Double Stochastic Quantization:Non perturbative approach to Stochastic Quantization $\lambda\varphi_d^{2n}, n \geq 2, d \geq 4$ model Eucledean Quantum Field Theory

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Abstract:In this paper, we show how the finite formulation of quantum field theory based on Langevin equations can be generalized to the case of *nonrenormalizable* theories.The 5th-time stochastic-quantization approach to field theory proposed by Parisi and Wu, is put in a path-integral form in [6]. The procedure of taking the limit $\tau \to \infty$ is analyzed and based on new grounds through the introduction of the vacuum-vacuum generating functional. In this paper non perturbative aprouch related to Parisi and Wu stochastic-quantization of the $\lambda\varphi_d^{2n}, n \geq 2, d \geq 4$ model quantum field theory is considered.

1.Introduction

Parisi and Wu' proposed the following alternative method to get the quantum averages [1]:

(i) Introduce a 5-th time τ, in addition to the usual four space-time t, and postulate the following Langevin equation for the dynamics of the field $\varphi(\tau,x)$ in this extra time τ

$$\begin{aligned} \frac{\partial\varphi(\tau,x)}{\partial\tau} &= -\frac{\delta S[\varphi]}{\delta\varphi(\tau,x)} + \eta(\tau,x), \\ \langle\eta(\tau,x)\rangle_\eta &= 0, \\ \langle\eta(\tau,x)\eta(\tau',x')\rangle_\eta &= 2\delta(\tau-\tau')(x-x'), \end{aligned} \tag{1.1}$$

where the angular brackets denote connected average with respect to the random variable η.
(ii) Evaluate the stochastic average of fields $\phi_\eta(\tau,x)$ satisfying Eq. (1.1), that means

$$\langle\phi_\eta(\tau_1,x_1)\phi_\eta(\tau_2,x_2)\ldots\phi_\eta(\tau_m,x_m)\rangle_\eta. \tag{1.2}$$

(iii) Put $\tau_1 = \tau_2 = \ldots = \tau_m = \tau$ in (1.2) and take the limit

$$\lim_{\tau\to\infty}\langle\phi_\eta(\tau,x_1)\phi_\eta(\tau,x_2)\ldots\phi_\eta(\tau,x_m)\rangle_\eta = G(x_1,x_2,\ldots,x_m) \tag{1.3}$$

It is possible to prove, at least perturbatively, that [6]

$$G(x_1,x_2,\ldots,x_m) = \frac{\int D[\varphi](\varphi(x_1)\varphi(x_2)\ldots\varphi(x_m))\exp\{-S[\varphi]\}}{\int D[\varphi]\exp\{-S[\varphi]\}}. \tag{1.4}$$

To understand this relation see ref.[6].
In this paper in particular we deal with double stochastic relaxation equations of the form:

$$\begin{aligned} \frac{\partial\varphi(\tau,x)}{\partial\tau} &= -\frac{\delta S[\varphi]}{\delta\varphi(\tau,x)} + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x), \\ \langle\eta(\tau,x)\rangle_\eta &= 0, \langle\widetilde{\eta}(\tau,x)\rangle_{\widetilde{\eta}_1} = 0 \\ \langle\eta(\tau,x)\eta(\tau',x')\rangle_\eta &= 2\delta(\tau-\tau')(x-x'), \\ \langle\widetilde{\eta}(\tau,x)\widetilde{\eta}(\tau',x')\rangle_{\widetilde{\eta}} &= 2\delta(\tau-\tau')(x-x'). \end{aligned} \tag{1.5}$$

Here $\eta(\tau,x) = \eta(\tau,x;\omega)$ is a space-time white noice on probability space $\Sigma = (\Omega,\mathcal{S},P)$ and $\widetilde{\eta}_{1,2}(\tau,x) = \widetilde{\eta}(\tau,x;\varpi)$ are space-time white noices on probability space $\widetilde{\Sigma} = (\widetilde{\Omega},\mathcal{S},P)$.
(ii) Evaluate the stochastic average of fields $\phi_\eta(\tau,x)$ satisfying Eq. (1.5), that means

$$\langle\phi_{\eta,\widetilde{\eta}}(\tau_1,x_1;\epsilon)\phi_{\eta,\widetilde{\eta}}(\tau_2,x_2;\epsilon)\ldots\phi_{\eta,\widetilde{\eta}}(\tau_m,x_m;\epsilon)\rangle_{\eta,\widetilde{\eta}}. \tag{1.6}$$

(iii) Put $\tau_1 = \tau_2 = \ldots = \tau_m = \tau$ in (1.2) and take the limit

$$\lim_{\tau\to\infty}\lim_{\epsilon\to 0}\langle\phi_{\eta,\widetilde{\eta}}(\tau,x_1;\epsilon)\phi_{\eta,\widetilde{\eta}}(\tau,x_2;\epsilon)\ldots\phi_{\eta,\widetilde{\eta}}(\tau,x_m;\epsilon)\rangle_{\eta,\widetilde{\eta}} = G(x_1,x_2,\ldots,x_m;\epsilon) \tag{1.7}$$

To understand this relation we have to introduce the notion of probability (density) $P(\varphi_\eta,\tau)$, that is, the probability (density) of having the system in the configuration $\varphi_\eta(\tau,x)$ at time τ. There exists for $P(\varphi_\eta,\tau)$ an equation that describes its evolution in the time τ. It is called the Stochastic Fokker-Planck (SFP)) equation and it has been derived in [8]:

$$\frac{\partial P[\varphi(\tau,x),\tau]}{\partial\tau} = \int d^4x \frac{\delta}{\delta\varphi(\tau,x)}\left[P[\varphi(\tau,x),\tau]\frac{\delta S^{\bigstar}[\varphi]}{\delta\varphi(\tau,x)}\right] + \int d^4x \frac{\delta^2 S^{\bigstar}[\varphi]}{\delta\varphi^2(\tau,x)},$$
$$S^{\bigstar}[\varphi] = S[\varphi] - \int[\varphi(\tau,x)\eta(\tau,x)]d^4x. \tag{1.8}$$

It is possible to rewrite this equation in a Schrodinger-type form:

$$\frac{\partial\Psi[\varphi(\tau,x),\tau]}{\partial\tau} = -2\mathbf{H}\Psi[\varphi(\tau,x),\tau],$$
$$\Psi = P[\varphi(\tau,x),\tau]\exp[S^{\bigstar}[\varphi(\tau,x)]/2], \tag{1.9}$$

where

$$\mathbf{H} = -\frac{1}{2}\frac{\delta^2}{\delta\varphi^2} + \frac{1}{8}\left[\frac{\delta S^{\bigstar}[\varphi]}{\delta\varphi}\right]^2 - \frac{1}{4}\frac{\delta^2 S^{\bigstar}[\varphi]}{\delta\varphi^2}. \tag{1.10}$$

It is a positive semi-definite operator $H\Psi_n = E_n\Psi_n$ whis a ground state E_0 is $\Psi_0[\varphi(\tau,x),\tau] = \exp[S^{\bigstar}[\varphi(\tau,x)]/2]$. The solution of Eq.(1.9) is

$$\Psi[\varphi(\tau,x),\tau] = \sum_{n=0}^{\infty} c_n\Psi_n[\varphi(\tau,x),\tau]\exp(-2E_n\tau), \tag{1.11}$$

where $\{c_n\}_{n=0}^{\infty}$ are normalizing constants. The probability density $P[\varphi(\tau,x),\tau]$ can be written as

$$P[\varphi(\tau,x),\tau] = \exp[S^{\bigstar}[\varphi(\tau,x)]/2]\sum_{n=0}^{\infty} c_n\Psi_n[\varphi(\tau,x),\tau]\exp(-2E_n\tau). \tag{1.12}$$

In the limit $\tau\to\infty$ the only term that does not disappear in this expression is $\Psi_0[\varphi(\tau,x),\tau]$, so finally we have

$$\lim_{\tau\to\infty} P[\varphi(\tau,x),\tau] = c_0\exp[-S^{\bigstar}[\varphi(\tau,x)]/2]\exp[-S^{\bigstar}[\varphi(\tau,x)]/2] =$$
$$= c_0\exp[-S^{\bigstar}[\varphi(\tau,x)]]. \tag{1.13}$$

This is the formal reason why Eq.(1.4) holds.

2.Non perturbative approach to Stochastic Quantization $\lambda\varphi^{2n}$ model Quantum Field Theory

2.1.The generating functional

In this paper we deal with a system of the double stochastic relaxation equations of the form:

$$\frac{\partial\varphi_{1,\epsilon}(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi_1,\varphi_2]}{\delta\varphi_1(\tau,x)}\bigg|_{\varphi_1=\varphi_{1,\epsilon},\varphi_2=\varphi_{2,\epsilon}} + \eta(\tau,x) + \epsilon\widetilde{\eta}_1(\tau,x),$$
$$\varphi_{1,\epsilon}(0,x) = 0, x\in\mathbb{R}^4, \tau\in\mathbb{R}_+,$$
$$\frac{\partial\varphi_{2,\epsilon}(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi_1,\varphi_2]}{\delta\varphi_2(\tau,x)}\bigg|_{\varphi_1=\varphi_{1,\epsilon},\varphi_2=\varphi_{2,\epsilon}} - \eta(\tau,x) + \epsilon\widetilde{\eta}_2(\tau,x),$$
$$\varphi_{2,\epsilon}(0,x) = 0, x\in\mathbb{R}^4, \tau\in\mathbb{R}_+,$$
$$S[\varphi_1,\varphi_2] = \tag{2.1.1}$$
$$\int\limits_{\mathbb{R}^4} d^4x\Big[\frac{1}{2}(\partial_\mu\varphi_1(\tau,x)\partial_\mu\varphi_1(\tau,x) + m^2\varphi_1(\tau,x)) + P(\varphi_1(\tau,x))\Big] +$$
$$\int\limits_{\mathbb{R}^4} d^4x\Big[\frac{1}{2}(\partial_\mu\varphi_2(\tau,x)\partial_\mu\varphi_2(\tau,x) + m^2\varphi_2(\tau,x)) + P(\varphi_2(\tau,x))\Big] +$$
$$+\int\limits_{\mathbb{R}^4} d^4x[\gamma\times\varphi_1(\tau,x)\varphi_2(\tau,x)], \gamma > 0.$$

where $0 < \epsilon \ll 1,\ P(\bullet)$ is a polinomial degree $2k, k \geq 2$

$$P(\varphi_{1,2}) = \quad (2.1.2)$$

and where $\eta(\tau,x;\omega), \widetilde{\eta}_1(\tau,x;\varpi)$,
$\widetilde{\eta}_2(\tau,x;\varpi)$ are Gaussian random variables such that

$$\langle\eta(\tau,x)\rangle_\eta = 0,$$
$$\langle\widetilde{\eta}_1(\tau,x)\rangle_{\widetilde{\eta}_1} = 0, \langle\widetilde{\eta}_2(\tau,x)\rangle_{\widetilde{\eta}_2} = 0, \tag{2.1.3}$$

and for the two-point correlation function associated with the random noises fields

$$\langle\eta(\tau,x)\eta(\tau',x')\rangle_\eta = 2\delta(\tau-\tau')(x-x'),$$
$$\langle\widetilde{\eta}_1(\tau,x)\widetilde{\eta}_1(\tau',x')\rangle_{\widetilde{\eta}} = 2\delta(\tau-\tau')(x-x'), \tag{2.1.4}$$
$$\langle\widetilde{\eta}_2(\tau,x)\widetilde{\eta}_2(\tau',x')\rangle_{\widetilde{\eta}} = 2\delta(\tau-\tau')(x-x').$$

Remark 2.1.1. Here $\eta(\tau,x) = \eta(\tau,x;\omega)$ is a space-time white noice on probability space $\Sigma = (\Omega,\mathcal{S},P)$ and $\widetilde{\eta}_{1,2}(\tau,x) = \widetilde{\eta}_{1,2}(\tau,x;\varpi)$ are space-time white noices on probability space $\widetilde{\Sigma} = (\widetilde{\Omega},\mathcal{S},P)$. The angular brackets denote connected average with respect to the random variables $\eta, \widetilde{\eta}_{1,2}$.
We want to build a generating functional $Z_\epsilon[J_1,J_2]$ from which the correlations

$$\langle\varphi_{1,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{1,\eta}(\tau_l,x_l;\omega)\varphi_{2,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{2,\eta}(\tau_l,x_r;\omega)\rangle_\eta$$
$$\frac{\partial\varphi_{1,\eta}(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi_1,\varphi_2]}{\delta\varphi_1(\tau,x)}\bigg|_{\varphi_1=\varphi_{1,\eta},\varphi_2=\varphi_{2,\eta}} + \eta(\tau,x),$$
$$\frac{\partial\varphi_{2,\eta}(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi_2,\varphi_2]}{\delta\varphi_2(\tau,x)}\bigg|_{\varphi_1=\varphi_{1,\eta},\varphi_2=\varphi_{2,\eta}} + \eta(\tau,x)$$

can be derived by the following fashion:

$$\langle\varphi_{1,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{1,\eta}(\tau_l,x_l;\omega)\varphi_{2,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{2,\eta}(\tau_l,x_r;\omega)\rangle_\eta =$$
$$\lim_{\epsilon\to 0}\langle\varphi_{1,\eta,\epsilon\widetilde{\eta}_1}(\tau_1,x_1;\omega,\varpi)\times\ldots\times\varphi_{1,\eta,\epsilon\widetilde{\eta}_1}(\tau_l,x_l;\omega,\varpi)\varphi_{2,\eta,\epsilon\widetilde{\eta}_2}(\tau_1,x_1;\omega,\varpi)\times\ldots$$
$$\ldots\times\varphi_{2,\eta,\epsilon\widetilde{\eta}_2}(\tau_l,x_r;\omega,\varpi)\rangle_\eta = \qquad (2.1.5)$$
$$=\lim_{\epsilon\to 0}\left.\frac{\delta^l Z_\epsilon[J_1,J_2;\omega]}{\delta J_1(\tau_1,x_1)\ldots\delta J_1(\tau_l,x_l)\delta J_2(\tau_1,x_1)\ldots\delta J_2(\tau_r,x_r)}\right|_{J_1=0,J_2=0}$$

By canonical definition [6] one obtains

$$Z_\epsilon[J_1,J_2;\omega] =$$
$$N\int D[\eta(\tau',x;\omega)]\Bigg(\Bigg\{\int D[\varphi_1(\tau',x;\omega)]D[\varphi_2(\tau',x;\omega)]D[\widetilde{\eta}_1(\tau',x;\varpi)]D[\widetilde{\eta}_2(\tau',x;\varpi)]$$
$$\delta(\varphi_1(0,x;\omega))\delta(\varphi_2(0,x;\omega))\delta(\varphi_1(\tau',x;\omega)-\varphi_{1,\eta,\epsilon\widetilde{\eta}_1,\epsilon\widetilde{\eta}_2})\delta(\varphi_2(\tau',x;\omega)-\varphi_{2,\eta,\epsilon\widetilde{\eta}_1,\epsilon\widetilde{\eta}_2})\times$$
$$\times\exp\left[-\int_0^\tau\int_{\mathbb{R}^4}J_1(\tau',x)\varphi_1(\tau',x;\omega)d^4xd\tau'\right]\exp\left[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\widetilde{\eta}_1^2(\tau',x;\varpi)d^4xd\tau'\right]\times \qquad (2.1.6)$$
$$\times\exp\left[-\int_0^\tau\int_{\mathbb{R}^4}J_2(\tau',x)\varphi_2(\tau',x;\omega)d^4xd\tau'\right]\exp\left[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\widetilde{\eta}_2^2(\tau',x;\varpi)d^4xd\tau'\right]\Bigg\}\times$$
$$\exp\left[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right]\Bigg),$$

where $\varphi_{1,2,\eta,\widetilde{\eta}} = \varphi_{1,2}(\tau',x;\omega,\varpi)$ that appears in Eq.(2.1.1)-Eq.(2.1.2) is the solution of the double stochastic Langevin equations (2.1.1), solved with zero initial condidion: $\varphi_{1,2,\eta,\widetilde{\eta}}(0,x;\omega,)\equiv 0$ and N is a normalizing constant and

$$D[\varphi_{1,2};\omega] = \lim_{M\to\infty}\prod_{i=0}^{M}D[\varphi_{1,2,\tau_i}(x;\omega)] \qquad (2.1.7)$$

where $\varphi_{1,2,\tau_i}(x;\omega)$, are the field configurations at the time τ_i, having sliced the interval 0 to τ in M infinitesimal parts ε with $\tau_i = i\varepsilon$ and $D[\varphi_{1,2,\tau_i}(x;\omega)]$ is path-integral random measure.

