\{\eta_\nu(\mathbf{k},\tau) + \frac{e}{(2\pi)^4}\int d^4\mathbf{p}\overline{\psi}(\mathbf{p},t)\gamma_\mu\psi(\mathbf{k}+\mathbf{p},\tau)\right\}, \tag{5.2.7$'$}$$

$$T_{\mu\nu}^{reg} = \delta_{\mu\nu} - \frac{\mathbf{k}_\mu\mathbf{k}_\nu}{\mathbf{k}^2+m_{reg}^2}, L_{\mu\nu}^{reg} = \frac{\mathbf{k}_\mu\mathbf{k}_\nu}{\mathbf{k}^2+m_{reg}^2}.$$

We derive Fourier transformed differential master equations directly from the Eq.(5.2.5$'$)-Eq.(5.2.7$'$) By the replacement

$$
\begin{aligned}
\psi(x,t)-\theta(t)\psi^{\star} &= \psi_{-}(x,t),\overline{\psi}(x,t)-\theta(t)\overline{\psi}^{\star}=\overline{\psi}_{-}(x,t),\\
\overline{\psi}(x,t) &= \overline{\psi}_{+}(x,t)+\theta(t)\psi^{\star},\overline{\psi}(x,t)=\overline{\psi}_{+}(x,t)-\theta(t)\overline{\psi}^{\star},\\
\psi(\mathbf{p},t) &= \mathcal{F}[\psi(x,t)](\mathbf{p},t)=\mathcal{F}[\psi_{-}(x,t)](\mathbf{p},t)+\theta(t)(2\pi)^{4}\delta(\mathbf{p})\psi^{\star}=\\
&= \psi_{-}(\mathbf{p},t)+\theta(t)(2\pi)^{4}\delta(\mathbf{p})\psi^{\star},\\
\overline{\psi}(\mathbf{p},t) &= \mathcal{F}[\overline{\psi}(x,t)](\mathbf{p},t)=\mathcal{F}[\overline{\psi}_{+}(x,t)](\mathbf{p},t)-\theta(t)(2\pi)^{4}\delta(\mathbf{p})\overline{\psi}^{\star}=\\
&= \overline{\psi}_{+}(\mathbf{p},t)-\theta(t)(2\pi)^{4}\delta(\mathbf{p})\overline{\psi}^{\star},\\
\frac{\partial\psi(x,t)}{\partial t} &= \frac{\partial\psi_{-}(x,t)}{\partial t}+\delta(t)\psi^{\star},\\
\frac{\partial\psi(\mathbf{p},t)}{\partial t} &= \frac{\partial\psi_{-}(\mathbf{p},t)}{\partial t}+\delta(t)(2\pi)^{4}\delta(\mathbf{p})\psi^{\star},\\
\frac{\partial\overline{\psi}(x,t)}{\partial t} &= \frac{\partial\overline{\psi}_{-}(x,t)}{\partial t}+\delta(t)\overline{\psi}^{\star},\\
\frac{\partial\overline{\psi}(\mathbf{p},t)}{\partial t} &= \frac{\partial\overline{\psi}_{+}(\mathbf{p},t)}{\partial t}-\delta(t)(2\pi)^{4}\delta(\mathbf{p})\overline{\psi}^{\star},\\
A_{\mu}(x,t)-\theta(t)a_{\mu} &= A_{\mu-}(x,t),\\
A_{\mu}(x,t) &= A_{\mu-}(x,t)+\theta(t)a_{\mu},\\
A_{\mu}(\mathbf{p},t) &= \mathcal{F}[A_{\mu-}(x,t)](\mathbf{p},t)=\mathcal{F}[A_{\mu-}(x,t)](\mathbf{p},t)+\theta(t)(2\pi)^{4}\delta(\mathbf{p})a_{\mu}=\\
&= A_{\mu-}(\mathbf{p},t)+\theta(t)(2\pi)^{4}a_{\mu},\\
A_{\mu}(x,t)+\theta(t)a_{\mu} &= A_{\mu+}(x,t),\\
A_{\mu}(x,t) &= A_{\mu+}(x,t)-\theta(t)a_{\mu},\\
\frac{\partial A_{\mu}(x,t)}{\partial t} &= \frac{\partial A_{\mu-}(x,t)}{\partial t}+\delta(t)a_{\mu},\\
\frac{\partial A_{\mu}(\mathbf{p},t)}{\partial t} &= \frac{\partial A_{\mu-}(\mathbf{p},t)}{\partial t}+\delta(t)(2\pi)^{4}\delta(\mathbf{p})a_{\mu},\\
\frac{\partial A_{\mu}(x,t)}{\partial t} &= \frac{\partial A_{\mu+}(x,t)}{\partial t}-\delta(t)a_{\mu},\\
\frac{\partial A_{\mu}(\mathbf{p},t)}{\partial t} &= \frac{\partial A_{\mu+}(\mathbf{p},t)}{\partial t}-\delta(t)(2\pi)^{4}\delta(\mathbf{p})a_{\mu}.
\end{aligned}
\tag{5.2.8}
$$

from Eq.(5.2.4) one obtains

$$
\begin{gathered}
\frac{\partial \psi_-(x,t)}{\partial t} = \\
-\delta(t)\psi^\star - (\underline{\not{\mathbf{D}}} - im)(\underline{\not{\mathbf{D}}} + im)[\psi_-(x,t) + \psi^\star] + i(\underline{\not{\mathbf{D}}} - im)\eta_1(x,t) + \eta_2(x,t) \\
\frac{\partial \overline{\psi}_-(x,t)}{\partial t} = \\
-\delta(t)\overline{\psi}^\star - (\underline{\not{\mathbf{D}}}' - im)^\top(\underline{\not{\mathbf{D}}}' + im)^\top[\overline{\psi}_-(x,t) + \overline{\psi}^\star] + i(\underline{\not{\mathbf{D}}}' - im)^\top\overline{\eta}_2(x,t) + \overline{\eta}_1(x,t) \\
\frac{\partial A_{\mu-}(x,t)}{\partial t} = \\
-\delta(t)a_\mu - \partial_\nu F_{\nu\mu-} - m_{reg}^2[A_{\mu-}(x,t) + a_\mu] + \\
e[\overline{\psi}_-(x,t) + \overline{\psi}^\star]\gamma_\mu[\psi_-(x,t) + \overline{\psi}^\star] + \eta_\mu(x,t) \\
\underline{\mathbf{D}}_\mu = \partial_\mu - ie(A_{\mu-} + a_\mu), \underline{\mathbf{D}}'_\mu = -\partial_\mu - ie(A_{\mu-} + a_\mu),
\end{gathered} \tag{5.2.9}
$$

From Eq.(5.2.9) and Eq.(5.2.5′) one obtains

$$
\begin{gathered}
\psi_-(\mathbf{k},t) = -(2\pi)^4\delta(\mathbf{k})\psi^\star \int_0^t d\tau\delta(\tau)e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)} + \\
+\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}} + m)\eta_1(\mathbf{k},\tau) + \eta_2(\mathbf{k},\tau) + \\
+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}\Big[\Big(\not{\mathbf{p}}\Big(\not{\mathbf{A}}_-(\mathbf{p},\tau) + (2\pi)^4\not{\delta}(\mathbf{p})\not{a}_\mu\Big) - 2(\mathbf{k}_\nu - \mathbf{p}_\nu)\Big(\mathbf{A}_{\nu-}(\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{p})a_\nu\Big)\Big) \times \\
\times\Big[\psi_-(\mathbf{k}-\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{k}-\mathbf{p})\psi^\star\Big] + \Big(\not{\mathbf{A}}_-(\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{p})\not{a}_\mu\Big)\eta_1(\mathbf{k}-\mathbf{p},\tau)\Big] - \\
-\frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q} \times \\
\Big(\mathbf{A}_{\nu-}(\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{p})a_\mu\Big)\Big(\mathbf{A}_{\nu-}(\mathbf{q},\tau) + (2\pi)^4\delta(\mathbf{p})a_\mu\Big)\Big[\psi_-(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau) + (2\pi)^4\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^\star\Big]\Big\} \\
= -(2\pi)^4\delta(\mathbf{k})e^{-\left(\mathbf{k}^2+m^2\right)t}\psi^\star + \int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}} + m)\eta_1(\mathbf{k},\tau) + \eta_2(\mathbf{k},\tau) + \\
+\frac{e}{(2\pi)^4}\int d^4\mathbf{p} \times \Big[\Big(\not{\mathbf{p}}\not{\mathbf{A}}_-(\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_\mu - 2(\mathbf{k}_\nu - \mathbf{p}_\nu)\mathbf{A}_{\nu-}(\mathbf{p},\tau) - 2(2\pi)^4(\mathbf{k}_\nu - \mathbf{p}_\nu)\delta(\mathbf{p})a_\nu\Big) \\
\times\Big[\psi_-(\mathbf{k}-\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{k}-\mathbf{p})\psi^\star\Big] + \Big(\not{\mathbf{A}}_-(\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{p})\not{a}_\mu\Big)\eta_1(\mathbf{k}-\mathbf{p},\tau)\Big] - \\
-\frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}\Big(\mathbf{A}_\nu(\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{p})a_\nu\Big)\Big(\mathbf{A}_\nu(\mathbf{q},\tau) + (2\pi)^4\delta(\mathbf{q})a_\nu\Big) \times \\
\times\Big(\psi_-(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau) + (2\pi)^4\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^\star\Big)\Big\}
\end{gathered}
$$

Note that

$$
\begin{gathered}
\Big(\not{\mathbf{p}}\not{\mathbf{A}}_-(\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_\mu-2(\mathbf{k}_\nu-\mathbf{p}_\nu)\mathbf{A}_{\nu-}(\mathbf{p},\tau)-2(2\pi)^4(\mathbf{k}_\nu-\mathbf{p}_\nu)\delta(\mathbf{p})a_\nu\Big)\\
\times\Big[\psi_-(\mathbf{k}-\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{k}-\mathbf{p})\psi^\star\Big]+\Big(\not{\mathbf{A}}_-(\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{p})\not{a}_\mu\Big)\eta_1(\mathbf{k}-\mathbf{p},\tau)=\\
=\Big(\not{\mathbf{p}}\not{\mathbf{A}}_-(\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_\mu\Big)\Big[\psi_-(\mathbf{k}-\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{k}-\mathbf{p})\psi^\star\Big]-\\
-\Big(2(\mathbf{k}_\nu-\mathbf{p}_\nu)\mathbf{A}_{\nu-}(\mathbf{p},\tau)+2(2\pi)^4(\mathbf{k}_\nu-\mathbf{p}_\nu)\delta(\mathbf{p})a_\nu\Big)\Big[\psi_-(\mathbf{k}-\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{k}-\mathbf{p})\psi^\star\Big]=\\
\underline{\not{\mathbf{p}}\not{\mathbf{A}}_-(\mathbf{p},\tau)\psi_-(\mathbf{k}-\mathbf{p},\tau)}+\\
+(2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_\mu\psi_-(\mathbf{k}-\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{k}-\mathbf{p})\not{\mathbf{p}}\not{\mathbf{A}}_-(\mathbf{p},\tau)\psi^\star-\\
-\underline{2(\mathbf{k}_\nu-\mathbf{p}_\nu)\mathbf{A}_{\nu-}(\mathbf{p},\tau)\psi_-(\mathbf{k}-\mathbf{p},\tau)}-\\
-2(2\pi)^4(\mathbf{k}_\nu-\mathbf{p}_\nu)\delta(\mathbf{p})a_\nu\psi_-(\mathbf{k}-\mathbf{p},\tau)-2(2\pi)^4(\mathbf{k}_\nu-\mathbf{p}_\nu)\mathbf{A}_{\nu-}(\mathbf{p},\tau)\delta(\mathbf{k}-\mathbf{p})\psi^\star-\\
-2(2\pi)^4(\mathbf{k}_\nu-\mathbf{p}_\nu)\delta(\mathbf{p})a_\nu\delta(\mathbf{k}-\mathbf{p})\psi^\star+(2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_\mu\delta(\mathbf{k}-\mathbf{p})\psi^\star+\\
+\not{\mathbf{A}}_-(\mathbf{p},\tau)\eta_1(\mathbf{k}-\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{p})\not{a}_\mu\eta_1(\mathbf{k}-\mathbf{p},\tau)
\end{gathered}
\tag{5.2.11}
$$

$$
\begin{gathered}
\Delta_1(\mathbf{k},\mathbf{p},\tau,e)=\underline{\not{\mathbf{p}}\not{\mathbf{A}}_-(\mathbf{p},\tau)\psi_-(\mathbf{k}-\mathbf{p},\tau)}-\underline{2(\mathbf{k}_\nu-\mathbf{p}_\nu)\mathbf{A}_{\nu-}(\mathbf{p},\tau)\psi_-(\mathbf{k}-\mathbf{p},\tau)}\\
\Delta_2(\mathbf{k},\mathbf{p},\tau,e)=(2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_\mu\psi_-(\mathbf{k}-\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{k}-\mathbf{p})\not{\mathbf{p}}\not{\mathbf{A}}_-(\mathbf{p},\tau)\psi^\star-\\
-2(2\pi)^4(\mathbf{k}_\nu-\mathbf{p}_\nu)\delta(\mathbf{p})a_\nu\delta(\mathbf{k}-\mathbf{p})\psi^\star+(2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_\mu\delta(\mathbf{k}-\mathbf{p})\psi^\star+\\
+\not{\mathbf{A}}_-(\mathbf{p},\tau)\eta_1(\mathbf{k}-\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{p})\not{a}_\mu\eta_1(\mathbf{k}-\mathbf{p},\tau)
\end{gathered}
\tag{5.2.12}
$$

Note that

$$
\begin{gathered}
\Big(\mathbf{A}_\nu(\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{p})a_\nu\Big)\Big(\mathbf{A}_\nu(\mathbf{q},\tau)+(2\pi)^4\delta(\mathbf{q})a_\nu\Big)\times\\
\times\Big(\psi_-(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)+(2\pi)^4\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^\star\Big)=\\
=\Big[\mathbf{A}_\nu(\mathbf{p},\tau)\mathbf{A}_\nu(\mathbf{q},\tau)+(2\pi)^4\delta(\mathbf{p})a_\nu\mathbf{A}_\nu(\mathbf{q},\tau)+(2\pi)^4\delta(\mathbf{q})a_\nu\mathbf{A}_\nu(\mathbf{p},\tau)+(2\pi)^8\delta(\mathbf{p})\delta(\mathbf{q})a_\nu^2\Big]\times\\
\times\Big[\psi_-(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)+(2\pi)^4\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^\star\Big]=\\
\underline{\mathbf{A}_\nu(\mathbf{p},\tau)\mathbf{A}_\nu(\mathbf{q},\tau)\psi_-(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)}+(2\pi)^4\delta(\mathbf{p})a_\nu\underline{\mathbf{A}_\nu(\mathbf{q},\tau)\psi_-(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)}+\\
+(2\pi)^4\delta(\mathbf{q})a_\nu\underline{\mathbf{A}_\nu(\mathbf{p},\tau)\psi_-(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)}+\\
\#+(2\pi)^8\delta(\mathbf{p})\delta(\mathbf{q})a_\nu^2\psi_-(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)+\\
+\Big[(2\pi)^4\delta(\mathbf{p})a_\nu\mathbf{A}_\nu(\mathbf{q},\tau)+(2\pi)^4\delta(\mathbf{q})a_\nu\mathbf{A}_\nu(\mathbf{p},\tau)+(2\pi)^8\delta(\mathbf{p})\delta(\mathbf{q})a_\nu^2\Big](2\pi)^4\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^\star=
\end{gathered}
\tag{5.2.13}
$$

$$
\begin{gathered}
=(2\pi)^8\delta(\mathbf{p})\delta(\mathbf{q})a_\nu^2\psi_-(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)+\\
+(2\pi)^8\delta(\mathbf{p})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^\star a_\nu\mathbf{A}_\nu(\mathbf{q},\tau)+(2\pi)^8\delta(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^\star a_\nu\mathbf{A}_\nu(\mathbf{p},\tau)+\\
+(2\pi)^{12}\delta(\mathbf{p})\delta(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^\star a_\nu^2
\end{gathered}
$$

$$\Delta_3(\mathbf{k},\mathbf{p},\tau,e) = \mathbf{A}_\nu(\mathbf{p},\tau)\mathbf{A}_\nu(\mathbf{q},\tau)\psi_-(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau) +$$
$$+(2\pi)^4\delta(\mathbf{p})a_\nu\mathbf{A}_\nu(\mathbf{q},\tau)\psi_-\mathbf{k}-\mathbf{p}-\mathbf{q},\tau) + (2\pi)^4\delta(\mathbf{q})a_\nu\mathbf{A}_\nu(\mathbf{p},\tau)\psi_-(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau) +$$

$$\Delta_4(\mathbf{k},\mathbf{p},\tau,e) = (2\pi)^8\delta(\mathbf{p})\delta(\mathbf{q})a_\nu^2\psi_-(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau) +$$
$$+(2\pi)^8\delta(\mathbf{p})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^\star a_\nu\mathbf{A}_\nu(\mathbf{q},\tau) + (2\pi)^8\delta(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^\star a_\nu\mathbf{A}_\nu(\mathbf{p},\tau) +$$
$$+(2\pi)^{12}\delta(\mathbf{p})\delta(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^\star a_\nu^2. \tag{5.2.14}$$

We rewrite Eq.(2.5.10) of the short form

$$\psi_-(\mathbf{k},t) =$$
$$-(2\pi)^4\delta(\mathbf{k})\psi^\star e^{-\left(\mathbf{k}^2+m^2\right)t} + \int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}}+m)\eta_1(\mathbf{k},\tau) + \eta_2(\mathbf{k},\tau) +$$
$$+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}[\Delta_1(\mathbf{k},\mathbf{p},\tau,e) + \Delta_2(\mathbf{k},\mathbf{p},\tau,e)] - \tag{5.2.15}$$
$$-\frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}[\Delta_3(\mathbf{k},\mathbf{p},\tau,e) + \Delta_4(\mathbf{k},\mathbf{p},\tau,e)]\Big\}$$

From Eq.(5.2.6′) by the Eq.(5.2.8) we obtain

$$\overline{\psi}_-(\mathbf{k},t) = \int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}}+m)\overline{\eta}_2(\mathbf{k},\tau) + \overline{\eta}_1(\mathbf{k},\tau) +$$
$$+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}\Big[\Big(\overline{\psi}(\mathbf{k}-\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{k}-\mathbf{p})\Big)\times$$
$$\Big(\Big(-\not{\mathbf{A}}(\mathbf{p},\tau)\not{\mathbf{p}} - (2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\Big) - 2(\mathbf{k}_\nu-\mathbf{p}_\nu)\mathbf{A}_\nu(\mathbf{p},\tau) - 2(2\pi)^4(\mathbf{k}_\nu-\mathbf{p}_\nu)\delta(\mathbf{p})\Big) \tag{5.2.16}$$
$$+\overline{\eta}_2(\mathbf{k}+\mathbf{p},\tau)\Big(\not{\mathbf{A}}(\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{p})\Big)\Big] -$$
$$-\frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}\Big(A_\nu(\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{p})\Big)\Big(A_\nu(\mathbf{q},\tau) + (2\pi)^4\delta(\mathbf{q})\Big)\times$$
$$\times\Big(\overline{\psi}(\mathbf{k}+\mathbf{p}+\mathbf{q},\tau) + +(2\pi)^4\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})\Big)\Big\}.$$

Note that

$$
\begin{gathered}
\Big(-\not{\mathbf{A}}_{-}(\mathbf{p},\tau)\not{\mathbf{p}} + (2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_{\mu} - 2(\mathbf{k}_{\nu}-\mathbf{p}_{\nu})\mathbf{A}_{\nu-}(\mathbf{p},\tau) - 2(2\pi)^4(\mathbf{k}_{\nu}-\mathbf{p}_{\nu})\delta(\mathbf{p})a_{\nu}\Big) \\
\times\Big[\psi_{-}(\mathbf{k}-\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{k}-\mathbf{p})\psi^{\star}\Big] + \Big(\not{\mathbf{A}}_{-}(\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{p})\not{a}_{\mu}\Big)\eta_1(\mathbf{k}-\mathbf{p},\tau) = \\
= \Big(\not{\mathbf{p}}\not{\mathbf{A}}_{-}(\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_{\mu}\Big)\Big[\psi_{-}(\mathbf{k}-\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{k}-\mathbf{p})\psi^{\star}\Big] - \\
-\Big(2(\mathbf{k}_{\nu}-\mathbf{p}_{\nu})\mathbf{A}_{\nu-}(\mathbf{p},\tau) + 2(2\pi)^4(\mathbf{k}_{\nu}-\mathbf{p}_{\nu})\delta(\mathbf{p})a_{\nu}\Big)\Big[\psi_{-}(\mathbf{k}-\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{k}-\mathbf{p})\psi^{\star}\Big] = \\
\underline{\not{\mathbf{p}}\not{\mathbf{A}}_{-}(\mathbf{p},\tau)\psi_{-}(\mathbf{k}-\mathbf{p},\tau)} + \\
+(2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_{\mu}\psi_{-}(\mathbf{k}-\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{k}-\mathbf{p})\not{\mathbf{p}}\not{\mathbf{A}}_{-}(\mathbf{p},\tau)\psi^{\star} - \\
-\underline{2(\mathbf{k}_{\nu}-\mathbf{p}_{\nu})\mathbf{A}_{\nu-}(\mathbf{p},\tau)\psi_{-}(\mathbf{k}-\mathbf{p},\tau)} - \\
-2(2\pi)^4(\mathbf{k}_{\nu}-\mathbf{p}_{\nu})\delta(\mathbf{p})a_{\nu}\psi_{-}(\mathbf{k}-\mathbf{p},\tau) - 2(2\pi)^4(\mathbf{k}_{\nu}-\mathbf{p}_{\nu})\mathbf{A}_{\nu-}(\mathbf{p},\tau)\delta(\mathbf{k}-\mathbf{p})\psi^{\star} - \\
-2(2\pi)^4(\mathbf{k}_{\nu}-\mathbf{p}_{\nu})\delta(\mathbf{p})a_{\nu}\delta(\mathbf{k}-\mathbf{p})\psi^{\star} + (2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_{\mu}\delta(\mathbf{k}-\mathbf{p})\psi^{\star} + \\
+\not{\mathbf{A}}_{-}(\mathbf{p},\tau)\eta_1(\mathbf{k}-\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{p})\not{a}_{\mu}\eta_1(\mathbf{k}-\mathbf{p},\tau)
\end{gathered}
\tag{5.2.11}
$$

Note that

$$
\begin{gathered}
\Big(\mathbf{A}_{\nu}(\mathbf{p},\tau) + (2\pi)^4\delta(\mathbf{p})a_{\nu}\Big)\Big(\mathbf{A}_{\nu}(\mathbf{q},\tau) + (2\pi)^4\delta(\mathbf{q})a_{\nu}\Big) \times \\
\times\Big(\psi_{-}(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau) + (2\pi)^4\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^{\star}\Big) = \\
= \Big[\mathbf{A}_{\nu}(\mathbf{p},\tau)\mathbf{A}_{\nu}(\mathbf{q},\tau) + (2\pi)^4\delta(\mathbf{p})a_{\nu}\mathbf{A}_{\nu}(\mathbf{q},\tau) + (2\pi)^4\delta(\mathbf{q})a_{\nu}\mathbf{A}_{\nu}(\mathbf{p},\tau) + (2\pi)^8\delta(\mathbf{p})\delta(\mathbf{q})a_{\nu}^2\Big] \times \\
\times\Big[\psi_{-}(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau) + (2\pi)^4\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^{\star}\Big] = \\
\underline{\mathbf{A}_{\nu}(\mathbf{p},\tau)\mathbf{A}_{\nu}(\mathbf{q},\tau)\psi_{-}(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)} + (2\pi)^4\delta(\mathbf{p})a_{\nu}\underline{\mathbf{A}_{\nu}(\mathbf{q},\tau)\psi_{-}(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)} + \\
+(2\pi)^4\delta(\mathbf{q})a_{\nu}\underline{\mathbf{A}_{\nu}(\mathbf{p},\tau)\psi_{-}(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)} + \\
\# + (2\pi)^8\delta(\mathbf{p})\delta(\mathbf{q})a_{\nu}^2\psi_{-}(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau) + \\
+\Big[(2\pi)^4\delta(\mathbf{p})a_{\nu}\mathbf{A}_{\nu}(\mathbf{q},\tau) + (2\pi)^4\delta(\mathbf{q})a_{\nu}\mathbf{A}_{\nu}(\mathbf{p},\tau) + (2\pi)^8\delta(\mathbf{p})\delta(\mathbf{q})a_{\nu}^2\Big](2\pi)^4\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^{\star} = \\
\\
= (2\pi)^8\delta(\mathbf{p})\delta(\mathbf{q})a_{\nu}^2\psi_{-}(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau) + \\
+(2\pi)^8\delta(\mathbf{p})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^{\star}a_{\nu}\mathbf{A}_{\nu}(\mathbf{q},\tau) + (2\pi)^8\delta(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^{\star}a_{\nu}\mathbf{A}_{\nu}(\mathbf{p},\tau) + \\
+(2\pi)^{12}\delta(\mathbf{p})\delta(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^{\star}a_{\nu}^2
\end{gathered}
\tag{5.2.13}
$$

We rewrite Eq.(2.5.16) of the short form

$$\overline{\psi}_{-}(\mathbf{k},t) =$$

$$-(2\pi)^4\delta(\mathbf{k})\psi^{\star}e^{-\left(\mathbf{k}^2+m^2\right)t} + \int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}}+m)\overline{\eta}_1(\mathbf{k},\tau) + \overline{\eta}_2(\mathbf{k},\tau) +$$

$$+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}[\Delta_1(\mathbf{k},\mathbf{p},\tau,e) + \Delta_2(\mathbf{k},\mathbf{p},\tau,e)] - \tag{5.2.17}$$

$$-\frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}[\Delta_3(\mathbf{k},\mathbf{p},\tau,e) + \Delta_4(\mathbf{k},\mathbf{p},\tau,e)]\Big\}$$

where

$$\underline{\Delta}_2(\mathbf{k},\mathbf{p},\tau,e) =$$
$$\underline{\Delta}_4(\mathbf{k},\mathbf{p},\tau,e) = \tag{5.2.18}$$

Therefore Fourier transformed differential master equations corresponding to Eq.(5.2.10) and Eq.(5.2.12) reads

$$\psi_{-}(\mathbf{k},t) = -(2\pi)^4\delta(\mathbf{k})\psi^{\star}e^{-\left(\mathbf{k}^2+m^2\right)t} +$$

$$+\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}}+m)\eta_1(\mathbf{k},\tau) + \eta_2(\mathbf{k},\tau) + \tag{5.2.19}$$

$$+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}\Delta_2(\mathbf{k},\mathbf{p},\tau,e) - \frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}\Delta_4(\mathbf{k},\mathbf{p},\tau,e)\Big\}$$

and

$$\overline{\psi}_{-}(\mathbf{k},t) = (2\pi)^4\delta(\mathbf{k})\overline{\psi}^{\star}e^{-\left(\mathbf{k}^2+m^2\right)t} +$$

$$+\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}}+m)\overline{\eta}_1(\mathbf{k},\tau) + \overline{\eta}_2(\mathbf{k},\tau) + \tag{5.2.20}$$

$$+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}\underline{\Delta}_2(\mathbf{k},\mathbf{p},\tau,e) - \frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}\underline{\Delta}_4(\mathbf{k},\mathbf{p},\tau,e)\Big\}$$

correspondingly.