Abbreviation 2.1.1. $D[\varphi_{1,2};\omega] \triangleq D[\varphi_1;\omega]D[\varphi_2;\omega], D\left[\widetilde{\eta}_{1,2}(\tau',x;\omega)\right] \triangleq D[\widetilde{\eta}_1(\tau',x;\omega)]\times D[\widetilde{\eta}_2(\tau',x;\omega)], \delta\left(\varphi_{1,2}-\varphi_{1,2,\eta,\epsilon\widetilde{\eta}_{1,2}}\right) \triangleq \delta(\varphi_1-\varphi_{1,\eta,\epsilon\widetilde{\eta}_1})\delta(\varphi_2-\varphi_{2,\eta,\epsilon\widetilde{\eta}_2})$,
$J_{1,2}(\tau',x)\varphi_{1,2}(\tau',x;\omega) = J_1(\tau',x)\varphi_1(\tau',x;\omega)+J_2(\tau',x)\varphi_2(\tau',x;\omega)$, etc.
The delta function $\delta(\varphi_{1,2}-\varphi_{1,2,\eta,\widetilde{\eta}})$ in Eq.(2.1.6) we can write as

$$\delta(\varphi_{1,2}-\varphi_{1,2,\eta,\widetilde{\eta}}) = \delta\left[\frac{\partial\varphi_{1,2,\epsilon}}{\partial\tau}+\left.\frac{\delta\widetilde{S}}{\delta\varphi_{1,2}}\right|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}-\epsilon\widetilde{\eta}_{1,2}\right]\left\|\frac{\delta(\epsilon\widetilde{\eta})_{1,2}}{\delta\varphi_{1,2,\epsilon}}\right\|, \qquad (2.1.8)$$

where $\widetilde{S} = S-\int_{\mathbb{R}^4}(\varphi_{1,2}\times\eta)d^4x$ and where $\|\delta\widetilde{\eta}_{1,2}/\delta\varphi_{1,2,\epsilon}\|$ is the Jacobian matrix of the transformation $\widetilde{\eta}_{1,2}\to\varphi_{1,2,\epsilon}$, that is

$$\left\| \frac{\delta\left(\epsilon\tilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| = N_\epsilon \left\| \frac{\delta\left(\tilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\|,$$

$$\det\left\| \frac{\delta\left(\tilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| = N_\epsilon^{-1}\det\left[\left[\partial_\tau + \frac{\delta^2\tilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\Bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\delta(\tau-\tau')\right], \qquad (2.1.9)$$

$$N_\epsilon = \prod_{i=1}^{\infty} x_i, x_i = \epsilon, i = 1,2,\ldots$$

From Eq.(2.1.9) by canonical calculation we get

$$\det\left\| \frac{\delta\left(\tilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| =$$

$$= N_\epsilon^{-1}\exp\left[\mathbf{tr}\ln\left[\left[\partial_\tau + \frac{\partial^2\tilde{S}}{\partial\varphi_{1,2}(\tau)\partial\varphi_{1,2}(\tau')}\Bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\delta(\tau-\tau')\right]\right] = \qquad (2.1.10)$$

$$= N_\epsilon^{-1}\exp\left[\mathbf{tr}\ln\partial_\tau\left[\delta(\tau-\tau') + \partial_\tau^{-1}\left(\frac{\partial^2\tilde{S}}{\partial\varphi_{1,2}(\tau)\partial\varphi_{1,2}(\tau')}\Bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right)\right]\right],$$

where ∂_τ^{-1} indicate the Geen's function $G(\tau-\tau')$ that satisfies

$$\partial_\tau G(\tau-\tau') = \delta(\tau-\tau'). \qquad (2.1.11)$$

The solutions of the Eq.(2.1.11) are: (i) if we choose propagation forward in time

$$G(\tau-\tau') = \theta(\tau-\tau') \qquad (2.1.12)$$

(ii) if we choose propagation backward in time

$$G(\tau-\tau') = -\theta(\tau-\tau') \qquad (2.1.13)$$

In case, propagation forward in time, we get

$$\det\left\| \frac{\delta\left(\tilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| =$$

$$N_\epsilon^{-1}\exp\left\{\mathbf{tr}\left[\ln\partial_\tau + \ln\left[\delta(\tau-\tau') + \theta(\tau-\tau')\frac{\delta^2\tilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\Bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\right]\right\} \qquad (2.1.14)$$

$$= N_\epsilon^{-1}\exp(\mathbf{tr}\ln\partial_\tau)\exp\left[\mathbf{tr}\ln\left[\delta(\tau-\tau') + \theta(\tau-\tau')\frac{\delta^2\tilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\Bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\right].$$

The term $\exp(\mathbf{tr}\ln\partial_\tau)$ can be dropped, as it cancels with the same term in the denominator of (2.1.6), once we normalize $\tilde{Z}_\epsilon[J_{1,2};\omega] = Z_\epsilon[J_{1,2};\omega]/Z_\epsilon[0,0;\omega]$.
Abbreviation 2.1.2.$Z_\epsilon[J_{1,2};\omega] \triangleq Z_\epsilon[J_1,J_2;\omega]$
So in Eq.(2.1.14) we are left with

$$\det\left\| \frac{\delta\left(\tilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| =$$

$$N_\epsilon^{-1}\exp\left[\mathbf{tr}\ln\left[\delta(\tau-\tau') + \theta(\tau-\tau')\frac{\delta^2\tilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\Bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\right]. \qquad (2.1.15)$$

By using the canonical expansion for the $\ln$, we obtain

$$
\det\left\| \frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| =
$$

$$
N_{\epsilon}^{-1}\exp\Bigg[\,\mathbf{tr}\Bigg[\,\theta(\tau-\tau')\frac{\delta^{2}\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\Bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} +
$$

$$
\theta(\tau-\tau')\theta(\tau'-\tau)\frac{\delta^{2}\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\frac{\delta^{2}\widetilde{S}}{\delta\varphi_{1,2}(\tau')\delta\varphi_{1,2}(\tau)}\Bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} +\ldots\Bigg]\Bigg] = \qquad (2.1.16)
$$

$$
= N_{\epsilon}^{-1}\exp\Bigg[\int d\tau\int_{\mathbb{R}^4} d^4x\theta(0)\frac{\delta^{2}\widetilde{S}}{\delta\varphi_{1,2}^{2}(\tau)}\Bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} +
$$

$$
\int d\tau'\int_{\mathbb{R}^4} d^4x\theta(\tau-\tau')\theta(\tau'-\tau)\frac{\delta^{2}\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\frac{\delta^{2}\widetilde{S}}{\delta\varphi_{1,2}(\tau')\delta\varphi_{1,2}(\tau)}\Bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} +\ldots\Bigg]
$$

The second term in this expression is zero because $\theta(\tau-\tau')\theta(\tau'-\tau) = 0$ and the

same for all the subsequent terins. The only one left is the first term and choosing $\theta(0) = 1/2$ we get

$$
\det\left\| \frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| = N_{\epsilon}^{-1}\exp\left[\frac{1}{2}\int d\tau'\frac{\delta^{2}\widetilde{S}}{\delta\varphi_{1,2,\epsilon}^{2}}\right]. \qquad (2.1.17)
$$

Inserting Eq.(2.1.17) and Eq.(2.1.8) into Eq.(2.1.6) and performing the $\widetilde{\eta}$ integration, we get

$$
\begin{aligned}
Z_{\epsilon}[J_1,J_2;\omega] &= N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\times\\
&\times\exp\left\{-\int_0^{\tau}\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau}+\frac{\delta\widetilde{S}}{\delta\varphi_1(\tau',x;\omega)}\right]^2-\frac{1}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_1^2(\tau',x;\omega)}\right]d^4xd\tau'\right\}\\
&\times\exp\left\{-\int_0^{\tau}\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau}+\frac{\delta\widetilde{S}}{\delta\varphi_2(\tau',x;\omega)}\right]^2-\frac{1}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_2^2(\tau',x;\omega)}\right]d^4xd\tau'\right\}\\
&\times\exp\left\{-\int_0^{\tau}\int_{\mathbb{R}^4}J_{1,2}(\tau',x)\varphi_{1,2}(\tau',x;\omega)d^4xd\tau'\right\}\exp\left[-\frac{1}{4}\int_0^{\tau}\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right].
\end{aligned} \qquad (2.1.18)
$$

From Eq.(2.1.18) finally we obtain

$$
\begin{aligned}
Z_{\epsilon}[J_{1,2};\omega] &= N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\times\\
&\times\exp\left\{-\int_0^{\tau}\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta\right]^2-\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_1^2}\right]d^4xd\tau'\right.\\
&-\int_0^{\tau}\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta\right]^2-\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_2^2}\right]d^4xd\tau'-\\
&\left.-\int_0^{\tau}\int_{\mathbb{R}^4}J_{1,2}(\tau',x)\varphi_{1,2}(\tau',x;\omega)d^4xd\tau'\right\}\times\exp\left[-\frac{1}{4}\int_0^{\tau}\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right].
\end{aligned} \qquad (2.1.19)
$$

If we want also to specify that we are interested only in the correlations at the same 5-th time τ_1, we have just to choose $J_{1,2}(x,\tau')$ of the form $J_{1,2}(x,\tau') = \bar{J}(x)\delta(\tau'-\tau_1),\ \tau_1 < \tau$ and Eq.(2.1.19) then becomes

$$Z_\epsilon[J_{1,2};\omega] = N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega)) \times$$
$$\times\exp\left\{-\int_0^\tau\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'} + \frac{\delta\widetilde{S}}{\delta\varphi_1(\tau',x;\omega)}\right]^2 - \frac{1}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_1^2(\tau',x;\omega)}\right]d^4xd\tau'\right.$$
$$-\int_0^\tau\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'} + \frac{\delta\widetilde{S}}{\delta\varphi_2(\tau',x;\omega)}\right]^2 - \frac{1}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_2^2(\tau',x;\omega)}\right]d^4xd\tau' - \qquad (2.1.20)$$
$$\left.-\int_{\mathbb{R}^4} J_{1,2}(\tau_1,x)\varphi_{1,2}(\tau_1,x;\omega)d^4x\right\} \times \exp\left[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right].$$

Remark 2.1.2. In all this we have to remember, of course, that once we set $\tau_1 \to \infty$ we have also to extend the interval of integration from $[0,\tau]$ to $[0,\infty]$.
From Eq.(2.1.21) with $\epsilon \ll 1$ for two-point correlation function $\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle$ defined by

$$\langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2)\rangle = \lim_{\epsilon\to 0}\left[\frac{\delta^l Z_\epsilon[J_1,J_2;\omega]}{\delta J_1(\tau_1,x_1)\delta J_2(\tau_2,x_2)}\right] \qquad (2.1.21')$$

for mutually two-point correlation function $\langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2)\rangle_\eta$ we get

$$\langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2)\rangle_\eta \asymp \langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2);\epsilon\rangle \triangleq$$
$$\triangleq N_\epsilon^{-1}\int D[\eta(\tau',x;\omega)]\exp\left[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right]\times$$
$$\left(\int D[\varphi_{1,2}(\tau',x;\omega)](\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega))\times\right.$$
$$\delta(\varphi_{1,2}(0,x;\omega))\times$$
$$\times\exp\left\{-\frac{1}{4\epsilon^2}\left[\int_0^\tau\int_{\mathbb{R}^4}\left[\frac{\partial\varphi(\tau',x;\omega)}{\partial\tau'} + \frac{\delta S}{\delta\varphi_1(\tau',x;\omega)} - \eta(\tau',x;\omega)\right]^2 - \right.\right. \qquad (2.1.22)$$
$$\left.\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_1^2(\tau',x;\omega)}\right]d^4xd\tau'\Bigg\}$$
$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^\tau\int_{\mathbb{R}^4}\left[\frac{\partial\varphi(\tau',x;\omega)}{\partial\tau'} + \frac{\delta S}{\delta\varphi_2(\tau',x;\omega)} - \eta(\tau',x;\omega)\right]^2 - \right.$$
$$\left.\left.\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_2^2(\tau',x;\omega)}\right]d^4xd\tau'\right\}\Bigg).$$

Performing the $\varphi_{1,2}(\tau',x;\omega)$ integration in Eq.(2.1.22) by using saddle point approximation we get

$$
\begin{gathered}
\langle\varphi_{1,\eta}(\tau_1,x_1)\varphi_{2,\eta}(\tau_2,x_2)\rangle_\eta \asymp \\
\det\left\|\frac{\delta(\varphi_1)}{\delta\eta}\right\|\det\left\|\frac{\delta(\varphi_2)}{\delta\eta}\right\|\int D[\eta(\tau',x;\omega)](\varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_2,x_2;\omega))\times \\
\times\exp\left\{-\int_0^\tau\left[-\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_1^2(\tau',x;\omega)}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\eta}}\right]d^4x d\tau'\right\}\times \\
\exp\left\{-\int_0^\tau\left[-\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_2^2(\tau',x;\omega)}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\eta}}\right]d^4x d\tau'\right\} \\
\exp\left[-\frac{1}{4}\int_0^\tau\eta^2(\tau',x;\omega)d^4x d\tau'\right] = \\
\int D[\eta(\tau',x;\omega)](\varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_2,x_2;\omega))\exp\left[-\frac{1}{4}\int_0^\tau\eta^2(\tau',x;\omega)d^4x d\tau'\right]\asymp \\
\asymp\langle\varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_1,x_2;\omega)\rangle_\eta
\end{gathered}
\tag{2.1.23}
$$

$\varphi_{1,2,\eta}(\tau_1,x_1;\omega)$ that appears in Eq.(2.1.23) is the solution of the Langevin equation (2.1.26), solved with zero initial condition. From Eq.(2.1.22)-Eq.(2.1.23) we get

$$
\begin{gathered}
\langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega)\rangle_\eta \simeq \\
N\int D[\varphi_{1,2}(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega)) \\
\left(\int D[\eta(\tau',x;\omega)](\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega))\exp\left[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4x d\tau'\right]\right)\times \\
\times\exp\left\{-\int_0^\tau\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2\right]d^4x d\tau'\right\} \\
\times\exp\left\{-\int_0^\tau\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2\right]d^4x d\tau'\right\}.
\end{gathered}
\tag{2.1.24}
$$

From Eq.(2.1.24) finally we get

$$
\begin{gathered}
\langle\varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_2,x_2;\omega)\rangle_\eta \simeq \langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2);\epsilon\rangle \triangleq \\
\triangleq N_\epsilon^{-1}\int D[\varphi_{1,2}(\tau',x;\omega)]\langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega)\rangle_\eta\times \\
\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_1}\int_{\mathbb{R}^4}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2 d^4x d\tau'\right\} \\
\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_2}\int_{\mathbb{R}^4}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2 d^4x d\tau'\right\}
\end{gathered}
\tag{2.1.25}
$$

where

$$
\begin{gathered}
\langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega)\rangle_\eta \triangleq \\
\int D[\eta(\tau',x;\omega)](\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega))\exp\left[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4x d\tau'\right]
\end{gathered}
$$

Proposition 2.1.1. It follows from Eq.(2.1.22)-Eq.(2.1.23) that in Eq.(2.1.22) we can interchange integration on variable $\eta(\tau',x;\omega)$ and integration on variables $\varphi_{1,2}(\tau',x;\omega)$ in Eq.(1.24)-Eq.(2.1.25)

Remark 2.1.3.Note that for any fixed values of parameters τ_1,x_1,τ_2,x_2 and $\gamma\simeq 0$ we get

$$\langle[\varphi_1(\tau_1,x_1;\omega)-a][\varphi_2(\tau_2,x_2;\omega)+a];\epsilon\rangle_\eta \simeq \langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega);\epsilon\rangle_\eta - a^2, \quad (2.1.26)$$

where by translation invariance

$$\langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega);\epsilon\rangle_\eta - a^2 \simeq 0 \Rightarrow a \simeq a(\tau_1,\tau_2,x_1-x_2). \quad (2.1.27)$$

From Eq.(2.1.25) by the replacement

$$\begin{aligned}
\varphi_1(\tau,x;\omega)-\theta(\tau)a &= \nu_-(\tau,x;\omega),\\
\varphi_2(\tau,x;\omega)+\theta(\tau)a &= \nu_+(\tau,x;\omega),\\
\varphi_1(\tau,x;\omega) &= \nu_-(\tau,x;\omega)+\theta(\tau)a,\\
\varphi_2(\tau,x;\omega) &= \nu_+(\tau,x;\omega)-\theta(\tau)a,\\
\frac{\partial\varphi_1(\tau,x;\omega)}{\partial\tau} &= \frac{\partial\nu_-(\tau,x;\omega)}{\partial\tau}+\delta(\tau)a,\\
\frac{\partial\varphi_2(\tau,x;\omega)}{\partial\tau} &= \frac{\partial\nu_+(\tau,x;\omega)}{\partial\tau}-\delta(\tau)a,
\end{aligned} \quad (2.1.28)$$

we obtain

$$\begin{aligned}
&\Omega(\tau_1,\tau_2,x_1-x_2,a;\epsilon) \triangleq\\
&N_\epsilon^{-1}\int D[\varphi_{1,2}(\tau',x;\omega)]\langle[\varphi_1(\tau_1,x_1;\omega)-a][\varphi_2(\tau_2,x_2;\omega)+a]\rangle_\eta \times\\
&\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_1}\int_{\mathbb{R}^4}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2 d^4x d\tau'\right\}\times\\
&\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_2}\int_{\mathbb{R}^4}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2 d^4x d\tau'\right\} =\\
&= N_\epsilon^{-1}\int D[\nu_-(\tau',x;\omega)]D[\nu_+(\tau',x;\omega)]\langle\nu_-(\tau_1,x_1)\nu_+(\tau_1,x_1)\rangle_\eta \times\\
&\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_1}\int_{\mathbb{R}^4}\left[\frac{\partial\nu_-(\tau',x;\omega)}{\partial\tau'}+\delta(\tau)a+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}\right|_{\varphi_1=\nu_-(\tau',x;\omega)+a_-}\right.\\
&\left.-\eta(\tau',x;\omega)d^4x d\tau']^2\right\}\times\\
&\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_2}\int_{\mathbb{R}^4}\left[\frac{\partial\nu_+(\tau',x;\omega)}{\partial\tau'}-\delta(\tau)a+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}\right|_{\varphi_2=\nu_+(\tau_1,x_1;\omega)-a}\right.\\
&\left.-\eta(\tau',x;\omega)d^4x d\tau'\right\}.
\end{aligned} \quad (2.1.29)$$

Remark 2.1.4. Note that

$$\lim_{\epsilon\to0}\Omega(\tau_1,\tau_2,x_1-x_2,a;\epsilon)=0 \Rightarrow \lim_{\epsilon\to0}\langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2);\epsilon\rangle - a^2 = 0. \quad (2.1.30)$$

Definition 2.1.1. Let $\nu_\mp(\tau,x;\omega)$ be the solution of the Langevin equations (2.1.31)

$$\begin{aligned}
\frac{\partial\nu_-(\tau,x;\omega)}{\partial\tau} &= -\delta(\tau)a-\frac{\delta S[\varphi_{1,2}]}{\delta\varphi_1(\tau,x;\omega)}\Bigg|_{\substack{\varphi_1=\nu_-(\tau',x;\omega)+a_-\\ \varphi_2=\nu_+(\tau_1,x_1;\omega)-a}}+\eta(\tau,x;\omega),\\
\frac{\partial\nu_+(\tau,x;\omega)}{\partial\tau} &= \delta(\tau)a-\frac{\delta S[\varphi_{1,2}]}{\delta\varphi_2(\tau,x;\omega)}\Bigg|_{\substack{\varphi_1=\nu_-(\tau',x;\omega)+a_-\\ \varphi_2=\nu_+(\tau_1,x_1;\omega)-a}}+\eta(\tau,x;\omega),\\
\nu_\mp(0,x;\omega) &= 0,
\end{aligned} \quad (2.1.31)$$

Linear stochastic differental *master equation* corresponding to the Langevin equations

(2.1.31) reads

$$\begin{aligned} \frac{\partial v_-(\tau,x,a;\omega)}{\partial \tau} &= -\delta(\tau)a - \mathcal{L}\left\{ \frac{\delta S[\varphi_{1,2}]}{\delta\varphi_1(\tau,x;\omega)} \Big|_{\varphi_1 = v_-(\tau',x;\omega)+a_-} \right\} + \eta(\tau,x;\omega), \\ \frac{\partial v_+(\tau,x,a;\omega)}{\partial \tau} &= \delta(\tau)a - \mathcal{L}\left\{ \frac{\delta S[\varphi_{1,2}]}{\delta\varphi_2(\tau,x;\omega)} \Big|_{\varphi_2 = v_+(\tau_1,x_1;\omega)-a} \right\} + \eta(\tau,x;\omega), \\ v_\mp(0,x,a;\omega) &= 0, \end{aligned} \tag{2.1.32}$$

where

$$\mathcal{L}\left\{ \delta S[\varphi_{1,2}]/\delta\varphi_1(\tau,x;\omega)\Big|_{\substack{\varphi_1 = v_-(\tau',x;\omega)+a_- \\ \varphi_2 = v_+(\tau_1,x_1;\omega)-a}} \right\} \tag{2.1.33}$$

is a linear part of variational derivative

$$\delta S[\varphi_{1,2}]/\delta\varphi_1(\tau,x;\omega)\Big|_{\substack{\varphi_1 = v_-(\tau',x;\omega)+a_- \\ \varphi_2 = v_+(\tau_1,x_1;\omega)-a}} \tag{2.1.34}$$

and where

$$\mathcal{L}\left\{ \delta S[\varphi_{1,2}]/\delta\varphi_2(\tau,x;\omega)\Big|_{\substack{\varphi_1 = v_-(\tau',x;\omega)+a_- \\ \varphi_2 = v_+(\tau_1,x_1;\omega)-a}} \right\}. \tag{2.1.35}$$

is a linear part of variational derivative

$$\delta S[\varphi_{1,2}]/\delta\varphi_2(\tau,x;\omega)\Big|_{\substack{\varphi_1 = v_-(\tau',x;\omega)+a_- \\ \varphi_2 = v_+(\tau_1,x_1;\omega)-a}} \tag{2.1.36}$$

2.2.Transcendental master equation corresponding to two-point Green function $G(x_1,x_2,\lambda)$.