By the replacement (5.2.8) from (5.2.7′) one obtains

$$
\begin{aligned}
&A_{\mu-}(\mathbf{k},t) = \\
&-(2\pi)^4 a_\mu \delta(\mathbf{k}) \times \int_0^t \delta(\tau) e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)} d\tau - m_{reg}^2 a_\mu \delta(\mathbf{k}) \int_0^t e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)} d\tau + \\
&+\int_0^t \left[e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)} T_{\mu\nu} + L_{\mu\nu} \right] \left\{ \eta_\nu(\mathbf{k},\tau) + \frac{e}{(2\pi)^4} \int d^4\mathbf{p} \left[\overline{\psi}_-(\mathbf{p},\tau) + (2\pi)^4 \delta(\mathbf{p}) \overline{\psi}^\star \right] \times \right. \\
&\left. \times \gamma_\mu \left[\psi_-(\mathbf{k}+\mathbf{p},\tau) + (2\pi)^4 \delta(\mathbf{k}+\mathbf{p}) \psi^\star \right] \right\} d\tau = \\
&-(2\pi)^4 a_\mu e^{-\left(\mathbf{k}^2+m^2\right)t} \delta(\mathbf{k}) \int_0^t \delta(\tau) e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)} d\tau - \\
&-m_{reg}^2 a_\mu \delta(\mathbf{k}) \int_0^t e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)} d\tau + \\
&+\int_0^t d\tau \left[e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)} T_{\mu\nu}^{reg} + L_{\mu\nu}^{reg} \right] \{ \eta_\nu(\mathbf{k},\tau) + \\
&+\frac{e}{(2\pi)^4} \int d^4\mathbf{p} \left[\overline{\psi}_-(\mathbf{p},\tau) \gamma_\mu \psi_-(\mathbf{k}+\mathbf{p},\tau) + (2\pi)^4 \delta(\mathbf{p}) \overline{\psi}^\star \psi_-(\mathbf{k}+\mathbf{p},\tau) + \right. \\
&\left. +(2\pi)^4 \overline{\psi}_-(\mathbf{p},\tau) \delta(\mathbf{k}+\mathbf{p}) \psi^\star + (2\pi)^8 \delta(\mathbf{p}) \delta(\mathbf{k}+\mathbf{p}) \overline{\psi}^\star \psi^\star \right] \\
&T_{\mu\nu}^{reg} = \delta_{\mu\nu} - \frac{\mathbf{k}_\mu \mathbf{k}_\nu}{\mathbf{k}^2 + m_{reg}^2}, L_{\mu\nu}^{reg} = \frac{\mathbf{k}_\mu \mathbf{k}_\nu}{\mathbf{k}^2 + m_{reg}^2}.
\end{aligned}
\tag{5.2.21}
$$

Therefore Fourier transformed linear differential master equation corresponding to Eq.(5.2.10) and Eq.(5.2.12) reads

$$
\begin{aligned}
&A_{\mu-}(\mathbf{k},t) = -(2\pi)^4 a_\mu e^{-\left(\mathbf{k}^2+m^2\right)t} \delta(\mathbf{k}) \int_0^t \delta(\tau) e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)} d\tau - \\
&-m_{reg}^2 a_\mu \delta(\mathbf{k}) \int_0^t e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)} d\tau + \int_0^t d\tau \left[e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)} T_{\mu\nu} + L_{\mu\nu} \right] \{ \eta_\nu(\mathbf{k},\tau) + \\
&+\frac{e}{(2\pi)^4} \int d^4\mathbf{p} \left[(2\pi)^4 \delta(\mathbf{p}) \overline{\psi}^\star \psi_-(\mathbf{k}+\mathbf{p},\tau) + \right. \\
&\left. +(2\pi)^4 \overline{\psi}_-(\mathbf{p},\tau) \delta(\mathbf{k}+\mathbf{p}) \psi^\star + (2\pi)^8 \delta(\mathbf{p}) \delta(\mathbf{k}+\mathbf{p}) \overline{\psi}^\star \psi^\star \right]
\end{aligned}
\tag{5.2.22}
$$

By the replacement (5.2.8) from (5.2.7′) one obtains

$$
\begin{aligned}
&A_{\mu+}(\mathbf{k},t) = \\
&(2\pi)^4 a_\mu \delta(\mathbf{k}) \times \int_0^t \delta(\tau) e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)} d\tau + m_{reg}^2 a_\mu \delta(\mathbf{k}) \int_0^t e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)} d\tau + \\
&+\int_0^t \left[e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)} T_{\mu\nu} + L_{\mu\nu} \right] \left\{ \eta_\nu(\mathbf{k},\tau) + \frac{e}{(2\pi)^4} \int d^4\mathbf{p} \left[\overline{\psi}_-(\mathbf{p},\tau) + (2\pi)^4 \delta(\mathbf{p}) \overline{\psi}^\star \right] \times \right. \\
&\left. \times \gamma_\mu \left[\psi_-(\mathbf{k}+\mathbf{p},\tau) + (2\pi)^4 \delta(\mathbf{k}+\mathbf{p}) \psi^\star \right] \right\} d\tau = \\
&(2\pi)^4 a_\mu e^{-\left(\mathbf{k}^2+m^2\right)t} \delta(\mathbf{k}) \int_0^t \delta(\tau) e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)} d\tau + \\
&+m_{reg}^2 a_\mu \delta(\mathbf{k}) \int_0^t e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)} d\tau + \qquad (5.2.16) \\
&+\int_0^t d\tau \left[e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)} T_{\mu\nu}^{reg} + L_{\mu\nu}^{reg} \right] \{ \eta_\nu(\mathbf{k},\tau) + \\
&+\frac{e}{(2\pi)^4} \int d^4\mathbf{p} \left[\overline{\psi}_-(\mathbf{p},\tau) \gamma_\mu \psi_-(\mathbf{k}+\mathbf{p},\tau) + (2\pi)^4 \delta(\mathbf{p}) \overline{\psi}^\star \psi_-(\mathbf{k}+\mathbf{p},\tau) + \right. \\
&\left. +(2\pi)^4 \overline{\psi}_-(\mathbf{p},\tau) \delta(\mathbf{k}+\mathbf{p}) \psi^\star + (2\pi)^8 \delta(\mathbf{p}) \delta(\mathbf{k}+\mathbf{p}) \overline{\psi}^\star \psi^\star \right] \\
&T_{\mu\nu}^{reg} = \delta_{\mu\nu} - \frac{\mathbf{k}_\mu \mathbf{k}_\nu}{\mathbf{k}^2 + m_{reg}^2}, L_{\mu\nu}^{reg} = \frac{\mathbf{k}_\mu \mathbf{k}_\nu}{\mathbf{k}^2 + m_{reg}^2}.
\end{aligned}
$$

We define now countable sequences $\{\psi_-^{(n)}(\mathbf{k},t,e)\}_{n\in\mathbb{N}}, \{\overline{\psi}_-^{(n)}(\mathbf{k},t,e)\}_{n\in\mathbb{N}}$, $\left\{A_{\mu-}^{(n)}(\mathbf{k},t,e)\right\}_{n\in\mathbb{N}}, \left\{A_{\mu+}^{(n)}(\mathbf{k},t,e)\right\}_{n\in\mathbb{N}}$ by

$$
\begin{aligned}
&\psi_-^{(n)}(\mathbf{k},t,e) = \Psi\left[\psi_-^{(n-1)}(\mathbf{k},t,e), \overline{\psi}_-^{(n-1)}(\mathbf{k},t,e), A_{\mu-}^{(n-1)}(\mathbf{k},t,e) \right] = \\
&-(2\pi)^4 \delta(\mathbf{k}) \psi^\star e^{-\left(\mathbf{k}^2+m^2\right)t} + \\
&+\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)} \{ (\not{\mathbf{k}} + m) \eta_1(\mathbf{k},\tau) + \eta_2(\mathbf{k},\tau) + \qquad (5.2.17) \\
&\left. +\frac{e}{(2\pi)^4} \int d^4\mathbf{p} \Delta_2^{(n-1)}(\mathbf{k},\mathbf{p},\tau,e) - \frac{e^2}{(2\pi)^8} \int d^4\mathbf{p} \int d^4\mathbf{q} \Delta_4^{(n-1)}(\mathbf{k},\mathbf{p},\tau,e) \right\}
\end{aligned}
$$

$$\overline{\psi}_{-}^{(n)}(\mathbf{k},t,e) = \Psi\Big[\psi_{-}^{(n-1)}(\mathbf{k},t,e),\overline{\psi}_{-}^{(n-1)}(\mathbf{k},t,e),A_{\mu-}^{(n-1)}(\mathbf{k},t,e)\Big] =$$

$$-(2\pi)^4\delta(\mathbf{k})\psi^{\star}e^{-\left(\mathbf{k}^2+m^2\right)t} +$$

$$+\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}}+m)\overline{\eta}_1(\mathbf{k},\tau)+\overline{\eta}_2(\mathbf{k},\tau)+ \tag{5.2.17'}$$

$$+\frac{e}{(2\pi)^4}\int d^4\mathbf{p}\underline{\Delta}_2^{(n-1)}(\mathbf{k},\mathbf{p},\tau,e)-\frac{e^2}{(2\pi)^8}\int d^4\mathbf{p}\int d^4\mathbf{q}\underline{\Delta}_4^{(n-1)}(\mathbf{k},\mathbf{p},\tau,e)\Bigg\}$$

where

$$\Delta_2^{(n-1)}(\mathbf{k},\mathbf{p},\tau,e) =$$
$$= (2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_{\mu}\psi_{-}^{(n-1)}(\mathbf{k}-\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{k}-\mathbf{p})\not{\mathbf{p}}\not{\mathbf{A}}_{-}^{(n-1)}(\mathbf{p},\tau)\psi^{\star}-$$
$$-2(2\pi)^4(\mathbf{k}_{\nu}-\mathbf{p}_{\nu})\delta(\mathbf{p})a_{\nu}\delta(\mathbf{k}-\mathbf{p})\psi^{\star}+(2\pi)^4\delta(\mathbf{p})\not{\mathbf{p}}\not{a}_{\mu}\delta(\mathbf{k}-\mathbf{p})\psi^{\star}+ \tag{5.2.12}$$
$$+\not{\mathbf{A}}_{-}^{(n-1)}(\mathbf{p},\tau)\eta_1(\mathbf{k}-\mathbf{p},\tau)+(2\pi)^4\delta(\mathbf{p})\not{a}_{\mu}\eta_1(\mathbf{k}-\mathbf{p},\tau)$$

$$\Delta_4^{(n-1)}(\mathbf{k},\mathbf{p},\tau,e) =$$
$$(2\pi)^8\delta(\mathbf{p})\delta(\mathbf{q})a_{\nu}^2\psi_{-}^{(n-1)}(\mathbf{k}-\mathbf{p}-\mathbf{q},\tau)+$$
$$(2\pi)^8\delta(\mathbf{p})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^{\star}a_{\nu}\mathbf{A}_{\nu-}^{(n-1)}(\mathbf{q},\tau)+ \tag{5.2.14}$$
$$(2\pi)^8\delta(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^{\star}a_{\nu}\mathbf{A}_{\nu-}^{(n-1)}(\mathbf{p},\tau)+$$
$$+(2\pi)^{12}\delta(\mathbf{p})\delta(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\psi^{\star}a_{\nu}^2.$$

$$\underline{\Delta}_2^{(n-1)}(\mathbf{k},\mathbf{p},\tau,e) = \qquad (5.2.14)$$
$$\underline{\Delta}_4^{(n-1)}(\mathbf{k},\mathbf{p},\tau,e) =$$

$$\psi_{-}^{(1)}(\mathbf{k},t,e) = \psi_{-}(\mathbf{k},t,e=0) =$$
$$-(2\pi)^4\delta(\mathbf{k})\psi^{\star}\int_0^t d\tau\delta(\tau)e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}-m^2\psi^{*}\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}+ \tag{5.2.18}$$
$$+\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}}+m)\eta_1(\mathbf{k},\tau)+\eta_2(\mathbf{k},\tau)\} =$$

$$\overline{\psi}_{-}^{(n)}(\mathbf{k},t,e) = \overline{\Psi}\Big[\psi_{-}^{(n-1)}(\mathbf{k},t,e),\overline{\psi}_{+}^{(n-1)}(\mathbf{k},t,e),A_{\mu-}^{(n-1)}(\mathbf{k},t,e)\Big] = \tag{5.2.19}$$

$$\overline{\psi}_{-}^{(1)}(\mathbf{k},t,e) = \overline{\psi}_{+}(\mathbf{k},t,e=0) =$$

$$(2\pi)^4\delta(\mathbf{k})\overline{\psi}^{\star}\int_0^t d\tau\delta(\tau)e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)} + m^2\overline{\psi}^{*}\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)} +$$

$$+\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\{(\not{\mathbf{k}}+m)\eta_2(\mathbf{k},\tau)+\overline{\eta}_1(\mathbf{k},\tau)\} = \tag{5.2.20}$$

$$A_{\mu-}^{(n)}(\mathbf{k},t,e) = \mathbf{A}_{\mu-}\left[\psi_{-}^{(n-1)}(\mathbf{k},t,e),\overline{\psi}_{+}^{(n-1)}(\mathbf{k},t,e),A_{\mu-}^{(n-1)}(\mathbf{k},t,e),\right] = \tag{5.2.21}$$

$$A_{\mu-}^{(1)}(\mathbf{k},t,e=0) =$$

$$= -(2\pi)^4 a_\mu\delta(\mathbf{k})\times\int_0^t \delta(\tau)e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)}d\tau - m_{reg}^2 a_\mu\delta(\mathbf{k})\int_0^t e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)}d\tau +$$

$$+\int_0^t\left[e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)}T_{\mu\nu}+L_{\mu\nu}\right] \tag{5.2.23}$$

$$A_{\mu-}^{(1)}(\mathbf{k},t,e=0) =$$

$$= -(2\pi)^4 a_\mu\delta(\mathbf{k})\times\int_0^t \delta(\tau)e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)}d\tau - m_{reg}^2 a_\mu\delta(\mathbf{k})\int_0^t e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)}d\tau +$$

$$+\int_0^t\left[e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)}T_{\mu\nu}^{reg}+L_{\mu\nu}^{reg}\right] \tag{5.2.24}$$

$$A_{\mu+}^{(n)}(\mathbf{k},t,e) = \mathbf{A}_{\mu-}\left[\psi_{-}^{(n-1)}(\mathbf{k},t,e),\overline{\psi}_{-}^{(n-1)}(\mathbf{k},t,e),A_{\mu+}^{(n-1)}(\mathbf{k},t,e),\right] = \tag{5.2.25}$$

$$A_{\mu+}^{(1)}(\mathbf{k},t,e=0)=$$

$$=(2\pi)^4 a_\mu\delta(\mathbf{k})\times\int_0^t \delta(\tau)e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)}d\tau+m_{reg}^2 a_\mu\delta(\mathbf{k})\int_0^t e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)}d\tau+ \tag{5.2.26}$$

$$+\int_0^t\left[e^{-\left(\mathbf{k}^2+m_{reg}^2\right)(t-\tau)}T_{\mu\nu}^{reg}+L_{\mu\nu}^{reg}\right].$$

From Eqs.(5.2.19) we find directly

$$\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\int_0^{t'} d\tau'\delta(\tau-\tau')e^{-\left(\mathbf{k}^2+m^2\right)(t'-\tau')}\langle\eta_1(\mathbf{k},t)\overline{\psi}_+(\mathbf{k}',t,e=0)\rangle=$$

$$=\langle\eta_1(\mathbf{k},t)\overline{\eta}_1(\mathbf{k}',t')\rangle=(2\pi)^4\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\int_0^{t'} d\tau'\delta(\tau-\tau')e^{-\left(\mathbf{k}^2+m^2\right)(t'-\tau')}\delta(\mathbf{k}+\mathbf{k}') \tag{5.2.27}$$

and

$$\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\int_0^{t'} d\tau'\delta(\tau-\tau')e^{-\left(\mathbf{k}^2+m^2\right)(t'-\tau')}\langle\eta_1(\mathbf{k},t)\psi_+(\mathbf{k}',t,e=0)\overline{\eta}_2(\mathbf{k},t)\rangle=$$

$$=\langle\eta_2(\mathbf{k},t)\overline{\eta}_2(\mathbf{k}',t')\rangle= \tag{5.2.28}$$

$$(2\pi)^4\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\int_0^{t'} d\tau'\delta(\tau-\tau')e^{-\left(\mathbf{k}^2+m^2\right)(t'-\tau')}\delta(\mathbf{k}+\mathbf{k}')$$

Therefore

$$\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\int_0^{t'} d\tau' e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau')}\langle\psi_-^{(1)}(\mathbf{k},t,0)\overline{\psi}_+^{(1)}(\mathbf{k}',t',0)\rangle=$$

$$=(2\pi)^4\psi^*\overline{\psi}^*\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\int_0^{t'} d\tau' e^{-\left(\mathbf{k}^2+m^2\right)(t'-\tau')}\delta(\mathbf{k})\delta(\mathbf{k}')- \tag{5.2.29}$$

$$-(2\pi)^4\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\int_0^{t'} d\tau'\delta(\tau-\tau')e^{-\left(\mathbf{k}^2+m^2\right)(t'-\tau')}(\not{\mathbf{k}}+m)\delta(\mathbf{k}+\mathbf{k}').$$

Then we easily find Fourier transcendental master equation to lowest order iteration $n=1$

$$\langle\psi_{-}^{(1)}(\mathbf{x},t,0)\overline{\psi}_{+}^{(1)}(\mathbf{x}',t',0)\rangle = \int d^4\mathbf{k}\int d^4\mathbf{k}' e^{i\pi\left(x\mathbf{k}+x\mathbf{k}'\right)}\langle\psi_{-}^{(1)}(\mathbf{k},t,0)\overline{\psi}_{+}^{(1)}(\mathbf{k}',t',0)\rangle =$$

$$= m^4\psi^*\overline{\psi}^*\int d^4\mathbf{k}\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\int_0^{t'} d\tau' e^{-\left(\mathbf{k}^2+m^2\right)(t'-\tau')}\int d^4\mathbf{k}' e^{i\pi\left(x\mathbf{k}+x\mathbf{k}'\right)}\delta(\mathbf{k})\delta(\mathbf{k}') -$$

$$-(2\pi)^4\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\int_0^{t'} d\tau'\delta(\tau-\tau')e^{-\left(\mathbf{k}^2+m^2\right)(t'-\tau')}\times \tag{5.2.30}$$

$$\times\int d^4\mathbf{k}\int d^4\mathbf{k}' e^{i\pi\left(\mathbf{x}\cdot\mathbf{k}+\mathbf{x}'\cdot\mathbf{k}'\right)}(\not{\mathbf{k}}+m)\delta(\mathbf{k}+\mathbf{k}') = 0$$

$$\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\int_0^{t'} d\tau' e^{-\left(\mathbf{k}^2+m^2\right)(t'-\tau')} = e^{-2\left(\mathbf{k}^2+m^2\right)(t+t')}\int_0^t d\tau e^{\left(\mathbf{k}^2+m^2\right)\tau}\int_0^{t'} d\tau' e^{\left(\mathbf{k}^2+m^2\right)\tau'} =$$

$$= \frac{e^{-2\left(\mathbf{k}^2+m^2\right)(t+t')}}{(\mathbf{k}^2+m^2)^2}\left[\left(e^{\left(\mathbf{k}^2+m^2\right)t}-1\right)\left(e^{\left(\mathbf{k}^2+m^2\right)t'}-1\right)\right] = \tag{5.2.31}$$

$$\frac{e^{-2\left(\mathbf{k}^2+m^2\right)(t+t')}}{(\mathbf{k}^2+m^2)^2}\left[e^{2\left(\mathbf{k}^2+m^2\right)(t+t')}-e^{\left(\mathbf{k}^2+m^2\right)t}-e^{\left(\mathbf{k}^2+m^2\right)t'}+1\right]\underset{t\to\infty,t'\to+\infty}{\to}$$

$$\underset{t\to\infty,t'\to+\infty}{\to}(\mathbf{k}^2+m^2)^{-2}$$

$$\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}\int_0^{t'} d\tau'\delta(\tau-\tau')e^{-\left(\mathbf{k}^2+m^2\right)(t'-\tau')} =$$

$$\int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)(t-\tau)}e^{-\left(\mathbf{k}^2+m^2\right)(t'-\tau)} = \int_0^t d\tau e^{-\left(\mathbf{k}^2+m^2\right)((t+t')-2\tau)} =$$

$$= e^{-\left(\mathbf{k}^2+m^2\right)(t+t')}\int_0^t d\tau e^{2\left(\mathbf{k}^2+m^2\right)\tau} = \frac{e^{-\left(\mathbf{k}^2+m^2\right)(t+t')}}{2(\mathbf{k}^2+m^2)}\left[e^{2\left(\mathbf{k}^2+m^2\right)t}-1\right] = \tag{5.2.32}$$

$$\frac{1}{2(\mathbf{k}^2+m^2)}\left[e^{-\left(\mathbf{k}^2+m^2\right)(t+t')}e^{2\left(\mathbf{k}^2+m^2\right)t}-1\right] =$$

$$\frac{1}{2(\mathbf{k}^2+m^2)}\left[e^{\left(\mathbf{k}^2+m^2\right)(t-t')}-e^{-\left(\mathbf{k}^2+m^2\right)(t+t')}\right]\underset{t=t'\to+\infty}{\to}\frac{1}{2(\mathbf{k}^2+m^2)}$$

From Eq.(5.2.30)-Eq.(5.2.32) we find directly transcendental master equation to lowest order iteration $n = 1$

$$\lim_{t=t'\to+\infty} \langle \psi_{-}^{(1)}(\mathbf{x},t,0)\overline{\psi}_{+}^{(1)}(\mathbf{x}',t',0)\rangle =$$
$$\psi^*\overline{\psi}^* - (2\pi)^{-4}\int d^4\mathbf{k}\exp[(\mathbf{x}-\mathbf{x}')\cdot\mathbf{k}]\frac{\not{\mathbf{k}}-m}{\mathbf{k}^2+m^2} = 0. \qquad (5.2.33)$$

Thus transcendental master equation corresponding to fermion propagator to lowest order iteration $n = 1$ gives the standard Euclidean fermion propagator:

$$\Delta(\mathbf{x}-\mathbf{x}') = (2\pi)^{-4}\int d^4\mathbf{k}\exp[(\mathbf{x}-\mathbf{x}')\cdot\mathbf{k}]\frac{\not{\mathbf{k}}-m}{\mathbf{k}^2+m^2}. \qquad (5.2.34)$$

From Eq.(5.2.24) and Eq.(5.2.26) we find directly transcendental master equation corresponding to photon propagator to lowest order iteration $n = 1$

$$\lim_{t=t'\to+\infty} \left\langle A_{\mu-}^{(1)}(\mathbf{x},t,0)A_{\nu+}^{(1)}(\mathbf{x}',t',0)\right\rangle =$$
$$a_\mu a_\nu - (2\pi)^{-4}\int d^4\mathbf{k}\frac{\exp[(\mathbf{x}-\mathbf{x}')\cdot\mathbf{k}]}{\mathbf{k}^2+m_{reg}^2}\left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{\mathbf{k}^2+m_{reg}^2}\right] = 0. \qquad (5.2.35)$$

Thus transcendental master equation corresponding to photon propagator to lowest order iteration $n = 1$ gives the standard Euclidean massive photon propagator:

(5.2.36)

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6.Double stochastic quantization of non-Abelian massive gauge field and Faddeev-Popov ghost effects.

A non-Abelian massive gauge field in the 4-dimensional Euclidean space is characterized by the action integral

$$S[A] = \frac{1}{4}\int d^4x\left[(\mathcal{F}^a_{\mu\nu}(x)\mathcal{F}^a_{\mu\nu}(x)) - \frac{1}{2}m_{reg}^2(A^a_\mu(x))^2\right] =$$
$$-\frac{1}{4}\int d^4x\left[(\mathcal{F}^a_{\mu\nu}(x)\mathcal{F}^a_{\mu\nu}(x)) + \frac{1}{2}m_{reg}^2(A^a_\mu(x))^2\right], \qquad (6.1.1)$$
$$\mathcal{F}^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) - g\times f^{abc}A^b_\mu(x)A^c_\nu(x),$$

where a, b and c stand for the color indices. From (6.1.1) we get the double stochastic

Langevin equation

$$\frac{d}{dt}A_\mu^a(k,t) = -((k^2+m^2)\delta_{\mu\nu} - k_\mu k_\nu)A_\nu^a(k,t) + \eta_\mu^a(\omega,k,t) + \epsilon\widetilde{\eta}_\mu^a(\varpi,k,t) + Y_\mu^a(k,t)$$

$$Y_\mu^a(k,t) =$$

$$+\frac{g}{(2\pi)^2}\int d^4k_1 d^4k_2 \delta^4(k-k_1-k_2)V_{\mu\kappa\lambda}^{abc}(k,-k_1,-k_2)A_\kappa^b(k_1,t)A_\lambda^d(k_2,t) + \tag{6.1.2}$$

$$+\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 d^4k_3 \delta^4(k-k_1-k_2-k_3)W_{\mu\nu\kappa\lambda}^{abcd}A_\nu^b(k_1,t)A_\kappa^c(k_2,t)A_\lambda^d(k_3,t),$$

where the Fourier transform $\widehat{f}(k)$ and it inverse $\check{f}(k)$ is defined as [21]-[22]:

$$\widehat{f}(k) = (2\pi)^{-2}\int f(x)e^{-ik\cdot x}d^4x,$$

$$\check{f}(k) = (2\pi)^{-2}\int f(x)e^{-ik\cdot x}d^4x, \tag{6.1.3}$$

and where

$$V_{\mu\kappa\lambda}^{abc}(k,k_1,k_2) = \left(-\frac{i}{2}\right)f^{abc}\times$$

$$\left[(k-k_1)_\lambda\delta_{\mu\kappa} + (k_1-k_2)_\mu\delta_{\kappa\lambda} + (k_2-k)_\kappa\delta_{\mu\lambda}\right].$$

$$W_{\mu\nu\kappa\lambda}^{abcd} = -\frac{1}{6}\left[f^{abe}f^{cde}(\delta_{\mu\kappa}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\kappa}) + \right. \tag{6.1.4}$$

$$+f^{ace}f^{bde}(\delta_{\mu\nu}\delta_{\kappa\lambda} - \delta_{\mu\lambda}\delta_{\nu\kappa}) +$$

$$+f^{ade}f^{ebe}(\delta_{\mu\kappa}\delta_{\nu\lambda} - \delta_{\mu\nu}\delta_{\kappa\lambda}).$$

From Eq.(6.1.2) we get

$$\frac{d}{dt}A_\mu^a(k,t) = -\delta(t)\delta^4(k)\chi_\mu^a -$$

$$-((k^2+m^2)\delta_{\mu\nu} - k_\mu k_\nu)A_\nu^a(k,t) + \eta_\mu^a(\omega,k,t) + \epsilon\eta_\mu^a(\varpi,k,t) + Y_\mu^a(k,t),$$

$$Y_\mu^a(k,t) =$$

$$+\frac{g}{(2\pi)^2}\int d^4k_1 V_{\mu\kappa\lambda}^{abc}(k,-k_1,-(k-k_1))A_\kappa^b(k_1,t)A_\lambda^c(k-k_1,t) + \tag{6.1.5}$$

$$+\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W_{\mu\nu\kappa\lambda}^{abcd}A_\nu^b(k_1,t)A_\kappa^c(k_2,t)A_\lambda^d(k-k_1-k_2,t),$$

$$V_{\mu\kappa\lambda}^{abc}(k,k_1,k-k_1) = \left(-\frac{i}{2}\right)f^{abc}\times$$

$$\times\left[(k-k_1)_\lambda\delta_{\mu\kappa} + (k_1-(k-k_1))_\mu\delta_{\kappa\lambda} + (-k_1)_\kappa\delta_{\mu\lambda}\right]$$

We define now the replacement

$$
\begin{gathered}
A^a_\mu(x,t) - \theta(t)\chi_\mu = A^a_{\mu-}(x,t),\\
A^a_\mu(x,t) = A^a_{\mu-}(x,t) + \theta(t)\chi^a_\mu,\\
A^a_\mu(k,t) = \mathcal{F}[A^a_\mu(x,t)](k,t) = \mathcal{F}[^a_{\mu-}(x,t)](k,t) + \theta(t)(2\pi)^4\delta^4(k)\chi^a_\mu =\\
= A^a_{\mu-}(p,t) + \theta(t)(2\pi)^4\chi^a_\mu,\\
A^a_\mu(x,t) + \theta(t)\chi^a_\mu = A^a_{\mu+}(x,t),\\
A^a_\mu(x,t) = A^a_{\mu+}(x,t) - \theta(t)\chi^a_\mu,\\
\frac{\partial A^a_\mu(x,t)}{\partial t} = \frac{\partial A^a_{\mu-}(x,t)}{\partial t} + \delta(t)\chi^a_\mu,\\
\frac{\partial A^a_\mu(k,t)}{\partial t} = \frac{\partial A^a_{\mu-}(k,t)}{\partial t} + \delta(t)(2\pi)^4\delta^4(k)\chi^a_\mu,\\
\frac{\partial A^a_\mu(x,t)}{\partial t} = \frac{\partial A^a_{\mu+}(x,t)}{\partial t} - \delta(t)\chi^a_\mu,\\
\frac{\partial A^a_\mu(k,t)}{\partial t} = \frac{\partial A^a_{\mu+}(k,t)}{\partial t} - \delta(t)(2\pi)^4\delta^4(k)\chi^a_\mu.
\end{gathered}
\tag{6.1.6}
$$

Eq.(6.1.5) by the replacement (6.1.6) we get

$$
\begin{gathered}
\frac{d}{dt}A^a_{\mu-}(k,t) = -\delta(t)\delta^4(k)\chi^a_\mu - ((k^2+m^2)\delta_{\mu\nu} - k_\mu k_\nu)\left[A^a_{\nu-}(k,t) + (2\pi)^2 m^2\delta^4(k)\chi^a_\nu\right] +\\
+\eta^a_\mu(\omega,k,t) + \epsilon\eta^a_\mu(\varpi,k,t) + Y^a_{\mu-}(k,t),\\
Y^a_{\mu-}(k,t) = \frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))Y^{a(1)}_{\mu-}(k,t) +\\
+\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda}Y^{a(2)}_{\mu-}(k,t) =\\
\frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))\left[A^b_{\kappa-}(k_1,t) + (2\pi)^2\delta^4(k_1)\chi^b_\kappa\right]\times\\
\times\left[A^c_{\lambda-}(k-k_1,t) + (2\pi)^2\delta^4(k-k_1)\chi^c_\lambda\right] +\\
+\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda}\left[A^b_{\nu-}(k_1,t) + (2\pi)^2\delta(k_1)\chi^b_\nu\right]\times\\
\times\left[A^c_{\kappa-}(k_2,t) + (2\pi)^2\delta(k_2)\chi^c_\kappa\right]\times\left[A^d_{\lambda-}(k-k_1-k_2,t) + (2\pi)^2\delta(k-k_1-k_2)\chi\right].
\end{gathered}
\tag{6.1.7}
$$

where

$$Y^{a(1)}_{\mu-}(k,t) = \left[A^b_{\kappa-}(k_1,t) + (2\pi)^2\delta^4(k_1)\chi^b_\kappa\right] \times \left[A^c_{\lambda-}(k-k_1,t) + (2\pi)^2\delta^4(k-k_1)\chi^c_\lambda\right],$$

$$Y^{a(2)}_{\mu-}(k,t) = \left[A^b_{\nu-}(k_1,t) + (2\pi)^2\delta^4(k_1)\chi^b_\nu\right] \times \left[A^c_{\kappa-}(k_2,t) + (2\pi)^2\delta^4(k_2)\chi^c_\kappa\right] \times \times\left[A^d_{\lambda-}(k-k_1-k_2,t) + (2\pi)^2\delta^4(k-k_1-k_2)\chi^d_\lambda\right]. \tag{6.1.8}$$