Definition 2.2.1.Let $v_\mp(\tau,x,a;\omega)$ be the solution of the *stochastic differental master equations* (2.1.32). Transcendental *master equation* corresponding to the stochastic Langevin equation (2.1.31) reads

$$\langle v_-(\tau_1,x_1,a;\omega)v_+(\tau_2,x_2,a;\omega)\rangle_\eta = 0. \tag{2.2.1}$$

Theorem 2.2.1.Let $a_{1,2}(\bar{\tau},\bar{x})$ be an solution of the equation (2.2.1) at fixed point $(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2) \in (\mathbb{R}_+ \times \mathbb{R}^4) \times (\mathbb{R}_+ \times \mathbb{R}^4)$ i.e.,

$$\langle v_-(\bar{\tau}_1,\bar{x}_1,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\omega)v_+(\bar{\tau}_2,\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\omega)\rangle_\eta = 0. \tag{2.2.2}$$

Let $\Delta(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2)$ be a set such that

$$\begin{aligned} a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2) \in \Delta(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2) \Leftrightarrow \\ \Leftrightarrow \langle v_-(\bar{\tau}_1,\bar{x}_1,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\omega)v_+(\bar{\tau}_2,\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\omega)\rangle_\eta = 0, \end{aligned} \tag{2.2.3}$$

and let $\widetilde{\Omega}(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2))$ be a set such that

$$\begin{aligned} a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2) \in \widetilde{\Omega}(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2)) \Leftrightarrow \\ \Leftrightarrow \underline{\lim}_{\epsilon\to 0}\Omega(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\epsilon) = 0, \end{aligned} \tag{2.2.4}$$

where the quantity $\Omega(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\epsilon)$ defined by Eq.(2.1.29). Then

$$\widetilde{\Omega}(\overline{\tau}_1,\overline{\tau}_2,\overline{x}_1-\overline{x}_2,a(\overline{\tau}_1,\overline{x}_1;\overline{\tau}_2,\overline{x}_2))\subseteq\Delta(\overline{\tau}_1,\overline{x}_1;\overline{\tau}_2,\overline{x}_2). \quad (2.2.5)$$

2.3. Double Stochastic Quantization the Free Scalar Fields

For a scalar field theory governed by the action in terms of the Euclidean spacetime is given by

$$S_E=\int d^4x\left[\frac{1}{2}(\partial\varphi(x))^2+\frac{1}{2}(m\varphi(x))^2\right] \quad (2.3.1)$$

differental *master equations* corresponding to the Langevin equations (2.1.31) reads

$$\begin{aligned}\frac{\partial v_-(x,\tau;\omega,\varpi)}{\partial\tau} &= (\partial^2-m^2)v_-(x,\tau;\omega,\varpi)+\eta(x,\tau;\omega),\\ \frac{\partial v_+(x,\tau;\omega,\varpi)}{\partial\tau} &= (\partial^2-m^2)v_+(x,\tau;\omega,\varpi)+\eta(x,\tau;\omega),\\ v_\mp(x,0;\omega,\varpi) &= 0.\end{aligned} \quad (2.3.2)$$

Fourier transformed stochastic differential equations (2.3.2) in k and τ given as

$$\begin{aligned}&\frac{\partial}{\partial\tau}\widehat{v}_-(k,\tau) =\\ &-(k^2+m^2)\widehat{v}_-(k,\tau)-(2\pi)^4a\delta(\tau)\delta^4(k)-(2\pi)^4m^2a\delta^4(k)+\widehat{\eta}(k,\tau;\omega),\\ &\frac{\partial}{\partial\tau}\widehat{v}_+(k,\tau) =\\ &-(k^2+m^2)\widehat{v}_+(k,\tau)-(2\pi)^4a\delta(\tau)\delta^4(k)+(2\pi)^4m^2a\delta^4(k)+\widehat{\eta}(k,\tau;\omega).\end{aligned} \quad (2.3.3)$$

Let us consider ODE

$$\dot{x}\,(\tau,\lambda)+\lambda x(\tau,\lambda) = g(\tau,\lambda), x(0) = 0. \quad (2.3.4)$$

The corresponding solution $x(\tau,\lambda)$ is

$$x(\tau,\lambda) = e^{-\lambda\tau}\int_0^\tau e^{\lambda\tau_1}g(\tau_1,\lambda)d\tau_1. \quad (2.3.5)$$

From Eq.(2.3.3)-Eq.(2.3.5) one obtains

$$
\begin{aligned}
&\widehat{v}_{-}(k,\tau,a) = \\
&e^{-(k^2+m^2)\tau}\int_0^{\tau} e^{(k^2+m^2)\tau_1}\Big[-(2\pi)^4 a\delta(\tau_1)\delta^4(k) - (2\pi)^4 m^2 a\delta^4(k) + \widehat{\eta}(k,\tau_1;\omega)\Big]d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} - (2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\int_0^{\tau} e^{(k^2+m^2)\tau_1}d\tau_1 + \\
&+e^{-(k^2+m^2)\tau}\int_0^{\tau} e^{(k^2+m^2)\tau_1}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} - (2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\int_0^{\tau} e^{(k^2+m^2)\tau_1}d\tau_1 + \\
&+\int_0^{\tau} e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} - \\
&(2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\left[\frac{e^{(k^2+m^2)\tau}}{k^2+m^2} - \frac{1}{k^2+m^2}\right] + \\
&+\int_0^{\tau} e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} - (2\pi)^4\frac{m^2 a\delta^4(k)}{k^2+m^2}\left[1 - e^{-(k^2+m^2)\tau}\right] + \\
&+\int_0^{\tau} e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m^2)\tau} + \frac{m^2}{k^2+m^2}\left[1 - e^{-(k^2+m^2)\tau}\right]\right] + \\
&+\int_0^{\tau} e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1.
\end{aligned}
\qquad (2.3.6)
$$

and

$$
\begin{aligned}
&\widehat{v}_{+}(k,\tau,a) = \\
&e^{-(k^2+m^2)\tau}\int_0^{\tau} e^{(k^2+m^2)\tau_1}\left[-(2\pi)^4 a\delta(\tau_1)\delta^4(k) + (2\pi)^4 m^2 a\delta^4(k) + \widehat{\eta}(k,\tau_1;\omega)\right]d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} - (2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\int_0^{\tau} e^{(k^2+m^2)\tau_1}d\tau_1 + \\
&+e^{-(k^2+m^2)\tau}\int_0^{\tau} e^{(k^2+m^2)\tau_1}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} + (2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\int_0^{\tau} e^{(k^2+m^2)\tau_1}d\tau_1 + \\
&+\int_0^{\tau} e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} - \\
&(2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\left[\frac{e^{(k^2+m^2)\tau}}{k^2+m^2} - \frac{1}{k^2+m^2}\right] + \\
&+\int_0^{\tau} e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} - (2\pi)^4\frac{m^2 a\delta^4(k)}{k^2+m^2}\left[1-e^{-(k^2+m^2)\tau}\right] + \\
&+\int_0^{\tau} e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m^2)\tau} + \frac{m^2}{k^2+m^2}\left[1-e^{-(k^2+m^2)\tau}\right]\right] + \\
&+\int_0^{\tau} e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1.
\end{aligned}
\tag{2.3.7}
$$

From Eq.(2.3.6)-Eq.(2.3.7) one obtains

$$
\begin{aligned}
&\widehat{v}_{-}(k_1,\tau,a)\widehat{v}_{+}(k_2,\tau',a') = \\
&\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right] + \right.\\
&\left.+\int_0^{\tau} e^{-(k_1^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right\} \times \\
&\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m^2)\tau'} + \frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_2^2+m^2)\tau'}\right]\right] + \right.\\
&\left.+\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_1)}\widehat{\eta}(k_2,\tau_1;\omega)d\tau_1\right\}
\end{aligned}
\tag{2.3.8}
$$

From Eq.(3.2.11) one obtains

$$
\begin{aligned}
&\langle \hat{v}_-(k_1,\tau,a)\hat{v}_+(k_2,\tau',a')\rangle_\eta = \\
&\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right]\right\} \times \\
&\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m^2)\tau'} + \frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_2^2+m^2)\tau'}\right]\right] + \\
&+\left\langle \left(\int_0^\tau e^{-(k_1^2+m^2)(\tau-\tau_1)}\hat{\eta}(k_1,\tau_1;\omega)d\tau_1\right)\left(\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_2)}\hat{\eta}(k_2,\tau_2;\omega)d\tau_2\right)\right\rangle_\eta = \\
&-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right] \times \\
&\left[e^{-(k_2^2+m^2)\tau'} + \frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_2^2+m^2)\tau'}\right]\right] + \\
&+\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_2)}\int_0^\tau e^{-(k_1^2+m^2)(\tau-\tau_1)}\langle\hat{\eta}(k_1,\tau_1;\omega)\hat{\eta}(k_2,\tau_2;\omega)\rangle_\eta d\tau_1 d\tau_2
\end{aligned} \tag{2.3.9}
$$

We set now $\tau' = \tau, a' = a$. Note that

$$
\begin{aligned}
&-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right] \times \\
&\left[e^{-(k_2^2+m^2)\tau'} + \frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau'}\right]\right]\Bigg|_{\tau'=\tau} = \\
&-(2\pi)^8 a^2\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right] \times \\
&\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right]
\end{aligned} \tag{2.3.10}
$$

and

$$
\begin{aligned}
&\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_2)}\int_0^\tau e^{-(k_1^2+m^2)(\tau-\tau_1)}\langle\hat{\eta}(k_1,\tau_1;\omega)\hat{\eta}(k_2,\tau_2;\omega)\rangle_\eta d\tau_1 d\tau_2\Bigg|_{\tau'=\tau} = \\
&\delta(k_1+k_2)\int_0^\tau e^{-(k_2^2+m^2)(\tau-\tau_2)}\int_0^\tau e^{-(k_1^2+m^2)(\tau-\tau_1)}\delta(\tau_1-\tau_2)d\tau_1 d\tau_2 = \\
&2(2\pi)^4\delta(k_1+k_2)\int_0^\tau e^{-(k_1^2+k_2^2+2m^2)(\tau-\tau_1)}d\tau_1 = \\
&2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m^2)\tau}\int_0^\tau e^{(k_1^2+k_2^2+2m^2)\tau_1}d\tau_1 = \\
&2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m^2)\tau}\left[\frac{1}{(k_1^2+k_2^2+2m^2)}e^{(k_1^2+k_2^2+m^2)\tau} - \frac{1}{(k_1^2+k_2^2+2m^2)}\right] = \\
&= \frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m^2} - 2(2\pi)^4\delta(k_1+k_2)\frac{e^{-(k_1^2+k_2^2+2m^2)\tau}}{(k_1^2+k_2^2+2m^2)}
\end{aligned} \tag{2.3.11}
$$

From Eq.(2.3.9)-Eq.(2.3.11) we get

$$\lim_{\tau\to\infty}\langle\hat{v}(k_1,\tau,a)\hat{v}(k_2,\tau,a)\rangle_\eta = (2\pi)^8 a^2\delta^4(k_1)\delta^4(k_2)\frac{m^4}{(k_1^2+m^2)(k_2^2+m^2)} - -\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2 \dotplus k_2^2+2m^2}. \tag{2.3.12}$$

Therefore

$$\begin{aligned}&\lim_{\tau\to\infty}\langle\hat{v}(x_1,\tau,a)\hat{v}(x_2,\tau,a)\rangle_\eta = \\ &\times a^2(2\pi)^{-8}\times\int d^4k_1 e^{ik_1x_1}\int d^4k_2 e^{ik_2x_2}\delta^4(k_1)\delta^4(k_2)\frac{m^4}{(k_1^2+m^2)\times(k_2^2+m^2)} - \\ &-(2\pi)^{-8}\times\int d^4k_1 e^{ik_1x_1}\int d^4k_2 e^{ik_2x_2}\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2\dotplus k_2^2+2m^2} = \\ &= \left[\frac{a^2m^4}{m^2\times m^2}+(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2}\right] = a^2-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2}\end{aligned} \tag{2.3.13}$$

Two point function $G(x_1,x_2)$ of euclidean QFT corresponding to the action (2.3.1) is

$$G(x_1,x_2) = \lim_{\tau\to\infty}\langle\varphi(x_1,\tau;\omega)\varphi(x_2,\tau;\omega)\rangle_\eta \tag{2.3.14}$$

Master equation corresponding to two-point function $G(x_1,x_2)$ reads

$$\lim_{\tau\to\infty}\langle\varphi(x_1,\tau;\omega)\varphi(x_2,\tau;\omega)\rangle - a^2 = \lim_{\tau\to\infty}\langle\hat{v}(k_1,\tau,a)\hat{v}(k_2,\tau,a)\rangle_\eta = 0. \tag{2.3.15}$$

From (2.3.15) we get

$$\lim_{\tau\to\infty}\langle\varphi(x_1,\tau;\omega)\varphi(x_2,\tau;\omega)\rangle_\eta = a^2. \tag{2.3.16}$$

From Eq.(2.3.13)-Eq.(2.3.16) we get

$$\lim_{\tau\to\infty}\langle\hat{v}(k_1,\tau,a)\hat{v}(k_2,\tau,a')\rangle_\eta = 0 \Leftrightarrow a^2-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2} = 0 \tag{2.3.17}$$

and therefore

$$a^2 = (2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2}. \tag{2.3.18}$$

From Eq.(2.3.16) and Eq.(2.3.21) finally we get desired result

$$\begin{aligned}&G_F(x_1,x_2) = \lim_{\tau\to\infty}\langle\varphi(x_1,\tau;\omega)\varphi(x_2,\tau;\omega)\rangle_\eta = \\ &(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2} = (2\pi)^{-2}\left(\frac{m}{|x|}\right)K_1(m|x_1-x_2|),\end{aligned} \tag{2.3.19}$$

where K_1 is the modified Bessel functions of the second kind, integer order 1, and where we used formula 6.566.2 of [12].

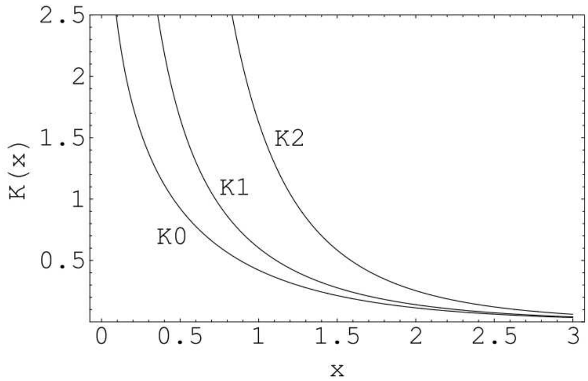

Figure 2.3.1.Plot of the modified Bessel functions of the second kind,integer order 1.

2.4.Double stochastic quantization the $\lambda\phi_d^4$ theory.

In this section we consider a neutral scalar field with a $\frac{\lambda}{4!}\phi_d^4$, $d \geq 4$, self-interaction, defined in a d-dimensional Minkowski spacetime. The vacuum persistence functional is the generating functional of all vacuum expectation value of time-ordered products of the theory. The Euclidean field theory can be obtained by analytic continuation to imaginary time supported by the positive energy condition for the relativistic field theory. In the Euclidean field theory, we have the Euclidean counterpart for the vacuum persistence functional, that is, the generating functional of complete Schwinger functions. Actually, the $(\lambda\varphi^4)_d$ Euclidean theory is defined by these Euclidean Green's functions. The Euclidean generating functional $Z[h]$ is formally defined by the following functional integral:

$$Z[h] = \int[d\varphi]\ \exp\Big(-S_0 - S_I + \int d^d x\, h(x)\varphi(x)\Big), \tag{2.4.1}$$

where the action that usually describes a free scalar field is

$$S_0[\varphi] = \int d^d x\left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m_0^2\varphi^2(x)\right), \tag{2.4.2}$$

and the interacting part, defined by the non-Gaussian contribution, is

$$S_I[\varphi] = \int d^d x\, \frac{\lambda}{4!}\,\varphi^4(x). \tag{2.4.3}$$

In Eq.(2.4.1), $[d\varphi]$ is a translational invariant measure, formally given by $[d\varphi] = \prod_{x\in\mathbb{R}^d} d\varphi(x)$. The terms λ and m_0^2 are respectively the bare coupling constant and the squared mass of the model. Finally, $h(x)$ is a smooth function that we introduce to generate the Schwinger functions of the theory by functional derivatives. In the weak-coupling perturbative expansion, which is the conventional procedure, we perform a formal perturbative expansion with respect to the non-Gaussian terms of the action. As a consequence of this formal expansion, all the n-point unrenormalized Schwinger functions are expressed in a powers series of the bare coupling constant λ.

The aim of this section is to discuss the double stochastic quantization of a free scalar field. It can be shown that it is equivalent to the usual path integral quantization. The starting point of the stochastic quantization to obtain the Euclidean field theory is a Markovian Langevin equation. Assume an Euclidean d-dimensional manifold, where we are choosing periodic boundary conditions for a scalar field and also a random noise. In other words, they are defined in a d-torus $\Omega \equiv T^d$. To implement the stochastic quantization we supplement the scalar field $\varphi(x)$ and the random noises $\eta(x)$ and $\hat{\eta}(\tau,x)$ with an extra coordinate τ, the Markov parameter, such that $\varphi(x) \to \varphi(\tau,x)$ and $\eta(x) \to \eta(\tau,x)$.
Therefore, the fields and the random noises $\eta(\tau,x)$ and $\widetilde{\eta}(\tau,x)$ are defined in a domain: $T^d \times R^{(+)}$. Let us consider that this dynamical system is out of equilibrium, being described by the following equation of evolution:

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -\frac{\delta S_0[\varphi]}{\delta\varphi(x)}\bigg|_{\varphi(x)=\varphi(\tau,x)} + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x), \qquad (2.4.4)$$

where τ is a Markov parameter, $\eta(\tau,x)$ is a random noise field and S_0 is the usual free action defined in Eq.(2.4.2). For a free scalar field, the double stochastic Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -(-\Delta + m_0^2)\varphi(\tau,x) + \eta(\tau,x) + \epsilon\hat{\eta}(\tau,x), \qquad (2.4.5)$$

where Δ is the d-dimensional Laplace operator. The Eq.(2.4.5) describes a Ornstein-Uhlenbeck process and we are assuming the Einstein relations, that is:

$$\begin{aligned} \langle\eta(\tau,x)\rangle_\eta &= 0, \\ \langle\widetilde{\eta}(\tau,x)\rangle_\eta &= 0, \end{aligned} \qquad (2.4.6)$$

and for the two-point correlation function associated with the random noise fields

$$\begin{aligned} \langle\eta(\tau,x)\,\eta(\tau',x')\rangle_\eta &= 2\delta(\tau-\tau')(x-x'), \\ \langle\widetilde{\eta}(\tau,x)\,\widetilde{\eta}(\tau',x')\rangle_{\hat{\eta}} &= 2\delta(\tau-\tau')(x-x'), \end{aligned} \qquad (2.4.7)$$

where $\langle\ldots\rangle_\eta$ means stochastic averages. In a generic way, the stochastic average for any functional of φ given by $F[\varphi]$ is defined by

$$\langle F[\varphi]\rangle_{\eta,\hat{\eta}} = \frac{\int D[\eta]D[\hat{\eta}]F[\varphi]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\eta^2(\tau,x,\omega)\Big]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\widetilde{\eta}^2(\tau,x,\widetilde{\omega})\Big]}{\Big(\int D[\eta]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\eta^2(\tau,x,\omega)\Big]\Big)\Big(\int D[\widetilde{\eta}]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\widetilde{\eta}^2(\tau,x,\widetilde{\omega})\Big]\Big)}. \qquad (2.4.8)$$

Let us define the retarded Green function for the diffusion problem that we call $G(\tau-\tau',x-x')$. The retarded Green function satisfies $G(\tau-\tau',x-x') = 0$ if $\tau-\tau' < 0$ and also

$$\Big[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2)\Big]G(\tau-\tau',x-x') = \delta^d(x-x')\delta^d(\tau-\tau'). \qquad (2.4.9)$$

Using the retarded Green function and the initial condition $\varphi(\tau,x)|_{\tau=0} = 0$, the solution for Eq.(3.5.5) reads

$$\varphi(\tau,x) = \int_0^\tau d\tau'\int_\Omega d^dx'\,G(\tau-\tau',x-x')[\eta(\tau',x') + \epsilon\hat{\eta}(\tau',x)]. \qquad (2.4.10)$$

In the following we are interested in calculating the quantity $\langle\varphi(\tau,x)\varphi(\tau',x')\rangle_{\eta,\hat{\eta}}$. Using

Eq.(2.4.6), Eq.(2.4.7) and Eq.(2.4.10), we have

$$\langle \varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle_{\eta,\hat{\eta}} = 2\int_0^{\min(\tau_1,\tau_2)} d\tau' \int_\Omega d^dx' G(\tau_1-\tau',x_1-x')\,G(\tau_2-\tau',x_2-x'), \tag{2.4.11}$$

where $\min(\tau_1,\tau_2)$ means the minimum of τ_1 and τ_2. Using a Fourier representation, the two-point correlation function $\langle \varphi(\tau,x)\varphi(\tau',x')\rangle_\eta \equiv D(\tau,x;\tau',x')$ is given by

$$D(\tau,x;\tau',x') = \frac{1}{(2\pi)^d}\int d^dp\,\frac{e^{-ip(x-x')}}{(p^2+m_0^2)}\,e^{-(p^2+m_0^2)(\tau-\tau')}. \tag{2.4.12}$$

It is not difficult to show that Eq.(2.4.12) can be written as:

$$D(\tau,x;\tau',x') = \frac{1}{(2\pi)^{d}}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(1-n)n!}(\tau-\tau')^n\left(\frac{m_0}{r}\right)^{\frac{d}{2}+n-1}K_{\frac{d}{2}+n-1}(m_0 r). \tag{2.4.13}$$

where $r = |\ x-x'\ |$ and K_ν is the modified Bessel function of order ν.
We can use the Fourier analysis to show that when the Markov parameters τ and τ' go to infinity we recover the standard Euclidean free field theory. Therefore let us define the Fourier transforms for the field and the noises given by $\varphi(\tau,k)$ and $\eta(\tau,k)$. We have respectively

$$\varphi(\tau,k) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int d^dx\,e^{-ikx}\,\varphi(\tau,x), \tag{2.4.14}$$

and

$$\begin{aligned}\hat{\eta}(\tau,k) &= \frac{1}{(2\pi)^{d/2}}\int e^{-ikx}\,\eta(\tau,x),\\ \hat{\hat{\eta}}(\tau,k) &= \frac{1}{(2\pi)^{d/2}}\int e^{-ikx}\,\hat{\hat{\eta}}(\tau,x)\end{aligned} \tag{2.4.15}$$

Substituting Eq.(2.4.14) in Eq.(2.4.2), the free action for the scalar field in the $(d+1)$-dimensional space writing in terms of the Fourier coefficients reads

$$S_0[\varphi(k)]|_{\varphi(k)=\varphi(\tau,k)} = \frac{1}{2}\int d^dk\,\varphi(\tau,k)(k^2+m_0^2)\varphi(\tau,k). \tag{2.4.16}$$

Substituting Eq.(2.4.14) and Eq.(2.4.15) in Eq.(2.4.5) we have that each Fourier coefficient satisfies a Langevin equation given by

$$\frac{\partial}{\partial\tau}\varphi(\tau,k) = -(k^2+m_0^2)\varphi(\tau,k)+\eta(\tau,k)+\epsilon\hat{\eta}(\tau,k). \tag{2.4.17}$$

The solution for this equation reads

$$\varphi(\tau,k) = \exp(-(k^2+m_0^2)\tau)\varphi(0,k)+\int_0^\tau d\tau'\exp(-(k^2+m_0^2)(\tau-\tau'))[\eta(\tau',k)+\epsilon\hat{\eta}(\tau',k)]. \tag{2.4.18}$$

Using the Einstein relation, we get that the Fourier coefficients for the random noise satisfies

$$\begin{aligned}\langle \eta(\tau,k)\rangle_\eta &= 0,\\ \langle \hat{\eta}(\tau,k)\rangle_\eta &= 0\end{aligned} \tag{2.4.19}$$

and

$$\begin{aligned} \langle \eta(\tau,k)\eta(\tau',k')\rangle_\eta &= 2(2\pi)^d\delta^d(\tau-\tau')(k+k'), \\ \langle \widehat{\widetilde{\eta}}(\tau,k)\widehat{\widetilde{\eta}}(\tau',k')\rangle_\eta &= 2(2\pi)^d\delta^d(\tau-\tau')(k+k') \end{aligned} \quad (2.4.20)$$

Before investigate the interacting field theory, let us calculate the Fourier representation for the two-point correlation function, i.e., $\langle \varphi(\tau,k)\varphi(\tau',k')\rangle_\eta$. Using Eq.(2.4.18), we obtain three contributions to the scalar two-point correlation function. The first one is given by

$$\exp(-(k^2+m_0^2)\,\tau + (k'^2+m_0^2)\,\tau')\varphi(0,k)\varphi(0,k'), \quad (2.4.21)$$

and decay to zero at long time. Let us assume that $\varphi(\tau,k)|_{\tau=0} = 0$. There are also two crossed terms, each first order in the noise Fourier component given by

$$2\,\varphi(0,k)\exp(-(k^2+m_0^2)\tau)\int_0^{\tau'} ds\,\exp(-(k'^2+m_0^2)(\tau'-s))\Big[\eta(s,k')+\epsilon\widehat{\widetilde{\eta}}(\sigma,k')\Big]. \quad (2.4.22)$$

Since we are assuming the Einstein relations, i.e., $\langle \eta(\tau,x)\rangle_\eta = 0, \langle \widetilde{\eta}(\tau,x)\rangle_\eta = 0$ on averaging on noise, these cross terms vanish. The final term is second-order in the noise Fourier component. Again, the solution subject to the initial condition $\varphi(\tau,k)|_{\tau=0} = 0$ can be used to give

$$\begin{aligned} &\left\{\int_0^\tau ds\,\exp(-(k^2+m_0^2)(\tau-s))\Big[\eta(s,k)+\epsilon\widehat{\widetilde{\eta}}(s,k)\Big]\right\}\times \\ &\left\{\int_0^{\tau'} d\sigma\,\exp(-(k'^2+m_0^2)(\tau'-\sigma))\Big[\eta(\sigma,k')+\epsilon\widehat{\widetilde{\eta}}(\sigma,k')\Big]\right\}. \end{aligned} \quad . \quad (2.4.23)$$

Again averaging on noises and using the Einstein relation given by Eq.(2.4.20) we have

that this term becomes

$$2\delta^d(k+k_0')\int_0^{\min(\tau,\tau')} ds\,\exp(-(k^2+m_0)(\tau+\tau'-2s)). \quad (2.4.24)$$

Assuming that $\tau=\tau'$ and using $\langle\varphi(\tau,k)\varphi(\tau',k')\rangle_\eta|_{\tau=\tau'} \equiv D(k,k';\tau,\tau')$ we have

$$D(k;\tau,\tau) = (2\pi)^d\delta^d(k+k')\frac{1}{(k^2+m_0^2)}(1-\exp(-2\tau(k^2+m_0^2))). \quad (2.4.25)$$

In the following, we are redefining the two-point correlation function as $D(k;\tau,\tau)\to(2\pi)^dD(k;\tau,\tau)$. In the limit when $\tau\to\infty$ we recover the standard two-point function of the Euclidean free field theory. Before going to the next section, we would like to mention the existence of more general Markovian Langevin equations. We can introduce a kernel defined in the d-torus. The kerneled Langevin equation reads:

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -\int d^dy\,K(x,y)\frac{\delta S_0}{\delta\varphi(y)}|_{\varphi(y)=\varphi(\tau,y)} + \eta(\tau,x)+\epsilon\widehat{\eta}(\tau,x). \quad (2.4.26)$$

The second moment of the noise fields will be modified to:

$$\begin{aligned} \langle \eta(\tau,x)\,\eta(\tau',x')\rangle_\eta &= 2\delta(\tau-\tau')\,K(x,x'), \\ \langle \widetilde{\eta}(\tau,x)\,\widetilde{\eta}(\tau',x')\rangle_\eta &= 2\delta(\tau-\tau')\,K(x,x'). \end{aligned} \quad (2.4.27)$$

Choosing an appropriate kernel, it can be shown that all the above conclusions remain unchanged.

The double stochastic Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi_\epsilon(\tau,x) = (\Delta - m_0^2)\varphi_\epsilon(\tau,x) - \frac{\lambda}{3!}\varphi_\epsilon^3(\tau,x) + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x). \quad (2.4.28)$$

By the replacements

$$\begin{aligned} \varphi_\epsilon(x,\tau;\omega,\varpi) &= v_{\epsilon-}(x,\tau;\omega,\varpi) + \theta(\tau)a, \\ \varphi_\epsilon(x,\tau;\omega,\varpi) &= v_{\epsilon+}(x,\tau;\omega,\varpi) - \theta(\tau)a, \\ \theta(\tau) &= \begin{cases} 0 & \text{if } \tau \leq 1 \\ 1 & \text{if } \tau > 1 \end{cases} \end{aligned} \quad (2.4.29)$$

we obtain from Eqs.(2.4.28)

$$\begin{gathered} \frac{\partial[v_{\epsilon-}(x,\tau;\omega,\varpi) + \theta(\tau)a]}{\partial\tau} = \frac{\partial v_{\epsilon-}(x,\tau;\omega,\varpi)}{\partial\tau} + a\delta(\tau) = \\ = (\partial^2 - m^2)[v_{\epsilon-}(x,\tau;\omega,\varpi) + \theta(\tau)a] - \frac{\lambda}{3!}[v_{\epsilon-}(x,\tau;\omega,\varpi) + a]^3 + \\ + \eta(x,\tau;\omega) + \epsilon\widetilde{\eta}(x,\tau;\varpi) = \\ (\partial^2 - m^2)[v_{\epsilon-}(x,\tau;\omega,\varpi) + \theta(\tau)a] - \frac{\lambda}{3!}(v_{\epsilon-}^3 + 3av_{\epsilon-}^2 + 3a^2 v_{\epsilon-} + a^3) + \\ + \eta(x,\tau;\omega) + \epsilon\widetilde{\eta}(x,\tau;\varpi) \\ \text{and} \\ \frac{\partial[v_{\epsilon+}(x,\tau;\omega,\varpi) - \theta(\tau)a]}{\partial\tau} = \frac{\partial v_{\epsilon+}(x,\tau;\omega,\varpi)}{\partial\tau} - a\delta(\tau) = \\ (\partial^2 - m^2)[v_{\epsilon+}(x,\tau;\omega,\varpi) - \theta(\tau)a] - \frac{\lambda}{3!}(v_{\epsilon+}^3 - 3av_{\epsilon+}^2 + 3a^2 v_{\epsilon+} - a^3) + \\ + \eta(x,\tau;\omega) + \epsilon\widetilde{\eta}(x,\tau;\varpi) \end{gathered} \quad (2.4.30)$$

Differential master equations corresponding to double stochastic Langevin equations (2.4.30) reads

$$\begin{gathered} \frac{\partial v_-(x,\tau;\omega)}{\partial\tau} = -a\delta(\tau) + [\partial^2 - (m^2 + 0.5\lambda a^2))]v_-(x,\tau;\omega) - \\ -\frac{\lambda}{6}a^3 - m^2 a + \eta(x,\tau;\omega) = \\ -a\delta(\tau) + [\partial^2 - m_1^2]v_-(x,\tau;\omega) - \left(\frac{\lambda}{6}a^3 + m^2 a\right) + \eta(x,\tau;\omega) \\ \text{and} \\ \frac{\partial v_+(x,\tau;\omega)}{\partial\tau} = a\delta(\tau) + [\partial^2 - (m^2 + 0.5\lambda a^2))]v_+(x,\tau;\omega) + \\ +\frac{\lambda}{6}a^3 + m^2 a + \eta(x,\tau;\omega) = \\ a\delta(\tau) + [\partial^2 - m_1^2]v_+(x,\tau;\omega) + \left(\frac{\lambda}{6}a^3 + m^2 a\right) + \eta(x,\tau;\omega) \\ m_1^2 = m^2 + 0.5\lambda a^2. \end{gathered} \quad (2.4.31)$$

Consider the Fourier transformed stochastic differential equation (2.4.31) in k and τ given as

$$\begin{gathered}\frac{\partial}{\partial\tau}\widehat{v}_{-}(k,\tau) = \\ -(k^2+m_1^2)\widehat{v}_{-}(k,\tau)-(2\pi)^4 a\delta(\tau)\delta^4(k)-(2\pi)^4\left(m^2 a+\frac{\lambda}{6}a^3\right)\delta^4(k)+\widehat{\eta}(k,\tau;\omega) \\ \text{and} \\ \frac{\partial}{\partial\tau}\widehat{v}_{+}(k,\tau) = \\ -(k^2+m_1^2)\widehat{v}_{-}(k,\tau)+(2\pi)^4 a\delta(\tau)\delta^4(k)+(2\pi)^4\left(m^2 a+\frac{\lambda}{6}a^3\right)\delta^4(k)+\widehat{\eta}(k,\tau;\omega)\end{gathered} \tag{2.4.32}$$

Let us consider ODE

$$\dot{x}\ (\tau,\lambda)+\lambda x(\tau,\lambda) = g(\tau,\lambda), x(0) = 0. \tag{2.4.33}$$

The corresponding solution $x(t,\lambda)$ reads

$$x(\tau,\lambda) = e^{-\lambda\tau}\int_0^\tau e^{\lambda\tau_1}g(\tau_1,\lambda)d\tau_1. \tag{2.4.34}$$

From Eq.(2.4.32)-Eq.(2.4.34) one obtains

$$\widehat{v}_{-}(k,\tau,a) = e^{-(k^2+m_1^2)\tau} \times$$

$$\times \int_0^{\tau} e^{(k^2+m_1^2)\tau_1}\left[-(2\pi)^4 a\delta(\tau_1)\delta^4(k) - (2\pi)^4\left(m^2 a + \frac{\lambda}{6}a^3\right)\delta^4(k) + \widehat{\eta}(k,\tau_1;\omega)\right]d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4\left(m^2 a + \frac{\lambda}{6}a^3\right)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^{\tau} e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+e^{-(k^2+m_1^2)\tau}\int_0^{\tau} e^{(k^2+m_1^2)\tau_1}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4 a\left(m^2 + \frac{\lambda}{6}a^2\right)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^{\tau} e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+\int_0^{\tau} e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} -$$

$$-(2\pi)^4 a\left(m^2 + \frac{\lambda}{6}a^2\right)\delta^4(k)e^{-(k^2+m^2)\tau}\left[\frac{e^{(k^2+m^2)\tau}}{k^2+m_1^2} - \frac{1}{k^2+m_1^2}\right] + \qquad (2.4.35)$$

$$+\int_0^{\tau} e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4\frac{\left(m^2 a + \frac{\lambda}{6}a^3\right)\delta^4(k)}{k^2+m_1^2}\left[1 - e^{-(k^2+m_1^2)\tau}\right] +$$

$$+\int_0^{\tau} e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m_1^2)\tau} + \frac{\left(m^2 + \frac{\lambda}{6}a^2\right)}{k^2+m_1^2}\left[1 - e^{-(k^2+m_1^2)\tau}\right]\right] +$$

$$+\int_0^{\tau} e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1,$$

and

$$\hat{v}_+(k,\tau,a) = e^{-(k^2+m_1^2)\tau} \times$$

$$\times \int_0^\tau e^{(k^2+m_1^2)\tau_1}\Big[(2\pi)^4 a\delta(\tau_1)\delta^4(k) - (2\pi)^4\Big(-m^2a - \frac{\lambda}{6}a^3\Big)\delta^4(k) \pm \hat{\eta}(k,\tau_1;\omega)\Big]d\tau_1 =$$

$$(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4\Big(-m^2a - \frac{\lambda}{6}a^3\Big)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}\hat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4 a\Big(m^2 + \frac{\lambda}{6}a^3\Big)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\hat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} +$$

$$+(2\pi)^4 a\Big(m^2 + \frac{\lambda}{6}a^2\Big)\delta^4(k)e^{-(k^2+m^2)\tau}\left[\frac{e^{(k^2+m^2)\tau}}{k^2+m_1^2} - \frac{1}{k^2+m_1^2}\right] + \qquad (2.4.35')$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\hat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4\frac{\Big(m^2 + \frac{\lambda}{6}a^3\Big)\delta^4(k)}{k^2+m_1^2}\Big[1 - e^{-(k^2+m_1^2)\tau}\Big] +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\hat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m_1^2)\tau} + \frac{\Big(m^2 + \frac{\lambda}{6}a^3\Big)}{k^2+m_1^2}\Big[1 - e^{-(k^2+m_1^2)\tau}\Big]\right] +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\hat{\eta}(k,\tau_1;\omega)d\tau_1.$$