Note that

$$\begin{gathered} Y^{a(1)}_{\mu-}(k,t) = \\ = \left[A^b_{\kappa-}(k_1,t) + (2\pi)^2\delta^4(k_1)\chi^b_\kappa\right] \times \left[A^c_{\lambda-}(k-k_1,t) + (2\pi)^2\delta^4(k-k_1)\chi^c_\lambda\right] = \\ \underline{A^b_{\kappa-}(k_1,t)A^c_\lambda(k-k_1,t)} + (2\pi)^2\delta^4(k_1)\chi^b_\kappa A^c_{\lambda-}(k-k_1,t) + (2\pi)^2\delta^4(k-k_1)\chi^c_\lambda A^b_{\kappa-}(k_1,t) + \\ +(2\pi)^4\delta^4(k_1)\delta^4(k-k_1)\chi^b_\kappa\chi^c_\lambda. \end{gathered} \tag{6.1.9}$$

$$\begin{gathered} Y^{a(2)}_{\mu-}(k,t) = \\ = \left[A^b_{\nu-}(k_1,t) + (2\pi)^2\delta^4(k_1)\chi^b_\nu\right] \times \left[A^c_{\kappa-}(k_2,t) + (2\pi)^2\delta^4(k_2)\chi^c_\kappa\right] \times \\ \times\left[A^d_{\lambda-}(k-k_1-k_2,t) + (2\pi)^2\delta^4(k-k_1-k_2)\chi^d_\lambda\right] = \\ \left[\underline{A^b_{\nu-}(k_1,t)A^c_{\kappa-}(k_2,t)} + (2\pi)^2\delta^4(k_1)\chi^b_\nu A^c_{\kappa-}(k_2,t) + \right. \\ \left. (2\pi)^2\delta(k_2)\chi^c_\kappa A^b_{\nu-}(k_1,t) + (2\pi)^4\delta^4(k_1)\delta^4(k_2)\chi^b_\nu\chi^c_\kappa\right] \times \\ \times\left[A^d_{\lambda-}(k-k_1-k_2,t) + (2\pi)^2\delta^4(k-k_1-k_2)\chi^d_\lambda\right] = \\ \left[\underline{A^b_{\nu-}(k_1,t)A^c_{\kappa-}(k_2,t)} + (2\pi)^2\delta^4(k_1)\chi^b_\nu A^c_{\kappa-}(k_2,t) + \right. \\ \left. (2\pi)^2\delta^4(k_2)\chi^c_\kappa A^b_{\nu-}(k_1,t) + (2\pi)^4\delta^4(k_1)\delta(k_2)\chi^b_\nu\chi^c_\kappa\right]A^d_{\lambda-}(k-k_1-k_2,t) + \\ \left[\underline{A^b_{\nu-}(k_1,t)A^c_{\kappa-}(k_2,t)} + (2\pi)^2\delta(k_1)\chi^b_\nu A^c_{\kappa-}(k_2,t) + \right. \\ \left. (2\pi)^2\delta^4(k_2)\chi^c_\kappa A^b_{\nu-}(k_1,t) + (2\pi)^4\delta^4(k_1)\delta(k_2)\chi^b_\nu\chi^c_\kappa\right] \times (2\pi)^2\delta^4(k-k_1-k_2)\chi^d_\lambda = \\ \underline{\underline{A^b_{\nu-}(k_1,t)A^c_{\kappa-}(k_2,t)A^d_\lambda(k-k_1-k_2,t)}} + \underline{(2\pi)^2\delta^4(k_1)\chi^b_\nu A^c_{\kappa-}(k_2,t)A^d_{\lambda-}(k-k_1-k_2,t)} + \\ \underline{(2\pi)^2\delta^4(k_2)\chi^c_\kappa A^b_{\nu-}(k_1,t)A^d_{\lambda-}(k-k_1-k_2,t)} + (2\pi)^4\delta^4(k_1)\delta^4(k_2)\chi^b_\nu\chi^c_\kappa A^d_{\lambda-}(k-k_1-k_2,t) + \\ \underline{(2\pi)^2\delta^4(k-k_1-k_2)\chi^d_\lambda A^b_{\nu-}(k_1,t)A^c_{\kappa-}(k_2,t)} + \\ (2\pi)^4\delta^4(k_1)\delta^4(k-k_1-k_2)\chi^d_\lambda\chi^c_\kappa A^c_{\kappa-}(k_1,t) + \\ +(2\pi)^4\delta^4(k_2)\delta^4(k-k_1-k_2)\chi^d_\lambda\chi^c_\kappa A^b_{\nu-}(k_1,t) + \\ +(2\pi)^6\delta^4(k_1)\delta^4(k_2)\delta^4(k-k_1-k_2)\chi^d_\lambda\chi^b_\nu\chi^c_\kappa \end{gathered} \tag{6.1.10}$$

$$
\begin{gathered}
Y^{a}_{\mu-}(k,t) = \\
= \widetilde{Y}^{a(1)}_{\mu-} + \widetilde{Y}^{a(2)}_{\mu-}(k,t) = \\
\frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))Y^{a(1)}_{\mu-}(k,t) + \\
+\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda} Y^{a(2)}_{\mu-}(k,t), \\
\widetilde{Y}^{a(1)}_{\mu-} = \\
\frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))Y^{a(1)}_{\mu-}(k,t) \\
\widetilde{Y}^{a(2)}_{\mu-}(k,t) = \\
\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda} Y^{a(2)}_{\mu-}(k,t)
\end{gathered}
\tag{6.1.11}
$$

From Eq.(6.1.9) we get

$$
\begin{aligned}
&\widetilde{Y}^{a(1)}_{\mu-} = \\
&= \frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))Y^{a(1)}_{\mu-}(k,t) = \\
&\frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))\underline{A^b_{\kappa-}(k_1,t)A^c_{\lambda-}(k-k_1,t)} + \\
&+\frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1)) \times \\
&\times\Big[(2\pi)^2\delta^4(k_1)\chi^b_\kappa A^c_{\lambda-}(k-k_1,t) + (2\pi)^2\delta^4(k-k_1)\chi^c_\lambda A^b_{\kappa-}(k_1,t) \\
&+(2\pi)^4\delta^4(k_1)\delta^4(k-k_1)\chi^b_\kappa\chi^c_\lambda\Big] = \\
&\frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))\underline{A^b_{\kappa-}(k_1,t)A^c_{\lambda-}(k-k_1,t)} + \\
&+g\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))\delta^4(k_1)\chi^b_\kappa A^c_{\lambda-}(k-k_1,t) + \\
&+g\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))\delta^4(k-k_1)\chi^c_\lambda A^b_{\kappa-}(k_1,t) + \\
&+(2\pi)^2 g\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))\delta^4(k_1)\delta^4(k-k_1)\chi^b_\kappa\chi^c_\lambda = \\
&\frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))\underline{A^b_{\kappa-}(k_1,t)A^c_{\lambda-}(k-k_1,t)} + \\
&+g\Big[V^{abc}_{\mu\kappa\lambda}(k,0,-k)\chi^b_\kappa A^c_{\lambda-}(k,t) + V^{abc}_{\mu\kappa\lambda}(k,-k,0)\chi^c_\lambda A^b_{\kappa-}(k,t)\Big] + \\
&+(2\pi)^2 g V^{abc}_{\mu\kappa\lambda}(k,-k,0)\delta^4(k)\chi^b_\kappa\chi^c_\lambda.
\end{aligned}
\tag{6.1.12}
$$

$$
V^{abc}_{\mu\kappa\lambda}(k,0,-k) = \left(-\frac{i}{2}\right)f^{abc} \times [k_\lambda\delta_{\mu\kappa} - k_\mu\delta_{\kappa\lambda}]
$$

$$
V^{abc}_{\mu\kappa\lambda}(k,-k,0) = \left(-\frac{i}{2}\right)f^{abc} \times
$$

From Eq.(6.1.10) we get

$$
\widetilde{Y}^{a(2)}_{\mu-}(k,t) =
$$

$$
= \frac{g^2}{(2\pi)^2}\int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda} Y^{a(2)}_{\mu-}(k,k_1,k_2,t) =
$$

$$
\begin{aligned}
&\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda}\Big[\underline{\underline{A^b_{\nu-}(k_1,t)A^c_{\kappa-}(k_2,t)A^d_{\lambda-}(k-k_1-k_2,t)}} + \\
&+\underline{(2\pi)^2\delta^4(k_1)\chi^b_\nu A^c_{\kappa-}(k_2,t)A^d_{\lambda-}(k-k_1-k_2,t)} + \\
&+\underline{(2\pi)^2\delta^4(k_2)\chi^c_\kappa A^b_{\nu-}(k_1,t)A^d_{\lambda-}(k-k_1-k_2,t)} + \\
&\underline{(2\pi)^2\delta^4(k-k_1-k_2)\chi^d_\lambda A^b_{\nu-}(k_1,t)A^c_{\kappa-}(k_2,t)} + \\
&\underline{(2\pi)^2\delta^4(k-k_1-k_2)\chi^d_\lambda} \underline{A^b_{\nu-}(k_1,t)A^c_{\kappa-}(k_2,t)}\Big] + \\
&+\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda}\Big[(2\pi)^4\delta^4(k_1)\delta^4(k_2)\chi^b_\nu\chi^c_\kappa A^d_{\lambda-}(k-k_1-k_2,t) + \\
&+(2\pi)^4\delta^4(k_2)\delta^4(k-k_1-k_2)\chi^d_\lambda\chi^c_\kappa A^c_{\kappa-}(k_1,t) + \\
&+(2\pi)^4\delta^4(k_2)\delta^4(k-k_1-k_2)\chi^d_\lambda\chi^c_\kappa A^b_{\nu-}(k_1,t) + \\
&+(2\pi)^8\delta^4(k_1)\delta^4(k_2)\delta^4(k-k_1-k_2)\chi^d_\lambda\chi^b_\nu\chi^c_\kappa\Big]
\end{aligned}
\qquad (6.1.13)
$$

From Eq.(6.11)-Eq.(6.13) we obtain linear differential master equations corresponding to $A^a_{\mu-}(k,t)$:

$$
\begin{aligned}
\frac{d}{dt}A^a_{\mu-}(k,t) &= -\delta(t)\delta^4(k)\chi^a_\mu - ((k^2+m^2)\delta_{\mu\nu} - k_\mu k_\nu)\Big[A^a_{\nu-}(k,t) + (2\pi)^2 m^2\delta^4(k)\chi^a_\nu\Big] + \\
&+\eta^a_\mu(\omega,k,t) + \widetilde{Y}^{\{\mathbf{L}\}a(1)}_{\mu-}(k,t) + \widetilde{Y}^{\{\mathbf{L}\}a(2)}_{\mu-}(k,t),
\end{aligned}
\qquad (6.1.14)
$$

where

$$
\widetilde{Y}^{\{\mathbf{L}\}a(1)}_{\mu-}(k,t) =
$$

$$
\begin{aligned}
&= gV^{abc}_{\mu\kappa\lambda}(k,0,-k)\chi^b_\kappa A^c_{\lambda-}(k,t) + (2\pi)^2\delta^4(k-k_1)\chi^c_\lambda A^b_{\kappa-}(k_1,t) + \\
&+(2\pi)^2 gV^{abc}_{\mu\kappa\lambda}(k,-k,0)\delta^4(k)\chi^b_\kappa\chi^c_\lambda.
\end{aligned}
\qquad (6.1.15)
$$

$$
V^{abc}_{\mu\kappa\lambda}(k,0,-k) = \left(-\frac{i}{2}\right)f^{abc}\times[k_\lambda\delta_{\mu\kappa} - k_\mu\delta_{\kappa\lambda}]
$$

and

$$
\begin{gathered}
\widetilde{Y}_{\mu-}^{\{\mathbf{L}\}a(2)}(k,t) = \\
\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W_{\mu\nu\kappa\lambda}^{abcd}\Big[(2\pi)^4\delta^4(k_1)\delta^4(k_2)\chi_\nu^b\chi_\kappa^c A_{\lambda-}^d(k-k_1-k_2,t) + \\
+(2\pi)^4\delta^4(k_1)\delta^4(k-k_1-k_2)\chi_\lambda^d\chi_\kappa^c A_{\kappa-}^c(k_1,t) + \\
+(2\pi)^4\delta^4(k_2)\delta^4(k-k_1-k_2)\chi_\lambda^d\chi_\kappa^c A_{\nu-}^b(k_1,t) + \\
+(2\pi)^8\delta^4(k_1)\delta^4(k_2)\delta^4(k-k_1-k_2)\chi_\lambda^d\chi_\nu^b\chi_\kappa^c\Big] = \\
g^2 W_{\mu\nu\kappa\lambda}^{abcd}[\chi_\nu^b\chi_\kappa^c A_{\lambda-}^d(k,t) + \chi_\lambda^d\chi_\kappa^c A_{\nu-}^b(k_1,t)] + g^2(2\pi)^4 W_{\mu\nu\kappa\lambda}^{abcd}\delta^4(k)\chi_\lambda^d\chi_\nu^b\chi_\kappa^c.
\end{gathered}
\tag{6.1.16}
$$

From Eq.(6.1.5) by the replacement (6.1.6) we get

$$
\begin{gathered}
\frac{d}{dt}A_{\mu+}^a(k,t) = \delta(t)\delta^4(k)\chi_\mu^a - ((k^2+m^2)\delta_{\mu\nu} - k_\mu k_\nu)\Big[A_{\nu+}^a(k,t) - (2\pi)^2 m^2\delta^4(k)\chi_\nu^a\Big] + \\
+\eta_\mu^a(\omega,k,t) + \epsilon\eta_\mu^a(\varpi,k,t) + Y_{\mu+}^a(k,t), \\
Y_{\mu+}^a(k,t) = \frac{g}{(2\pi)^2}\int d^4k_1 V_{\mu\kappa\lambda}^{abc}(k,-k_1,-(k-k_1))Y_{\mu+}^{a(1)}(k,t) + \\
+\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W_{\mu\nu\kappa\lambda}^{abcd} Y_{\mu+}^{a(2)}(k,t) = \\
\frac{g}{(2\pi)^2}\int d^4k_1 V_{\mu\kappa\lambda}^{abc}(k,-k_1,-(k-k_1))\Big[A_{\kappa+}^b(k_1,t) - (2\pi)^2\delta^4(k_1)\chi_\kappa^b\Big]\times \\
\times\Big[A_{\lambda+}^c(k-k_1,t) - (2\pi)^2\delta^4(k-k_1)\chi_\lambda^c\Big] + \\
+\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W_{\mu\nu\kappa\lambda}^{abcd}\Big[A_{\nu+}^b(k_1,t) - (2\pi)^2\delta^4(k_1)\chi_\nu^b\Big]\times \\
\times\Big[A_{\kappa+}^c(k_2,t) - (2\pi)^2\delta^4(k_2)\chi_\kappa^c\Big]\times\Big[A_{\lambda+}^d(k-k_1-k_2,t) - (2\pi)^2\delta^4(k-k_1-k_2)\chi\Big].
\end{gathered}
\tag{6.1.17}
$$

where

$$
\begin{gathered}
Y_{\mu+}^{a(1)}(k,t) = \\
\Big[A_{\kappa+}^b(k_1,t) - (2\pi)^2\delta^4(k_1)\chi_\kappa^b\Big]\times\Big[A_{\lambda+}^c(k-k_1,t) - (2\pi)^2\delta^4(k-k_1)\chi_\lambda^c\Big], \\
\\
Y_{\mu+}^{a(2)}(k,t) = \\
\Big[A_{\nu+}^b(k_1,t) - (2\pi)^2\delta^4(k_1)\chi_\nu^b\Big]\times\Big[A_{\kappa+}^c(k_2,t) - (2\pi)^2\delta^4(k_2)\chi_\kappa^c\Big]\times \\
\times\Big[A_{\lambda+}^d(k-k_1-k_2,t) - (2\pi)^2\delta^4(k-k_1-k_2)\chi_\lambda^d\Big].
\end{gathered}
\tag{6.1.18}
$$

Note that

$$
\begin{gathered}
Y_{\mu+}^{a(1)}(k,t) = \\
= \left[A_{\kappa+}^{b}(k_1,t) - (2\pi)^2\delta^4(k_1)\chi_\kappa^b\right] \times \left[A_{\lambda+}^{c}(k-k_1,t) - (2\pi)^2\delta^4(k-k_1)\chi_\lambda^c\right] = \\
\underline{A_{\kappa+}^{b}(k_1,t)A_{\lambda+}^{c}(k-k_1,t)} - (2\pi)^2\delta^4(k_1)\chi_\kappa^b A_{\lambda+}^{c}(k-k_1,t) - (2\pi)^2\delta^4(k-k_1)\chi_\lambda^c A_{\kappa+}^{b}(k_1,t) + \\
+(2\pi)^4\delta^4(k_1)\delta^4(k-k_1)\chi_\kappa^b\chi_\lambda^c
\end{gathered}
\tag{6.1.19}
$$

$$
\begin{gathered}
Y_{\mu+}^{a(2)}(k,t) = \\
= \left[A_{\nu+}^{b}(k_1,t) - (2\pi)^2\delta^4(k_1)\chi_\nu^b\right] \times \left[A_{\kappa+}^{c}(k_2,t) - (2\pi)^2\delta^4(k_2)\chi_\kappa^c\right] \times \\
\times\left[A_{\lambda+}^{d}(k-k_1-k_2,t) - (2\pi)^2\delta^4(k-k_1-k_2)\chi_\lambda^d\right] = \\
\Big[\underline{A_{\nu+}^{b}(k_1,t)A_{\kappa+}^{c}(k_2,t)} - (2\pi)^2\delta^4(k_1)\chi_\nu^b A_{\kappa+}^{c}(k_2,t) - \\
-(2\pi)^2\delta(k_2)\chi_\kappa^c A_{\nu+}^{b}(k_1,t) + (2\pi)^4\delta^4(k_1)\delta^4(k_2)\chi_\nu^b\chi_\kappa^c\Big] \times \\
\times\left[A_{\lambda+}^{d}(k-k_1-k_2,t) - (2\pi)^2\delta^4(k-k_1-k_2)\chi_\lambda^d\right] = \\
\Big[\underline{A_{\nu+}^{b}(k_1,t)A_{\kappa+}^{c}(k_2,t)} - (2\pi)^2\delta^4(k_1)\chi_\nu^b A_{\kappa+}^{c}(k_2,t) - \\
-(2\pi)^2\delta^4(k_2)\chi_\kappa^c A_{\nu+}^{b}(k_1,t) + (2\pi)^4\delta^4(k_1)\delta^4(k_2)\chi_\nu^b\chi_\kappa^c\Big] \times A_{\lambda+}^{d}(k-k_1-k_2,t) + \\
\Big[\underline{A_{\nu+}^{b}(k_1,t)A_{\kappa+}^{c}(k_2,t)} - (2\pi)^2\delta(k_1)\chi_\nu^b A_{\kappa+}^{c}(k_2,t) - \\
-(2\pi)^2\delta(k_2)\chi_\kappa^c A_{\nu+}^{b}(k_1,t) + (2\pi)^4\delta^4(k_1)\delta^4(k_2)\chi_\nu^b\chi_\kappa^c\Big] \times (2\pi)^2\delta^4(k-k_1-k_2)\chi_\lambda^d = \\
= \underline{\underline{A_{\nu+}^{b}(k_1,t)A_{\kappa+}^{c}(k_2,t)A_{\lambda+}^{d}(k-k_1-k_2,t)}} - \\
-\underline{(2\pi)^2\delta^4(k_1)\chi_\nu^b A_{\kappa+}^{c}(k_2,t)A_{\lambda+}^{d}(k-k_1-k_2,t)} - \\
-\underline{(2\pi)^2\delta(k_2)\chi_\kappa^c A_{\nu+}^{b}(k_1,t)A_{\lambda+}^{d}(k-k_1-k_2,t)} + \\
+(2\pi)^4\delta(k_1)\delta(k_2)\chi_\nu^b\chi_\kappa^c A_{\lambda+}^{d}(k-k_1-k_2,t) + \\
+\underline{(2\pi)^2\delta(k-k_1-k_2)\chi_\lambda^d A_{\nu+}^{b}(k_1,t)A_{\kappa-}^{c}(k_2,t)} - \\
-(2\pi)^4\delta^4(k_1)\delta^4(k-k_1-k_2)\chi_\lambda^d\chi_\kappa^c A_{\kappa+}^{c}(k_1,t) - \\
-(2\pi)^4\delta^4(k_2)\delta^4(k-k_1-k_2)\chi_\lambda^d\chi_\kappa^c A_{\nu+}^{b}(k_1,t) + \\
+(2\pi)^6\delta^4(k_1)\delta^4(k_2)\delta^4(k-k_1-k_2)\chi_\lambda^d\chi_\nu^b\chi_\kappa^c
\end{gathered}
\tag{6.1.20}
$$

$$
\begin{gathered}
Y^{a}_{\mu+}(k,t) = \\
= \widetilde{Y}^{a(1)}_{\mu+} + \widetilde{Y}^{a(2)}_{\mu+}(k,t) = \\
\frac{g}{(2\pi)^2} \int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))Y^{a(1)}_{\mu+}(k,t) + \\
+\frac{g^2}{(2\pi)^4} \int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda} Y^{a(2)}_{\mu+}(k,t), \\
\widetilde{Y}^{a(1)}_{\mu+} = \\
\frac{g}{(2\pi)^2} \int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))Y^{a(1)}_{\mu+}(k,t) \\
\widetilde{Y}^{a(2)}_{\mu+}(k,t) = \\
\frac{g^2}{(2\pi)^4} \int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda} Y^{a(2)}_{\mu+}(k,t)
\end{gathered}
\tag{6.1.21}
$$

From Eq.(6.1.19) we get

$$
\begin{aligned}
&\widetilde{Y}^{a(1)}_{\mu+} = \\
&= \frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))Y^{a(1)}_{\mu+}(k,t) = \\
&\frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))\underline{A^b_{\kappa+}(k_1,t)A^c_{\lambda+}(k-k_1,t)} + \\
&+\frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))\times \\
&\times\left[(2\pi)^2\delta^4(k_1)\chi^b_\kappa A^c_{\lambda+}(k-k_1,t) + (2\pi)^4\delta^4(k_1)\delta^4(k-k_1)\chi^b_\kappa\chi^c_\lambda\right] = \\
&\frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))\underline{A^b_{\kappa+}(k_1,t)A^c_{\lambda+}(k-k_1,t)} + \\
&-g\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))[\delta(k_1)\chi^b_\kappa A^c_{\lambda+}(k-k_1,t) + \\
&+\delta^4(k-k_1)\chi^c_\lambda A^b_{\kappa+}(k_1,t)] + \\
&+(2\pi)^2 g\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))\delta^4(k_1)\delta^4(k-k_1)\chi^b_\kappa\chi^c_\lambda = \\
&\frac{g}{(2\pi)^2}\int d^4k_1 V^{abc}_{\mu\kappa\lambda}(k,-k_1,-(k-k_1))\underline{A^b_{\kappa+}(k_1,t)A^c_{\lambda+}(k-k_1,t)} - \\
&-gV^{abc}_{\mu\kappa\lambda}(k,0,-k)[\chi^b_\kappa A^c_{\lambda+}(k,t) + \chi^c_\lambda A^b_{\kappa+}(k,t)] + \\
&+(2\pi)^2 gV^{abc}_{\mu\kappa\lambda}(k,-k,0)\delta^4(k)\chi^b_\kappa\chi^c_\lambda.
\end{aligned}
\tag{6.1.22}
$$

$$
V^{abc}_{\mu\kappa\lambda}(k,0,-k) = \left(-\frac{i}{2}\right)f^{abc}\times[k_\lambda\delta_{\mu\kappa} - k_\mu\delta_{\kappa\lambda}]
$$

From Eq.(6.1.20) we get

$$
\begin{aligned}
&\widetilde{Y}^{a(2)}_{\mu+}(k,t) = \\
&= \frac{g^2}{(2\pi)^2}\int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda} Y^{a(2)}_{\mu-}(k,k_1,k_2,t) = \\
&\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda}\Big[\underline{\underline{A^b_{\nu+}(k_1,t)A^c_{\kappa+}(k_2,t)A^d_{\lambda+}(k-k_1-k_2,t)}} + \\
&+\underline{(2\pi)^2\delta^4(k_1)\chi^b_\nu A^c_{\kappa+}(k_2,t)A^d_{\lambda+}(k-k_1-k_2,t)} + \\
&+\underline{(2\pi)^2\delta^4(k_2)\chi^c_\kappa A^b_{\nu+}(k_1,t)A^d_{\lambda+}(k-k_1-k_2,t)} + \\
&\underline{(2\pi)^2\delta^4(k-k_1-k_2)\chi^d_\lambda A^b_{\nu+}(k_1,t)A^c_{\kappa-}(k_2,t)} + \\
&\underline{(2\pi)^2\delta^4(k-k_1-k_2)\chi^d_\lambda A^b_{\nu+}(k_1,t)A^c_{\kappa+}(k_2,t)}\Big] + \\
&+\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda}\Big[(2\pi)^4\delta(k_1)\delta(k_2)\chi^b_\nu\chi^c_\kappa A^d_{\lambda+}(k-k_1-k_2,t) + \\
&-(2\pi)^4\delta^4(k_2)\delta^4(k-k_1-k_2)\chi^d_\lambda\chi^c_\kappa A^c_{\kappa+}(k_1,t) - \\
&-(2\pi)^4\delta^4(k_2)\delta^4(k-k_1-k_2)\chi^d_\lambda\chi^c_\kappa A^b_{\nu+}(k_1,t) + \\
&+(2\pi)^8\delta^4(k_1)\delta^4(k_2)\delta(k-k_1-k_2)\chi^d_\lambda\chi^b_\nu\chi^c_\kappa\Big]
\end{aligned}
\qquad (6.1.23)
$$

From Eq.(6.21)-Eq.(6.23) we obtain linear differential master equations corresponding to $A^a_{\mu+}(k,t)$:

$$
\begin{aligned}
\frac{d}{dt}A^a_{\mu+}(k,t) &= \delta(t)\delta^4(k)\chi^a_\mu - ((k^2+m^2)\delta_{\mu\nu} - k_\mu k_\nu)\Big[A^a_{\nu+}(k,t) - (2\pi)^2 m^2\delta^4(k)\chi^a_\nu\Big] + \\
&+\eta^a_\mu(\omega,k,t) + \widetilde{Y}^{\{\mathbf{L}\}a(1)}_{\mu+}(k,t) + \widetilde{Y}^{\{\mathbf{L}\}a(2)}_{\mu+}(k,t),
\end{aligned}
\qquad (6.1.24)
$$

where

$$
\begin{aligned}
&\widetilde{Y}^{\{\mathbf{L}\}a(1)}_{\mu+}(k,t) = \\
&= gV^{abc}_{\mu\kappa\lambda}(k,0,-k)[\chi^b_\kappa A^c_{\lambda+}(k,t) + \chi^c_\lambda A^b_{\kappa+}(k,t)] + \\
&+(2\pi)^2 gV^{abc}_{\mu\kappa\lambda}(k,-k,0)\delta^4(k)\chi^b_\kappa\chi^c_\lambda. \\
&V^{abc}_{\mu\kappa\lambda}(k,0,-k) = \left(-\frac{i}{2}\right)f^{abc}\times[k_\lambda\delta_{\mu\kappa} - k_\mu\delta_{\kappa\lambda}]
\end{aligned}
\qquad (6.1.25)
$$

and

$$\widetilde{Y}^{\{\mathbf{L}\}a(2)}_{\mu+}(k,t) =$$

$$\frac{g^2}{(2\pi)^4}\int d^4k_1 d^4k_2 W^{abcd}_{\mu\nu\kappa\lambda}\Big[(2\pi)^4\delta^4(k_1)\delta^4(k_2)\chi^b_\nu\chi^c_\kappa A^d_{\lambda-}(k-k_1-k_2,t) -$$
$$-(2\pi)^4\delta^4(k_1)\delta^4(k-k_1-k_2)\chi^d_\lambda\chi^c_\kappa A^c_{\kappa+}(k_1,t) -$$
$$-(2\pi)^4\delta^4(k_2)\delta^4(k-k_1-k_2)\chi^d_\lambda\chi^c_\kappa A^b_{\nu+}(k_1,t) + \tag{6.1.26}$$
$$+(2\pi)^8\delta^4(k_1)\delta^4(k_2)\delta^4(k-k_1-k_2)\chi^d_\lambda\chi^b_\nu\chi^c_\kappa\Big] =$$
$$-g^2 W^{abcd}_{\mu\nu\kappa\lambda}[\chi^d_\lambda\chi^c_\kappa A^c_{\kappa+}(k,t) + \chi^d_\lambda\chi^c_\kappa A^b_{\nu+}(k,t)] +$$
$$+g^2(2\pi)^4 W^{abcd}_{\mu\nu\kappa\lambda}\delta^4(k)\chi^d_\lambda\chi^b_\nu\chi^c_\kappa$$

From Eq.(6.14)-Eq.(6.16) we obtain linear differential master equations corresponding
to $A^a_{\mu-}(k,t)$:

$$\frac{d}{dt}A^a_{\mu-}(k,t) = -\delta(t)\delta(k)\chi^a_\mu - ((k^2+m^2)\delta_{\mu\nu} - k_\mu k_\nu)\Big[A^a_{\nu-}(k,t) + (2\pi)^2\delta^4(k)m^2\chi^a_\nu\Big] +$$
$$+\eta^a_\mu(\omega,k,t) + \widetilde{Y}^{ab\{\mathbf{L}\}\alpha(1)}_{\mu-}(k,t) + \widetilde{Y}^{ab\{\mathbf{L}\}\alpha(2)}_{\mu-}(k,t), \tag{6.1.27}$$

where

$$\widetilde{Y}^{a\{\mathbf{L}\}\alpha(1)}_{\mu-}(k,t) =$$
$$= gV^{abc}_{\mu\kappa\lambda}(k,0,-k)[\chi^b_\kappa A^c_{\lambda-}(k,t) + \chi^c_\lambda A^b_{\kappa-}(k,t)] +$$
$$+(2\pi)^2 gV^{abc}_{\mu\kappa\lambda}(k,-k,0)\delta^4(k)\chi^b_\kappa\chi^c_\lambda. \tag{6.1.28}$$
$$\widetilde{Y}^{a\{\mathbf{L}\}\alpha(2)}_{\mu-}(k,t) =$$
$$g^2 W^{abcd}_{\mu\nu\kappa\lambda}[\chi^b_\nu\chi^c_\kappa A^d_{\lambda-}(k,t) + \chi^d_\lambda\chi^c_\kappa A^b_{\nu-}(k_1,t)] + g^2(2\pi)^4 W^{abcd}_{\mu\nu\kappa\lambda}\delta(k)\chi^d_\lambda\chi^b_\nu\chi^c_\kappa.$$
$$V^{abc}_{\mu\kappa\lambda}(k,0,-k) = \Big(-\frac{i}{2}\Big)f^{abc}\times[k_\lambda\delta_{\mu\kappa} - k_\mu\delta_{\kappa\lambda}].$$

From Eq.(6.27-6.28) we obtain linear integral master equation

$$A^{a}_{\mu-}(k,t) = -\chi^{a}_{\mu}\int_{0}^{t} G^{ab}_{\mu\nu}(k;t,t',m)\delta(t')\delta^{4}(k)dt' + (2\pi)^{2}m^{2}\chi^{a}_{\nu}\int_{0}^{t} G^{ab}_{\mu\nu}(k;t,t',m)\delta(k)dt' +$$

$$+\int_{0}^{t} G^{ab}_{\mu\nu}(k;t,t',m)\eta^{a}_{\nu}(\omega,k,t)dt' +$$

$$+\int_{0}^{t} G^{ab}_{\mu\nu'}(k;t,t',m)\widetilde{Y}^{b\{\mathbf{L}\}\alpha(1)}_{\nu'-}(k,t')dt' + \int_{0}^{t} G^{ab}_{\mu\nu'}(k;t,t',m)\widetilde{Y}^{b\{\mathbf{L}\}\alpha(2)}_{\nu'-}(k,t') =$$

$$-\chi^{a}_{\mu}\delta^{4}(k)G^{ab}_{\mu\nu}(k;t,0,m) + (2\pi)^{2}m^{2}\chi^{a}_{\nu}\int_{0}^{t} G^{ab}_{\mu\nu}(k;t,t',m)\delta(k)dt' + \tag{6.1.29}$$

$$+\int_{0}^{t} G^{ab}_{\mu\nu'}(k;t,t')\widetilde{Y}^{b\{\mathbf{L}\}\alpha(1)}_{\nu'-}(k,t')dt' + \int_{0}^{t} G^{ab}_{\mu\nu'}(k;t,t')\widetilde{Y}^{b\{\mathbf{L}\}\alpha(2)}_{\nu'-}(k,t')dt',$$

$$\widetilde{Y}^{b\{\mathbf{L}\}\alpha(1)}_{\nu'-}(k,t') = \widetilde{Y}^{b\{\mathbf{L}\}\alpha(1)}_{\mu-}(k,t')\Big|_{\mu\triangleq\nu'}$$

$$\widetilde{Y}^{b\{\mathbf{L}\}\alpha(2)}_{\nu'-}(k,t') = \widetilde{Y}^{b\{\mathbf{L}\}\alpha(2)}_{\mu-}(k,t')\Big|_{\mu\triangleq\nu'}$$

where

$$G^{ab}_{\mu\nu}(k;t,t',m) = \delta_{ab}G_{\mu\nu}(k;t,t',m),$$

$$G_{\mu\nu}(k;t,t',m) = \left\{\left(\delta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^{2}+m^{2}}\right)e^{-(k^{2}+m^{2})(t-t')}\right\}, \tag{6.1.29'}$$

where $G^{ab}_{\mu\nu}(k;t,t') = \delta_{ab}G_{\mu\nu}(k;t,t')$ is the zeroth order Green function, see Apendix 7. From Eq.(6.24)-Eq.(6.26) we obtain the linear integral master equation

$$A^a_{\mu+}(k,t) = \int_0^t G^{ab}_{\mu\nu}(k;t,t')\delta(t')\delta^4(k)dt' - (2\pi)^2 m^2 \chi^a_\nu \int_0^t G^{ab}_{\mu\nu}(k;t,t')\delta^4(k)dt' +$$

$$+\int_0^t G^{ab}_{\mu\nu}(k;t,t')\eta^a_\mu(\omega,k,t)dt' +$$

$$+\int_0^t G^{ab}_{\mu\nu'}(k;t,t')\widetilde{Y}^{b\{\mathbf{L}\}\alpha(1)}_{\nu'+}(k,t')dt' + \int_0^t G^{ab}_{\mu\nu'}(k;t,t')\widetilde{Y}^{b\{\mathbf{L}\}\alpha(2)}_{\nu'+}(k,t') =$$

$$= \chi^a_\mu \delta^4(k) G^{ab}_{\mu\nu}(k;t,0) - (2\pi)^2 m^2 \chi^a_\nu \int_0^t G^{ab}_{\mu\nu}(k;t,t')\delta^4(k)dt' + \qquad (6.1.30)$$

$$+\int_0^t G^{ab}_{\mu\nu'}(k;t,t')\widetilde{Y}^{b\{\mathbf{L}\}\alpha(1)}_{\nu'+}(k,t')dt' + \int_0^t G^{ab}_{\mu\nu'}(k;t,t')\widetilde{Y}^{b\{\mathbf{L}\}\alpha(2)}_{\nu'+}(k,t'),$$

$$\widetilde{Y}^{b\{\mathbf{L}\}\alpha(1)}_{\nu'+}(k,t') = \widetilde{Y}^{b\{\mathbf{L}\}\alpha(1)}_{\mu+}(k,t')\Big|_{\mu \triangleq \nu'}$$

$$\widetilde{Y}^{b\{\mathbf{L}\}\alpha(2)}_{\nu'+}(k,t') = \widetilde{Y}^{b\{\mathbf{L}\}\alpha(2)}_{\mu+}(k,t')\Big|_{\mu \triangleq \nu'}$$

The linear equations (6.1.29) and (6.1.30) admit exact solutions denoted by

$$A^a_{\mu-}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\}), A^a_{\mu+}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\}). \qquad (6.1.31)$$

Therefore the system of the transcendental master equations reads

$$\langle [A^a_{\mu-}(x,t;\{\eta^a_\mu\},\{\chi^a_\mu\})] A^a_{\mu+}(x,t;\{\eta^a_\mu\},\{\chi^a_\mu\})\rangle_\eta = 0. \qquad (6.1.32)$$

where

$$A^a_{\mu-}(x,t;\{\eta^a_\mu\},\{\chi^a_\mu\}) = \mathcal{F}^{-1}[A^a_{\mu-}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\})],$$
$$A^a_{\mu+}(x,t;\{\eta^a_\mu\},\{\chi^a_\mu\}) = \mathcal{F}^{-1}[A^a_{\mu+}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\})]. \qquad (6.1.33)$$

For a sufficiently small $g < 1$, equations (6.1.29) and (6.1.30) can be solved iteratively
as an expansion in g, see Apendix 3, Theorem 3.2.We define now corresponding countable sequences $\left\{A^{a(n)}_{\mu-}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\})\right\}_{n\in\mathbb{N}}$ and $\left\{A^{a(n)}_{\mu+}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\})\right\}_{n\in\mathbb{N}}$ of the
iterations by recursions

$$A^{a(n+1)}_{\mu-}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\}) =$$

$$-\chi^a_\mu\delta^4(k)G^{ab}_{\mu\nu}(k;t,0) + (2\pi)^2 m^2\chi^a_{\nu'}\int_0^t G^{ab}_{\mu\nu'}(k;t,t')\delta(k)dt' +$$

$$+\int_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt' +$$

$$+\int_0^t G^{ab}_{\mu\nu'}(k;t,t')\widetilde{Y}^{b\{\mathbf{L}\}\alpha(1)}_{\nu'-}(k,t';n)dt' + \int_0^t G^{ab}_{\mu\nu'}(k;t,t')\widetilde{Y}^{b\{\mathbf{L}\}\alpha(2)}_{\nu'-}(k,t';n),$$

$$\widetilde{Y}^{b\{\mathbf{L}\}\alpha(1)}_{\nu'-}(k,t';n) = \tag{6.1.34}$$

$$= gV^{abc}_{\nu'\kappa\lambda}(k,0,-k)[\chi^b_\kappa A^c_{\lambda-}(k,t;\{\eta^c_\lambda\},\{\chi^c_\lambda\}) + \chi^c_\lambda A^b_{\kappa-}(k,t;\{\eta^b_\kappa\},\{\chi^b_\kappa\})] +$$

$$+(2\pi)^2 gV^{abc}_{\nu'\kappa\lambda}(k,-k,0)\delta(k)\chi^b_\kappa\chi^c_\lambda.$$

$$\widetilde{Y}^{\{\mathbf{L}\}a(2)}_{\nu'-}(k,t;n) =$$

$$g^2 W^{abcd}_{\mu\nu\kappa\lambda}[\chi^b_\nu\chi^c_\kappa A^d_{\lambda-}(k,t) + \chi^d_\lambda\chi^c_\kappa A^b_{\nu-}(k_1,t)] + g^2(2\pi)^4 W^{abcd}_{\mu\nu\kappa\lambda}\delta^4(k)\chi^d_\lambda\chi^b_\nu\chi^c_\kappa.$$

$$V^{abc}_{\mu\kappa\lambda}(k,0,-k) = \left(-\frac{i}{2}\right)f^{abc}\times[k_\lambda\delta_{\mu\kappa} - k_\mu\delta_{\kappa\lambda}].$$

We set now

$$A^{a(1)}_{\mu-}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\}) =$$

$$= -\chi^{a4}_\mu\delta(k)G^{ab}_{\mu\nu}(k;t,0,m) + (2\pi)^2 m^2\chi^a_{\nu'}\int_0^t G^{ab}_{\mu\nu'}(k;t,t',m)\delta(k)dt' +$$

$$+\int_0^t G^{ab}_{\mu\nu'}(k;t,t',m)\eta^a_{\nu'}(\omega,k,t)dt'. \tag{6.1.35}$$

$$G^{ab}_{\mu\nu'}(k;t,t',m) =$$

$$G^{ab}_{\mu\nu}(k;t,t',m) = \delta_{ab}G_{\mu\nu}(k;t,t',m),$$

$$G_{\mu\nu}(k;t,t',m) = \left\{\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+m^2}\right)e^{-(k^2+m^2)(t-t')}\right\},$$

and

$$A_{\mu+}^{a(n+1)}(k,t;\{\eta_\mu^a\},\{\chi_\mu^a\}) =$$

$$= \chi_\mu^a \delta^4(k) G_{\mu\nu}^{ab}(k;t,0,m) - (2\pi)^2 m^2 \chi_{\nu'}^a \int_0^t G_{\mu\nu'}^{ab}(k;t,t',m)\delta^4(k)dt' +$$

$$+\int_0^t G_{\mu\nu'}^{ab}(k;t,t',m)\eta_{\nu'}^a(\omega,k,t)dt' +$$

$$+\int_0^t G_{\mu\nu'}^{ab}(k;t,t',m)\widetilde{Y}_{\nu'+}^{b\{\mathbf{L}\}\alpha(1)}(k,t';n)dt' + \int_0^t G_{\mu\nu'}^{ab}(k;t,t',m)\widetilde{Y}_{\nu'+}^{b\{\mathbf{L}\}\alpha(2)}(k,t';n),$$

$$\widetilde{Y}_{\nu'+}^{b\{\mathbf{L}\}\alpha(1)}(k,t';n) =$$

$$gV_{\nu'\kappa\lambda}^{abc}(k,0,-k)\Big[\chi_\kappa^b A_{\lambda+}^{c(n)}(k,t;\{\eta_\lambda^c\},\{\chi_\lambda^c\}) + \chi_\lambda^c A_{\kappa+}^{b(n)}(k,t;\{\eta_\kappa^b\},\{\chi_\kappa^b\})\Big] +$$

$$+(2\pi)^2 g V_{\nu'\kappa\lambda}^{abc}(k,-k,0)\delta^4(k)\chi_\kappa^b\chi_\lambda^c, \tag{6.1.36}$$

$$V_{\nu'\kappa\lambda}^{abc}(k,0,-k) = \left(-\frac{i}{2}\right) f^{abc} \times [k_\lambda \delta_{\nu'\kappa} - k_{\nu'}\delta_{\kappa\lambda}].$$

$$\widetilde{Y}_{\nu'+}^{\{\mathbf{L}\}a(2)}(k,t;n) =$$

$$= -g^2 W_{\nu'\nu\kappa\lambda}^{abcd}\Big[\chi_\lambda^d\chi_\kappa^c A_{\kappa+}^{c(n)}(k,t;\{\eta_\kappa^c\},\{\chi_\kappa^c\}) + \chi_\lambda^d\chi_\kappa^c A_{\nu+}^{b(n)}(k,t;\{\eta_\nu^b\},\{\chi_\nu^b\})\Big] +$$

$$+g^2(2\pi)^4 W_{\nu'\nu\kappa\lambda}^{abcd}\delta^4(k)\chi_\lambda^d\chi_\nu^b\chi_\kappa^c$$

$$W_{\nu'\nu\kappa\lambda}^{abcd} = -\frac{1}{6}\Big[f^{abe}f^{cde}(\delta_{\nu'\kappa}\delta_{\nu\lambda} - \delta_{\nu'\lambda}\delta_{\nu\kappa}) +$$

$$+f^{ace}f^{bde}(\delta_{\nu'\nu}\delta_{\kappa\lambda} - \delta_{\nu'\lambda}\delta_{\nu\kappa}) +$$

$$+f^{ade}f^{ebe}(\delta_{\nu'\kappa}\delta_{\nu\lambda} - \delta_{\nu'\nu}\delta_{\kappa\lambda}).$$

We set now

$$A^{a(1)}_{\mu+}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\}) =$$

$$(2\pi)^2\chi^a_\mu\delta^4(k)G^{ab}_{\mu\nu'}(k;t,0,m) - (2\pi)^2 m^2\chi^a_{\nu'}\int\limits_0^t G^{ab}_{\mu\nu'}(k;t,t',m)\delta(k)dt' +$$

$$+\int\limits_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt'. \qquad (6.1.37)$$

$$G^{ab}_{\mu\nu'}(k;t,t',m) = \delta_{ab}G_{\mu\nu'}(k;t,t',m),$$

$$G_{\mu\nu'}(k;t,t',m) = \left(\delta_{\mu\nu'} - \frac{k_\mu k_{\nu'}}{k^2+m^2}\right)e^{-(k^2+m^2)(t-t')}$$

From Eq.(6.1.29)-Eq.(6.1.29$'$) we get

$$A^{a(1)}_{\mu-}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\}) = -(2\pi)^2\chi^a_\mu\delta^4(k)\delta_{ab}\left(\delta_{\mu\nu'} - \frac{k_\mu k_{\nu'}}{k^2+m^2}\right)e^{-(k^2+m^2)t} +$$

$$+(2\pi)^2 m^2\delta^4(k)\delta_{ab}\chi^a_{\nu'}\int\limits_0^t\left(\delta_{\mu\nu'} - \frac{k_\mu k_{\nu'}}{k^2+m^2}\right)e^{-(k^2+m^2)(t-t')}dt' +$$

$$+\int\limits_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt' =$$

$$= -(2\pi)^2\chi^a_\mu\delta^4(k)\left(1 - \frac{k^2_\mu}{k^2+m^2}\right)e^{-(k^2+m^2)t} + \qquad (6.1.38)$$

$$+(2\pi)^2\chi^a_\mu\delta^4(k)\frac{m^2}{k^2+m^2}\left(1 - \frac{k^2_\mu}{k^2+m^2}\right)\left(1 - e^{-(k^2+m^2)t}\right) +$$

$$+\int\limits_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt'.$$

From Eq.(6.1.37) and Eq.(6.1.29$'$) we get

$$
\begin{gathered}
A^{a(1)}_{\mu+}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\}) = \\
(2\pi)^2\chi^a_\mu\delta^4(k)\delta_{ab}\left(\delta_{\mu\nu'} - \frac{k_\mu k_{\nu'}}{k^2+m^2}\right)e^{-(k^2+m^2)t} - \\
-(2\pi)^2 m^2\chi^a_{\nu'}\int_0^t G^{ab}_{\mu\nu'}(k;t,t',m)\delta^4(k)dt' + \\
+\int_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt'.
\end{gathered}
\tag{6.1.39}
$$

From Eq.(6.1.39)-Eq.(6.1.29′) we get

$$
\begin{gathered}
A^{a(1)}_{\mu+}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\}) = -(2\pi)^2\chi^a_\mu\delta^4(k)\delta_{ab}\left(\delta_{\mu\nu'} - \frac{k_\mu k_{\nu'}}{k^2+m^2}\right)e^{-(k^2+m^2)t} - \\
-(2\pi)^2 m^2\delta^4(k)\delta_{ab}\chi^a_{\nu'}\int_0^t\left(\delta_{\mu\nu'} - \frac{k_\mu k_{\nu'}}{k^2+m^2}\right)e^{-(k^2+m^2)(t-t')}dt' + \\
+\int_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt' = \\
= -(2\pi)^2\chi^a_\mu\delta^4(k)\left(1 - \frac{k^2_\mu}{k^2+m^2}\right)e^{-(k^2+m^2)t} - \\
-(2\pi)^2\chi^a_\mu\delta^4(k)\frac{m^2}{k^2+m^2}\left(1 - \frac{k^2_\mu}{k^2+m^2}\right)\left(1 - e^{-(k^2+m^2)t}\right) + \\
+\int_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt'.
\end{gathered}
\tag{6.1.40}
$$

From Eq.(6.1.38) by using inverse Fourier transform we get

$$
\begin{aligned}
&A^{a(1)}_{\mu-}(x,t;\{\eta^a_\mu\},\{\chi^a_\mu\}) = \mathcal{F}^{-1}\Big[A^{a(1)}_{\mu-}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\})\Big](x,t) = \\
&= -\chi^a_\mu \int e^{ikx}\delta(k)\left(1-\frac{k_\mu^2}{k^2+m^2}\right)e^{-(k^2+m^2)t}d^4k + \\
&+\chi^a_\mu \int e^{ikx}\delta(k)\frac{m^2}{k^2+m^2}\left(1-\frac{k_\mu^2}{k^2+m^2}\right)\left(1-e^{-(k^2+m^2)t}\right)d^4k + \\
&+\mathcal{F}^{-1}\left[\int\limits_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt'\right](x,t) = \\
&= -\chi^a_\mu e^{-m^2t} + \chi^a_\mu\left(1-e^{-m^2t}\right) + \mathcal{F}^{-1}\left[\int\limits_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt'\right](x,t).
\end{aligned}
\tag{6.1.41}
$$

From Eq.(6.1.40) by using inverse Fourier transform we get

$$
\begin{aligned}
&A^{a(1)}_{\mu+}(x,t;\{\eta^a_\mu\},\{\chi^a_\mu\}) = \mathcal{F}^{-1}\Big[A^{a(1)}_{\mu+}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\})\Big](x,t) = \\
&= \chi^a_\mu \int e^{ikx}\delta(k)\left(1-\frac{k_\mu^2}{k^2+m^2}\right)e^{-(k^2+m^2)t}d^4k - \\
&-\chi^a_\mu \int e^{ikx}\delta(k)\frac{m^2}{k^2+m^2}\left(1-\frac{k_\mu^2}{k^2+m^2}\right)\left(1-e^{-(k^2+m^2)t}\right)d^4k + \\
&+\mathcal{F}^{-1}\left[\int\limits_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt'\right](x,t) = \\
&= \chi^a_\mu e^{-m^2t} - \chi^a_\mu\left(1-e^{-m^2t}\right) + \mathcal{F}^{-1}\left[\int\limits_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt'\right](x,t).
\end{aligned}
\tag{6.1.42}
$$

From Eq.(6.1.41)-Eq.(6.1.42) in the limit $t \to \infty$ we get

$$
\begin{aligned}
&\lim_{t\to\infty}\left\langle A^{a(1)}_{\mu-}(x,t;\{\eta^a_\mu\},\{\chi^a_\mu\})A^{a(1)}_{\nu+}(x',t;\{\eta^a_\mu\},\{\chi^a_\mu\})\right\rangle_\eta = \\
&= -(\chi^a_\mu)^2 + \int d^4k e^{ik(x-x')}\frac{1}{k^2+m^2}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2+m^2}\right).
\end{aligned}
\tag{6.1.43}
$$

The system of the transcendental master equations corresponding to correlations $\langle[A^a_{\mu-}(x,t;\{\eta^a_\mu\})]A^a_{\nu+}(x,t;\{\eta^a_\nu\})\rangle_\eta$ reads

$$
\langle[A^a_{\mu-}(x,t;\{\eta^a_\mu\},\{\chi^a_\mu\})]A^a_{\nu+}(x,t;\{\eta^a_\nu\},\{\chi^a_\nu\})\rangle_\eta = 0. \tag{6.1.44}
$$

where

$$
\begin{aligned}
A^{a}_{\mu-}(x,t;\{\eta^{a}_{\mu}\},\{\chi^{a}_{\mu}\}) &= \mathcal{F}^{-1}[A^{a}_{\mu-}(k,t;\{\eta^{a}_{\mu}\},\{\chi^{a}_{\mu}\})],\\
A^{a}_{\nu-}(x,t;\{\eta^{a}_{\nu}\},\{\chi^{a}_{\nu}\}) &= \mathcal{F}^{-1}[A^{a}_{\nu-}(k,t;\{\eta^{a}_{\nu}\},\{\chi^{a}_{\nu}\})],\\
A^{a}_{\mu+}(x,t;\{\eta^{a}_{\mu}\},\{\chi^{a}_{\mu}\}) &= \mathcal{F}^{-1}[A^{a}_{\mu+}(k,t;\{\eta^{a}_{\mu}\},\{\chi^{a}_{\mu}\})],\\
A^{a}_{\nu+}(x,t;\{\eta^{a}_{\nu}\},\{\chi^{a}_{\nu}\}) &= \mathcal{F}^{-1}[A^{a}_{\nu+}(k,t;\{\eta^{a}_{\nu}\},\{\chi^{a}_{\nu}\})].
\end{aligned}
\tag{6.1.45}
$$

For a sufficiently small $g < 1$, Eqs.(6.1.44)-(6.1.45) can be solved iteratively as an expansion in g, see Apendix 3, Theorem 3.2.
From Eq.(6.1.34) and Eq.(6.1.38) we get

$$
\begin{aligned}
A^{a(n+1)}_{\mu-}(k,t;\{\eta^{a}_{\mu}\},\{\chi^{a}_{\mu}\}) &= -(2\pi)^2\chi^{a}_{\mu}\delta^4(k)\delta_{ab}\left(\delta_{\mu\nu'}-\frac{k_\mu k_{\nu'}}{k^2+m^2}\right)e^{-(k^2+m^2)t}+\\
&+(2\pi)^2\chi^{a}_{\mu}\delta^4(k)\frac{m^2}{k^2+m^2}\left(1-\frac{k^2_\mu}{k^2+m^2}\right)\left(1-e^{-(k^2+m^2)t}\right)+\\
&+\int_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^{a}_{\nu'}(\omega,k,t)dt'+\\
&+g\int_0^t dt' G^{ab}_{\mu\nu'}(k;t,t')V^{abc}_{\nu'\kappa\lambda}(k,0,-k)\times\\
&\times\left[\chi^{b}_{\kappa}A^{c(n)}_{\lambda-}(k,t;\{\eta^{c}_{\lambda}\},\{\chi^{c}_{\lambda}\})+\chi^{c}_{\lambda}A^{b(n)}_{\kappa-}(k,t;\{\eta^{b}_{\kappa}\},\{\chi^{b}_{\kappa}\})\right]+\\
&+(2\pi)^2 g\int_0^t dt' G^{ab}_{\mu\nu'}(k;t,t')V^{abc}_{\nu'\kappa\lambda}(k,-k,0)\delta(k)\chi^{b}_{\kappa}\chi^{c}_{\lambda}+\\
&+g^2\int_0^t dt' G^{ab}_{\mu\nu'}(k;t,t')W^{abcd}_{\mu\nu'\kappa\lambda}\left\{\left[\chi^{b}_{\nu'}\chi^{c}_{\kappa}A^{d(n)}_{\lambda-}(k,t)+\chi^{d}_{\lambda}\chi^{c}_{\kappa}A^{b(n)}_{\nu'-}(k_1,t)\right]+\right.\\
&\left.+(2\pi)^4\delta^4(k)\chi^{d}_{\lambda}\chi^{b}_{\nu'}\chi^{c}_{\kappa}\right\}.
\end{aligned}
\tag{6.1.46}
$$

From Eq.(6.1.36) and Eq.(6.1.38) we get

$$
\begin{aligned}
A^{a(n+1)}_{\mu+}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\}) &= (2\pi)^2\chi^a_\mu\delta^4(k)\delta_{ab}\left(\delta_{\mu\nu'}-\frac{k_\mu k_{\nu'}}{k^2+m^2}\right)e^{-(k^2+m^2)t}- \\
&-(2\pi)^2\chi^a_\mu\delta^4(k)\frac{m^2}{k^2+m^2}\left(1-\frac{k^2_\mu}{k^2+m^2}\right)\left(1-e^{-(k^2+m^2)t}\right)+ \\
&+\int_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt'+ \\
&+g\int_0^t dt' G^{ab}_{\mu\nu'}(k;t,t')V^{abc}_{\nu'\kappa\lambda}(k,0,-k)\times \\
&\times\left[\chi^b_\kappa A^{c(n)}_{\lambda-}(k,t;\{\eta^c_\lambda\},\{\chi^c_\lambda\})+\chi^c_\lambda A^{b(n)}_{\kappa-}(k,t;\{\eta^b_\kappa\},\{\chi^b_\kappa\})\right]+ \\
&+(2\pi)^2 g\int_0^t dt' G^{ab}_{\mu\nu'}(k;t,t')V^{abc}_{\nu'\kappa\lambda}(k,-k,0)\delta(k)\chi^b_\kappa\chi^c_\lambda+ \\
&+g^2\int_0^t dt' G^{ab}_{\mu\nu'}(k;t,t')W^{abcd}_{\mu\nu'\kappa\lambda}\left\{\left[\chi^b_{\nu'}\chi^c_\kappa A^{d(n)}_{\lambda-}(k,t)+\chi^d_\lambda\chi^c_\kappa A^{b(n)}_{\nu'-}(k_1,t)\right]+\right. \\
&\left.+(2\pi)^4\delta^4(k)\chi^d_\lambda\chi^b_{\nu'}\chi^c_\kappa\right\}.
\end{aligned}
\tag{6.1.47}
$$

From Eq.(6.1.46) by using iverse Fourier transform we obtain

$$
\begin{aligned}
A^{a(n+1)}_{\mu-}(x,t;\{\eta^a_\mu\},\{\chi^a_\mu\}) &= \mathcal{F}^{-1}\left[A^{a(n+1)}_{\mu-}(k,t;\{\eta^a_\mu\},\{\chi^a_\mu\})\right] = \\
&= -\chi^a_\mu\int d^4k e^{ikx}\delta^4(k)\delta_{ab}\left(\delta_{\mu\nu'}-\frac{k_\mu k_{\nu'}}{k^2+m^2}\right)e^{-(k^2+m^2)t} + \\
&+\chi^a_\mu\int d^4k e^{ikx}\delta^4(k)\frac{m^2}{k^2+m^2}\left(1-\frac{k^2_\mu}{k^2+m^2}\right)\left(1-e^{-(k^2+m^2)t}\right) + \\
&+\int d^4k e^{ikx}\int_0^t G^{ab}_{\mu\nu'}(k;t,t')\eta^a_{\nu'}(\omega,k,t)dt' + \\
&+g\int d^4k e^{ikx}\int_0^t dt' G^{ab}_{\mu\nu'}(k;t,t')V^{abc}_{\nu'\kappa\lambda}(k,0,-k)\times \\
&\times\left[\chi^b_\kappa A^{c(n)}_{\lambda-}(k,t;\{\eta^c_\lambda\},\{\chi^c_\lambda\})+\chi^c_\lambda A^{b(n)}_{\kappa-}(k,t;\{\eta^b_\kappa\},\{\chi^b_\kappa\})\right] + \\
&+g\int d^4k e^{ikx}\int_0^t dt' G^{ab}_{\mu\nu'}(k;t,t')V^{abc}_{\nu'\kappa\lambda}(k,-k,0)\delta^4(k)\chi^b_\kappa\chi^c_\lambda + \\
&+g^2\int d^4k e^{ikx}\int_0^t dt' G^{ab}_{\mu\nu'}(k;t,t')W^{abcd}_{\mu\nu'\kappa\lambda}\left\{\left[\chi^b_{\nu'}\chi^c_\kappa A^{d(n)}_{\lambda-}(k,t)+\chi^d_\lambda\chi^c_\kappa A^{b(n)}_{\nu'-}(k_1,t)\right] + \right. \\
&\left. +(2\pi)^2\int d^4k e^{ikx}\delta^4(k)\chi^d_\lambda\chi^b_{\nu'}\chi^c_\kappa\right\}.
\end{aligned}
\tag{6.1.48}
$$

From Eq.(6.1.47) by using iverse Fourier transform we obtain

$$
\begin{aligned}
&A^{a(n+1)}_{\mu+}(x',t;\{\eta^a_\mu\},\{\chi^a_\mu\}) = \mathcal{F}^{-1}\Big[A^{a(n+1)}_{\mu-}(k',t;\{\eta^a_\mu\},\{\chi^a_\mu\})\Big] = \\
&= \chi^a_\mu \int d^4k' e^{ik'x'}\delta^4(k')\delta_{ab}\left(\delta_{\mu\nu'} - \frac{k'_\mu k'_{\nu'}}{k'^2+m^2}\right)e^{-\left(k'^2+m^2\right)t} - \\
&-\chi^a_\mu \int d^4k' e^{ik'x}\delta^4(k')\frac{m^2}{k'^2+m^2}\left(1-\frac{k'^2_\mu}{k'^2+m^2}\right)\left(1-e^{-\left(k'^2+m^2\right)t}\right) + \\
&+\int d^4k' e^{ik'x'}\int_0^t G^{ab}_{\mu\nu'}(k';t,t')\eta^a_{\nu'}(\omega,k',t)dt' + \\
&+g\int d^4k e^{ik'x'}\int_0^t dt' G^{ab}_{\mu\nu'}(k';t,t')V^{abc}_{\nu'\kappa\lambda}(k',0,-k)\times \\
&\times\Big[\chi^b_\kappa A^{c(n)}_{\lambda-}(k',t;\{\eta^c_\lambda\},\{\chi^c_\lambda\}) + \chi^c_\lambda A^{b(n)}_{\kappa-}(k',t;\{\eta^b_\kappa\},\{\chi^b_\kappa\})\Big] + \\
&+g\int d^4k' e^{ik'x'}\int_0^t dt' G^{ab}_{\mu\nu'}(k';t,t')V^{abc}_{\nu'\kappa\lambda}(k',-k',0)\delta^4(k')\chi^b_\kappa\chi^c_\lambda + \\
&+g^2\int d^4k' e^{ik'x'}\int_0^t dt' G^{ab}_{\mu\nu'}(k';t,t')W^{abcd}_{\mu\nu'\kappa\lambda}\Big\{\Big[\chi^b_{\nu'}\chi^c_\kappa A^{d(n)}_{\lambda-}(k',t) + \chi^d_\lambda\chi^c_\kappa A^{b(n)}_{\nu'-}(k'_1,t)\Big] + \\
&+(2\pi)^2\int d^4k' e^{ik'x'}\delta^4(k')\chi^d_\lambda\chi^b_{\nu'}\chi^c_\kappa\Big\}.
\end{aligned}
\tag{6.1.49}
$$