From Eq.(2.4.35)-Eq.(2.4.35$'$) one obtains

$$
\begin{gathered}
\widehat{v}_{-}(k_1,\tau,a)\widehat{v}_{+}(k_2,\tau',a') = \\
\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m_1^2)\tau} + \frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]+\right. \\
\left.+\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right\}\times \\
\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right]+\right. \\
\left.+\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_1)}\widehat{\eta}(k_2,\tau_1;\omega)d\tau_1\right\}
\end{gathered}
\tag{2.4.36}
$$

From Eq.(2.4.36) one obtains

$$
\begin{gathered}
\langle\widehat{v}_{-}(k_1,\tau,a)\widehat{v}_{+}(k_2,\tau',a')\rangle_{\eta} = \\
\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\right\}\times \\
\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right]\right\}- \\
-\left\langle\left(\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right)\left(\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_2)}\widehat{\eta}(k_2,\tau_2;\omega)d\tau_2\right)\right\rangle_{\eta} = \\
-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m_1^2)\tau} + \frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times \\
\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right]- \\
-\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\langle\widehat{\eta}(k_1,\tau_1;\omega)\widehat{\eta}(k_2,\tau_2;\omega)\rangle_{\eta}d\tau_1 d\tau_2
\end{gathered}
\tag{2.4.37}
$$

Note that

$$-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau}+\frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times$$

$$\left[e^{-(k_2^2+m^2)\tau'}+\frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau'}\right]\right]\Bigg|_{\tau'=\tau,a'=a}=$$

$$-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m_1^2)\tau}+\frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times$$

$$\times\left[e^{-(k_1^2+m^2)\tau}+\frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right] \quad (2.4.38)$$

and note that

$$\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m^2)(\tau-\tau_1)}\langle\hat{\eta}(k_1,\tau_1;\omega)\hat{\eta}(k_2,\tau_2;\omega)\rangle_\eta d\tau_1 d\tau_2\Bigg|_{\tau'=\tau,a'=a}=$$

$$\delta(k_1+k_2)\int_0^{\tau} e^{-(k_2^2+m_1^2)(\tau-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\delta(\tau_1-\tau_2)d\tau_1 d\tau_2=$$

$$2(2\pi)^4\delta(k_1+k_2)\int_0^{\tau} e^{-(k_1^2+k_2^2+2m_1^2)(\tau-\tau_1)}d\tau_1=$$

$$2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m_1^2)\tau}\int_0^{\tau} e^{(k_1^2+k_2^2+2m_1^2)\tau_1}d\tau_1= \quad (2.4.39)$$

$$2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m_1^2)\tau}\left[\frac{1}{(k_1^2+k_2^2+2m_1^2)}e^{(k_1^2+k_2^2+m_1^2)\tau}-\frac{1}{(k_1^2+k_2^2+2m_1^2)}\right]=$$

$$=\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m_1^2}-2(2\pi)^4\delta(k_1+k_2)\frac{e^{-(k_1^2+k_2^2+2m_1^2)\tau}}{(k_1^2+k_2^2+2m_1^2)}$$

From Eq.(2.4.38)-Eq.(2.4.39) we get

$$\lim_{\tau\to\infty}\langle\hat{v}_-(k_1,\tau,a)\hat{v}_+(k_2,\tau,a)\rangle_\eta=(2\pi)^8 a^2\delta^4(k_1)\delta^4(k_2)\frac{\left(m^2+\frac{\lambda}{6}a^2\right)^2}{(k_1^2+m_1^2)(k_2^2+m_1^2)}-$$

$$-\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m_1^2} \quad (2.4.40)$$

$$m_1^2=m^2+0.5\lambda a^2$$

Therefore

$$\lim_{\tau\to\infty}\langle\hat{v}_-(x_1,\tau,a)\hat{v}_+(x_2,\tau,a)\rangle_\eta =$$

$$-a^2(2\pi)^{-8}\times\int d^4k_1 e^{ik_1x_1}\int d^4k_2 e^{ik_2x_2}\delta^4(k_1)\delta^4(k_2)\frac{\left(m^2+\frac{\lambda}{6}a^2\right)^2}{(k_1^2+m_1^2)\times(k_2^2+m_1^2)}+$$

$$(2\pi)^{-8}\times\int d^4k_1 e^{ik_1x_1}\int d^4k_2 e^{ik_2x_2}\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m_1^2} =$$

$$= \left[\frac{a^2\left(m^2+\frac{\lambda}{6}a^2\right)^2}{m_1^2\times m_1^2}-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m_1^2}\right] = \tag{2.4.41}$$

$$\frac{a^2\left(m^2+\frac{\lambda}{6}a^2\right)^2}{m_1^2\times m_1^2}-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m_1^2}$$

$$m_1^2 = m^2+0.5\lambda a^2$$

Transcendental master equation corresponding to two-point Eucledian Green function $G_{(4)}(x_1,x_2,m,\lambda)$ in Euclidean space E_4 with $\dim E_4 = 4$ reads

$$\frac{\left[a(x_1-x_2)^2\right]\left(m^2+\frac{\lambda}{6}a(x_1-x_2)^2\right)^2}{\left(m^2+0.5\lambda a(x_1-x_2)^2\right)^2}-$$
$$-(2\pi)^{-4}\int d^4k\frac{\theta_\delta(|x_1-x_2|)e^{ik(x_1-x_2)}}{k^2+m^2+0.5\lambda a(x_1-x_2)^2} = 0. \tag{2.4.41}$$

Transcendental master equation corresponding to two-point Eucledian Green function $G_{(D)}(x_1,x_2,m,\lambda)$ in Euclidean space E_D with $\dim E_D = D$ reads

$$\frac{\left[a(x_1-x_2)^2\right]\left(m^2+\frac{\lambda}{6}a(x_1-x_2)^2\right)^2}{\left(m^2+0.5\lambda a(x_1-x_2)^2\right)^2}-$$
$$-(2\pi)^{-D}w\text{-}\lim_{\eta\to0_+}\int d^Dk\frac{e^{-\eta k+ik(x_1-x_2)}}{k^2+m^2+0.5\lambda a(x_1-x_2)^2} = 0. \tag{2.4.42}$$
$$\eta>0,$$

where weak limit taken in $\mathcal{L}'(\mathbb{R}_x^D)$, see Apendix 1.
Note that

$$\int d^4k\frac{e^{ik(x_1-x_2)}}{\left[k^2+m^2+0.5\lambda a(x_1-x_2)^2\right]^2} =$$
$$(4\pi)^{-2}\left(\frac{m_1}{|x_1-x_2|}\right)K_0(m_1|x_1-x_2|) \tag{2.4.43}$$
$$m_1 = \sqrt{m^2+0.5\lambda a(x_1-x_2)^2},$$
$$K_0(m_1|x_1-x_2|) \asymp -\ln\left(\frac{m_1|x_1-x_2|}{2}\right)-\gamma \text{ as } m_1|x_1-x_2|\to 0$$

see Apendix 1. In order to derive Eq.(2.4.43) we applied Eq.(2.4.44) (see formula

13.6 (2) from [13]).

$$\int_0^\infty \frac{y^{\nu+1}J_\nu(by)dy}{(y^2+p^2)^{\mu+1}} = \frac{b^\mu p^{\nu-\mu}}{2^\mu\Gamma(\mu+1)}K_{\nu-\mu}(bp) \qquad (2.4.44)$$
$$-1 < \mathrm{Re}\,\nu < 2\,\mathrm{Re}\,\mu + 1.5.$$

2.5.Double stochastic quantization of the $\lambda\phi_4^6$ theory.

In this section we consider a neutral scalar field with a $\frac{\lambda}{6!}\phi_d^6$, $d \geq 4$, self-interaction,

defined in a 4-dimensional Minkowski spacetime. It well known that all these theories is

nonrenormalizable [14]. The vacuum persistence functional is the generating functional

of all vacuum expectation value of time-ordered products of the theory. Thus we deal now with simple *nonrenormalizable* theory with the Lagrangian

$$\mathcal{L} = S_0[\varphi] + S_I[\varphi], \qquad (2.5.1)$$

where the action that usually describes a free scalar field is

$$S_0[\varphi] = \int d^4x\left(\frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi) + \frac{1}{2}m_0^2\varphi^2(x)\right), \qquad (2.5.2)$$

and the interacting part, defined by the non-Gaussian contribution, is

$$S_I[\varphi] = \int d^4x\,\frac{\lambda}{6!}\,\varphi^6(x). \qquad (2.5.3)$$

The double stochastic Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi_\epsilon(\tau,x) = (\Delta - m_0^2)\varphi_\epsilon(\tau,x) - \frac{\lambda}{5!}\varphi_\epsilon^5(\tau,x) + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x). \qquad (2.5.4)$$

By the replacements

$$\begin{aligned}\varphi_\epsilon(x,\tau;\omega,\varpi) &= v_{\epsilon-}(x,\tau;\omega,\varpi) + \theta(\tau)a,\\ \varphi_\epsilon(x,\tau;\omega,\varpi) &= v_{\epsilon+}(x,\tau;\omega,\varpi) - \theta(\tau)a,\\ \theta(\tau) &= \begin{cases} 0 & \text{if } \tau \leq 1\\ 1 & \text{if } \tau > 1\end{cases}\end{aligned} \qquad (2.5.5)$$

we obtain from Eq.(2.5.4)

$$\frac{\partial[v_{\epsilon-}(x,\tau;\omega,\varpi)+\theta(\tau)a]}{\partial\tau} = \frac{\partial v_{\epsilon-}(x,\tau;\omega,\varpi)}{\partial\tau} + a\delta(\tau) =$$

$$= (\partial^2 - m^2)[v_{\epsilon-}(x,\tau;\omega,\varpi)+\theta(\tau)a] - \frac{\lambda}{5!}[v_{\epsilon-}(x,\tau;\omega,\varpi)+a]^5 +$$

$$+\eta(x,\tau;\omega)+\epsilon\widetilde{\eta}(x,\tau;\varpi) =$$

$$(\partial^2 - m^2)[v_{\epsilon-}(x,\tau;\omega,\varpi)+\theta(\tau)a] -$$

$$-\frac{\lambda}{5!}(a^5 + 5a^4 v_{\epsilon-} + 10a^3 v_{\epsilon-}^2 + 10a^2 v_{\epsilon-}^3 + 5a v_{\epsilon-}^4 + v_{\epsilon-}^5) +$$

$$+\eta(x,\tau;\omega)+\epsilon\widetilde{\eta}(x,\tau;\varpi) \qquad (2.5.6)$$

and

$$\frac{\partial[v_{\epsilon+}(x,\tau;\omega,\varpi)-\theta(\tau)a]}{\partial\tau} = \frac{\partial v_{\epsilon+}(x,\tau;\omega,\varpi)}{\partial\tau} - a\delta(\tau) =$$

$$(\partial^2 - m^2)[v_{\epsilon+}(x,\tau;\omega,\varpi)-\theta(\tau)a] -$$

$$-\frac{\lambda}{5!}(-a^5 + 5a^4 v_{\epsilon+} - 10a^3 v_{\epsilon+}^2 + 10a^2 v_{\epsilon+}^3 - 5a v_{\epsilon+}^4 + v_{\epsilon+}^5) +$$

$$+\eta(x,\tau;\omega)+\epsilon\widetilde{\eta}(x,\tau;\varpi)$$

Differential master equations corresponding to double stochastic Langevin equations (2.5.6) reads

$$\frac{\partial v_{-}(x,\tau;\omega)}{\partial\tau} = -a\delta(\tau) + \left[\partial^2 - \left(m^2 + \frac{\lambda}{4!}a^4\right)\right)\Big]v_{-}(x,\tau;\omega) -$$

$$-\frac{\lambda}{5!}a^5 - m^2 a + \eta(x,\tau;\omega) =$$

$$-a\delta(\tau) + [\partial^2 - m_1^2]v_{-}(x,\tau;\omega) - \left(\frac{\lambda}{5!}a^5 + m^2 a\right) + \eta(x,\tau;\omega)$$

and

$$\frac{\partial v_{+}(x,\tau;\omega)}{\partial\tau} = a\delta(\tau) + \left[\partial^2 - \left(m^2 + \frac{\lambda}{5!}a^5\right)\right)\Big]v_{+}(x,\tau;\omega) + \qquad (2.5.7)$$

$$+\frac{\lambda}{5!}a^5 + m^2 a + \eta(x,\tau;\omega) =$$

$$a\delta(\tau) + [\partial^2 - m_1^2]v_{+}(x,\tau;\omega) + \left(\frac{\lambda}{5!}a^5 + m^2 a\right) + \eta(x,\tau;\omega)$$

$$m_1^2 = m^2 + \frac{\lambda}{4!}a^4.$$

Consider the Fourier transformed stochastic differential equation (2.5.7) in k given as

$$\frac{\partial}{\partial\tau}\widehat{v}_{-}(k,\tau) =$$

$$-(k^2+m_1^2)\widehat{v}_{-}(k,\tau) - (2\pi)^4 a\delta(\tau)\delta^4(k) - (2\pi)^4\left(m^2 a + \frac{\lambda}{5!}a^5\right)\delta^4(k) + \widehat{\eta}(k,\tau;\omega)$$

and

$$\frac{\partial}{\partial\tau}\widehat{v}_{+}(k,\tau) = \qquad (2.4.32)$$

$$-(k^2+m_1^2)\widehat{v}_{-}(k,\tau) + (2\pi)^4 a\delta(\tau)\delta^4(k) + (2\pi)^4\left(m^2 a + \frac{\lambda}{5!}a^5\right)\delta^4(k) + \widehat{\eta}(k,\tau;\omega)$$

Let us consider ODE

$$\dot{x}\,(\tau,\lambda) + \lambda x(\tau,\lambda) = g(\tau,\lambda), x(0) = 0. \qquad (2.5.8)$$

The corresponding solution $x(t,\lambda)$ reads

$$x(\tau,\lambda) = e^{-\lambda\tau}\int_0^{\tau} e^{\lambda\tau_1} g(\tau_1,\lambda)d\tau_1. \tag{2.5.9}$$

From Eq.(2.4.32)-Eq.(2.4.34) one obtains

$$\begin{aligned}
&\widehat{v}_-(k,\tau,a) = e^{-(k^2+m_1^2)\tau} \times \\
&\times\int_0^{\tau} e^{(k^2+m_1^2)\tau_1}\left[-(2\pi)^4 a\delta(\tau_1)\delta^4(k) - (2\pi)^4\left(m^2 a + \frac{\lambda}{5!}a^5\right)\delta^4(k) + \widehat{\eta}(k,\tau_1;\omega)\right]d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4\left(m^2 a + \frac{\lambda}{5!}a^5\right)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^{\tau} e^{(k^2+m_1^2)\tau_1}d\tau_1 + \\
&+e^{-(k^2+m_1^2)\tau}\int_0^{\tau} e^{(k^2+m_1^2)\tau_1}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4 a\left(m^2 + \frac{\lambda}{5!}a^4\right)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^{\tau} e^{(k^2+m_1^2)\tau_1}d\tau_1 + \\
&+\int_0^{\tau} e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - \\
&-(2\pi)^4 a\left(m^2 + \frac{\lambda}{5!}a^4\right)\delta^4(k)e^{-(k^2+m^2)\tau}\left[\frac{e^{(k^2+m^2)\tau}}{k^2+m_1^2} - \frac{1}{k^2+m_1^2}\right] + \\
&+\int_0^{\tau} e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4\frac{a\left(m^2 + \frac{\lambda}{5!}a^4\right)\delta^4(k)}{k^2+m_1^2}\left[1 - e^{-(k^2+m_1^2)\tau}\right] + \\
&+\int_0^{\tau} e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
&-(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m_1^2)\tau} + \frac{\left(m^2 + \frac{\lambda}{5!}a^4\right)}{k^2+m_1^2}\left[1 - e^{-(k^2+m_1^2)\tau}\right]\right] + \\
&+\int_0^{\tau} e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1, \\
&m_1^2 = m^2 + \frac{\lambda}{4!}a^4,
\end{aligned} \tag{2.5.10}$$

and

$$\widehat{v}_+(k,\tau,a) = e^{-(k^2+m_1^2)\tau} \times$$

$$\times\int_0^\tau e^{(k^2+m_1^2)\tau_1}\left[(2\pi)^4 a\delta(\tau_1)\delta^4(k) - (2\pi)^4\left(-m^2 a - \frac{\lambda}{5!}a^5\right)\delta^4(k) + \widehat{\eta}(k,\tau_1;\omega)\right]d\tau_1 =$$

$$(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4\left(-m^2 a - \frac{\lambda}{5!}a^5\right)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4 a\left(m^2 + \frac{\lambda}{5!}a^4\right)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} +$$

$$+(2\pi)^4 a\left(m^2 + \frac{\lambda}{5!}a^4\right)\delta^4(k)e^{-(k^2+m^2)\tau}\left[\frac{e^{(k^2+m^2)\tau}}{k^2+m_1^2} - \frac{1}{k^2+m_1^2}\right] + \qquad (2.5.11')$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4\frac{a\left(m^2+\frac{\lambda}{5!}a^4\right)\delta^4(k)}{k^2+m_1^2}\left[1-e^{-(k^2+m_1^2)\tau}\right] +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m_1^2)\tau} + \frac{\left(m^2+\frac{\lambda}{5!}a^4\right)}{k^2+m_1^2}\left[1-e^{-(k^2+m_1^2)\tau}\right]\right] +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1.$$

$$m_1^2 = m^2 + \frac{\lambda}{4!}a^4$$

From Eq.(2.5.10)-Eq.(2.5.11) one obtains

$$\begin{gathered}
\widehat{v}_{-}(k_1,\tau,a)\widehat{v}_{+}(k_2,\tau',a') = \\
\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m_1^2)\tau} + \frac{m^2+\frac{\lambda}{5!}a^4}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right] + \right. \\
\left. +\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1 \right\} \times \\
\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{5!}a^4}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right] + \right. \\
\left. +\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_1)}\widehat{\eta}(k_2,\tau_1;\omega)d\tau_1 \right\}
\end{gathered} \tag{2.5.12}$$

From Eq.(2.5.12) one obtains

$$\begin{gathered}
\langle \widehat{v}_{-}(k_1,\tau,a)\widehat{v}_{+}(k_2,\tau',a')\rangle_{\eta} = \\
\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2+\frac{\lambda}{5!}a^4}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\right\} \times \\
\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{5!}a'^4}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right]\right\} - \\
-\left\langle \left(\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right)\left(\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_2)}\widehat{\eta}(k_2,\tau_2;\omega)d\tau_2\right)\right\rangle_{\eta} = \\
-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m_1^2)\tau} + \frac{m^2+\frac{\lambda}{5!}a^4}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right] \times \\
\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{5!}a'^4}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right] - \\
-\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\langle\widehat{\eta}(k_1,\tau_1;\omega)\widehat{\eta}(k_2,\tau_2;\omega)\rangle_{\eta}d\tau_1 d\tau_2
\end{gathered} \tag{2.5.13}$$

Note that

$$-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau}+\frac{m^2+\frac{\lambda}{5!}a^4}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times$$

$$\left[e^{-(k_2^2+m^2)\tau'}+\frac{m^2+\frac{\lambda}{5!}a'^4}{k_2^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau'}\right]\right]\Bigg|_{\tau'=\tau,a'=a}=$$

$$-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m_1^2)\tau}+\frac{m^2+\frac{\lambda}{5!}a^4}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times \tag{2.5.14}$$

$$\times\left[e^{-(k_1^2+m^2)\tau}+\frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]$$

$$m_1^2=m^2+\frac{\lambda}{4!}a^4$$

and note that

$$\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\langle\hat{\eta}(k_1,\tau_1;\omega)\hat{\eta}(k_2,\tau_2;\omega)\rangle_\eta d\tau_1 d\tau_2\Bigg|_{\tau'=\tau,a'=a}=$$

$$\delta(k_1+k_2)\int_0^{\tau} e^{-(k_2^2+m_1^2)(\tau-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\delta(\tau_1-\tau_2)d\tau_1 d\tau_2=$$

$$2(2\pi)^4\delta(k_1+k_2)\int_0^{\tau} e^{-(k_1^2+k_2^2+2m_1^2)(\tau-\tau_1)}d\tau_1=$$

$$2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m_1^2)\tau}\int_0^{\tau} e^{(k_1^2+k_2^2+2m_1^2)\tau_1}d\tau_1= \tag{2.5.15}$$

$$2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m_1^2)\tau}\left[\frac{1}{(k_1^2+k_2^2+2m_1^2)}e^{(k_1^2+k_2^2+m_1^2)\tau}-\frac{1}{(k_1^2+k_2^2+2m_1^2)}\right]=$$

$$=\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m_1^2}-2(2\pi)^4\delta(k_1+k_2)\frac{e^{-(k_1^2+k_2^2+2m_1^2)\tau}}{(k_1^2+k_2^2+2m_1^2)}$$

$$m_1^2=m^2+\frac{\lambda}{4!}a^4$$

Transcendental master equation corresponding to two-point Eucledian Green function $G_{(4)}(x_1,x_2,m,\lambda)$ in Euclidean space E_4 with $\dim E_4=4$ reads

$$\frac{\left[a(x_1-x_2)^2\right]\left(m^2+\frac{\lambda}{5!}a(x_1-x_2)^4\right)^2}{\left(m^2+\frac{\lambda}{4!}a(x_1-x_2)^4\right)^2}- \tag{2.5.16}$$

$$-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2+\frac{\lambda}{4!}a(x_1-x_2)^4}=0.$$

Transcendental master equation corresponding to two-point Eucledian Green function $G_{(D)}(x_1,x_2,m,\lambda)$ in Euclidean space E_D with $\dim E_D=D$ reads

$$\frac{\left[a(x_1-x_2)^2\right]\left(m^2+\frac{\lambda}{5!}a(x_1-x_2)^4\right)^2}{\left(m^2+\frac{\lambda}{4!}a(x_1-x_2)^4\right)^2}-$$

$$-(2\pi)^{-D}w\text{-}\lim_{\eta\to 0_+}\int d^Dk\frac{e^{-\eta k+ik(x_1-x_2)}}{k^2+m^2+\frac{\lambda}{4!}a(x_1-x_2)^4}=0. \qquad (2.5.17)$$

$$\eta>0,$$

where a weak limit taken in $\mathcal{L}'(\mathbb{R}_x^D)$, see Apendix 1.
Note that

$$\int d^4k\frac{e^{ik(x_1-x_2)}}{\left[k^2+m^2+\frac{\lambda}{4!}a(x_1-x_2)^4\right]^2}=$$

$$(4\pi)^{-2}\left(\frac{m_1}{|x_1-x_2|}\right)K_0(m_1|x_1-x_2|) \qquad (2.5.18)$$

$$m_1=\sqrt{m^2+\frac{\lambda}{4!}a(x_1-x_2)^4},$$

$$K_0(m_1|x_1-x_2|)\asymp-\ln\left(\frac{m_1|x_1-x_2|}{2}\right)-\gamma \text{ as } m_1|x_1-x_2|\to 0$$

see Apendix 1. In order to derive Eq.(2.5.18) we applied Eq.(2.5.19) (see formula 13.6 (2) from [13]).

$$\int_0^\infty\frac{y^{\nu+1}J_\nu(by)dy}{(y^2+p^2)^{\mu+1}}=\frac{b^\mu p^{\nu-\mu}}{2^\mu\Gamma(\mu+1)}K_{\nu-\mu}(bp) \qquad (2.5.19)$$

$$-1<\operatorname{Re}\nu<2\operatorname{Re}\mu+1.5.$$

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3.Weak coupling. Nonperturbative result.

3.1.Weak coupling.The $\lambda\phi_d^4$ theory

We assume now that

$$\varepsilon=\frac{\lambda\theta_\delta(|x|)a(x_1-x_2)^2}{m^2}\ll 1, \qquad (3.1.1)$$

where $\theta_\delta(|x|)=\theta_\delta(|x_1-x_2|)=\theta(|x_1-x_2|-\delta), x=|x_1-x_2|$.
From Eq.(2.4.41) and Eq.(3.1.1) we get

$$\theta_\delta(|x|)a^2(x)\frac{\left(1+\frac{\lambda a^2(x)}{6m^2}\right)^2}{\left(1+\frac{0.5\lambda a^2(x)}{m^2}\right)^2}-(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)\left[1+\frac{0.5\lambda a^2(x)}{m^2+k^2}\right]}. \qquad (3.1.2)$$

and

$$\theta_\delta(|x|)a^2(x) = (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}\left[1-\frac{0.5\lambda a^2(x)}{m^2+k^2}+\left(\frac{0.5\lambda a^2(x)}{m^2+k^2}\right)^2+\ldots\right] =$$

$$(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)} - 0.5\lambda(2\pi)^{-4}a^2(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^2} + \tag{3.1.3}$$

$$+0.25\lambda^2(2\pi)^{-4}a^4(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^3}+\ldots$$

Therefore under condition (3.1.41) we get

$$a^2(x)\left[1+0.5\lambda(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^2}\right] = (2\pi)^{-4}\int\frac{\theta_\delta(|x|)d^4ke^{ikx}}{(m^2+k^2)}+O(\lambda^2), \tag{3.1.4}$$

where constant in symbol $O(\lambda^2)$ depend on m^2 and δ.

Remark 3.1.1. Note that for a given values of the parameters m^2 and δ we can choose value of the parameter λ such that the inequality (3.1.1) is satisfied.

Thus finally for two-point for ϕ_4^4 theory in the Euclidean QFT we obtain non perturbative result

$$\theta_\delta(|x_1-x_2|)G(x_1-x_2) = \theta_\delta(|x_1-x_2|)a^2(x_1-x_2) =$$

$$(2\pi)^{-4}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)}\times$$

$$\left[1+0.5\lambda(2\pi)^{-4}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)^2}\right]^{-1} = \tag{3.1. 5}$$

$$= (2\pi)^{-4}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)} -$$

$$-0.5\lambda(2\pi)^{-8}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)^2}\times$$

$$\left(\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)}\right)+o(\varepsilon).$$

Remark 3.1.2. To first order in λ, and in coordinate space, the two point function $G_{(4)}(x_1-x_2;\delta) = \theta_\delta(|x_1-x_2|)G_{(4)}(x_1-x_2)$ bounded on region $\mathbb{R}^4\backslash[-\delta,\delta]^4$ in Euclidean space with $\dim = 4$ is

$$G_{(4)}(x_1-x_2;\delta) = \int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(2\pi)^4(m^2+k^2)} -$$

$$-\frac{\lambda}{2}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(2\pi)^4(m^2+k^2)^2}\times\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(2\pi)^4(m^2+k^2)} = \tag{3.1.6}$$

$$\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(2\pi)^4(m^2+k^2)}\left\{1-\frac{\lambda}{2}\frac{G_{(4)F}(x_1-x_2;\delta)}{(m^2+k^2)}\right\},$$

where $G_{(4)F}(x_1-x_2;\delta) = \theta_\delta(|x_1-x_2|)G_{(4)F}(x_1-x_2)$. For $\lambda \ll 1$ we get

$$\begin{aligned}\left\{1-\frac{\lambda}{2}\frac{G_{(4)F}(x_1-x_2;\delta)}{(m^2+k^2)}\right\} &\simeq \left\{1+\frac{0.5\lambda G_{(4)F}(x_1-x_2;\delta)}{(m^2+k^2)}\right\}^{-1}\\ &= \frac{m^2+k^2}{m^2+0.5\lambda G_{(4)F}(x_1-x_2;\delta)+k^2}.\end{aligned} \tag{3.1.7}$$

From Eq.(3.1.6) and Eq.(3.1.7) we get

$$G_{(4)}(x_1-x_2;\delta) = \frac{1}{(2\pi)^4}\int\frac{d^4k e^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{m^2+0.5\lambda G_{(4)F}(x_1-x_2;\delta)+k^2}. \tag{3.1.8}$$

From Eq.(3.1.8) finally we get

$$G_{(4)}(x_1-x_2;\delta) \simeq \frac{1}{(2\pi)^4}\int\frac{d^4k e^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{m^2+0.5\lambda G_{(4)F}(\delta;\delta)+k^2}. \tag{3.1.9}$$

This expression leads us to define $\mu^2_{\mathbf{ren}}$ by

$$\mu^2_{\mathbf{ren}} = m^2+0.5\lambda G_{(4)F}(\delta;\delta) = m^2+\delta m^2. \tag{3.1.10}$$

Note that in contrast with canonical perturbative calculation (see Apendix 2,eq.2.28) $\delta m^2 = 0.5\lambda G_{(4)F}(\delta;\delta)$ is finite.

Remark 3.1.3. To first order in λ, and in coordinate space, the two point function $G_{(D)}(x_1-x_2;\delta) \upharpoonright \mathbb{R}^D\backslash[-\delta,\delta]^D = \theta_\delta(|x_1-x_2|)G_{(4)}(x_1-x_2)$ bounded on region $\mathbb{R}^D\backslash[-\delta,\delta]^D$ in Euclidean space E_D with $\dim E_D = D$ is

$$\begin{gathered}G_{(D)}(x_1-x_2;\delta) = \int\frac{d^Dk e^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(2\pi)^D(m^2+k^2)} - \\ -\frac{\lambda}{2}\int\frac{d^Dk e^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(2\pi)^D(m^2+k^2)^2}\times\int\frac{d^Dk e^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(2\pi)^D(m^2+k^2)} = \\ \int\frac{d^Dk e^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(2\pi)^D(m^2+k^2)}\left\{1-\frac{\lambda}{2}\frac{G_{(D)F}(x_1-x_2;\delta)}{(m^2+k^2)}\right\},\end{gathered} \tag{3.1.11}$$

where $G_{(D)F}(x_1-x_2;\delta) = \theta_\delta(|x_1-x_2|)G_{(D)F}(x_1-x_2)$. For $\lambda \ll 1$ we get

$$\begin{aligned}\left\{1-\frac{\lambda}{2}\frac{G_{(D)F}(x_1-x_2;\delta)}{(m^2+k^2)}\right\} &\simeq \left\{1+\frac{0.5\lambda G_{(D)F}(x_1-x_2;\delta)}{(m^2+k^2)}\right\}^{-1}\\ &= \frac{m^2+k^2}{m^2+0.5\lambda G_{(D)F}(x_1-x_2;\delta)+k^2}.\end{aligned} \tag{3.1.12}$$

From Eq.(3.1.11) and Eq.(3.1.12) we get

$$G_{(D)}(x_1-x_2;\delta) = \frac{1}{(2\pi)^D}\int\frac{d^Dk e^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{m^2+0.5\lambda G_{(D)F}(x_1-x_2;\delta)+k^2}. \tag{3.1.13}$$

From Eq.(3.1.13) finally we get

$$G_{(D)}(x_1-x_2;\delta) \simeq \frac{1}{(2\pi)^D}\int\frac{d^Dk e^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{m^2+0.5\lambda G_{(D)F}(\delta;\delta)+k^2}. \tag{3.1.14}$$

This expression leads us to define $\mu^2_{\mathbf{ren}}$ by

$$\mu^2_{\mathbf{ren}} = m^2+0.5\lambda G_{(D)F}(\delta;\delta) = m^2+\delta m^2. \tag{3.1.15}$$

Note that in contrast with canonical perturbative calculation (see Apendix 2,eq.2.28) $\delta m^2 = 0.5\lambda G_{(D)F}(\delta;\delta)$ is finite.

3.2.Weak coupling.The $\lambda\phi_d^4$ theory.Exact nonperturbative solution.

Trancendental master equation corresponding to $\lambda\phi_d^4$ theory [see Eq.(3.1.2)] reads

$$\theta_\delta(|x|)a^2(x)\frac{\left(1+\dfrac{\lambda a^2(x)}{6m^2}\right)^2}{\left(1+\dfrac{0.5\lambda a^2(x)}{m^2}\right)^2}-(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)\left[1+\dfrac{0.5\lambda a^2(x)}{m^2+k^2}\right]}. \tag{3.2.1}$$

Assuming now for simplisity that $\lambda/m^2 \ll 1$, then from Eq.(3.2.1) we obtain

$$\theta_\delta(|x|)a^2(x) \simeq (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)\left[1+\dfrac{0.5\lambda a^2(x)}{m^2+k^2}\right]}. \tag{3.2.2}$$

Note that

$$\frac{1}{1+z} = 1-z+z^2-z^3+z^4-\ldots 1+\sum_{n=1}^{\infty}(-1)^n z^n \tag{3.2.3}$$
$$z>0, z<1.$$

From Eq.(3.2.3) by setting

$$z := \frac{0.5\lambda a^2(x)}{m^2+k^2} \tag{3.2.4}$$

we obtain

$$\left(1+\frac{0.5\lambda a^2(x)}{m^2+k^2}\right)^{-1} = 1-\frac{0.5\lambda a^2(x)}{m^2+k^2}+\left(\frac{0.5\lambda a^2(x)}{m^2+k^2}\right)^2-\ldots= \\ 1+\sum_{n=1}^{\infty}(-1)^n\frac{0.5\lambda a^2(x)}{m^2+k^2}. \tag{3.2.5}$$

From Eq.(3.2.2) using Eq.(3.2.5) we obtain

$$\theta_\delta(|x|)a^2(x) = (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}\left[1-\frac{0.5\lambda a^2(x)}{m^2+k^2}+\left(\frac{0.5\lambda a^2(x)}{m^2+k^2}\right)^2+\ldots\right]= \\ (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}-0.5\lambda(2\pi)^{-4}a^2(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^2}+ \\ -0.25\lambda^2(2\pi)^{-4}a^4(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^3}+\ldots= \\ =(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}+(2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n a^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}. \tag{3.2.6}$$

Let $\mathbb{R}^1$ be metric space equipped with distance $\rho[x_1,y_1] = |x_1-y_1|, x_1,y_1 \in \mathbb{R}$.
Define now a map $\mathcal{F}(x,a(x)) : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$\begin{aligned}&\mathcal{F}(x,a^2(x)) = \\ &(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)} + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n a^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} = \\ &(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)} - (2\pi)^{-4}\lambda/2\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n2}} + \\ &+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n a^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\end{aligned} \tag{3.2.7}$$

We rewrite now Eq.(3.2.7) of the form

$$\begin{aligned}&\mathcal{F}(x,b(x)) = \\ &g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n b^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} = \\ &g_0(x) - \lambda g_1(x)b(x) + \\ &+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n b^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\},\end{aligned} \tag{3.2.8}$$

where we let for a shortness

$$\begin{aligned}b(x) &= a^2(x), \\ g_0(x) &= (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}, \\ g_1(x) &= (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{2(m^2+k^2)^2}.\end{aligned} \tag{3.2.9}$$

In Eq.(3.2.8) - Eq.(3.2.9) we set for simplisity but without loss of generality $m^2 = 1$.
We define now infinite sequence by

$$\begin{aligned}q_n(x) &= \mathcal{F}(x,q_{n-1}(x)), n = 1,2,\ldots \\ q_0(x) &= g_0(x).\end{aligned} \tag{3.2.11}$$

From Eq.(3.2.8) and Eq.(3.2.9) we obtain

$$\begin{aligned}&q_1(x) = \mathcal{F}(x,g_0(x)) = \\ &g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n g_0^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} = \\ &g_0(x) - \lambda g_1(x)g_0(x) + \\ &+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n g_0^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}.\end{aligned} \tag{3.2.12}$$

From Eq.(3.2.12) we obtain

$$\begin{aligned}
&\rho[q_1(x),g_0(x)] = \rho[\mathcal{F}(x,g_0(x)),g_0(x)] = |q_1(x)-q_0(x)| = \\
&(2\pi)^{-4}\left|\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n g_0^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right| = \\
&\left|-\lambda g_1(x)g_0(x)+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n g_0^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right| \leq \\
&\lambda g_1(x)g_0(x)+(2\pi)^{-4}\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n g_0^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right| \leq \\
&\lambda g_1(x)g_0(x)+(2\pi)^{-4}\left(\sum_{n=2}^{\infty}(\lambda/2)^n g_0^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right) \leq \\
&\lambda g_1(x)g_0(x)+(2\pi)^{-4}\left(\sum_{n=2}^{\infty}(\lambda/2)^n g_0^n(x)\theta_\delta(|x|)\int\frac{d^4k}{(m^2+k^2)^{n+1}}\right).
\end{aligned}\tag{3.2.13}$$

From Eq.(3.2.8) we obtain

$$\begin{aligned}
&q_2(x) = \mathcal{F}(x,q_1(x)) = \\
&g_0(x)+(2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n q_1^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} = \\
&= g_0(x)-\lambda g_1(x)q_1(x)+ \\
&+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_1^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}.
\end{aligned}\tag{3.2.14}$$