From Eq.(6.1.48)-Eq.(6.1.49) we obtain

$$
\begin{aligned}
&\left\langle A_{\mu-}^{a(n+1)}(x,t;\{\eta_\mu^a\},\{\chi_\mu^a\})A_{\mu+}^{a(n+1)}(x',t;\{\eta_\mu^a\},\{\chi_\mu^a\})\right\rangle_\eta = O\left(e^{-m^2t}\right) - (\chi_\mu^a)^2 + \\
&+\left\langle\left(\int d^4k e^{ikx}\int_0^t G_{\mu\nu'}^{ab}(k;t,t')\eta_{\nu'}^a(\omega,k,t)dt'\right)\times\right.\\
&\left.\times\left(\int d^4k' e^{ik'x'}\int_0^t G_{\mu\nu'}^{ab}(k';t,t')\eta_{\nu'}^a(\omega,k',t)dt'\right)\right\rangle_\eta + \\
&g^2\left\langle\left(\int d^4k e^{ikx}\int_0^t dt' G_{\mu\nu'}^{ab}(k;t,t')V_{\nu'\kappa\lambda}^{abc}(k',0,-k)\times\right.\right. \\
&\left.\left[\chi_\kappa^b A_{\lambda-}^{c(n)}(k,t;\{\eta_\lambda^c\},\{\chi_\lambda^c\}) + \chi_\lambda^c A_{\kappa-}^{b(n)}(k,t;\{\eta_\kappa^b\},\{\chi_\kappa^b\})\right]\right)\times \\
&\left(\int d^4k' e^{ik'x'}\int_0^t dt' G_{\mu\nu'}^{ab}(k';t,t')V_{\nu'\kappa\lambda}^{abc}(k',0,-k)\times\right.\\
&\left.\left.\times\left[\chi_\kappa^b A_{\lambda+}^{c(n)}(k',t;\{\eta_\lambda^c\},\{\chi_\lambda^c\}) + \chi_\lambda^c A_{\kappa+}^{b(n)}(k',t;\{\eta_\kappa^b\},\{\chi_\kappa^b\})\right]\right)\right\rangle_\eta + O(g^4).
\end{aligned}
\tag{6.1.50}
$$

By using Eq.(6.1.43) we obtain

$$
\begin{aligned}
&\left\langle A_{\mu-}^{a(n+1)}(x,t;\{\eta_\mu^a\},\{\chi_\mu^a\})A_{\mu+}^{a(n+1)}(x',t;\{\eta_\mu^a\},\{\chi_\mu^a\})\right\rangle_\eta = \\
&= O\left(e^{-m^2t}\right) - (\chi_\mu^a)^2 + \int d^4k e^{ik(x-x')}\frac{1}{k^2+m^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+m^2}\right) + \\
&g^2\left\langle\left(\int d^4k e^{ikx}\int_0^t dt' G_{\mu\nu'}^{ab}(k;t,t')V_{\nu'\kappa\lambda}^{abc}(k',0,-k)\times\right.\right. \\
&\left.\left[\chi_\kappa^b A_{\lambda-}^{c(n)}(k,t;\{\eta_\lambda^c\},\{\chi_\lambda^c\}) + \chi_\lambda^c A_{\kappa-}^{b(n)}(k,t;\{\eta_\kappa^b\},\{\chi_\kappa^b\})\right]\right)\times \\
&\left(\int d^4k' e^{ik'x'}\int_0^t dt' G_{\mu\nu'}^{ab}(k';t,t')V_{\nu'\kappa\lambda}^{abc}(k',0,-k)\times\right.\\
&\left.\left.\times\left[\chi_\kappa^b A_{\lambda+}^{c(n)}(k',t;\{\eta_\lambda^c\},\{\chi_\lambda^c\}) + \chi_\lambda^c A_{\kappa+}^{b(n)}(k',t;\{\eta_\kappa^b\},\{\chi_\kappa^b\})\right]\right)\right\rangle_\eta + O(g^4).
\end{aligned}
\tag{6.1.51}
$$

Therefore the system of the transcendental master equations (6.1.44) corresponding to correlations $\langle[A_{\mu-}^a(x,t;\{\eta_\mu^a\})]A_{\mu+}^a(x,t;\{\eta_\nu^a\})\rangle_\eta$ in the limit $t\to\infty$ reads

$$
\langle[A_{\mu-}^a(x,t;\{\eta_\mu^a\})]A_{\mu+}^a(x,t;\{\eta_\nu^a\})\rangle_\eta = 0. \tag{6.1.51'}
$$

Thus we get

$$
\begin{gathered}
-(\chi_\mu^a)^2 + \int d^4k e^{ik(x-x')} \frac{1}{k^2+m^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+m^2}\right) + \\
g^2\left\langle\left(\left(\int d^4k e^{ikx}\int_0^\infty dt' G_{\mu\nu'}^{ab}(k;t,t') V_{\nu'\kappa\lambda}^{abc}(k',0,-k) \times\right.\right.\right. \\
\left.\left[\chi_\kappa^b A_{\lambda-}^{c(n)}(k,t;\{\eta_\lambda^c\},\{\chi_\lambda^c\}) + \chi_\lambda^c A_{\kappa-}^{b(n)}(k,t;\{\eta_\kappa^b\},\{\chi_\kappa^b\})\right]\right) \times \qquad (6.1.52) \\
\left(\int d^4k' e^{ik'x'}\int_0^\infty dt' G_{\mu\nu'}^{ab}(k';t,t') V_{\nu'\kappa\lambda}^{abc}(k',0,-k) \times\right. \\
\left.\left.\left.\times\left[\chi_\kappa^b A_{\lambda+}^{c(n)}(k',t;\{\eta_\lambda^c\},\{\chi_\lambda^c\}) + \chi_\lambda^c A_{\kappa+}^{b(n)}(k',t;\{\eta_\kappa^b\},\{\chi_\kappa^b\})\right]\right)\right)\right\rangle_\eta + O(g^4) \simeq 0.
\end{gathered}
$$

For a sufficiently small $g < 1$, equations (6.1.52) can be solved iteratively as an expansion in g, see Apendix 3, Theorem 3.2. We define now corresponding countable sequences $\left\{\chi_\mu^{a(n)}\right\}_{n\in\mathbb{N}\cup\{0\}} \triangleq \left\{\chi_\mu^{a(n)}(g)\right\}_{n\in\mathbb{N}\cup\{0\}}, \chi_\mu^{a(0)} \triangleq \chi_\mu^a(g=0)$ of the iterations by the recursions:

$$
\begin{gathered}
\left(\chi_\mu^{a(n+1,n+1)}(g)\right)^2 = \\
= \int d^4k e^{ik(x-x')} \frac{1}{k^2+m^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+m^2}\right) + \\
+g^2\left\langle\left(\left(\int d^4k e^{ikx}\int_0^\infty dt' G_{\mu\nu'}^{ab}(k;t,t') V_{\nu'\kappa\lambda}^{abc}(k',0,-k) \times\right.\right.\right. \\
\left.\left[\chi_\kappa^{b(n)} A_{\lambda-}^{c(n)}\left(k,t;\{\eta_\lambda^c\},\left\{\chi_\lambda^{c(n)}\right\}\right) + \chi_\lambda^{c(n)} A_{\kappa-}^{b(n)}\left(k,t;\{\eta_\kappa^b\},\left\{\chi_\kappa^{b(n)}\right\}\right)\right]\right) \times \qquad (6.1.53) \\
\left(\int d^4k' e^{ik'x'}\int_0^\infty dt' G_{\mu\nu'}^{ab}(k';t,t') V_{\nu'\kappa\lambda}^{abc}(k',0,-k) \times\right. \\
\left.\left.\left.\times\left[\chi_\kappa^{b(n)} A_{\lambda+}^{c(n)}\left(k',t;\{\eta_\lambda^c\},\left\{\chi_\lambda^{c(n)}\right\}\right) + \chi_\lambda^{c(n)} A_{\kappa+}^{b(n)}\left(k',t;\{\eta_\kappa^b\},\left\{\chi_\kappa^{b(n)}\right\}\right)\right]\right)\right)\right\rangle_\eta + \\
+O(g^4) \simeq 0.
\end{gathered}
$$

Where we set

$$
\left(\chi_\mu^{a(0)}\right)^2 = \left(\chi_\mu^{a(0)}(0)\right)^2 = \int d^4k e^{ik(x-x')} \frac{1}{k^2+m^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+m^2}\right). \qquad (6.1.54)
$$

$$A_{\mu-}^{a(0)}\left(x,t;\{\eta_\mu^a\},\left\{\chi_\mu^{a(0)}\right\}\right) = \mathcal{F}^{-1}\left[A_{\mu-}^{a(0)}\left(k,t;\{\eta_\mu^a\},\left\{\chi_\mu^{a(0)}\right\}\right)\right]$$

$$\begin{gathered}A_{\mu-}^{a(0)}\left(k,t;\{\eta_\mu\},\left\{\chi_\mu^{a(0)}\right\}\right) = -(2\pi)^2\chi_\mu^a\delta^4(k)\delta_{ab}\left(\delta_{\mu\nu'} - \frac{k_\mu k_{\nu'}}{k^2+m^2}\right)e^{-(k^2+m^2)t} + \\ +(2\pi)^2\chi_\mu^a\delta^4(k)\frac{m^2}{k^2+m^2}\left(1-\frac{k_\mu^2}{k^2+m^2}\right)\left(1-e^{-(k^2+m^2)t}\right) + \\ +\int_0^t G_{\mu\nu'}^{ab}(k;t,t')\eta_{\nu'}^a(k,k,t)dt'\end{gathered} \quad (6.1.55)$$

$$A_{\mu+}^{a(0)}\left(x,t;\{\eta_\mu^a\},\left\{\chi_\mu^{a(0)}\right\}\right) = \mathcal{F}^{-1}\left[A_{\mu+}^{a(0)}\left(k,t;\{\eta_\mu^a\},\left\{\chi_\mu^{a(0)}\right\}\right)\right]$$

$$\begin{gathered}A_{\mu+}^{a(0)}\left(k,t;\{\eta_\mu\},\left\{\chi_\mu^{a(0)}\right\}\right) = (2\pi)^2\chi_\mu^a\delta^4(k)\delta_{ab}\left(\delta_{\mu\nu'} - \frac{k_\mu k_{\nu'}}{k^2+m^2}\right)e^{-(k^2+m^2)t} - \\ -(2\pi)^2\chi_\mu^a\delta^4(k)\frac{m^2}{k^2+m^2}\left(1-\frac{k_\mu^2}{k^2+m^2}\right)\left(1-e^{-(k^2+m^2)t}\right) + \\ +\int_0^t G_{\mu\nu'}^{ab}(k;t,t')\eta_{\nu'}^a(k,k,t)dt'.\end{gathered} \quad (6.1.56)$$

Apendix1.Two point function of the free scalar fields. Main properties.

$$G_{(D)F}(x_1-x_2;m) = (2\pi)^{-4}\int d^Dk\frac{e^{ik(x_1-x_2)}}{|k|^2+m^2}, \quad (1)$$

where $d^Dk = dk_0dk_1\cdots dk_{D-1}, |k| = \left(\sum_{i=0}^{D-1}k_i^2\right)^{1/2}$.
In spherical coordinate system with $r = |k| = \left(\sum_{i=0}^{3}k_i^2\right)^{1/2}, \varphi_1$- angle between vector x_1-x_2 and vector k we obtain

$$
\begin{gathered}
G_{(4)F}(x_1 - x_2; m) = (2\pi)^{-4} \int d^4k \frac{e^{ik(x_1-x_2)}}{|k|^2 + m^2} = \\
(2\pi)^{-4} \int_0^{\infty}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} \frac{e^{i|x_1-x_2|\times r\cos\varphi_1}}{r^2+m^2} \times r^3 dr \sin^2\varphi_1 \sin\varphi_2 d\varphi_1 d\varphi_2 d\varphi_3 = \\
= \frac{1}{\pi(2\pi)^{3/2}\Gamma(3/2)} \int_0^{\infty}\int_0^{\pi} \frac{e^{i|x_1-x_2|\times r\cos\varphi_1}}{r^2+m^2} \times r^3 dr \sin^2\varphi_1 d\varphi_1.
\end{gathered} \tag{2}
$$

Note that

$$
\int_0^{\pi} e^{i|x_1-x_2|\times r\cos\varphi_1} \sin^2\varphi_1 d\varphi_1 = 2\sqrt{\pi}\,\Gamma(3/2) \frac{J_1(r|x_1-x_2|)}{r|x_1-x_2|}, \tag{3}
$$

where $J_1(z)$ is the Bessel function of the first kind, integer order 1, (see Definition 1) and

$$
\int_0^{\infty} \frac{r^2}{r^2+m^2} J_1(r|x_1-x_2|) dr = mK_1(m|x_1-x_2|), \tag{4}
$$

where $K_1(z)$ is the modified Bessel function of the second kind, integer order 1,(see Definition 2) **and** where we used formula 6.566.2 of [12]. Thus finally we get

$$
\begin{gathered}
G_{(4)F}(x_1 - x_2) = \lim_{\tau\to\infty}\langle \phi(x_1,\tau;\omega)\phi(x_2,\tau;\omega)\rangle = \\
(2\pi)^{-2} \int d^4k \frac{e^{ik(x_1-x_2)}}{k^2+m^2} = (2\pi)^{-2}\left(\frac{m}{|x|}\right) K_1(m|x_1-x_2|).
\end{gathered} \tag{5}
$$

where K_1 is the modified Bessel functions of the second kind, integer order 1.

Definition 1. The Bessel function of the first kind $J_1(z)$, integer order 1 is defined by the power series:

$$
J_1(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!}\left(\frac{z}{2}\right)^{2k+1}, \tag{6}
$$

where $z \in \mathbb{C}$. For $z \to 0$ we have:

$$
J_1(z) \asymp z/2. \tag{7}
$$

Definition 2.The modified Bessel functions of the first kind $I_\nu(z)$ with $\nu \in \mathbb{R}$ and $z \in \mathbb{C}$ are defined by the power series

$$
I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{2k+\nu}. \tag{8}
$$

The modified Bessel functions of the second kind $K_\nu(z)$ are defined by

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}. \tag{9}$$

Remark 1. Note that

$$\begin{gathered}
1. K_\nu(m|x|) \asymp -\ln\left(\frac{m|x|}{2}\right) - \gamma \text{ as } m|x| \to 0, \nu = 0, \\
2. K_\nu(m|x|) \asymp \left(\frac{2}{m|x|}\right)^\nu \text{ as } m|x| \to 0, \nu > 0, \\
3. K_1(m|x|) \asymp \frac{1}{m|x|} \text{ as } m|x| \to 0, \\
\text{For } z \to \infty \text{ with } |\arg(z)| < \pi/2 \text{ and for any } \nu \text{ we have :} \\
K_\nu(z) = \frac{1}{\sqrt{2\pi z}} e^{-z}, \\
K_1(m|x|) \asymp \frac{1}{\sqrt{2\pi m|x|}} e^{-m|x|}
\end{gathered} \tag{10}$$

where γ is the Euler–Mascheroni constant. From Eq.(5) and Eq.(10) we get

$$\begin{gathered}
G_{(4)F}(x_1 - x_2) = \lim_{\tau\to\infty}\langle\phi(x_1,\tau;\omega)\phi(x_2,\tau;\omega)\rangle = \\
(2\pi)^{-2}\int d^4k \frac{e^{ik(x_1-x_2)}}{k^2+m^2} = (2\pi)^{-2}\left(\frac{m}{|x|}\right)K_1(m|x_1-x_2|) \asymp (2\pi)^{-2}|x|^{-2}
\end{gathered} \tag{11}$$

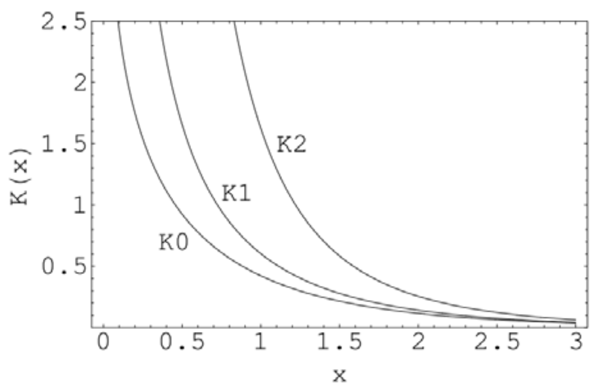

Figure 3.2.1.Plot of the modified Bessel functions of the second kind,integer order 0,1,2.

Definition3.

$$\begin{gathered}
G_{(D)F}(x_1 - x_2) = \lim_{\tau\to\infty}\langle\phi(x_1,\tau;\omega)\phi(x_2,\tau;\omega)\rangle = \\
= (2\pi)^{-D}\int d^Dk \frac{e^{ik(x_1-x_2)}}{k^2+m^2}.
\end{gathered} \tag{12}$$

The integrand in Eq.(12) is not radial but one can choose a frame where

$k_\mu x^\mu = -kr\cos\theta, \quad k \equiv |k_\mu k^\mu| = \sqrt{\sum_{\mu=1}^{D} k_\mu^2}, r = |x_\mu x^\mu| = \sqrt{\sum_{\mu=1}^{D} x_\mu^2}, \mu = 1,2,\ldots,D$ and the angular integral
reads

$$\int d\Omega_D\, e^{-ik\cdot x} = \Omega_{D-1}\int_0^\pi d\theta(\sin\theta)^{D-2}e^{ikr\cos\theta}$$
$$= \Omega_{D-1}\sqrt{\pi}\,\Gamma\left(\frac{D-1}{2}\right)\left(\frac{2}{kr}\right)^{\frac{D}{2}-1}J_{\frac{D}{2}-1}(kr), \tag{13}$$

where $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$ is the volume of the unit D-ball. In the last line we used formula 3.915.5 of [12] Then,

$$G_{(D)}(k;m) = \Gamma\left(\frac{D}{2}\right)\left(\frac{2}{k}\right)^{\frac{D}{2}-1}\int_0^{+\infty} dr\, r^{\frac{D}{2}}[\Omega_D G(r)]J_{\frac{D}{2}-1}(kr). \tag{14}$$

Now we take the massive propagator in momentum space:

$$G_{(D)}(k;m) = -\frac{\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D\alpha}{2}\right)}\left(\frac{2}{k}\right)^{\frac{D}{2}-1}\left(\frac{m}{2}\right)^{\frac{D}{2}-1}\times$$
$$\times\int_0^{+\infty} dr\, rK_{\frac{D}{2}-1}(mr)J_{\frac{D}{2}-1}(kr) == -m^{-2}F\left(\frac{D}{2},1;\frac{D}{2};-\frac{k^2}{m^2}\right) = \tag{15}$$
$$= -\frac{1}{k^2+m^2}$$

where we used formula 6.576.3 and formula 9.121.1 of [12] with $D = 4$, and F is the hypergeometric function

$$F(a,b;c;z) = \sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+n)}\frac{z^n}{n!}. \tag{16}$$

Remark 2.Note that when $D = 4$, we exactly obtain massive propagator $G_{(4)}(k;m)$ in momentum space, see Eqs.(1-5).
Remark 3. If $D \geq 5$ Definition 3 no longer holds, since integral in RHS of Eq.(12) diverges as $|k'|^{\frac{D}{2}-1}, k' \to \infty$ since the estimate (17) holds

$$\int d^Dk\frac{e^{ik(x_1-x_2)}}{k^2+m^2} \sim \int_0^\infty |k|^{\frac{D}{2}-2}d|k|. \tag{17}$$

Remark 4. In order to avoid these difficultness mentioned above, we replace Eq.(12) in
def.3 by the weak limit taiken in space $\mathcal{L}'(\mathbb{R}_x^D), D \geq 5$

$$G_{(D)F}(x_1-x_2) = w\text{-}\lim_{\eta\to 0_+} G_{(D)}(k;m,\eta) = w\text{-}\lim_{\eta\to 0_+}\int d^Dk\frac{e^{-\eta|k|+ik(x_1-x_2)}}{k^2+m^2}$$
$$\eta > 0 \tag{18}$$

Notice the definition by Eq.(18) is correct since in space $\mathcal{L}'(\mathbb{R}_k^D)$:

$$w\text{-}\lim_{\eta\to 0_+} e^{-\eta|k|}(k^2+m^2)^{-1} = (k^2+m^2)^{-1}, \tag{19}$$

and $\int (k^2+m^2)^{-1} d^D k < \infty$.

Apendix 2.Correlation functions for the $(\lambda\varphi^4)_d$ scalar theory in canonical Parisi and Wu stochastic quantization.

In this section we will analyze the canonical stochastic quantization for the $(\lambda\varphi^4)_d$ self-interaction scalar theory in comparison with double stochastic quantization for the $(\lambda\varphi^4)_d$ scalar theory. In this case the Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -(-\Delta+m_0^2)\varphi(\tau,x) - \frac{\lambda}{3!}\varphi^3(\tau,x) + \eta(\tau,x). \tag{2.1}$$

The two-point correlation function associated with the random field is given by the Einstein relation, while the other connected correlation functions vanish, i.e.,
The two-point correlation function associated with the random field is given by the Einstein relation, while the other connected correlation functions vanish, i.e.,

$$\langle \eta(\tau_1,x_1)\eta(\tau_2,x_2)\ldots\eta(\tau_{2k-1},x_{2k-1})\rangle_\eta = 0, \tag{2.2}$$

and also

$$\langle\eta(\tau_1,x_1)\ldots\eta(\tau_{2k},x_{2k})\rangle_\eta = \sum\langle\eta(\tau_1,x_1)\eta(\tau_2,x_2)\rangle_\eta\langle\eta(\tau_k,x_k)\eta(\tau_l,x_l)\rangle_\eta\ldots, \tag{2.3}$$

where the sum is to be taken over all the different ways in which the $2k$ labels can be divided into k parts, i.e., into k pairs. Performing Gaussian averages over the white random noise, it is possible to prove that perturbatively [6]

$$\lim_{\tau\to\infty}\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\ldots\varphi(\tau_n,x_n)\rangle_\eta = \frac{\int D[\varphi]\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\,e^{-S(\varphi)}}{\int D[\varphi]\,e^{-S(\varphi)}}, \tag{2.4}$$

where $S(\varphi) = S_0(\varphi) + S_I(\varphi)$ is the d-dimensional action. This result leads us to consider
the Euclidean path integral measure a stationary distribution of a stochastic process. Note that the solution of the Langevin equation needs a given initial condition. As for example

$$\varphi(\tau,x)|_{\tau=0} = \varphi_0(x). \tag{2.5}$$

Let us use the Langevin equation to perturbatively solve the interacting field theory. One way to handle the Eq.(2.1) is with the method of Green's functions. We defined the retarded Green function for the diffusion problem in the Eq.(2.1). Let us assume that the coupling constant is a small quantity. Therefore to solve the Langevin equation
in the case of a interacting theory we use a perturbative series in λ. Therefore we can
write

$$\varphi(\tau,x) = \varphi^{(0)}(\tau,x) + \lambda\varphi^{(1)}(\tau,x) + \lambda^2\varphi^{(2)}(\tau,x) + \ldots \tag{2.6}$$

Substituting the Eq.(2.6) in the Eq.(2.1), and if we equate terms of equal power in λ, the resulting equations are

$$\left[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2)\right]\varphi^{(0)}(\tau,x) = \eta(\tau,x), \tag{2.7}$$

$$\left[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2)\right]\varphi^{(1)}(\tau,x) = -\frac{1}{3!}(\varphi^{(0)}(\tau,x))^3, \tag{2.8}$$

$$\left[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2)\right]\varphi^{(1)}(\tau,x) = -\frac{1}{3!}(\varphi^{(0)}(\tau,x))^3, \tag{2.9}$$

and so on. Using the retarded Green function and assuming that $\varphi^{(q)}(\tau,x)|_{\tau=0} = 0$, $\forall\, q$, the solution to the first equation given by Eq.(2.7) can be written formally as

$$\varphi^{(0)}(\tau,x) = \int_0^\tau d\tau' \int_\Omega d^dx'\, G(\tau-\tau',x-x')\eta(\tau',x'). \tag{2.10}$$

The second equation given by Eq.(2.8) can also be solved using the above result. We obtain

$$\begin{aligned}\varphi^{(1)}(\tau,x) = -\frac{1}{3!}\int_0^\tau d\tau_1 \int_\Omega d^dx_1\, G(\tau-\tau_1,x-x_1) + \\ +\left(\int_0^{\tau_1} d\tau' \int_\Omega d^dx'\, G(\tau_1-\tau',x_1-x')\eta(\tau',x')\right)^3.\end{aligned} \tag{2.11}$$

We have seen that we can generate all the tree diagrams with the noise field contributions. We can also consider the n-point correlation function $\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\ldots\varphi(\tau_n,x_n)\rangle_\eta$. Substituting the above results in the n-point correlation function, and taking the random averages over the white noise field using the Wick-decomposition property defined by Eq.(2.3) we generate the stochastic diagrams.

Each of these stochastic diagrams has the form of a Feynman diagram, apart from the fact that we have to take into account that we are joining together two white random noise fields many times.

As simple examples let us show how to derive the two-point function in the zeroth order $\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle_\eta^{(0)}$, and also the first order correction to the scalar two-point-function given by $\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle_\eta^{(1)}$. Using the Eq.(2.10) and the Einstein relations we have

$$\begin{aligned}\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle_\eta^{(0)} = \\ 2\int_0^{\min(\tau_1,\tau_2)}\int d\tau' \int_\Omega d^dx'\, G(\tau_1-\tau',x_1-x')\,G(\tau_2-\tau',x_2-x'),\end{aligned} \tag{2.12}$$

which is just Eq.(2.13). For the first order correction we get:

$$\langle \varphi(X_1)\varphi(X_2)\rangle_\eta^{(1)} = -\frac{\lambda}{3!}\langle \int dX_3 \int dX_4 (G(X_1 - X_4)G(X_2 - X_3) + G(X_1 - X_3)G(X_2 - X_4))\eta(X_3)\left(\int dX_5\, G(X_4 - X_5)\eta(X_5)\right)^3 \rangle_\eta. \quad (2.13)$$

where, for simplicity, we have introduced a compact notation:

$$\int_0^\tau d\tau \int_\Omega d^d x \equiv \int dX, \quad (2.14)$$

and also $\varphi(\tau,x) \equiv \varphi(X)$ and finally $\eta(\tau,x) \equiv \eta(X)$.

The process can be repeated and therefore the stochastic quantization can be used as an alternative approach to describe scalar quantum fields. We stress here that the stochastic quantization is based in the fact that although one starts with the system out of equilibrium, the Markovian Langevin equation forces it into equilibrium. Moreover, when the thermodynamic equilibrium is reached, the stochastic expectation values will coincide with the Schwinger functions of the Euclidean field theory.

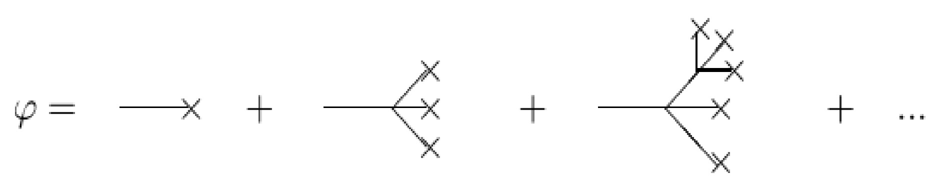

Figure1:Perturbative expansion for the scalar field where crosses denote noise fields.

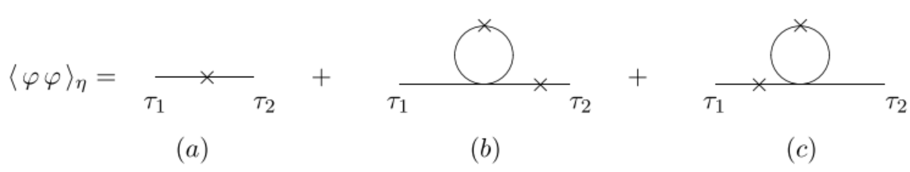

Figure2:The corrections up to one-loop to the two-point correlation function.

We can represent Eq.(6) graphically as figure (1) (the random noise field is represented by a cross). Using this diagrammatical expansion, it is possible to show that the two-point correlation function up to one-loop level is given by figure (2), where we represent the retarded Green function by a line and the free two-point function by a crossed line. The rules to obtain the algebraic values of the stochastic diagrams are similar to the usual Feynman rules. For instance the two-point function at one-loop level is given by

$$(b) = -\frac{\lambda}{2}\,\delta^d(k_1+k_2)\int d^dk\int_0^{\tau_1} d\tau\, G(k_1;\tau_1-\tau)D(k;\tau,\tau)D(k_2;\tau_2,\tau). \tag{2.15}$$

$$(c) = -\frac{\lambda}{2}\,\delta^d(k_1+k_2)\int d^dk\int_0^{\tau_2} d\tau\, G(k_2;\tau_2-\tau)D(k;\tau,\tau)D(k_1;\tau_1,\tau). \tag{2.16}$$

A simple computation shows that we recover the correct equilibrium result at equal asymptotic Markov parameters ($\tau_1 = \tau_2 \to \infty$):

$$(b)|_{\tau_1=\tau_2\to\infty} = -\frac{\lambda}{2}\,\delta^d(k_1+k_2)\frac{1}{(k_2^2+m_0^2)}\,\frac{1}{(k_1^2+k_2^2+2m_0^2)}\int\frac{d^dk}{(k^2+m_0^2)}. \tag{2.17}$$

Obtaining the Schwinger functions in the asymptotic limit does not guarantee that we

gain a finite physical theory. The next step is to implement a suitable regularization scheme. A crucial point to find a satisfactory regularization scheme is to use one that preserves the symmetries of the original model. In the stochastic regularization method the symmetries of the physical theory is maintained. There are in general two different ways to implement the stochastic regularization. The first one is to start from a Langevin equation with a memory kernel. It is known from the literature [9] that this method can at best only remove two degrees of divergence. Another possibility is to smear only the noise field in the probability functional [10]-[11]:

$$\langle F[\varphi]\rangle_\eta = \frac{\int D[\eta]F[\varphi]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\int d\tau'\,\eta(\tau,x)K_\Lambda^{-1}\eta(\tau',x)\Big]}{\int D[\eta]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\int d\tau'\,\eta(\tau,x)K_\Lambda^{-1}\eta(\tau',x)\Big]}, \tag{2.18}$$

where K_Λ is a memory kernel. In this case we change the Einstein relations of the noise

field to:

$$\langle\,\eta(\tau,x)\,\eta(\tau',x')\,\rangle_\eta = 2K_\Lambda(\tau,\tau'^{d}(x-x'). \tag{2.19}$$

The smearing function should be chosen such that, when $\Lambda\to\infty$:

$$\lim_{\Lambda\to\infty}K_\Lambda(\tau-\tau') = \delta(\tau-\tau'), \tag{2.20}$$

recovering the usual theory.