From Eq.(3.2.12) and Eq.(3.2.14) we obtain

$$\rho[q_2(x),q_1(x)] = \rho[\mathcal{F}(x,q_1(x)),\mathcal{F}(x,q_0(x))] =$$

$$\rho[\mathcal{F}(x,q_1(x)),q_1(x)] = |\mathcal{F}(x,q_1(x)) - q_1(x)| =$$

$$\left|\left(g_0(x) - \lambda g_1(x)q_1(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_1^n(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right) -\right.$$

$$\left.-\left(g_0(x) - \lambda g_1(x)q_0(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_0^n(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right)\right| =$$

$$|\lambda g_1(x)(q_0(x) - q_1(x)) +$$

$$+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n[q_1^n(x) - q_0^n(x)]\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} \Big] =$$

$$|\lambda g_1(x)(q_0(x) - q_1(x)) -$$

$$-(2\pi)^{-4}\lambda(q_0(x) - q_1(x)) \times$$

$$\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_0^{n-1}(x) + q_0^{n-2}(x)q_1(x) + \ldots + q_0(x)q_1^{n-2}(x) + q_1^{n-1}(x)] \times\right. \quad (3.2.15)$$

$$\left.\times\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Bigg| =$$

$$\sqrt{\lambda}\,|(q_0(x) - q_1(x))|\Big|\Big[\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\ \times$$

$$\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_0^{n-1}(x) + q_0^{n-2}(x)q_1(x) + \ldots + q_0(x)q_1^{n-2}(x) + q_1^{n-1}(x)] \times\right.$$

$$\left.\times\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Big]\Bigg| \leq$$

$$\sqrt{\lambda}\,|(q_0(x) - q_1(x))|\left(\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\right)\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}(n-1)q_0^{n-1}(x)\right| \leq$$

$$\sqrt{\lambda}\,|(q_0(x) - q_1(x))|O(\lambda), O(\lambda) < 1.$$

In the last lines we applied formulas

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots\ + ab^{n-2} + b^{n-1}),$$

$$\int_0^\infty \frac{dr^3}{(m^2+r^2)^{n+1}} = \left(\frac{-1}{2(n-1)(m^2+r^2)^{n-1}} + \frac{m^2}{2n(m^2+r^2)^n}\right)\Bigg|_0^\infty = \quad (3.2.16)$$

$$\frac{1}{2(n-1)m^{2(n-1)}} - \frac{1}{2nm^{2(n-1)}} = \frac{1}{2m^{2(n-1)}}\left(\frac{1}{(n-1)} - \frac{1}{n}\right),$$

and Euler–Maclaurin sum formula

$$\sum_{n=a}^{\infty} f(n) = \int_a^\infty f(x)dx + 0.5f(a) - \sum_{i=2}^{k}\frac{b_i}{i!}f^{(i-1)}(a) - \int_a^\infty \frac{B_k(\{1-t\})}{k!}dt \quad (3.2.17)$$

where b_i is Bernoulli's numbers and $\{x\}$ is the fractional part of x. Note that

$$|q_1(x)| = |\mathcal{F}(x,g_0(x))| \leq g_0(x) + \lambda g_1(x)g_0(x) + \frac{(2\pi)^{-4}(\lambda/2)^2 g_0^2(x)}{1-(\lambda/2)g_0(x)}. \quad (3.2.18)$$

From Eq.(3.2.8) we obtain

$$
\begin{gathered}
|\mathcal{F}(x,b(x))| = \\
\left| g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n b^n(x)\int\frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right| \leq \\
g_0(x) + \lambda g_1(x)b(x) + \\
+(2\pi)^{-4}\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n b^n(x)\int\frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right| \leq \\
g_0(x) + \lambda g_1(x)b(x) + (2\pi)^{-4}\sum_{n=2}^{\infty}(\lambda/2)^n b^n(x)\int\frac{\theta_\delta(|x|)d^4k}{(m^2+k^2)^{n+1}} \leq \qquad (3.2.19) \\
g_0(x) + \lambda g_1(x)b(x) + (2\pi)^{-4}\sum_{n=2}^{\infty}(\lambda/2)^n b^n(x) = \\
g_0(x) + \lambda g_1(x)b(x) + (2\pi)^{-4}(\lambda/2)^2 b^2(x)\sum_{n=0}^{\infty}(\lambda/2)^n b^n(x) = \\
= g_0(x) + \lambda g_1(x)b(x) + \frac{(2\pi)^{-4}(\lambda/2)^2 b^2(x)}{1-(\lambda/2)b(x)}.
\end{gathered}
$$

From Eq.(3.2.8) and Eq.(3.2.14) we obtain

$$
\begin{gathered}
\rho[q_3(x), q_2(x)] = \rho[\mathcal{F}(x, q_2(x)), \mathcal{F}(x, q_1(x))] = \\
|\mathcal{F}(x, q_2(x)) - \mathcal{F}(x, q_1(x))| = \\
\left| \left(g_0(x) - \lambda g_1(x) q_2(x) + (2\pi)^{-4} \left\{ \sum_{n=2}^{\infty} (-1)^n (\lambda/2)^n q_2^n(x) \int \frac{d^4 k e^{ikx} \theta_\delta(|x|)}{(m^2 + k^2)^{n+1}} \right\} \right) - \right. \\
\left. - \left(g_0(x) - \lambda g_1(x) q_1(x) + (2\pi)^{-4} \left\{ \sum_{n=2}^{\infty} (-1)^n (\lambda/2)^n q_1^n(x) \int \frac{d^4 k e^{ikx} \theta_\delta(|x|)}{(m^2 + k^2)^{n+1}} \right\} \right) \right| = \\
|\lambda g_1(x)(q_2(x) - q_1(x)) + \\
+ (2\pi)^{-4} \left\{ \sum_{n=2}^{\infty} (-1)^n (\lambda/2)^n [q_2^n(x) - q_1^n(x)] \int \frac{d^4 k e^{ikx} \theta_\delta(|x|)}{(m^2 + k^2)^{n+1}} \right\} \Bigg| = \\
|\lambda g_1(x)(q_2(x) - q_1(x)) - \\
-(2\pi)^{-4} \lambda (q_2(x) - q_1(x)) \times \\
\left\{ \sum_{n=2}^{\infty} (-1)^n (\lambda/2)^{n-1} [q_1^{n-1}(x) + q_1^{n-2}(x) q_2(x) + \ldots + q_1(x) q_2^{n-2}(x) + q_2^{n-1}(x)] \times \right. \\
\left. \times \int \frac{d^4 k e^{ikx} \theta_\delta(|x|)}{(m^2 + k^2)^{n+1}} \right\} \Bigg| = \\
\sqrt{\lambda}\, |(q_2(x) - q_1(x))| \Big| \Big[\sqrt{\lambda}\, g_1(x) - (2\pi)^{-4} \sqrt{\lambda} \times \\
\left\{ \sum_{n=2}^{\infty} (-1)^n (\lambda/2)^{n-1} [q_1^{n-1}(x) + q_1^{n-2}(x) q_2(x) + \ldots + q_1(x) q_2^{n-2}(x) + q_2^{n-1}(x)] \times \right. \\
\left. \times \int \frac{d^4 k e^{ikx} \theta_\delta(|x|)}{(m^2 + k^2)^{n+1}} \right\} \Bigg] \Bigg| \leq \\
\sqrt{\lambda}\, |(q_1(x) - q_2(x))| \left(\sqrt{\lambda}\, g_1(x) - (2\pi)^{-4} \sqrt{\lambda} \right) \left| \sum_{n=2}^{\infty} (-1)^n (\lambda/2)^{n-1} (n-1) q_1^{n-1}(x) \right| \leq \\
\sqrt{\lambda}\, |(q_1(x) - q_2(x))| O(\lambda), O(\lambda) < 1.
\end{gathered} \tag{3.2.20}
$$

Processing inductively we get

$$\rho[q_{n+1}(x),q_n(x)] = \rho[\mathcal{F}(x,q_{n+1}(x)),\mathcal{F}(x,q_n(x))] =$$
$$|\mathcal{F}(x,q_{n+1}(x)) - \mathcal{F}(x,q_n(x))| =$$
$$\left|\left(g_0(x) - \lambda g_1(x)q_{n+1}(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_{n+1}^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right) -\right.$$
$$\left.-\left(g_0(x) - \lambda g_1(x)q_n(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_n^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right)\right| =$$
$$|\lambda g_1(x)(q_{n+1}(x) - q_n(x)) +$$
$$\left. +(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n[q_{n+1}^n(x) - q_n^n(x)]\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\ \right| =$$
$$|\lambda g_1(x)(q_{n+1}(x) - q_n(x)) -$$
$$-(2\pi)^{-4}\lambda(q_{n+1}(x) - q_n(x)) \times$$
$$\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_n^{n-1}(x) + q_n^{n-2}(x)q_{n+1}(x) + \ldots + q_n(x)q_{n+1}^{n-2}(x) + q_{n+1}^{n-1}(x)] \times\right. \tag{3.2.21}$$
$$\left.\times\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Bigg| =$$
$$\sqrt{\lambda}\,|(q_{n+1}(x) - q_n(x))|\Big|\Big[\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\ \times$$
$$\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_n^{n-1}(x) + q_n^{n-2}(x)q_{n+1}(x) + \ldots + q_n(x)q_{n+1}^{n-2}(x) + q_{n+1}^{n-1}(x)] \times\right.$$
$$\left.\times\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Big]\Big| \le$$
$$\sqrt{\lambda}\,|(q_{n+1}(x) - q_n(x))|\left(\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\right)\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}(n-1)q_n^{n-1}(x)\right| \le$$
$$\sqrt{\lambda}\,|(q_{n+1}(x) - q_n(x))|O(\lambda),\ O(\lambda) < 1.$$

Theorem 3.2.1.Infinite sequence defined by

$$q_n(x) = \mathcal{F}(x,q_{n-1}(x)),\ n = 1,2,\ldots$$
$$q_0(x) = g_0(x) \tag{3.2.22}$$

for any $x,|x| > \delta$ has a limit: $\lim_{n\to\infty} q_n(x) = \bar{b}(x)$ such that $\mathcal{F}(x,\bar{b}(x)) = \bar{b}(x)$,where

$$\mathcal{F}(x,b(x)) =$$
$$g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n b^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} =$$
$$g_0(x) - \lambda g_1(x)b(x) + \tag{3.2.23}$$
$$+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n b^n(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\},$$

and where

$$b(x) = a^2(x),$$
$$g_0(x) = (2\pi)^{-4}\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)},$$
$$g_1(x) = (2\pi)^{-4}\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{2(m^2+k^2)^2},$$
$$|x| > \delta. \quad (3.2.24)$$

Proof. Immediate from (3.2.21) by Generalized Banach fixed-point theorem, see Apendix3 Theorem 3.2..

3.3.Weak coupling.The $\lambda\phi_d^{2n}, n > 2$ theory.Exact nonperturbative solution.

Transcendental master equation (2.5.16) corresponding to two-point Eucledian Green function $G_{(4)}(x_1,x_2,m,\lambda)$ in Euclidean space E_4 with $\dim E_4 = 4$ reads

$$\frac{\left[a(x_1-x_2)^2\right]\left(m^2+\frac{\lambda}{5!}a(x_1-x_2)^4\right)^2}{\left(m^2+\frac{\lambda}{4!}a(x_1-x_2)^4\right)^2} - $$
$$-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2+\frac{\lambda}{4!}a(x_1-x_2)^4} = 0. \quad (3.3.1)$$

Assuming now for simplisity that $\lambda/m^2 \ll 1$, then from Eq.(3.3.1) we obtain

$$\theta_\delta(|x|)a^2(x) \simeq (2\pi)^{-4}\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)\left[1+\frac{\lambda a^4(x)}{4!(m^2+k^2)}\right]}. \quad (3.3.2)$$

Note that

$$\frac{1}{1+z} = 1-z+z^2-z^3+z^4-\ldots 1+\sum_{n=1}^{\infty}(-1)^n z^n$$
$$z > 0, z < 1. \quad (3.3.3)$$

From Eq.(3.3.3) by setting

$$z := \frac{\lambda a^4(x)}{4!(m^2+k^2)} \quad (3.3.4)$$

we obtain

$$\left(1+\frac{\lambda a^4(x)}{4!(m^2+k^2)}\right)^{-1} = 1-\frac{\lambda a^4(x)}{4!(m^2+k^2)}+\left(\frac{\lambda a^4(x)}{4!(m^2+k^2)}\right)^2-\ldots=$$
$$= 1+\sum_{n=1}^{\infty}(-1)^n\frac{\lambda a^4(x)}{4!(m^2+k^2)}. \quad (3.3.5)$$

From Eq.(3.3.2) using Eq.(3.3.5) we obtain

$$\theta_\delta(|x|)a^2(x) =$$
$$(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}\left[1-\frac{\lambda a^4(x)}{4!(m^2+k^2)}+\left(\frac{\lambda a^4(x)}{4!(m^2+k^2)}\right)^2+\ldots\right]=$$
$$(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{4!(m^2+k^2)}-\lambda(2\pi)^{-4}a^4(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{4!(m^2+k^2)^2}+ \tag{3.3.6}$$
$$-\lambda^2(2\pi)^{-4}a^8(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(4!)^2(m^2+k^2)^3}+\ldots=$$
$$=(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}+(2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/4!)^n a^{4n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}.$$

Let $\mathbb{R}^1$ be metric space equipped with distance $\rho[x_1,y_1]=|x_1-y_1|,x_1,y_1\in\mathbb{R}$.
Define now a map $\mathcal{F}(x,a(x)):\mathbb{R}^1\to\mathbb{R}^1$ by

$$\mathcal{F}(x,a^2(x)) =$$
$$(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}+(2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/4!)^n a^{4n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}=$$
$$(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}-(2\pi)^{-4}(\lambda/4!)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n2}}+ \tag{3.3.7}$$
$$+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/4!)^n a^{4n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}$$

We rewrite now Eq.(3.3.7) of the form

$$\mathcal{F}(x,b(x)) =$$
$$g_0(x)+(2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/4!)^n b^{4n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}=$$
$$g_0(x)-(\lambda/4!)g_1(x)b(x)+ \tag{3.2.8}$$
$$+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n b^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\},$$

where we let for a shortness

$$b(x)=a^2(x),$$
$$g_0(x)=(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}, \tag{3.3.9}$$
$$g_1(x)=(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{2(m^2+k^2)^2}.$$

In Eq.(3.3.8) - Eq.(3.3.9) we set for simplisity but without loss of generality $m^2=1$.
We define now infinite sequence by

$$q_n(x)=\mathcal{F}(x,q_{n-1}(x)),n=1,2,\ldots \tag{3.3.11}$$
$$q_0(x)=g_0(x).$$

From Eq.(3.3.8) and Eq.(3.3.9) we obtain

$$q_1(x) = \mathcal{F}(x, g_0(x)) =$$
$$g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/4!)^n g_0^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} =$$
$$g_0(x) - (\lambda/4!)g_1(x)g_0^2(x) +$$
$$+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/4!)^n g_0^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}. \quad (3.3.12)$$

From Eq.(3.3.12) we obtain

$$\rho[q_1(x), g_0(x)] = \rho[\mathcal{F}(x, g_0(x)), g_0(x)] = |q_1(x) - q_0(x)| = |q_1(x) - g_0(x)| =$$
$$(2\pi)^{-4}\left|\sum_{n=1}^{\infty}(-1)^n(\lambda/4!)^n g_0^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right| =$$
$$\left|-(\lambda/4!)g_1(x)g_0^2(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/4!)^n g_0^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right| \leq$$
$$(\lambda/4!)g_1(x)g_0^2(x) + (2\pi)^{-4}\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/4!)^n g_0^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right| \leq \quad (3.3.13)$$
$$(\lambda/4!)g_1(x)g_0^2(x) + (2\pi)^{-4}\left(\sum_{n=2}^{\infty}(\lambda/4!)^n g_0^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right) \leq$$
$$(\lambda/4!)g_1(x)g_0^2(x) + (2\pi)^{-4}\left(\sum_{n=2}^{\infty}(\lambda/4!)^n g_0^{2n}(x)\theta_\delta(|x|)\int\frac{d^4k}{(m^2+k^2)^{n+1}}\right).$$

From Eq.(3.3.8) we obtain

$$q_2(x) = \mathcal{F}(x, q_1(x)) =$$
$$g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n q_1^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} =$$
$$= g_0(x) - \lambda g_1(x)q_1^2(x) +$$
$$+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_1^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}. \quad (3.3.14)$$

From Eq.(3.3.12) and Eq.(3.3.14) we obtain

$$\rho[q_2(x), q_1(x)] = \rho[\mathcal{F}(x, q_1(x)), \mathcal{F}(x, q_0(x))] =$$

$$= \rho[\mathcal{F}(x, q_1(x)), q_1(x)] = |\mathcal{F}(x, q_1(x)) - q_1(x)| =$$

$$\left|\left(g_0(x) - \lambda g_1(x) q_1^2(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_1^{2n}(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right) -\right.$$

$$\left.-\left(g_0(x) - \lambda g_1(x) q_0^2(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_0^{2n}(x)\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right)\right| =$$

$$|\lambda g_1(x)(q_0^2(x) - q_1^2(x)) +$$

$$\left.+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n[q_1^{2n}(x) - q_0^{2n}(x)]\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\ \right| =$$

$$|\lambda g_1(x)(q_0^2(x) - q_1^2(x)) -$$

$$-(2\pi)^{-4}\lambda(q_0(x) - q_1(x)) \times \tag{3.3.15}$$

$$\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_0^{2n-1}(x) + q_0^{2n-2}(x)q_1(x) + \ldots + q_0(x)q_1^{2n-2}(x) + q_1^{2n-1}(x)] \times\right.$$

$$\left.\times\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right| =$$

$$\sqrt{\lambda}\,|(q_0(x) - q_1(x))||(q_0(x) + q_1(x))|\left|\left[\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\ \times\right.\right.$$

$$\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_0^{2n-1}(x) + q_0^{2n-2}(x)q_1(x) + \ldots + q_0(x)q_1^{2n-2}(x) + q_1^{2n-1}(x)] \times\right.$$

$$\left.\left.\left.\times\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\ \right]\right| \le \sqrt{\lambda}\,|(q_1(x) - q_0(x))| \times c_1,$$

$$c_1 \le 1.$$

In the last lines we applied formula

$$a^{2n} - b^{2n} = (a-b)(a^{2n-1} + a^{2n-2}b + a^{2n-3}b^2 + \ldots\ + ab^{2n-2} + b^{2n-1}) \tag{3.3.16}$$

and Euler–Maclaurin sum formula

$$\sum_{n=a}^{\infty} f(n) = \int_a^{\infty} f(x)dx + 0.5f(a) - \sum_{i=2}^{k}\frac{b_i}{i!}f^{(i-1)}(a) - \int_a^{\infty}\frac{B_k(\{1-t\})}{k!}dt \tag{3.3.17}$$

where b_i is Bernoulli's numbers and $\{x\}$ is the fractional part of x. Note that

$$|q_1(x)| = |\mathcal{F}(x, g_0(x))| \le g_0(x) + \lambda g_1(x) g_0^2(x) + \frac{(2\pi)^{-4}(\lambda/2)^2 g_0^2(x)}{1 - (\lambda/2)g_0^2(x)}. \tag{3.3.18}$$