Since the Langevin equation is unaffected by the stochastic regularization, the physical field is the same as in the regularized case. However, the zeroth-order two-point correlation function is given by:

$$D(k;\tau,\tau') = 2\delta^d(k+k')\int_0^{\tau} ds \int_0^{\tau'} ds'\, G(k;\tau-s)\, G(k;\tau'-s')K_\Lambda(s-s') =$$
$$2\delta^d(k+k')\int_0^{\tau} ds \int_0^{\tau'} ds' \exp[-(\tau+\tau'-s-s'^2+m_0^2)]K_\Lambda(s-s'). \qquad (2.21)$$

It is possible to prove that a necessary condition that the regularization function K_Λ should satisfy in order to render the divergent loops finite is $K_\Lambda(\tau)\ |_{\tau=0}\ = 0$. The following series of kernels obeying this condition were proposed:

$$K_\Lambda^{(n)}(\tau) = \frac{1}{2n!}\Lambda^2(\Lambda^2\ |\ \tau\ |)^n \exp[-\Lambda^2\ |\ \tau\ |]. \qquad (2.22)$$

For the case $n = 0$ we obtain, for the free two-point correlation function:

$$\lim_{\tau\to\infty} D(k;\tau,\tau) = \frac{\delta^d(k+k')}{(k^2+m_0^2)}\frac{\Lambda^2}{(\Lambda^2+k^2+m_0^2)}. \qquad (2.23)$$

Since the stochastic diagrams contains crossed lines in its loops, we have that the ultraviolet divergences can be regularized choosing an appropriate n. Note that it is possible to use a different regulator of the type $K_\sigma(\tau) = \frac{1}{2}\sigma\tau^{\sigma-1}$.

We now return to φ_4^4 theory. To first order in λ, and in momentum space, the two point
function in Euclidean space is

$$G(p,-p) = \frac{1}{p^2+m^2} + \frac{-\lambda}{2}\left(\int\frac{d^4q}{(2\pi)}\frac{1}{p^2+m^2}\right)\left(\frac{1}{p^2+m^2}\right) + O(\lambda^2). \qquad (2.24)$$

Let us define now by $\mu_{\mathbf{ren}}^2$ the effective or renormalized mass (squared) such that

$$\frac{1}{p^2+\mu_{\mathbf{ren}}^2} = \frac{1}{p^2+m^2}\left\{1-\frac{\lambda}{2}\left(\int\frac{d^4q}{(2\pi)}\frac{1}{p^2+m^2}\right)\left(\frac{1}{p^2+m^2}\right) + O(\lambda^2)\right\}. \qquad (2.25)$$

Again, to first order in λ, we can write the equivalent equation

$$G(p,-p) = \frac{1}{p^2+m^2+\frac{\lambda}{2}\int\frac{d^4q}{(2\pi)}\frac{1}{p^2+m^2}} + O(\lambda^2). \qquad (2.26)$$

This equation leads us to define $\mu_{\mathbf{ren}}^2$ by

$$\mu_{\mathbf{ren}}^2 = m^2 + \frac{\lambda}{2}\int\frac{d^4q}{(2\pi)}\frac{1}{p^2+m^2} = m^2+\delta m^2, \qquad (2.27)$$

and the quantity $\mu_{\mathbf{ren}}$ represents the physical (or renormalized) mass of the particle. Thus, the interaction changes the mass of the particle. However $\delta m^2 = \infty$ since that

$$\int\frac{d^4q}{(2\pi)}\frac{1}{p^2+m^2} \simeq \int_0^\infty\frac{q^3dq}{q^2} = \int_0^\infty qdq = \infty. \qquad (2.28)$$

This is an example of a typical “ultraviolet” divergence in canonical perturbative expansion in quantum field theory.

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Apendix. 3.Generalized Banach fixed-point theorem.

Theorem 3.1. [15] Let (X,d) be a complete metric space and $T : X \to X$ be a contraction
mapping with the Lipschitz constant $\alpha \in [0,1)$. Then:
(1) T has a unique fixed point $\bar{x}$ in X.
(2) For an *arbitrary* point $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ generated by the Picard iteration
process (defined by $x_{n+1} = T(x_n), n \in \mathbb{N} \cup \{0\}$) converges to $\bar{x}$.
(3) $d(x_n,\bar{x}) \leq \frac{\alpha^n}{1-\alpha} d(x_0,x_1)$ for all $n \in \mathbb{N}$.

Definition 3.1. [15] We say that $\bar{x} \in X$ is a contractive fixed point (abbr. CFP) of T if $\bar{x} = T(\bar{x})$ and the Picard iterates $T_n(x)$ converge to $\bar{x}$ as $n \to \infty$ for all $x \in X$.

The operator T from Definition 3.1 became known as a Picard operator (shortly PO),

Definition 3.2. We say that $\bar{x} \in X$ is a quasy contractive fixed point (abbr. QCFP) of T
if $\bar{x} = T(\bar{x})$ and there is an $x_0 \in X$ such that the Picard iterates $T_n(x_0)$ converge to $\bar{x}$ as $n \to \infty$.

Definition 3.3.The operator T from Definition 3.2 became known as a quasy Picard operator (abbr. QPO)

Theorem 3.2.(Generalized Banach fixed-point theorem). Let (X,d) be a complete metric
space and $T : X \to X$ is continuous.Assume that there is an $y_0 \in X$ such that the inequalities are satisfied

$$\begin{gathered} d(y_1,y_0) = d(T(y_0),y_0) \leq \alpha \times const, \\ d(y_2,y_1) = d(T(y_1),T(y_0)) \leq \alpha d(y_1,y_0), \\ d(y_3,y_2) = d(T(y_2),T(y_1)) \leq \alpha d(y_2,y_1), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ d(y_{n+1},y_n) = d(T(y_n),T(y_{n-1})) \leq \alpha d(y_n,y_{n-1}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{gathered} \tag{3.1}$$

where $\alpha < 1,\ y_n = T(y_{n-1})$. Ten there is a quasy contractive fixed point $\bar{x}$ of T, such that the Picard iterates $T_n(y_0)$ converge to $\bar{y} \in X$ as $n \to \infty$.

Proof. From inequalities (3.1) we obtain

$$\begin{gathered} d(y_1,y_0) = d(T(y_0),y_0) \leq \alpha \times const, \\ d(y_2,y_1) = d(T(y_1),T(y_0)) \leq \alpha d(y_1,y_0), \\ d(y_3,y_2) = d(T(y_2),T(y_1)) \leq \alpha d(y_2,y_1) \leq \alpha^2 d(y_1,y_0), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ d(y_{n+1},y_n) = d(T(y_n),T(y_{n-1})) \leq \alpha d(y_n,y_{n-1}) \leq \alpha^n d(y_1,y_0), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{gathered} \tag{3.2}$$

From inequalities (3.2) by using triangle inequality we obtain for some $\varepsilon \ll 1$

$$\begin{aligned} d(y_n, y_{n+m}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{n+m-1}, y_{n+m}) \leq \\ &\leq (\alpha^n + \alpha^{n+1} + \ldots + \alpha^{n+m-1}) d(y_1, y_0) = \alpha^n \frac{1 - \alpha^{n+m}}{1 - \alpha} d(y_1, y_0) < \\ &< \frac{\alpha^n}{1 - \alpha} d(y_1, y_0) < \varepsilon. \end{aligned} \tag{3.3}$$

Thus a sequence $\{y_n\}_{n=0}^{\infty} \subset X$ is a fundamental sequence in X and there is exists $\lim_{n\to\infty} y_n = \bar{y} \in X.$ We assume now that $T(\bar{y}) = \bar{\bar{y}},$ by using triangle inequality we obtain

$$d(\bar{y}, \bar{\bar{y}}) \leq d(\bar{y}, y_n) + d(y_n, y_{n+1}) + d(y_{n+1}, \bar{\bar{y}}). \tag{3.4}$$

For any $\varepsilon > 0$ we can choose $N = N(\varepsilon)$ such that for any $n \geq N$

(i) $d(\bar{y}, y_n) < \varepsilon/3,$ since that is $\bar{y} = \lim_{n\to\infty} y_n;$

(ii) $d(y_n, y_{n+1}) < \varepsilon/3,$ since that $\{y_n\}_{n=0}^{\infty} \subset X$ is a fundamental sequence;

(iii) $d(y_{n+1}, \bar{\bar{y}}) = d(T(y_n), T(\bar{y})) \leq \varepsilon/3,$ since that is $\lim_{n\to\infty} T(y_n) = T(\lim_{n\to\infty} y_n) = T(\bar{y}).$

Therefore for any ϵ such that $0 < \epsilon < \varepsilon : d(\bar{y}, \bar{\bar{y}}) \leq \epsilon$ it follows that $d(\bar{y}, \bar{\bar{y}}) = 0$ and consequently $\bar{\bar{y}} = \bar{y}.$

Apendix. 4.1.Integration over random interval.

Definition 4.1.1.Let $q_1(\omega)$ and $q_2(\omega)$ are $\mathbb{R}$- valued random variables defined on a probability space $\Sigma = (\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$,i.e., $q_{1,2}(\omega) : \mathbf{\Omega} \to \mathbb{R}$. Assume that a.s.$-\infty < q_1(\omega) < q_2(\omega) < \infty.$

Let $\Xi(\Sigma, \mathbb{R})$ be a set of the all $\mathbb{R}$- valued random variables defined on a probability space $\Sigma = (\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$. Closed random interval $[q_1(\omega), q_2(\omega)]$ that is a subset $[q_1(\omega), q_2(\omega)] \subset \Xi(\Sigma, \mathbb{R})$ such that $-\infty < q_1(\omega) < q_2(\omega) < \infty$ and

$$\forall q(\omega)\{q(\omega) \in [q_1(\omega), q_2(\omega)] \Leftrightarrow \{q(\omega) \in \Xi(\Sigma, \mathbb{R}) | a.s.(q_1(\omega) < q(\omega) < q_2(\omega))\}\} \tag{4.1.1}$$

Definition 4.1.2. The lengths $l([q_1(\omega), q_2(\omega)])$ of the random interval $[q_1(\omega), q_2(\omega)]$ is defined by

$$l([q_1(\omega), q_2(\omega)]) = \mathbf{ess}\sup q_3(\omega), \tag{4.1.2}$$

where $q_3(\omega) = q_2(\omega) - q_1(\omega).$

Notation 4.1.2. Assume that a.s.$q_1(\omega) < q_2(\omega)$. Then we will write: $q_1(\omega) \prec q_2(\omega).$ Assume that a.s.$q_1(\omega) \leq q_2(\omega)$. Then we will write: $q_1(\omega) \preccurlyeq q_2(\omega).$

Definition 4.1.3. A partition P of a closed random interval $[q_1(\omega), q_2(\omega)]$ is a finite system of random variables $x_0(\omega), \ldots, x_n(\omega)$ such that $q_1(\omega) = x_0(\omega) \prec x_1(\omega) \prec \ldots x_{n-1}(\omega) \prec x_n(\omega) = q_2(\omega).$

The closed random intervals $[x_{i-1}(\omega), x_i(\omega)],\ (i = 1, \ldots, n)$ are called the intervals of the partition P.

Definition 4.1.4.The largest of the lengths of the intervals of the partition P, denoted $\lambda(P),$ is called the *mesh* of the partition.

Definition 4.1.5. We speak of a partition with distinguished points (P,ξ) on the closed random interval $[q_1(\omega),q_2(\omega)]$ if we have a partition P of $[q_1(\omega),q_2(\omega)]$ and a point $\xi_1(\omega) \in [x_{i-1}(\omega),x_i(\omega)]$ has been chosen in each of the intervals of the partition $[x_{i-1}(\omega),x_i(\omega)]$, $(i = 1,\ldots,n)$.

We denote the set of points $(\xi_1(\omega),\ldots,\xi_n(\omega))$ by the single letter $\xi(\omega)$.

Definition 4.1.6. In the set $\widetilde{P}$ of partitions with distinguished points on a given random interval $[q_1(\omega),q_2(\omega)]$, we consider the following base $\mathbf{B} = \{B_d\}$. The element $B_d, d > 0$, of the base B consists of all partitions with distinguished points (P,ξ) on $[q_1(\omega),q_2(\omega)]$ for which $\lambda(P) < d$.

Proposition 4.1.1. Let us verify that $\{B_d\}, d > 0$ is actually a base in $\widetilde{P}$.

Proof.First $B_d \neq \varnothing$. In fact, for any number $d > 0$, it is obvious that there exists a partition P of $[q_1(\omega),q_2(\omega)]$ with mesh $\lambda(P) < d$ (for example, a partition into n congruent closed random intervals). But then there also exists a partition $(P,\xi(\omega))$ with distinguished points for which $\lambda(P) < d$.

Second, if $d_1 > 0, d_2 > 0$, and $d = \min\{d_1,d_2\}$, it is obvious that $B_{d_1} \cap B_{d_2} = B_d \in \mathbf{B}$. Hence $\mathbf{B} = \{B_d\}$ is indeed a base in $\widetilde{P}$.

Definition 4.1.7. (**Random Riemann Sums**) (i) If a function $f : \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ is defined on the closed random interval $[q_1(\omega),q_2(\omega)]$ and $(P,\xi(\omega))$ is a partition with distinguished points on this closed random interval, the sum

$$\sigma[f,P,\xi(\omega)] = \sum_{i=1}^{n} f(\xi_i(\omega))\Delta x_i(\omega), \tag{4.1.3}$$

where $\Delta x_i(\omega) = x_i(\omega) - x_{i-1}(\omega)$, is the random Riemann sum of the function $f : \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ corresponding to the partition $(P,\xi(\omega))$ with distinguished points on $[q_1(\omega),q_2(\omega)]$. Thus, when the function f is fixed, the random Riemann sum $\sigma[f,P,\xi(\omega)]$ is a function $\Phi(p) = \sigma[f,P]$ on the set $\widetilde{P}$ of all partitions $p = (P,\xi(\omega))$ with distinguished points on the closed interval $[q_1(\omega),q_2(\omega)]$.

Since there is a base $\mathbf{B}$ in $\widetilde{P}$, one can ask about the limit of the function $\Phi(p(\omega))$ over that base.

(ii) If a function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ is defined on the closed random interval $[q_1(\omega),q_2(\omega)]$ and $(P,\xi(\omega))$ is a partition with distinguished points on this closed random interval, the sum

$$\sigma[f,P,\xi(\omega)] = \sum_{i=1}^{n} f(\omega,\xi_i(\omega))\Delta x_i(\omega), \tag{4.1.3$'$}$$

where $\Delta x_i(\omega) = x_i(\omega) - x_{i-1}(\omega)$, is the random Riemann sum of the function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ corresponding to the partition $(P,\xi(\omega))$ with distinguished points on $[q_1(\omega),q_2(\omega)]$. Thus, when the function $f : \Omega \times [q_1(\omega),q_2(\omega)] \to \Xi(\Sigma,\mathbb{R})$ is fixed, the random Riemann sum $\sigma[f,P,\xi(\omega)]$ is a function $\Phi(p) = \sigma[f,P]$ on the set $\widetilde{P}$ of all partitions $p = (P,\xi(\omega))$ with distinguished points on the closed interval $[q_1(\omega),q_2(\omega)]$.

Definition 4.1.8. (**Riemann Integral on a random interval**)

(i) Let $f : \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ be a function restricted on a closed random interval $[q_1(\omega),q_2(\omega)]$.

The random variable $I(\omega)$ is the Riemann integral of the function $f : \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ on the closed random interval $[q_1(\omega),q_2(\omega)]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a.s.:\ \left|I(\omega) - \sum_{i=1}^{n} f(\xi_i(\omega))\Delta x_i(\omega)\right| < \varepsilon \tag{4.1.4}$$

for any partition $(P,\xi(\omega))$ with distinguished points on $[q_1(\omega),q_2(\omega)]$ whose mesh $\lambda(P)$ is less than δ.

(ii) Let $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ be a function restricted on a closed random interval $[q_1(\omega),q_2(\omega)]$ such that $f : \Omega \times [q_1(\omega),q_2(\omega)] \to \Xi(\Sigma,\mathbb{R})$.

The random variable $I(\omega)$ is the Riemann integral of the function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ on the closed random interval $[q_1(\omega),q_2(\omega)]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a.s.:\ \left|I(\omega) - \sum_{i=1}^{n} f(\omega,\xi_i(\omega))\Delta x_i(\omega)\right| < \varepsilon. \tag{4.1.4$'$}$$

Since the partitions $p = (P,\xi(\omega))$ for which $\lambda(P) < \delta$ form the element B_δ of the base $\mathbf{B}$ introduced above in the set $\widetilde{P}$ of partitions with distinguished points, Definition 4.1.8 is equivalent to the statement

$$a.s.:\ I(\omega) = \lim_{\mathbf{B}} \Phi(p(\omega)). \tag{4.1.5}$$

that is, the integral $I(\omega)$ is the limit over $\mathbf{B}$ of the Riemann sums of the function f corresponding to partitions with distinguished points on $[q_1(\omega),q_2(\omega)]$.

It is natural to denote the base $\mathbf{B}$ by $\lambda(P) \to 0$, and then the definition of the Riemann integral on a random interval can be rewritten as

$$\begin{gathered} a.s.:\ I(\omega) = \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\xi_i(\omega))\Delta x_i(\omega), \\ a.s.:\ I(\omega) = \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\omega,\xi_i(\omega))\Delta x_i(\omega). \end{gathered} \tag{4.1.6}$$

Notation 4.1.2.The integral of $f(x(\omega))$ ($f(\omega,x(\omega))$) over $[q_1(\omega),q_2(\omega)]$ is denoted

$$\int_{q_1(\omega)}^{q_2(\omega)} f(x(\omega))d[x(\omega)],$$
$$\int_{q_1(\omega)}^{q_2(\omega)} f(\omega, x(\omega))d[x(\omega)], \tag{4.1.7}$$

in which the random variables $q_1(\omega)$ and $q_2(\omega)$ are called respectively the lower and upper limits of integration. The function f is called the integrand, $f(x(\omega))d[x(\omega)]$ ($f(\omega,x(\omega))d[x(\omega)]$) is called the differential form, and $x(\omega)$ is the random variable of integration. Thus

$$a.s.: \int_{q_1(\omega)}^{q_2(\omega)} f(x(\omega))d[x(\omega)] = \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\xi_i(\omega))\Delta x_i(\omega),$$
$$a.s.: \int_{q_1(\omega)}^{q_2(\omega)} f(\omega, x(\omega))d[x(\omega)] = \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\omega, \xi_i(\omega))\Delta x_i(\omega). \tag{4.1.8}$$

Definition 4.1.9. A function $f : \Omega \times \Xi(\Sigma, \mathbb{R}) \to \Xi(\Sigma, \mathbb{R})$ is Riemann integrable on the closed interval $[q_1(\omega), q_2(\omega)]$ if the limit of the Riemann sums in (3.1.8) exists $a.s.$ as $\lambda(P) \to 0$ (that is, the Riemann integral of f is defined).

Notation 4.1.3.The set of Riemann-integrable functions on a closed random interval $[q_1(\omega), q_2(\omega)]$ will be denoted $\Re[q_1(\omega), q_2(\omega)]$

By the definition of the integral (Definition 4.1.8) and its reformulation in the forms (4.1.5) and (4.1.8), an integral is the limit of a certain special function $\Phi(p(\omega)) = \sigma[f, P, \xi(\omega)]$ the random Riemann sum, defined on the set $\widetilde{P}$ of partitions $p(\omega) = (P, \xi(\omega))$ with distinguished points on $[q_1(\omega), q_2(\omega)]$. This limit is taken with respect to the base $\mathbf{B}$ in $\widetilde{P}$ that we have denoted $\lambda(P) \to 0$.

Thus the integrability or nonintegrability of a function f on $[q_1(\omega), q_2(\omega)]$ depends on the existence of this limit. By the Cauchy criterion, this limit exists a.s., if and only if for every $\varepsilon > 0$ there exists an element $B_\delta \in \mathbf{B}$ in the base such that

$$a.s.: |\Phi(p'(\omega)) - \Phi(p''(\omega))| < \varepsilon \tag{4.1.9}$$

for any two points $p'(\omega), p''(\omega)$ in B_δ. In more detailed notation, what has just been said means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a.s.: |\sigma[f, P, \xi(\omega)] - \sigma[f, P, \xi(\omega)]| < \varepsilon \tag{4.1.10}$$

or, what is the same,

$$a.s.: \left| \sum_{i=1}^{n'} f(\omega, \xi_i'(\omega))\Delta x_i'(\omega) - \sum_{i=1}^{n''} f(\omega, \xi_i''(\omega))\Delta x_i''(\omega) \right| < \varepsilon \tag{4.1.11}$$

for any partitions $(P', \xi'(\omega))$ and $(P'', \xi''(\omega))$ with distinguished points on the random interval $[q_1(\omega), q_2(\omega)]$ with $\lambda(P') < \delta$ and $\lambda(P'') < \delta$.

Proposition 4.1.2. A necessary condition for a function $f(\omega, x(\omega))$ defined on $\Omega \times [q_1(\omega), q_2(\omega)]$ to be Riemann integrable on $[q_1(\omega), q_2(\omega)]$ is that f be bounded a.s. on $\Omega \times [q_1(\omega), q_2(\omega)]$.

Proof. If f is not bounded a.s. on $[q_1(\omega), q_2(\omega)]$, then for any partition (P, ξ) of $[q_1(\omega), q_2(\omega)]$ the function f is unbounded on at least one of the intervals $[x_{i-1}(\omega), x_i(\omega)]$ of (P, ξ). This means that, by choosing the point
$\xi_i(\omega) \in [x_{i-1}(\omega), x_i(\omega)]$
in different ways, we can make the quantity $|f(\omega, \xi_i(\omega))\Delta x_i(\omega)|$ a.s. as large as desired.

But then the Riemann sum $\sum_{i=1}^{n} f(\omega, \xi_i(\omega))\Delta x_i(\omega)$ can also be made as large as desired in absolute value by changing only the point $\xi_i(\omega)$ in this interval.

We agree that when a partition P

$$q_1(\omega) = x_0(\omega) \prec x_1(\omega) \prec \ldots x_{n-1}(\omega) \prec x_n(\omega) = q_2(\omega).$$

is given on the closed random interval $[q_1(\omega), q_2(\omega)]$, we shall use the symbol $\Delta_i(\omega)$ to
denote the interval $[x_{i-1}(\omega), x_i(\omega)]$ along with $\Delta x_i(\omega)$ as a notation for the difference $x_i(\omega) - x_{i-1}(\omega)$.

If a partition $P^\star$ of the closed random interval $[q_1(\omega), q_2(\omega)]$ is obtained from the partition P by the adjunction of new points to P, we call $P^\star$ a *refinement* of P.

When a refinement $P^\star$ of a partition P is constructed, some (perhaps all) of the closed random intervals $\Delta_i(\omega) = [x_{i-1}(\omega), x_i(\omega)]$ of the partition P themselves undergo partitioning: $x_{i-1}(\omega) = x_{i_0}(\omega) \prec \ldots \prec x_{i_{n_i}}(\omega) = x_i(\omega)$. In that connection, it will be
useful for us to label the points of $P^\star$ by double indices. In the notation $x_{i_j}(\omega)$ the first index means that $x_{i_j}(\omega) \in \Delta_i(\omega)$, and the second index is the ordinal number of the point on the closed random interval $\Delta_i(\omega)$. It is now natural to set $\Delta x_{i_j}(\omega)$
$= x_{i_j} - x_{i_{j-1}}$
and $\Delta_{i_j}(\omega)$. Thus $\Delta x_i(\omega) = \Delta x_{i_1}(\omega) + \ldots + \Delta x_{i_{n_i}}(\omega)$.

As an example of a partition that is a refinement of both the partition P' and P'' one can take $P^\star = P' \cup P''$, obtained as the union of the points of the two partitions P' and P''.

We recall finally that $\Omega(f(\omega, x(\omega)), E, \omega')$ denotes the oscillation of the function $f(\omega, x)$ on
the random set $E(\omega)$, that is

$$\Omega(f(\omega, x(\omega)), E(\omega)) = \sup_{a.s.:\ x_1,(\omega) x_2(\omega) \in E(\omega)} |f(\omega, x_1(\omega)) - f(\omega, x_2(\omega))|. \qquad (4.1.12)$$

In particular, $\Omega(f(\omega, x(\omega)), \Delta_i(\omega))$ is the oscillation of $f(\omega, x(\omega))$ on the closed random interval $[x_{i-1}(\omega), x_i(\omega)]$.

This oscillation is necessarily a.s. finite if $f(\omega,x)$ is a.s. bounded function of variable x.

Proposition 4.1.3. A sufficient condition for a.s. bounded function $f(\omega,x)$ to be integrable on a closed random interval $[q_1(\omega),q_2(\omega)]$ is that for every $\varepsilon > 0$ there exist a number $\delta > 0$ such that

$$\sum_{i=1}^{n} \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega) < \varepsilon \tag{4.1.13}$$

for any partition P of $[q_1(\omega),q_2(\omega)]$ with mesh $\lambda(P) < \delta$.

Proof Let P be a partition of $[q_1(\omega),q_2(\omega)]$ and $P^\star$ a refinement of P. Let us estimate the difference between the random Riemann sums $\sigma(f,P^\star,\xi^\star) - \sigma(f,P,\xi)$. Using the notation introduced above, we can write

$$\begin{aligned}
&|\sigma(f,P^\star,\xi^\star(\omega)) - \sigma(f,P,\xi(\omega))| = \\
&\left|\sum_{i=1}^{n}\sum_{j=1}^{n_i} f(\omega,\xi_{i_j}(\omega))\Delta x_{i_j}(\omega) - \sum_{i=1}^{n} f(\omega,\xi_i(\omega))\Delta x_i(\omega)\right| = \\
&\left|\sum_{i=1}^{n}\sum_{j=1}^{n_i} f(\omega,\xi_{i_j}(\omega))\Delta x_{i_j}(\omega) - \sum_{i=1}^{n} f(\omega,\xi_i(\omega))\Delta x_{i_j}(\omega)\right| = \\
&\left|\sum_{i=1}^{n}\sum_{j=1}^{n_i} [f(\omega,\xi_{i_j}(\omega)) - f(\omega,\xi_i(\omega))]\Delta x_{i_j}(\omega)\right| \leq \\
&\sum_{i=1}^{n}\sum_{j=1}^{n_i} |[f(\omega,\xi_{i_j}(\omega)) - f(\omega,\xi_i(\omega))]|\Delta x_{i_j}(\omega) \leq \\
&\sum_{i=1}^{n}\sum_{j=1}^{n_i} \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_{i_j}(\omega) = \\
&\sum_{i=1}^{n} \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega).
\end{aligned} \tag{4.1.14}$$

In this computation we have used the relation $\Delta x_i(\omega) = \sum_{j=1}^{n_i} \Delta x_{i_j}(\omega)$ and the inequality

$$a.s.: |f(\omega,\xi_{i_j}(\omega)) - f(\omega,\xi_i(\omega))| \leq \Omega(f(\omega,x(\omega)),\Delta_i(\omega)), \tag{4.1.15}$$

which holds because $a.s.: \xi_{i_j}(\omega) \in \Delta_{i_j}(\omega) \subset \Delta_i(\omega)$ and $a.s.: \xi_i(\omega) \in \Delta_i(\omega)$.

It follows from the estimate for the difference of the random Riemann sums that if the function satisfies the sufficient condition given in the statement of Proposition 4.1.3, then for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$a.s.: |\sigma(f,P^\star,\xi^\star(\omega)) - \sigma(f,P,\xi(\omega))| < \varepsilon/2 \tag{4.1.16}$$

for any partition P of $[q_1(\omega), q_2(\omega)]$ with mesh $\lambda(P) < \delta$, any refinement $P^\star$ of P, and any choice of the sets of distinguished points $\xi(\omega)$ and $\xi^\star(\omega)$.

Now if (P', ξ') and (P'', ξ'') are arbitrary partitions with distinguished points on $[q_1(\omega), q_2(\omega)]$ whose meshes satisfy $\lambda(P') < \delta$ and $\lambda(P'') < \delta$, then, by what has just been proved, the partition $P^\star = P' \cup P''$, which is a refinement of both of them, must satisfy

$$\begin{aligned} a.s.: &|\sigma(f, P^\star, \xi^\star(\omega)) - \sigma(f, P', \xi'(\omega))| < \varepsilon/2, \\ a.s.: &|\sigma(f, P^\star, \xi^\star(\omega)) - \sigma(f, P'', \xi''(\omega))| < \varepsilon/2. \end{aligned} \tag{4.1.17}$$

It follows that

$$|\sigma(f, P', \xi'(\omega)) - \sigma(f, P'', \xi''(\omega))| < \varepsilon, \tag{4.1.18}$$

provided $\lambda(P') < \delta$ and $\lambda(P'') < \delta$. Therefore, by the Cauchy criterion, the limit of the random Riemann sums exists $a.s.$:

$$a.s.: \ \exists \lim_{\lambda(P) \to 0} \sum_{i=1}^{n} f(\omega, \xi_i(\omega)) \Delta x_i(\omega), \tag{4.1.19}$$

that is $f \in \Re[q_1(\omega), q_2(\omega)]$.

Definition 4.1.10. (i) A function $f(\omega, x(\omega)) : \Omega \times \Xi(\Sigma, \mathbb{R}) \to \Xi(\Sigma, \mathbb{R})$ is a.s. continuous at point $x_0(\omega) \in [q_1(\omega), q_2(\omega)]$ if

1.$f(\omega, x_0(\omega))$ is defined, so that $x_0(\omega)$ is in the domain of $f(\omega, x(\omega))$.

2. $\lim_{x(\omega) \to x_0(\omega)} f(\omega, x(\omega))$ a.s. exists for $x(\omega)$ in the domain of $f(\omega, x(\omega))$.

3. $a.s.: \lim_{x(\omega) \to x_0(\omega)} f(\omega, x(\omega)) = f(\omega, x_0(\omega))$.

(ii) A function $f(\omega, x(\omega)) : \Omega \times \Xi(\Sigma, \mathbb{R}) \to \Xi(\Sigma, \mathbb{R})$ is a.s. continuouson on the closed random interval $[q_1(\omega), q_2(\omega)]$ if $f(\omega, x(\omega))$ is a.s. continuous at any point $x(\omega) \in [q_1(\omega), q_2(\omega)]$

Notation 4.1.4.The set of a.s. continuous functions on a closed random interval $[q_1(\omega), q_2(\omega)]$ will be denoted $C_{a.s.}[q_1(\omega), q_2(\omega)]$

Definition 4.1.11.

Corollary 4.1.1. $f(\omega, x(\omega)) \in C_{a.s.}[q_1(\omega), q_2(\omega)] \Rightarrow f(\omega, x(\omega)) \in \Re[q_1(\omega), q_2(\omega)]$, that is, every a.s. continuous function $f(\omega, x(\omega))$ on a closed random interval $[q_1(\omega), q_2(\omega)]$ is integrable on that closed random interval.

Proof. If a function is continuous on a closed random interval, it is a.s. bounded there, so that the necessary condition for integrability is satisfied in this case. But a.s. continuous function on a closed random interval $[q_1(\omega), q_2(\omega)]$ is uniformly a.s. continuous on that interval. Therefore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$a.s.: \Omega(f(\omega,x(\omega)),\Delta(\omega)) < \dfrac{\varepsilon}{q_2(\omega)-q_1(\omega)}$ on any closed interval
$\Delta(\omega) \subset [q_1(\omega),q_2(\omega)]$
of length less than δ. Then for any partition P with mesh $\lambda(P) < \delta$ we have that

$$a.s.: \sum_{i=1}^{n}\Omega(f(\omega,x(\omega)),\Delta_i(\omega)) < \frac{\varepsilon}{q_2(\omega)-q_1(\omega)}\sum_{i=1}^{n}\Delta_i(\omega) = \\ = \frac{\varepsilon[q_2(\omega)-q_1(\omega)]}{q_2(\omega)-q_1(\omega)} = \varepsilon. \tag{4.1.20}$$

By Proposition 3.1.3, we can now conclude that $f \in \Re[q_1(\omega),q_2(\omega)]$.

Corollary 4.1.2. If a.s. bounded function f on a closed random interval $[q_1(\omega),q_2(\omega)]$ is a.s. continuous everywhere except at a finite set of random points $x_i(\omega), i = 1,\ldots,k$ then $f \in \Re[q_1(\omega),q_2(\omega)]$.

Proof. Let $a.s.: \Omega(f(\omega,x(\omega)),[q_1(\omega),q_2(\omega)]) < C < oo$, and suppose f has k points of discontinuity on $[q_1(\omega),q_2(\omega)]$. We shall verify that the sufficient condition for integrability of the function f is satisfied.

For a given $\varepsilon > 0$ we choose the number $\delta_1 = \dfrac{\varepsilon}{8C \times k}$ and construct the δ_1-neighborhood of each of the k points of a.s. discontinuity of f on $[q_1(\omega),q_2(\omega)]$. The
complement of the union of these neighborhoods in $[q_1(\omega),q_2(\omega)]$ consists of a finite number of closed random intervals, on each of which f is a.s. continuous and hence a.s. uniformly continuous. Since the number of these intervals is finite, given $\varepsilon > 0$ there exists $\delta_2 > 0$ such that on each interval $\Delta_i(\omega) = [x_i(\omega),x_{i-1}(\omega)], i = 2,\ldots,k$ whose
length $l[\Delta_i(\omega)] = x_i(\omega) - x_{i-1}(\omega)$ a.s. is less than δ_2 and which is entirely contained in one of the closed random intervals just mentioned, on which f is a.s. continuous, we have

$$a.s.: \Omega(f(\omega,x(\omega)),\Delta_i(\omega)) < \frac{\varepsilon}{2[q_2(\omega)-q_1(\omega)]}$$

We now choose $\delta = \min\{\delta_1,\delta_2\}$. Let P be an arbitrary partition of $[q_1(\omega),q_2(\omega)]$ for which $\lambda(P) < \delta$. We break the sum $\sum_{i=1}^{n}\Omega(f(\omega,x(\omega)),\Delta_i(\omega))$ corresponding to the partition P into two parts:

$$\sum_{i=1}^{n}\Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega) = \sum{}'\,\Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega) + \\ \sum{}''\,\Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega). \tag{4.1.21}$$

The sum $\sum'$ contains the terms corresponding to random intervals $\Delta_i(\omega)$ of the partition having no points in common with any of the δ_1-neighborhoods of the points of discontinuity. For these random intervals $\Delta_i(\omega)$ we have

$$a.s.: \ \Omega(f(\omega,x(\omega)),\Delta_i(\omega)) < \frac{\varepsilon}{2(q_2(\omega)-q_1(\omega))}$$

and so

$$a.s.:\ \sum{}' \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega) < \frac{\varepsilon}{2(q_2(\omega)-q_1(\omega))}\sum{}' \Delta x_i(\omega) \leq \frac{\varepsilon}{2(q_2(\omega)-q_1(\omega))}(q_2(\omega)-q_1(\omega)) = \frac{\varepsilon}{2}. \quad (4.1.22)$$

The sum of the lengths of the remaining intervals of the partition P, as one can easily see, is at most $(\delta+2\delta_1+\delta) \leq \frac{4\varepsilon}{8C\times k}k = \varepsilon/2C$, and therefore

$$a.s.:\ \sum{}'' \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega) \leq C\frac{\varepsilon}{2C} = \frac{\varepsilon}{2}. \quad (4.1.23)$$

Thus we find that for $\lambda(P) < \delta$:

$$a.s.:\ \sum_{i=1}^{n} \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega) < \varepsilon. \quad (4.1.24)$$

that is, the sufficient condition for integrability holds, and so $f \in \Re[q_1(\omega),q_2(\omega)]$.

Corollary 4.1.3. Any a.s. monotonic function on a closed random interval $[q_1(\omega),q_2(\omega)]$ is integrable on that random interval.

Proof. It follows from the a.s. monotonicity of $f(\omega,x(\omega))$ on $[q_1(\omega),q_2(\omega)]$ that $a.s.:\ \sum_{i=1}^{n} \Omega(f(\omega,x(\omega)),[q_1(\omega),q_2(\omega)]) = |f(\omega,q_2(\omega))-f(\omega,q_1(\omega))|$. Suppose $\varepsilon > 0$ is given. We set $\delta(\omega) = \frac{\varepsilon}{|f(\omega,q_2(\omega))-f(\omega,q_1(\omega))|}$ We assume that $a.s.:\ f(\omega,q_2(\omega))-f(\omega,q_1(\omega)) \neq 0$, since otherwise f a.s. is constant, and there is no doubt as to its integrability. Let P be an arbitrary partition of $[q_1(\omega),q_2(\omega)]$ with mesh $a.s.:\ \lambda(P) < \delta(\omega)$. Then, taking account of the a.s.monotonicity of f, we have

$(3.1.25)$

Thus / satisfies the sufficient condition for integrability, and therefore $f \in \Re[q_1(\omega),q_2(\omega)]$.

Remark 4.1.1. A monotonic function may have a (countably) infinite set of discontinu ities on a closed random $[q_1(\omega),q_2(\omega)]$interval. For example, the function defined by the relations

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Apendix 4.2.Integration over generalized random interval.

Definition 4.2.1.Let for any $\omega \in \Omega$: $q_1(\omega,\omega')$ and $q_2(\omega,\omega')$ are $\Omega' = \Xi(\Sigma,\mathbb{R})$- valued random variables defined on a generalized probability space $\Sigma = \left(\Omega,\mathcal{F},\widetilde{\mathbf{P}}\right)$,i.e., $\forall\omega[\omega \in \Omega \Rightarrow q_{1,2}(\omega,\omega') : \Omega \to \Omega' = \mathbf{\Xi}(\mathbf{\Sigma},\mathbb{R})]$.

Let $\widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$ be a set of the all $\Xi(\Sigma,\mathbb{R})$- valued random variables defined on a probability space $\Sigma = \left(\Omega,\mathcal{F},\widetilde{\mathbf{P}}\right)$,thus $q_{1,2}(\omega,\omega') \in \widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$.

Definition 4.2.2.Let $q_1(\omega,\omega')$ and $q_2(\omega,\omega')$ are $\Xi(\Sigma,\mathbb{R})$- valued random variables

defined on a generalized probability space $\Sigma = \left(\Omega, \mathcal{F}, \widetilde{\mathbf{P}}\right)$, i.e.,
$q_{1,2}(\omega,\omega') : \Omega \to \Xi(\mathbf{\Sigma},\mathbb{R})$. Assume that a.s.:$-\infty < q_1(\omega,\omega') < q_2(\omega,\omega') < \infty$.
Let $\widetilde{\widetilde{\Xi}}(\Omega,\Sigma,\mathbb{R})$ be a set of the all $\Xi(\Sigma,\mathbb{R})$- valued random variables defined on a generalized probability space $\Sigma = \left(\Omega, \mathcal{F}, \widetilde{\mathbf{P}}\right)$. Closed generalized random interval $[q_1(\omega,\omega'), q_2(\omega,\omega')]$ that is a subset $[q_1(\omega,\omega'), q_2(\omega,\omega')] \subset \widetilde{\widetilde{\Xi}}(\Omega,\Sigma,\mathbb{R})$ such that a.s.:$-\infty < q_1(\omega,\omega') < q_2(\omega,\omega') < \infty$ and

$$\begin{gathered}\forall q(\omega,\omega')\{q(\omega,\omega') \in [q_1(\omega,\omega'), q_2(\omega,\omega')] \Leftrightarrow \\ \{q(\omega,\omega') \in \Xi(\Sigma,\mathbb{R}) | a.s.(q_1(\omega,\omega') < q(\omega,\omega') < q_2(\omega,\omega'))\}\}.\end{gathered} \tag{4.2.1}$$

Notation 4.2.1. Assume that a.s.:$q_1(\omega,\omega') < q_2(\omega,\omega')$. Then we will write:

$$q_1(\omega,\omega') \prec q_2(\omega,\omega').$$

Assume that a.s.$q_1(\omega,\omega') \leq q_2(\omega,\omega')$. Then we will write:

$$q_1(\omega,\omega') \preccurlyeq q_2(\omega,\omega').$$

Definition 4.2.3. Let E be a separable complete metric space and let Σ be its Borel σ-algebra. Generalized random measure it is a function $\mu : \Sigma \to \Xi(\Sigma,\mathbb{R})$, that satisfies:

(1) If $E_1 \subset E_2$, then a.s.:$\mu(E_1) < \mu(E_2)$.

(2) If $E_n \in \mathcal{S}$, $n = 1,2,\ldots$ and $E_i \cap E_j = \varnothing$ $(i \neq j)$, then a.s.:$\mu(\cup_n E_n) = \sum_n \mu(E_n)$.

If we further impose a condition $\mu(\Omega) = 1$, then the generalized random measure so defined is called the generalized random probability measure and is usually denoted by $\widetilde{\mathbf{P}}$.

Definition 4.2.4. The tuple $(\Omega,\mathcal{S},\widetilde{\mathbf{P}})$ is then called the *generalized probability space*. Suppose $(\Omega,\mathcal{S})$ and $(\Omega',\mathcal{S}')$ are two measurable spaces. The function $X : \Omega \to \Omega'$ is called a *generalized random element*, if for every $A \in \mathcal{S}'$, $X^{-1}(A) \in \mathcal{S}$. If the first measurable space is equipped with a generalized random probability measure $\widetilde{\mathbf{P}}$, then the generalized random element $X : \Omega \to \Omega'$ induces a generalized random probability measure on the second space $(\Omega',\mathcal{S}')$ and given by $\mathcal{P} \equiv \widetilde{\mathbf{P}} \circ X^{-1}$, called the generalized random distribution of X.

Definition 4.2.5. If the second measurable space $(\Omega',\mathcal{S}')$ is taken to be $(\Xi(\Sigma,\mathbb{R}), \mathfrak{B}_{\Xi(\Sigma,\mathbb{R})})$, where $\mathfrak{B}_{\Xi(\Sigma,\mathbb{R})}$ is the Borel σ-algebra on $\Xi(\Sigma,\mathbb{R})$ the function $X(\omega) : \Omega \to \Omega' = (\Xi(\Sigma,\mathbb{R}), \mathfrak{B}_{\Xi(\Sigma,\mathbb{R})})$ is called a *generalized random variable*. For this case, the generalized random distribution of X is completely determined by the *random distribution function* defined as

$$F_{X(\omega)}(x(\omega,\omega')) = \widetilde{\mathbf{P}}(X(\omega) \preccurlyeq x(\omega,\omega')) \tag{4.2.2}$$

Definition 4.2.6. The lengths $l([q_1(\omega),q_2(\omega)])$ of the random interval $[q_1(\omega),q_2(\omega)]$ is defined by

$$l([q_1(\omega,\omega'),q_2(\omega,\omega')]) = \mathbf{ess}\sup q_3(\omega,\omega'), \tag{4.2.2}$$

where $q_3(\omega,\omega') = q_2(\omega,\omega') - q_1(\omega,\omega')$.

Notation 4.2.2. Assume that a.s.$q_1(\omega,\omega') < q_2(\omega,\omega')$. Then we will write:

$$q_1(\omega,\omega') \prec q_2(\omega,\omega').$$

Assume that a.s.$q_1(\omega,\omega') \leq q_2(\omega,\omega')$. Then we will write:

$$q_1(\omega,\omega') \preccurlyeq q_2(\omega,\omega').$$

Definition 4.2.7. A partition P of a closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ is a finite system of random variables $x_0(\omega,\omega'),\ldots,x_n(\omega,\omega')$ such that

$$q_1(\omega,\omega') = x_0(\omega,\omega') \prec x_1(\omega,\omega') \prec\ldots x_{n-1}(\omega,\omega') \prec x_n(\omega,\omega') = q_2(\omega,\omega').$$

The closed random intervals $[x_{i-1}(\omega),x_i(\omega)]$, $(i = 1,\ldots,n)$ are called the intervals of the partition P.

Definition 4.2.8.The largest of the lengths of the intervals of the partition P, denoted $\lambda(P)$, is called the *mesh* of the partition.

Definition 4.2.9. We speak of a partition with distinguished points (P,ξ) on the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ if we have a partition P of $[q_1(\omega,\omega'),q_2(\omega,\omega')]$and a point $\xi_1(\omega,\omega') \in [x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$ has been chosen in each of the intervals of the partition $[x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$, $(i = 1,\ldots,n)$.

We denote the set of points $(\xi_1(\omega,\omega'),\ldots,\xi_n(\omega,\omega'))$ by the single letter $\xi(\omega,\omega')$.

Definition 4.2.10. In the set $\widetilde{P}$ of partitions with distinguished points on a given random

interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$, we consider the following base $\mathbf{B} = \{B_d\}$. The element $B_d, d > 0$, of the base B consists of all partitions with distinguished points (P,ξ) on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ for which $\lambda(P) < d$.

Proposition 4.2.1. Let us verify that $\{B_d\}, d > 0$ is actually a base in $\widetilde{P}$.

Proof.First $B_d \neq \varnothing$. In fact, for any number $d > 0$, it is obvious that there exists a partition P of $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ with mesh $\lambda(P) < d$ (for example, a partition into n

congruent closed random intervals). But then there also exists a partition $(P,\xi(\omega,\omega'))$ with distinguished points for which $\lambda(P) < d$.

Second, if $d_1 > 0, d_2 > 0$, and $d = \min\{d_1,d_2\}$, it is obvious that $B_{d_1} \cap B_{d_2}$ $= B_d \in \mathbf{B}$. Hence $\mathbf{B} = \{B_d\}$ is indeed a base in $\widetilde{P}$.

Definition 4.2.11. (**Random Riemann Sums**) (i) If a function $f : \widetilde{\Xi}(\Sigma,\mathbb{R}) \to \widetilde{\Xi}(\Sigma,\mathbb{R})$ is defined on the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ and $(P,\xi(\omega))$ is a partition with distinguished points on this closed random interval, the sum

$$\sigma[f,P,\xi(\omega,\omega')] = \sum_{i=1}^{n} f(\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'), \tag{4.2.3}$$

where $\Delta x_i(\omega,\omega') = x_i(\omega,\omega') - x_{i-1}(\omega,\omega')$, is the random Riemann sum of the function $f : \tilde{\tilde{\Xi}}(\Sigma,\mathbb{R}) \to \tilde{\tilde{\Xi}}(\Sigma,\mathbb{R})$ corresponding to the partition $(P,\xi(\omega,\omega'))$ with distinguished points on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$. Thus, when the function f is fixed, the random Riemann sum $\sigma[f,P,\xi(\omega,\omega')]$ is a function $\Phi(p) = \sigma[f,P]$ on the set $\tilde{P}$ of all partitions $p = (P,\xi(\omega,\omega'))$ with distinguished points on the closed interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$. Since there is a base $\mathbf{B}$ in $\tilde{P}$, one can ask about the limit of the function $\Phi(p(\omega,\omega'))$ over that base.

(ii) If a function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \tilde{\tilde{\Xi}}(\Sigma,\mathbb{R})$ is defined on the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ and $(P,\xi(\omega,\omega'))$ is a partition with distinguished points on this closed random interval, the sum

$$\sigma[f,P,\xi(\omega,\omega')] = \sum_{i=1}^{n} f(\omega,\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'), \qquad (4.2.3')$$

where $\Delta x_i(\omega,\omega') = x_i(\omega,\omega') - x_{i-1}(\omega,\omega')$, is the random Riemann sum of the function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ corresponding to the partition $(P,\xi(\omega,\omega'))$ with distinguished points on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$. Thus, when the function

$$f : \Omega \times [q_1(\omega,\omega'),q_2(\omega,\omega')] \to \tilde{\tilde{\Xi}}(\Omega,\Sigma,\mathbb{R})$$

is fixed, the random Riemann sum $\sigma[f,P,\xi(\omega,\omega')]$ is a function $\Phi(p) = \sigma[f,P]$ on the set $\tilde{P}$ of all partitions $p = (P,\xi(\omega,\omega'))$ with distinguished points on the closed interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$.

Definition 4.2.12. (**Riemann Integral on a generalized random interval**)

(i) Let $f : \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ be a function restricted on a closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$.

The random variable $I(\omega,\omega')$ is the Riemann integral of the function $f : \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ on the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a.s.: \left| I(\omega,\omega') - \sum_{i=1}^{n} f(\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega') \right| < \varepsilon \qquad (4.2.4)$$

for any partition $(P,\xi(\omega,\omega'))$ with distinguished points on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ whose mesh $\lambda(P)$ is less than δ.

(ii) Let $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ be a function restricted on a closed random interval $[q_1(\omega),q_2(\omega)]$ such that $f : \Omega \times [q_1(\omega,\omega'),q_2(\omega,\omega')] \to \Xi(\Sigma,\mathbb{R})$.

The random variable $I(\omega,\omega')$ is the Riemann integral of the function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ on the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a.s.: \left| I(\omega,\omega') - \sum_{i=1}^{n} f(\omega,\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega') \right| < \varepsilon. \qquad (4.2.4')$$

Since the partitions $p = (P,\xi(\omega,\omega'))$ for which $\lambda(P) < \delta$ form the element B_δ

of the base $\mathbf{B}$ introduced above in the set $\widetilde{P}$ of partitions with distinguished points, Definition 3.2.12 is equivalent to the statement

$$a.s.:\ I(\omega,\omega') = \lim_{\mathbf{B}} \Phi(p(\omega,\omega')). \tag{4.2.5}$$

that is, the integral $I(\omega,\omega')$ is the limit over $\mathbf{B}$ of the Riemann sums of the function f corresponding to partitions with distinguished points on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$.

It is natural to denote the base $\mathbf{B}$ by $\lambda(P) \to 0$, and then the definition of the Riemann
integral on a random interval can be rewritten as

$$\begin{aligned} a.s.:\ I(\omega,\omega') &= \lim_{\lambda(P)\to 0} \sum\nolimits_{i=1}^{n} f(\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'),\\ a.s.:\ I(\omega,\omega') &= \lim_{\lambda(P)\to 0} \sum\nolimits_{i=1}^{n} f(\omega,\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'). \end{aligned} \tag{4.2.6}$$

Notation 4.2.2. The integral of $f(x(\omega,\omega'))$ over $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ is denoted

$$\begin{aligned} &\int_{q_1(\omega)}^{q_2(\omega)} f(x(\omega,\omega'))d[x(\omega,\omega')],\\ &\int_{q_1(\omega)}^{q_2(\omega)} f(\omega,x(\omega,\omega'))d[x(\omega,\omega')], \end{aligned} \tag{4.2.7}$$

in which the random variables $q_1(\omega,\omega')$ and $q_2(\omega,\omega')$ are called respectively the lower and upper limits of integration. The function f is called the integrand, $f(x(\omega,\omega'))d[x(\omega,\omega')]$ is called the differential form, and $x(\omega,\omega')$ is the random variable
of integration. Thus

$$\begin{aligned} a.s.:\ &\int_{q_1(\omega,\omega')}^{q_2(\omega,\omega')} f(x(\omega,\omega'))d[x(\omega,\omega')] = \lim_{\lambda(P)\to 0} \sum\nolimits_{i=1}^{n} f(\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'),\\ a.s.:\ &\int_{q_1(\omega,\omega')}^{q_2(\omega,\omega')} f(\omega,x(\omega,\omega'))d[x(\omega,\omega')] = \lim_{\lambda(P)\to 0} \sum\nolimits_{i=1}^{n} f(\omega,\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'). \end{aligned} \tag{4.2.8}$$

Definition 4.2.13. A function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ is Riemann integrable on the closed interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ if the limit of the Riemann sums in (4.2.8) exists $a.s.$ as $\lambda(P) \to 0$ (that is, the Riemann integral of f is defined).

Notation 4.2.3. The set of Riemann-integrable functions on a closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ will be denoted $\Re[q_1(\omega,\omega'),q_2(\omega,\omega')]$

By the definition of the integral (Definition 4.2.8) and its reformulation in the forms

(4.2.5) and (4.2.8), an integral is the limit of a certain special function $\Phi(p(\omega,\omega')) = \sigma[f,P,\xi(\omega,\omega')]$ the random Riemann sum, defined on the set $\widetilde{P}$ of partitions $p(\omega,\omega') = (P,\xi(\omega,\omega'))$ with distinguished points on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$. This limit is taken with respect to the base $\mathbf{B}$ in $\widetilde{P}$ that we have denoted $\lambda(P) \to 0$. Thus the integrability or nonintegrability of a function f on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ depends on the existence of this limit. By the Cauchy criterion, this limit exists a.s., if and only if for every $\varepsilon > 0$ there exists an element $B_\delta \in \mathbf{B}$ in the base such that

$$a.s.: |\Phi(p'(\omega,\omega')) - \Phi(p''(\omega,\omega'))| < \varepsilon \tag{4.2.9}$$

for any two points $p'(\omega,\omega'),p''(\omega,\omega')$ in B_δ. In more detailed notation, what has just been said means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a.s.: |\sigma[f,P,\xi(\omega,\omega')] - \sigma[f,P,\xi(\omega,\omega')]| < \varepsilon \tag{4.2.10}$$

or, what is the same,

$$a.s.: \left|\sum_{i=1}^{n'} f(\omega,\xi_i'(\omega,\omega'))\Delta x_i'(\omega(\omega,\omega')) - \sum_{i=1}^{n''} f(\omega,\xi_i''(\omega,\omega'))\Delta x_i''(\omega,\omega')\right| < \varepsilon \tag{4.2.11}$$

for any partitions $(P',\xi'(\omega,\omega'))$ and $(P'',\xi''(\omega,\omega'))$ with distinguished points on the random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ with $\lambda(P') < \delta$ and $\lambda(P'') < \delta$.

Proposition 4.2.2. A necessary condition for a function $f(\omega,x(\omega,\omega'))$ defined on $\Omega \times [q_1(\omega,\omega'),q_2(\omega,\omega')]$ to be Riemann integrable on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ is that f be bounded a.s. on $\Omega \times [q_1(\omega,\omega'),q_2(\omega,\omega')]$.

Proof. If f is not bounded a.s. on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$, then for any partition (P,ξ) of $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ the function f is unbounded on at least one of the intervals $[x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$ of (P,ξ). This means that, by choosing the point $\xi_i(\omega,\omega') \in [x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$ in different ways, we can make the quantity $|f(\omega,\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega')|$ a.s. as large as desired.But then the Riemann sum $\sum_{i=1}^{n} f(\omega,\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega')$ can also be made as large as desired in absolute value by changing only the point $\xi_i(\omega,\omega')$ in this interval.

We agree that when a partition P

$$q_1(\omega,\omega') = x_0(\omega,\omega') \prec x_1(\omega,\omega') \prec \ldots x_{n-1}(\omega,\omega') \prec x_n(\omega,\omega') = q_2(\omega,\omega').$$

is given on the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$, we shall use the symbol $\Delta_i(\omega,\omega')$ to denote the interval $[x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$ along with $\Delta x_i(\omega,\omega')$ as a notation for the difference $x_i(\omega,\omega') - x_{i-1}(\omega,\omega')$.

If a partition $P^\star$ of the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ is obtained from the partition P by the adjunction of new points to P, we call $P^\star$ a *refinement* of P. When a refinement $P^\star$ of a partition P is constructed, some (perhaps all)

of the closed random intervals $\Delta_i(\omega,\omega') = [x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$ of the partition P themselves undergo partitioning: $x_{i-1}(\omega,\omega') = x_{i_0}(\omega,\omega') \prec \ldots \prec x_{i_{n_i}}(\omega,\omega') = x_i(\omega,\omega')$.
In that connection, it will be useful for us to label the points of $P^\star$ by double indices.
In the notation $x_{i_j}(\omega,\omega')$ the first index means that $x_{i_j}(\omega,\omega') \in \Delta_i(\omega,\omega')$, and the second
index is the ordinal number of the point on the closed random interval $\Delta_i(\omega)$. It is now
natural to set $\Delta x_{i_j}(\omega,\omega') = x_{i_j}(\omega,\omega') - x_{i_{j-1}}(\omega,\omega')$ and $\Delta_{i_j}(\omega,\omega')$. Thus

$$\Delta x_i(\omega,\omega') = \Delta x_{i_1}(\omega,\omega') + \ldots + \Delta x_{i_{n_i}}(\omega,\omega').$$

As an example of a partition that is a refinement of both the partition $P'$and P'' one can
take $P^\star = P' \cup P''$, obtained as the union of the points of the two partitions P' and P''.
We recall finally that $\Omega(f(\omega,x(\omega,\omega')),E,\omega')$ denotes the oscillation of the function $f(\omega,x)$ on the random set $E(\omega,\omega')$, that is

$$\Omega(f(\omega,x(\omega,\omega')),E(\omega,\omega')) = \sup_{a.s.:\, x_1,(\omega,\omega'),x_2(\omega,\omega')\in E(\omega,\omega')} |f(\omega,x_1(\omega,\omega')) - f(\omega,x_2(\omega,\omega'))|. \quad (4.2.12)$$

In particular, $\Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))$ is the oscillation of $f(\omega,x(\omega,\omega'))$ on the closed random interval $[x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$.
This oscillation is necessarily a.s. finite if $f(\omega,x)$ is a.s. bounded function of variable x.

Proposition 4.1.3. A sufficient condition for a.s. bounded function $f(\omega,x)$ to be integrable on a closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ is that for every $\varepsilon > 0$ there
exist a number $\delta > 0$ such that

$$\sum_{i=1}^{n} \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))\Delta x_i(\omega,\omega') < \varepsilon \quad (4.2.13)$$

for any partition P of $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ with mesh $\lambda(P) < \delta$.
Proof Let P be a partition of $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ and $P^\star$ a refinement of P. Let us estimate the difference between the random Riemann sums $\sigma(f,P^\star,\xi^\star) - \sigma(f,P,\xi)$.
Using the notation introduced above, we can write

$$
\begin{aligned}
&|\sigma(f,P^\star,\xi^\star(\omega,\omega')) - \sigma(f,P,\xi(\omega,\omega'))| = \\
&\left|\sum_{i=1}^{n}\sum_{j=1}^{n_i} f(\omega,\xi_{i_j}(\omega,\omega'))\Delta x_{i_j}(\omega,\omega') - \sum_{i=1}^{n} f(\omega,\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega')\right| = \\
&\left|\sum_{i=1}^{n}\sum_{j=1}^{n_i} f(\omega,\xi_{i_j}(\omega,\omega'))\Delta x_{i_j}(\omega,\omega') - \sum_{i=1}^{n} f(\omega,\xi_i(\omega,\omega'))\Delta x_{i_j}(\omega,\omega')\right| = \\
&\left|\sum_{i=1}^{n}\sum_{j=1}^{n_i} [f(\omega,\xi_{i_j}(\omega,\omega')) - f(\omega,\xi_i(\omega,\omega'))]\Delta x_{i_j}(\omega,\omega')\right| \leq \\
&\sum_{i=1}^{n}\sum_{j=1}^{n_i} |[f(\omega,\xi_{i_j}(\omega)) - f(\omega,\xi_i(\omega))]|\Delta x_{i_j}(\omega,\omega') \leq \\
&\sum_{i=1}^{n}\sum_{j=1}^{n_i} \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))\Delta x_{i_j}(\omega,\omega') = \\
&\sum_{i=1}^{n} \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))\Delta x_i(\omega,\omega').
\end{aligned}
\tag{4.2.14}
$$

In this computation we have used the relation $\Delta x_i(\omega,\omega') = \sum_{j=1}^{n_i} \Delta x_{i_j}(\omega,\omega')$ and the inequality

$$a.s.: |f(\omega,\xi_{i_j}(\omega,\omega')) - f(\omega,\xi_i(\omega,\omega'))| \leq \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega')), \tag{4.2.15}$$

which holds because $a.s.: \xi_{i_j}(\omega) \in \Delta_{i_j}(\omega) \subset \Delta_i(\omega)$ and $a.s.: \xi_i(\omega) \in \Delta_i(\omega)$.
It follows from the estimate for the difference of the random Riemann sums that if the function satisfies the sufficient condition given in the statement of Proposition 4.2.3, then for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$a.s.: |\sigma(f,P^\star,\xi^\star(\omega,\omega')) - \sigma(f,P,\xi(\omega,\omega'))| < \varepsilon/2 \tag{4.2.16}$$

for any partition P of $[q_1(\omega),q_2(\omega)]$ with mesh $\lambda(P) < \delta$, any refinement $P^\star$ of P, and any choice of the sets of distinguished points $\xi(\omega,\omega')$ and $\xi^\star(\omega,\omega')$.
Now if (P',ξ') and (P'',ξ'') are arbitrary partitions with distinguished points on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ whose meshes satisfy $\lambda(P') < \delta$ and $\lambda(P'') < \delta$, then, by what has just been proved, the partition $P^\star = P' \cup P''$, which is a refinement of both of them, must satisfy

$$
\begin{aligned}
&a.s.: |\sigma(f,P^\star,\xi^\star(\omega,\omega')) - \sigma(f,P',\xi'(\omega,\omega'))| < \varepsilon/2, \\
&a.s.: |\sigma(f,P^\star,\xi^\star(\omega,\omega')) - \sigma(f,P'',\xi''(\omega,\omega'))| < \varepsilon/2.
\end{aligned}
\tag{4.2.17}
$$

It follows that

$$|\sigma(f,P',\xi'(\omega,\omega')) - \sigma(f,P'',\xi''(\omega,\omega'))| < \varepsilon, \tag{4.2.18}$$

provided $\lambda(P') < \delta$ and $\lambda(P'') < \delta$. Therefore, by the Cauchy criterion, the limit of the random Riemann sums exists $a.s.$:

$$a.s.:\ \exists \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\omega,\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'), \tag{4.2.19}$$

that is $f \in \Re[q_1(\omega,\omega'),q_2(\omega,\omega')]$.

Definition 4.2.10. (i) A function $f(\omega,x(\omega,\omega')) : \Omega \times \tilde{\tilde{\Xi}}(\Omega,\Sigma,\mathbb{R}) \to \tilde{\tilde{\Xi}}(\Omega,\Sigma,\mathbb{R})$ is a.s. continuous at point $x_0(\omega,\omega') \in [q_1(\omega,\omega'),q_2(\omega,\omega')]$ if

1.$f(\omega,x_0(\omega,\omega'))$ is defined, so that $x_0(\omega,\omega')$ is in the domain of $f(\omega,x(\omega,\omega'))$.

2. $\lim_{x(\omega)\to x_0(\omega)} f(\omega,x(\omega))$ a.s. exists for $x(\omega)$ in the domain of $f(\omega,x(\omega))$.

3. $a.s.: \lim_{x(\omega)\to x_0(\omega)} f(\omega,x(\omega)) = f(\omega,x_0(\omega))$.

(ii) A function $f(\omega,x(\omega,\omega')) : \Omega \times \tilde{\tilde{\Xi}}(\Omega,\Sigma,\mathbb{R}) \to \tilde{\tilde{\Xi}}(\Omega,\Sigma,\mathbb{R})$ is a.s. continuouson on the closed random interval $[q_1(\omega),q_2(\omega)]$ if $f(\omega,x(\omega))$ is a.s. continuous at any point $x(\omega,\omega') \in [q_1(\omega,\omega'),q_2(\omega,\omega')]$

Notation 4.2.4.The set of a.s. continuous functions on a closed random interval $[q_1(\omega),q_2(\omega)]$ will be denoted $C_{a.s.}[q_1(\omega),q_2(\omega)]$

Definition 4.2.11.

Corollary 4.2.1. $f(\omega,x(\omega,\omega')) \in C_{a.s.}[q_1(\omega,\omega'),q_2(\omega,\omega')] \Rightarrow$
$f(\omega,x(\omega,\omega')) \in \Re[q_1(\omega,\omega'),q_2(\omega,\omega')]$, that is, every a.s. continuous function
$f(\omega,x(\omega))$
on a closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ is integrable on that closed random interval.

Proof. If a function is continuous on a closed random interval, it is a.s. bounded there,
so that the necessary condition for integrability is satisfied in this case. But
a.s. continuous function on a closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ is uniformly
a.s. continuous on that interval. Therefore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$a.s.: \Omega(f(\omega,x(\omega,\omega')),\Delta(\omega,\omega')) < \dfrac{\varepsilon}{q_2(\omega,\omega') - q_1(\omega,\omega')}$ on any closed interval
$\Delta(\omega,\omega') \subset [q_1(\omega,\omega'),q_2(\omega,\omega')]$ of length less than δ. Then for any partition P with
mesh $\lambda(P) < \delta$ we have that

$$\begin{gathered} a.s.: \sum_{i=1}^{n} \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega')) < \frac{\varepsilon}{q_2(\omega,\omega') - q_1(\omega,\omega')} \sum_{i=1}^{n} \Delta_i(\omega,\omega') = \\ = \frac{\varepsilon[q_2(\omega,\omega') - q_1(\omega,\omega')]}{q_2(\omega,\omega') - q_1(\omega,\omega')} = \varepsilon. \end{gathered} \tag{4.2.20}$$

By Proposition 4.2.3, we can now conclude that $f \in \Re[q_1(\omega,\omega'),q_2(\omega,\omega')]$.

Corollary 4.2.2. If a.s. bounded function f on a closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$
is a.s. continuous everywhere except at a finite set of random points $x_i(\omega,\omega')$, $i = 1,\ldots,k$ then $f \in \Re[q_1(\omega,\omega'),q_2(\omega,\omega')]$.

Proof. Let $a.s.: \Omega(f(\omega,x(\omega,\omega')),[q_1(\omega,\omega'),q_2(\omega,\omega')]) < C < oo$, and suppose f has k points of discontinuity on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$. We shall verify that the sufficient condition for integrability of the function f is satisfied.

For a given $\varepsilon > 0$ we choose the number $\delta_1 = \frac{\varepsilon}{8C \times k}$ and construct the δ_1-neighborhood of each of the k points of a.s. discontinuity of f on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$.

The complement of the union of these neighborhoods in $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ consists
of a finite number of closed random intervals, on each of which f is a.s. continuous and
hence a.s. uniformly continuous. Since the number of these intervals is finite, given $\varepsilon > 0$ there exists $\delta_2 > 0$ such that on each interval $\Delta_i(\omega,\omega') = [x_i(\omega,\omega'),x_{i-1}(\omega)]$, $i = 2,\ldots,k$ whose length $l[\Delta_i(\omega,\omega')] = x_i(\omega,\omega') - x_{i-1}(\omega,\omega')$ a.s. is less than δ_2 and which is entirely contained in one of the closed random intervals just mentioned, on which f is a.s. continuous, we have

$$a.s.: \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega')) < \frac{\varepsilon}{2[q_2(\omega,\omega') - q_1(\omega,\omega')]}.$$

We now choose $\delta = \min\{\delta_1,\delta_2\}$. Let P be an arbitrary partition of $[q_1(\omega),q_2(\omega)]$ for which $\lambda(P) < \delta$. We break the sum $\sum_{i=1}^{n} \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega))$ corresponding to the
partition P into two parts:

$$\begin{aligned}\sum_{i=1}^{n} \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))\Delta x_i(\omega,\omega') = \\ \sum{}' \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))\Delta x_i(\omega,\omega') + \\ \sum{}'' \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))\Delta x_i(\omega,\omega').\end{aligned} \quad (4.2.21)$$

The sum $\sum{}'$ contains the terms corresponding to random intervals $\Delta_i(\omega)$ of the partition having no points in common with any of the δ_1-neighborhoods of the points of discontinuity. For these random intervals $\Delta_i(\omega,\omega')$ we have

$$a.s.: \ \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega)) < \frac{\varepsilon}{2(q_2(\omega,\omega') - q_1(\omega,\omega'))}$$

and so

$$a.s.: \sum{}' \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))\Delta x_i(\omega,\omega') < \frac{\varepsilon}{2(q_2(\omega,\omega')-q_1(\omega,\omega'))}\sum{}' \Delta x_i(\omega,\omega') \leq \frac{\varepsilon}{2(q_2(\omega,\omega')-q_1(\omega,\omega'))}(q_2(\omega,\omega')-q_1(\omega,\omega')) = \frac{\varepsilon}{2}. \tag{4.2.22}$$

The sum of the lengths of the remaining intervals of the partition P, as one can easily see, is at most $(\delta+2\delta_1+\delta) \leq \frac{4\varepsilon}{8C\times k}k = \varepsilon/2C$, and therefore

$$a.s.: \sum{}'' \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))\Delta x_i(\omega,\omega') \leq C\frac{\varepsilon}{2C} = \frac{\varepsilon}{2}. \tag{4.2.23}$$

Thus we find that for $\lambda(P) < \delta$:

$$a.s.: \sum_{i=1}^{n} \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))\Delta x_i(\omega,\omega') < \varepsilon. \tag{4.2.24}$$

that is, the sufficient condition for integrability holds, and so $f \in \Re[q_1(\omega,\omega'),q_2(\omega,\omega')]$.

Corollary 4.2.3. Any a.s. monotonic function on a closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ is integrable on that random interval.

Proof. It follows from the a.s. monotonicity of $f(\omega,x(\omega,\omega'))$ on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ that

$a.s.: \sum_{i=1}^{n} \Omega(f(\omega,x(\omega,\omega')),[q_1(\omega,\omega'),q_2(\omega,\omega')]) = |f(\omega,q_2(\omega,\omega')) - f(\omega,q_1(\omega,\omega'))|$.

Suppose $\varepsilon > 0$ is given. We set

$$\delta(\omega,\omega') = \frac{\varepsilon}{|f(\omega,q_2(\omega,\omega')) - f(\omega,q_1(\omega,\omega'))|}$$

We assume that $a.s.: f(\omega,q_2(\omega,\omega')) - f(\omega,q_1(\omega,\omega')) \neq 0$, since otherwise f a.s. is constant, and there is no doubt as to its integrability. Let P be an arbitrary partition of $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ with mesh $a.s.: \lambda(P) < \delta(\omega,\omega')$. Then, taking account of the a.s.monotonicity of f, we have

(4.2.25)

Thus / satisfies the sufficient condition for integrability, and therefore $f \in \Re[q_1(\omega,\omega'),q_2(\omega,\omega')]$.

Remark 4.1.1. A monotonic function may have a (countably) infinite set of discontinu ities on a closed random $[q_1(\omega,\omega'),q_2(\omega,\omega')]$interval. For example, the function

defined by the relations

Apendix 5.Generalized random variables and double stochastic processes.

Definition 5.1..Let $\Xi(\Sigma,\mathbb{R})$ be a set of the all $\mathbb{R}$- valued random variables defined on a generalized probability space $\Sigma = \left(\Omega,\mathcal{F},\widetilde{\mathbf{P}}\right)$, see def.5.4.

Definition 5.2.Let $q_1(\omega,\omega')$ and $q_2(\omega,\omega')$ are $\Xi(\Sigma,\mathbb{R})$- valued random variables defined on a generalized probability space $\Sigma = \left(\Omega,\mathcal{F},\widetilde{\mathbf{P}}\right)$, i.e., $q_{1,2}(\omega,\omega') : \Omega \to \mathbf{\Omega}' = \Xi(\mathbf{\Sigma},\mathbb{R})$. Assume that a.s.:$-\infty < q_1(\omega,\omega') < q_2(\omega,\omega') < \infty$. Let $\widetilde{\widetilde{\Xi}}(\Omega,\Sigma,\mathbb{R})$ be a set of the all $\Xi(\Sigma,\mathbb{R})$- valued random variables defined on a generalized probability space $\Sigma = \left(\Omega,\mathcal{F},\widetilde{\mathbf{P}}\right)$. Closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ that is a subset $[q_1(\omega,\omega'),q_2(\omega,\omega')] \subset \widetilde{\widetilde{\Xi}}(\Omega',\Sigma,\mathbb{R})$ such that a.s.:$-\infty < q_1(\omega,\omega') < q_2(\omega,\omega') < \infty$ and

$$\forall\omega\{q(\omega,\omega') \in [q_1(\omega,\omega'),q_2(\omega,\omega')] \Leftrightarrow \left\{q(\omega,\omega') \in \widetilde{\widetilde{\Xi}}(\Omega',\Sigma,\mathbb{R}) \wedge a.s.: (q_1(\omega,\omega') < q(\omega,\omega') < q_2(\omega,\omega'))\right\}\}. \tag{5.1}$$

Notation 5.1.Assume that a.s.:$q_1(\omega,\omega') < q_2(\omega,\omega')$. Then we will write:

$$q_1(\omega,\omega') \prec q_2(\omega,\omega').$$

Assume that a.s.$q_1(\omega) \leq q_2(\omega)$. Then we will write:

$$q_1(\omega,\omega') \preccurlyeq q_2(\omega,\omega').$$

Definition 5.3.Let E be a separable complete metric space and let Σ be its Borel σ-algebra. <u>*Generalized*</u> random measure it is a function $\mu : \Sigma \to \Xi(\Sigma,\mathbb{R})$, that satisfies:

(1) If $E_1 \subset E_2$, then a.s.:$\mu(E_1) < \mu(E_2)$.

(2) If $E_n \in \mathcal{S}$, $n = 1,2,\ldots$ and $E_i \cap E_j = \varnothing$ $(i \neq j)$, then a.s.:$\mu(\cup_n E_n) = \sum_n \mu(E_n)$.

If we further impose a condition $\mu(\Omega) = 1$, then the generalized random measure so defined is called the generalized random probability measure and is usually denoted by $\widetilde{\mathbf{P}}$.

Definition 5.4.The tuple $(\Omega,\mathcal{S},\widetilde{\mathbf{P}})$ is then called the *generalized probability space*. Suppose $(\Omega,\mathcal{S})$ and $(\Omega',\mathcal{S}')$ are two measurable spaces. The function $X : \Omega \to \Omega'$ is called a <u>*generalized*</u> <u>*random element*</u>, if for every $A \in \mathcal{S}'$, $X^{-1}(A) \in \mathcal{S}$. If the first measurable space is equipped with a generalized random probability measure $\widetilde{\mathbf{P}}$, then the generalized random element $X : \Omega \to \Omega'$ induces a generalized random probability measure on the second space $(\Omega',\mathcal{S}')$ and given by $\mathcal{P} \equiv \widetilde{\mathbf{P}} \circ X^{-1}$, called the generalized random distribution of X.

Definition 5.5.If the second measurable space $(\Omega',\mathcal{S}')$ is taken to be $(\Xi(\Sigma,\mathbb{R}),\mathfrak{B}_{\Xi(\Sigma,\mathbb{R})})$, where $\mathfrak{B}_{\Xi(\Sigma,\mathbb{R})}$ is the Borel σ-algebra on $\Xi(\Sigma,\mathbb{R})$ the function $X(\omega) : \Omega \to \Omega'$ is called a *generalized random variable*. For this case, the generalized random distribution of X is completely determined by the *random distribution function* defined as

$$F_{X(\omega)}(x(\omega,\omega')) = \widetilde{\mathbf{P}}(X(\omega) \preccurlyeq x(\omega,\omega')) \tag{5.2}$$

where $X \preccurlyeq x$ is defined as the set $\{X(\omega) \preccurlyeq x(\omega,\omega')\} = \{\omega \in \Omega \mid X(\omega) \preccurlyeq x(\omega,\omega')\}$. The distribution function is continuously non-decreasing and gives the generalized random measure associated with $(x(\omega,\omega'),\, x(\omega') + d[x(\omega,\omega')]) \in \mathfrak{B}$ as $d[F_{X(\omega)}(x(\omega,\omega')))]$.

Definition 5.6. A special case is when the distribution function can be written in the form

$$F(x(\omega,\omega')) = \int_{-\infty(\omega,\omega')}^{x(\omega,\omega')} p(x(\omega,\omega'))d[x(\omega,\omega')] \tag{5.3}$$

In this case, the function $p(x(\omega,\omega')) : \widetilde{\Xi}(\Omega,\Sigma,\mathbb{R}) \to \widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$ is called the *random probability density function* of generalized random variable $X(\omega,\omega')$, and the generalized random measure associated with $d[x(\omega,\omega')]$ is $d[F_{X(\omega)}(x(\omega,\omega'))] = p(x(\omega,\omega'))d[x(\omega,\omega')]$.

Definition 5.7. Gaussian random distribution is a type of continuous random probability distribution for a real-valued generalized random variable. The general form of its random probability density function is

$$p(x(\omega,\omega')) = \frac{1}{\sqrt{2\pi(\omega,\omega')\sigma^2}} \exp\left[-\frac{(x(\omega,\omega') - \mu)^2}{2\sigma^2}, \right] \tag{5.3}$$

where a.s.: $\pi(\omega,\omega') = \pi$

With the knowledge of the distribution function, one can define the expectation value of any function $h(X)$ as

$$\mathbf{E}[h(X(\omega,\omega'))] = \int_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')} h(x(\omega,\omega'))d[F_{X(\omega)}(x(\omega,\omega'))]). \tag{5.4}$$

Definition 5.8. A double stochastic process is an indexed set of generalized random variables $X_t(\omega),\ t \in T,$ where T is called the index set and is usually taken (in the continuous case) to be $T = [0,\infty)$. In a physical setting, the index is time and a stochastic process can be interpreted as, for each $t \in T$, $X_t(\omega)$ picks an event $E \in \mathcal{S}$ with random probability $\widetilde{\mathbf{P}}(E)$, and returns $X_t(E)$.

Consider generalized probability space $(\Omega,\mathcal{F},\widetilde{\mathbf{P}})$ and an 1-dimensional time-dependent generalized random variable $X_t(\omega) : \Omega \subset \mathbb{R} \to \Xi(\Sigma,\mathbb{R})$, i.e. a $\Xi(\Sigma,\mathbb{R})$-valued function that maps elements of the sample space to $\Xi(\Sigma,\mathbb{R})$.

Assume now that the random variable is time-dependent: $X(t,\omega)\,:\,\Omega\subset\mathbb{R}\to\Omega'=\Xi(\Sigma,\mathbb{R})$,

then, given a sequence of timesteps $t_1,t_2,\ldots,t_N$ with $t_1<t_2<\ldots<t_N$, we can write

$$\{X(t_1,\omega)=x_1(\omega,\omega'),X(t_2,\omega)=x_2(\omega,\omega'),\ldots,X(t_N,\omega)=x_N(\omega,\omega')\},\qquad(5.5)$$

where $x_1(\omega,\omega'),x_2(\omega,\omega'),\ldots,x_N(\omega,\omega')\in\widetilde{\widetilde{\Xi}}(\Omega,\Sigma,\mathbb{R})$.

The sequence defined in Eq.(5.5) is a *double stochastic process* if the joint random probability density $p_\omega(x_1(\omega,\omega'),t_1,;x_2(\omega,\omega'),t_2,;\ldots x_N,t_N(\omega,\omega'))$, fully describes the system.

Depending on how the joint random probability density is defined, we can classify the double stochastic processes. Here, we consider two cases.

1.Purely double stochastic process or separable stochastic process. If successive values of $X(t)$ are statistically independent, then we the joint random probability density is written as

$$a.s.:\;\; p(x_1(\omega,\omega'),t_1,;x_2(\omega,\omega'),t_2,;\ldots x_N(\omega,\omega'),t_N)=\prod\nolimits_{i=1}^{N}p(x_i(\omega,\omega'),t_i).\qquad(5.6)$$

The underlying idea is that the random probability of an event $x_i(\omega,\omega')$ occurring at a time t_i does not depend on the past and in no way determines the future. In terms of conditional random probabilities, we can write

$$\begin{aligned}a.s.:\; &p(x_N(\omega,\omega'),t_N|x_1(\omega,\omega'),t_1,;x_2(\omega,\omega'),t_2,;\ldots x_{N-1}(\omega,\omega'),t_{N-1})\\ &=p(x_N(\omega,\omega'),t_N).\end{aligned}\qquad(5.7)$$

2.A second example of a stochastic process is the Markov process, whose joint probability density is written as

$$\begin{aligned}&p(x_1(\omega,\omega'),t_1,;x_2(\omega,\omega'),t_2,;\ldots x_N(\omega,\omega'),t_N)=\\ &=\prod\nolimits_{i=2}^{N}p(x_i(\omega,\omega'),t_i|x_{i-1}(\omega,\omega'),t_{i-1})p(x_1(\omega,\omega'),t_1),\end{aligned}\qquad(5.8)$$

or in terms of conditional probabilities as

$$\begin{aligned}&p(x_N(\omega,\omega'),t_N|x_1(\omega,\omega'),t_1,;x_2(\omega,\omega'),t_2,;\ldots x_{N-1}(\omega,\omega'),t_{N-1})=\\ &p(x_N(\omega,\omega'),t_N|x_{N-1}(\omega,\omega'),t_{N-1}),\end{aligned}\qquad(5.9)$$

i.e., double stochastic Markovian process is a process without memory, whose temporal evolution depends only on the present state, not on the past.

Apendix 6.Double stochastic space-time white noise.

Definition 6.1. A distribution valued double stochastic Gaussian process with almost surely mean zero $\{\dot{W}(t,x;\omega,\omega') : t \in [0,T], x \in \mathbb{R}^d\}$ is a *double stochastic space-time white noise* if

$$\mathbb{E}(\dot{W}(t,x;\omega,\omega')\dot{W}(s,y;\omega,\omega')) = \delta(t-s)\delta(x-y). \tag{6.1}$$

More precisely :

Definition 6.2.We denote by $\mathcal{D}((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ the space of the infinitely differentiable functions with compact support in $(0,T)\times\mathbb{R}^d$ and values in $\mathbb{R}^d$.

1.For any $\xi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ the random variable $\dot{W}(\xi)$ is double stochastic Gaussian variable with almost surely mean zero

$$\mathbb{E}(\dot{W}(\xi) = 0 \tag{2.6.10}$$

For any $\xi_1(t,x),\xi_2(t,x) \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ the double stochastic random variables $\dot{W}(\xi_1),\dot{W}(\xi_2)$ almost surely have covariance

- $$\mathbb{E}(\dot{W}(\xi_1)\dot{W}(\xi_2)) = \int_{[0,T]\times\mathbb{R}^d}\xi_1(t,x)\cdot\xi_2(t,x)dxdt. \tag{2.6.11}$$

It is not difficult to construct a space-time white noise. In fact, let $\{f_j : j \in \mathbb{N}\}$ be a complete orthonormal basis of $L^2([0,T]\times\mathbb{R}^d)$ and $\{Z_j(\omega,\omega') : j \in \mathbb{N}\}$ be a family of independent Gaussian generalized random variables with mean zero and variance one. Then

$$\dot{W}(t,x;\omega,\omega') := \sum_{j=1}^{\infty} f_j(t,x)Z_j(\omega,\omega') \tag{2.6.12}$$

is a space-time white noise, where the action is defined as

$$(\dot{W},\xi)_{\mathcal{D}'} = \sum_{j=1}^{\infty}(\xi,f_j)Z_j(\omega,\omega'). \tag{2.6.13}$$

Remark 2.6.1.It is well know that the last action can be extended to $\xi \in L^2([0,T]\times\mathbb{R}^d)$ using Itô isometry.

Let $\rho : \mathbb{R}^d \to [0,\infty)$ be an infinitely differentiable symmetric function with compact support such that $\int_{\mathbb{R}^d}\rho(x)\,dx = 1$. We will consider the mollifiers $\rho_n(x) = n^d\rho(nx)$, with $n \in \mathbb{N}$.

Definition 2.6.3.The regularization by ρ of the space-time white noise $\dot{W}(t,x;\omega,\omega')$, denoted by $\dot{W}_{\rho_n}$, are defined to be

$$\dot{W}_{\rho_n}(t,x;\omega,\omega') := \rho_n * \dot{W}(t,x;\omega,\omega'). \tag{2.6.14}$$

Remark 2.6.2.Note that $\dot{W}_{\rho_n}$ is white in time and colored in space, in fact we have that $\dot{W}_{\rho_n}(t,x)$ is a distribution valued Gaussian process with mean zero and covariance,

$$\mathbb{E}(\dot{W}_{\rho_n}(t,x;\omega,\omega')\dot{W}_{\rho_n}(s,y;\omega,\omega')) = \delta(t-s)h_n(x-y) \tag{2.6.15}$$

where $h_n : \mathbb{R}^d \to \mathbb{R}$ is given by

$$h_n(z) = \int_{\mathbb{R}^d}\rho_n(u)\rho_n(u+z)du. \tag{2.6.16}$$

In terms of the expansion (2.6.13) we have that

$$\dot{W}_{\rho_n}(t,x;\omega,\omega') := \sum_{j=1}^{\infty} \rho_n * f_j(t,x) Z_j(\omega,\omega'). \quad (2.6.17)$$

The mollified cylindrical Wiener process $W_{\rho_n,t}(x)$ associated with the space-time white
noise $\dot{W}(t,x)$ is defined by

$$W_{\rho_n,t}(x;\omega,\omega') := W_t(\rho_n(x-\bullet)). \quad (2.6.18)$$

The distributional time derivative of $W_{\rho_n,t}(x;\omega,\omega')$ is $\dot{W}_{\rho_n}(t,x;\omega,\omega')$. We have that $W_{\rho_n}(x;\omega,\omega')$ is a double stochastic Brownian motion with quadratic variation,

$$\langle W_{\rho_n}(x;\omega,\omega')\rangle_t = \|\rho\|^2 n^d \bullet t. \quad (2.6.19)$$

Proposition 2.6.1.$(W_{\rho_n}(x))$ is a good weak approximation to the cylindrical double stochastic Wiener process W.

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Apendix 6.The Dirac equation.

The Dirac equation in the form originally proposed by Dirac reads

$$\left(\beta mc^2 + c\sum_{n=1}^{3} \alpha_n p_n\right)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t} \quad (6.1)$$

where $\psi(x,t)$ is the wave function for an electron of rest mass m with spacetime coordinates $(x,t), x = (x_1,x_2,x_3)$,

$$p_1 = -i\hbar\frac{\partial}{\partial x_1}, p_2 = -i\hbar\frac{\partial}{\partial x_2}, p_3 = -i\hbar\frac{\partial}{\partial x_3} \quad (6.2)$$

are the components of the momentum, understood to be the momentum operator in the Schrödinger equation. c is the speed of light, and ħ is the reduced Planck constant;
these fundamental physical constants reflect special relativity and quantum mechanics,
respectively.α_n and β are 4×4 gamma matrices.

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Apendix 7.Double stochastic quantization of massive free Abelian gauge field

Free Abelian gauge field in the 4-dimensional Euclidean space is characterized by the
action integral

$$S[A] = \frac{1}{2}\int d^4x[A_\mu(x)(\Box\delta_{\mu\nu} + \delta_{\mu\nu}m^2 + \partial_\mu\partial_\nu)A_\nu(x)] \tag{7.1.1}$$

From (7.1.1) in terms of Fourier transforms $A_\mu(k,t)$ and $\eta(\omega,k,t),\widetilde{\eta}(\varpi,k,t)$ we have the double stochastic Langevin equation

$$\frac{\partial}{\partial t}A^a_\mu(k,t) = -(k^2+m^2)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+m^2}\right)A^a_\mu(k,t) + \eta^a_\mu(\omega,k,t) + \widetilde{\eta}^a_\mu(\varpi,k,t) \tag{7.1.2}$$

where

$$\begin{gathered}\langle\eta^a_\mu(\omega,k,t)\rangle_\eta = \left\langle\widetilde{\eta}^a_\mu(\varpi,k,t)\right\rangle_{\widetilde{\eta}} = 0,\\ \langle\eta^a_\mu(\omega,k,t)\eta^b_\nu(\omega,k,t)\rangle_\eta = 2\delta_{ab}\delta_{\mu\nu}\delta^4(k+k')\delta(t-t'),\\ \langle\eta^a_\mu(\varpi,k,t)\eta^b_\nu(\varpi,k,t)\rangle_{\widetilde{\eta}} = 2\delta_{ab}\delta_{\mu\nu}\delta^4(k+k')\delta(t-t').\end{gathered} \tag{7.1.3}$$

Differential master equation corresponding to double stochastic Langevin equation (7.1.2) reads

$$\frac{\partial}{\partial t}A^a_\mu(k,t) = -(k^2+m^2)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+m^2}\right)A^a_\mu(k,t) + \eta^a_\mu(\omega,k,t). \tag{7.1.4}$$

Solving (7.1.4) under a special initial condition $A_\mu(k,0) = 0$, one obtains

$$A^a_\mu(k,t) = \int\limits_0^\infty G^{ab}_{\mu\nu}(k,t-t')\eta^b_\nu(\omega,k,t')dt', \tag{7.1.5}$$

where $G^{ab}_{\mu\nu}(k,t-t')$ is the Green function given by

$$\begin{gathered}G^{ab}_{\mu\nu}(k,t-t') = \delta_{ab}G_{\mu\nu}(k,t-t'),\\ G_{\mu\nu}(k,t-t') = \left\{\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+m^2}\right)e^{-(k^2+m^2)(t-t')} + \frac{k_\mu k_\nu}{k^2+m^2}\right\}\theta(t-t'),\end{gathered} \tag{7.1.6}$$

where $G_{\mu\nu}(k,t-t')$ is the solution of the equation

$$\begin{gathered}\left[\delta_{\mu\kappa}\frac{\partial}{\partial t} + ((k^2+m^2)\delta_{\mu\kappa} - k_\mu k_\nu)\right]G_{\mu\nu}(k,t-t') = \delta_{\mu\nu}\delta(t-t'),\\ G_{\mu\nu}(k,0_-) = G_{\mu\nu}(k,0_+) = 0.\end{gathered} \tag{7.1.7}$$

Thus correlation functions reads

$$\langle A^a_\mu(k,t)A^b_\nu(k',t')\rangle_\eta = (2\pi)^4\delta_{ab}\delta^4(k+k')D^{(0)}_{\mu\nu}(k,t;k',t'), \tag{7.1.8}$$

where

$$\begin{gathered}D^{(0)}_{\mu\nu}(k,t;k',t') =\\ \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+m^2}\right)e^{-(k^2+m^2)(t+t')}\frac{e^{2(k^2+m^2)\min(t,t')} - 1}{k^2+m^2} + \frac{2k_\mu k_\nu}{k^2+m^2}\min(t,t').\end{gathered} \tag{7.1.9}$$

Thus

$$\langle A_\mu^a(k,t)A_\nu^b(k',t')\rangle_\eta \to_{t=t'\to\infty} (2\pi)^4\delta_{ab}\delta^4(k+k') \times \times\left\{\frac{1}{k^2+m^2}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2+m^2}\right)+2t\frac{k_\mu k_\nu}{k^2+m^2}\right\}. \tag{7.1.10}$$

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