From Eq.(3.3.8) we obtain

$$|\mathcal{F}(x,b(x))| =$$

$$\left| g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n b^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right| \leq$$

$$g_0(x) + \lambda g_1(x)b^2(x) +$$

$$+(2\pi)^{-4}\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n b^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right| \leq$$

$$g_0(x) + \lambda g_1(x)b^2(x) + (2\pi)^{-4}\sum_{n=2}^{\infty}(\lambda/2)^n b^{2n}(x)\int\frac{\theta_\delta(|x|)d^4k}{(m^2+k^2)^{n+1}} \leq \qquad (3.3.19)$$

$$g_0(x) + \lambda g_1(x)b^2(x) + (2\pi)^{-4}\sum_{n=2}^{\infty}(\lambda/2)^n b^{2n}(x) =$$

$$g_0(x) + \lambda g_1(x)b(x) + (2\pi)^{-4}(\lambda/2)^2 b^2(x)\sum_{n=0}^{\infty}(\lambda/2)^n b^{2n}(x) =$$

$$= g_0(x) + \lambda g_1(x)b^2(x) + \frac{(2\pi)^{-4}(\lambda/2)^2 b^2(x)}{1-(\lambda/2)b^2(x)}$$

Processing inductively we get

$$\rho[q_{n+1}(x),q_n(x)] = \rho[\mathcal{F}(x,q_{n+1}(x)),\mathcal{F}(x,q_n(x))] =$$
$$|\mathcal{F}(x,q_{n+1}(x)) - \mathcal{F}(x,q_n(x))| =$$
$$\left|\left(g_0(x) - \lambda g_1(x)q_{n+1}^2(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_{n+1}^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right) -\right.$$
$$\left.-\left(g_0(x) - \lambda g_1(x)q_n^2(x) + (2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n q_n^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\right)\right| =$$
$$|\lambda g_1(x)(q_{n+1}(x) - q_n(x))(q_{n+1}(x) + q_n(x)) +$$
$$+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n[q_{n+1}^{2n}(x) - q_n^{2n}(x)]\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Bigg| =$$
$$|\lambda g_1(x)(q_{n+1}(x) - q_n(x))(q_{n+1}(x) + q_n(x)) -$$
$$-(2\pi)^{-4}\lambda(q_{n+1}(x) - q_n(x)) \times$$
$$\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_n^{2n-1}(x) + q_n^{2n-2}(x)q_{n+1}(x) + \ldots + q_n(x)q_{n+1}^{2n-2}(x) + q_{n+1}^{2n-1}(x)] \times\right. \tag{3.3.21}$$
$$\left.\times\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Bigg| =$$
$$\sqrt{\lambda}\,|(q_{n+1}(x) - q_n(x))||(q_{n+1}(x) + q_n(x))|\Big|\Big[\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\ \times$$
$$\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}[q_n^{2n-1}(x) + q_n^{2n-2}(x)q_{n+1}(x) + \ldots + q_n(x)q_{n+1}^{2n-2}(x) + q_{n+1}^{2n-1}(x)] \times\right.$$
$$\left.\times\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\}\Big]\Big| \leq$$
$$\sqrt{\lambda}\,|(q_{n+1}(x) - q_n(x))| \times |(q_{n+1}(x) + q_n(x))| \times$$
$$\left(\sqrt{\lambda}\,g_1(x) - (2\pi)^{-4}\sqrt{\lambda}\right)\left|\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^{n-1}(n-1)q_n^{2n-1}(x)\right| \leq$$
$$\sqrt{\lambda}\,|(q_{n+1}(x) - q_n(x))|c_n,$$
$$c_n \leq 1.$$

Theorem 3.3.1.Infinite sequence defined by

$$q_n(x) = \mathcal{F}(x,q_{n-1}(x)), n = 1,2,\ldots$$
$$q_0(x) = g_0(x) \tag{3.3.22}$$

for any $x, |x| > \delta$ has a limit: $\lim_{n\to\infty} q_n(x) = \bar{b}(x)$ such that $\mathcal{F}(x,\bar{b}(x)) = \bar{b}(x)$, where

$$\mathcal{F}(x,b(x)) =$$
$$g_0(x) + (2\pi)^{-4}\left\{\sum_{n=1}^{\infty}(-1)^n(\lambda/2)^n b^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\} =$$
$$g_0(x) - \lambda g_1(x)b^2(x) + \tag{3.3.23}$$
$$+(2\pi)^{-4}\left\{\sum_{n=2}^{\infty}(-1)^n(\lambda/2)^n b^{2n}(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^{n+1}}\right\},$$

and where

$$
\begin{aligned}
b(x) &= a^2(x),\\
g_0(x) &= (2\pi)^{-4}\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{(m^2+k^2)},\\
g_1(x) &= (2\pi)^{-4}\int \frac{d^4k e^{ikx}\theta_\delta(|x|)}{2(m^2+k^2)^2},\\
|x| &> \delta.
\end{aligned}
\tag{3.3.24}
$$

Proof. Immediate from (3.3.21) by Generalized Banach fixed-point theorem, see Apendix3 Theorem 3.2..

Apendix1.

$$
G_{(D)F}(x_1-x_2;m) = (2\pi)^{-4}\int d^Dk\frac{e^{ik(x_1-x_2)}}{|k|^2+m^2}, \tag{1}
$$

where $d^Dk = dk_0dk_1\bullet\bullet\bullet dk_{D-1}, |k| = \left(\sum_{i=0}^{D-1}k_i^2\right)^{1/2}$.

In spherical coordinate system with $r = |k| = \left(\sum_{i=0}^{3}k_i^2\right)^{1/2}, \varphi_1$- angle between vector x_1-x_2 and vector k we obtain

$$
\begin{aligned}
&G_{(4)F}(x_1-x_2;m) = (2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{|k|^2+m^2} = \\
&(2\pi)^{-4}\int_0^\infty\int_0^\pi\int_0^\pi\int_0^{2\pi}\frac{e^{i|x_1-x_2|\times r\cos\varphi_1}}{r^2+m^2}\times r^3dr\sin^2\varphi_1\sin\varphi_2 d\varphi_1 d\varphi_2 d\varphi_3 = \\
&= \frac{1}{\pi(2\pi)^{3/2}\Gamma(3/2)}\int_0^\infty\int_0^\pi\frac{e^{i|x_1-x_2|\times r\cos\varphi_1}}{r^2+m^2}\times r^3dr\sin^2\varphi_1 d\varphi_1.
\end{aligned}
\tag{2}
$$

Note that

$$
\int_0^\pi e^{i|x_1-x_2|\times r\cos\varphi_1}\sin^2\varphi_1 d\varphi_1 = 2\sqrt{\pi}\,\Gamma(3/2)\frac{J_1(r|x_1-x_2|)}{r|x_1-x_2|}, \tag{3}
$$

where $J_1(z)$ is the Bessel function of the first kind, integer order 1, (see Definition 1) and

$$
\int_0^\infty\frac{r^2}{r^2+m^2}J_1(r|x_1-x_2|)dr = mK_1(m|x_1-x_2|), \tag{4}
$$

where $K_1(z)$ is the modified Bessel function of the second kind, integer order 1,(see Definition 2) **and** where we used formula 6.566.2 of [12]. Thus finally we get

$$
\begin{aligned}
&G_{(4)F}(x_1-x_2) = \lim_{\tau\to\infty}\langle\phi(x_1,\tau;\omega)\phi(x_2,\tau;\omega)\rangle = \\
&(2\pi)^{-2}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2} = (2\pi)^{-2}\left(\frac{m}{|x|}\right)K_1(m|x_1-x_2|).
\end{aligned}
\tag{5}
$$

where K_1 is the modified Bessel functions of the second kind, integer order 1.

Definition 1. The Bessel function of the first kind $J_1(z)$, integer order 1 is defined by the power series:

$$J_1(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!}\left(\frac{z}{2}\right)^{2k+1}, \tag{6}$$

where $z \in \mathbb{C}$. For $z \to 0$ we have:

$$J_1(z) \asymp z/2. \tag{7}$$

Definition 2.The modified Bessel functions of the first kind $I_\nu(z)$ with $\nu \in \mathbb{R}$ and $z \in \mathbb{C}$ are defined by the power series

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{2k+\nu}. \tag{8}$$

The modified Bessel functions of the second kind $K_\nu(z)$ are defined by

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}. \tag{9}$$

Remark 1. Note that

$$\begin{gathered} 1. K_\nu(m|x|) \asymp -\ln\left(\frac{m|x|}{2}\right) - \gamma \text{ as } m|x| \to 0, \nu = 0, \\ 2. K_\nu(m|x|) \asymp \left(\frac{2}{m|x|}\right)^\nu \text{ as } m|x| \to 0, \nu > 0, \\ 3. K_1(m|x|) \asymp \frac{1}{m|x|} \text{ as } m|x| \to 0, \end{gathered} \tag{10}$$

where γ is the Euler–Mascheroni constant. From Eq.(5) and Eq.(10) we get

$$\begin{gathered} G_{(4)F}(x_1 - x_2) = \lim_{\tau\to\infty}\langle\phi(x_1,\tau;\omega)\phi(x_2,\tau;\omega)\rangle = \\ (2\pi)^{-2}\int d^4k \frac{e^{ik(x_1-x_2)}}{k^2+m^2} = (2\pi)^{-2}\left(\frac{m}{|x|}\right)K_1(m|x_1-x_2|) \asymp (2\pi)^{-2}|x|^{-2} \end{gathered} \tag{11}$$

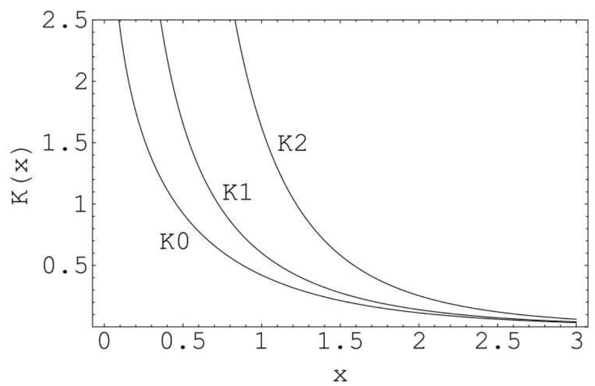

Figure 3.2.1.Plot of the modified Bessel functions of the second kind,integer order 0,1,2.

Definition3.

$$G_{(D)F}(x_1 - x_2) = \lim_{\tau\to\infty}\langle\phi(x_1,\tau;\omega)\phi(x_2,\tau;\omega)\rangle = \\ = (2\pi)^{-D}\int d^D k \frac{e^{ik(x_1-x_2)}}{k^2+m^2}. \tag{12}$$

The integrand in Eq.(12) is not radial but one can choose a frame where $k_\mu x^\mu = -kr\cos\theta,\quad k \equiv |k_\mu k^\mu| = \sqrt{\sum_{\mu=1}^{D} k_\mu^2}, r = |x_\mu x^\mu| = \sqrt{\sum_{\mu=1}^{D} x_\mu^2}, \mu = 1,2,\ldots,D$ and the angular integral reads

$$\int d\Omega_D\, e^{-ik\cdot x} = \Omega_{D-1}\int_0^\pi d\theta(\sin\theta)^{D-2}e^{ikr\cos\theta} \\ = \Omega_{D-1}\sqrt{\pi}\,\Gamma\left(\frac{D-1}{2}\right)\left(\frac{2}{kr}\right)^{\frac{D}{2}-1} J_{\frac{D}{2}-1}(kr), \tag{13}$$

where $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$ is the volume of the unit D-ball. In the last line we used formula 3.915.5 of [12] Then,

$$G_{(D)}(k;m) = \Gamma\left(\frac{D}{2}\right)\left(\frac{2}{k}\right)^{\frac{D}{2}-1}\int_0^{+\infty} dr\, r^{\frac{D}{2}}[\Omega_D G(r)] J_{\frac{D}{2}-1}(kr). \tag{14}$$

Now we take the massive propagator in momentum space:

$$G_{(D)}(k;m) = -\frac{\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D\alpha}{2}\right)}\left(\frac{2}{k}\right)^{\frac{D}{2}-1}\left(\frac{m}{2}\right)^{\frac{D}{2}-1}\times \\ \times\int_0^{+\infty} dr\, r K_{\frac{D}{2}-1}(mr) J_{\frac{D}{2}-1}(kr) == -m^{-2}F\left(\frac{D}{2},1;\frac{D}{2};-\frac{k^2}{m^2}\right) = \\ = -\frac{1}{k^2+m^2} \tag{15}$$

where we used formula 6.576.3 and formula 9.121.1 of [12] with $D = 4$, and F is the hypergeometric function

$$F(a,b;c;z) = \sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+n)}\frac{z^n}{n!}. \tag{16}$$

Remark 2.Note that when $D = 4$, we exactly obtain massive propagator $G_{(4)}(k;m)$ in momentum space, see Eqs.(1-5).

Remark 3. If $D \geq 5$ Definition 3 no longer holds, since integral in RHS of Eq.(12) diverges as $|k'|^{\frac{D}{2}-1}, k' \to \infty$ since the estimate (17) holds

$$\int d^D k \frac{e^{ik(x_1-x_2)}}{k^2+m^2} \sim \int_0^\infty |k|^{\frac{D}{2}-2} d|k|. \tag{17}$$

Remark 4. In order to avoid these difficultness mentioned above, we replace Eq.(12) in def.3 by the weak limit taiken in space $\mathcal{L}'(\mathbb{R}_x^D), D \geq 5$

$$G_{(D)F}(x_1 - x_2) = w\text{-}\lim_{\eta\to 0_+} G_{(D)}(k;m,\eta) = w\text{-}\lim_{\eta\to 0_+}\int d^D k \frac{e^{-\eta|k| + ik(x_1-x_2)}}{k^2+m^2} \\ \eta > 0 \tag{18}$$

Notice the definition by Eq.(18) is correct since in space $\mathcal{L}'(\mathbb{R}_k^D)$:

$$w\text{-}\lim_{\eta\to 0_+} e^{-\eta|k|}(k^2+m^2)^{-1} = (k^2+m^2)^{-1}, \tag{19}$$

and $\int (k^2+m^2)^{-1} d^D k < \infty$.

Apendix 2.Correlation functions for the $(\lambda\varphi^4)_d$ scalar theory in canonical Parisi and Wu stochastic quantization.

In this section we will analyze the canonical stochastic quantization for the $(\lambda\varphi^4)_d$ self-interaction scalar theory in comparison with double stochastic quantization for the $(\lambda\varphi^4)_d$ scalar theory. In this case the Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -(-\Delta + m_0^2)\varphi(\tau,x) - \frac{\lambda}{3!}\varphi^3(\tau,x) + \eta(\tau,x). \qquad (2.1)$$

The two-point correlation function associated with the random field is given by the Einstein relation, while the other connected correlation functions vanish, i.e.,

The two-point correlation function associated with the random field is given by the Einstein relation, while the other connected correlation functions vanish, i.e.,

$$\langle\,\eta(\tau_1,x_1)\eta(\tau_2,x_2)\ldots\eta(\tau_{2k-1},x_{2k-1})\,\rangle_\eta = 0, \qquad (2.2)$$

and also

$$\langle\eta(\tau_1,x_1)\ldots\eta(\tau_{2k},x_{2k})\,\rangle_\eta = \sum\langle\eta(\tau_1,x_1)\eta(\tau_2,x_2)\,\rangle_\eta\langle\,\eta(\tau_k,x_k)\eta(\tau_l,x_l)\,\rangle_\eta\ldots, \qquad (2.3)$$

where the sum is to be taken over all the different ways in which the $2k$ labels can be divided into k parts, i.e., into k pairs. Performing Gaussian averages over the white random noise, it is possible to prove that perturbatively [6]

$$\lim_{\tau\to\infty}\langle\,\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\ldots\varphi(\tau_n,x_n)\,\rangle_\eta = \frac{\int D[\varphi]\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\,e^{-S(\varphi)}}{\int D[\varphi]\,e^{-S(\varphi)}}, \qquad (2.4)$$

where $S(\varphi) = S_0(\varphi) + S_I(\varphi)$ is the d-dimensional action. This result leads us to consider the Euclidean path integral measure a stationary distribution of a stochastic process. Note that the solution of the Langevin equation needs a given initial condition. As for example

$$\varphi(\tau,x)|_{\tau=0} = \varphi_0(x). \qquad (2.5)$$

Let us use the Langevin equation to perturbatively solve the interacting field theory. One way to handle the Eq.(2.1) is with the method of Green's functions. We defined the retarded Green function for the diffusion problem in the Eq.(2.1). Let us assume that the coupling constant is a small quantity. Therefore to solve the Langevin equation
in the case of a interacting theory we use a perturbative series in λ. Therefore we can write

$$\varphi(\tau,x) = \varphi^{(0)}(\tau,x) + \lambda\varphi^{(1)}(\tau,x) + \lambda^2\varphi^{(2)}(\tau,x) + \ldots \qquad (2.6)$$

Substituting the Eq.(2.6) in the Eq.(2.1), and if we equate terms of equal power in λ, the resulting equations are

$$\left[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2)\right]\varphi^{(0)}(\tau,x) = \eta(\tau,x), \qquad (2.7)$$

$$\left[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2)\right]\varphi^{(1)}(\tau,x) = -\frac{1}{3!}(\varphi^{(0)}(\tau,x))^3, \qquad (2.8)$$

$$\left[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2)\right]\varphi^{(1)}(\tau,x) = -\frac{1}{3!}(\varphi^{(0)}(\tau,x))^3, \qquad (2.9)$$

and so on. Using the retarded Green function and assuming that $\varphi^{(q)}(\tau,x)|_{\tau=0} = 0$,

$\forall\, q$, the solution to the first equation given by Eq.(2.7) can be written formally as

$$\varphi^{(0)}(\tau,x) = \int_0^\tau d\tau' \int_\Omega d^dx'\, G(\tau-\tau', x-x')\eta(\tau',x'). \qquad (2.10)$$

The second equation given by Eq.(2.8) can also be solved using the above result. We obtain

$$\varphi^{(1)}(\tau,x) = -\frac{1}{3!}\int_0^\tau d\tau_1 \int_\Omega d^dx_1\, G(\tau-\tau_1, x-x_1) + \\ +\Big(\int_0^{\tau_1} d\tau' \int_\Omega d^dx'\, G(\tau_1-\tau', x_1-x')\eta(\tau',x')\Big)^3. \qquad (2.11)$$

We have seen that we can generate all the tree diagrams with the noise field contributions. We can also consider the n-point correlation function $\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\ldots\varphi(\tau_n,x_n)\rangle_\eta$. Substituting the above results in the n-point correlation function, and taking the random averages over the white noise field using the Wick-decomposition property defined by Eq.(2.3) we generate the stochastic diagrams.

Each of these stochastic diagrams has the form of a Feynman diagram, apart from the fact that we have to take into account that we are joining together two white random noise fields many times.

As simple examples let us show how to derive the two-point function in the zeroth order $\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle_\eta^{(0)}$, and also the first order correction to the scalar two-point-function given by $\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle_\eta^{(1)}$. Using the Eq.(2.10) and the Einstein relations we have

$$\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle_\eta^{(0)} = \\ 2\int_0^{\min(\tau_1,\tau_2)} \int d\tau' \int_\Omega d^dx'\, G(\tau_1-\tau', x_1-x')\, G(\tau_2-\tau', x_2-x'), \qquad (2.12)$$

which is just Eq.(2.13). For the first order correction we get:

$$\langle\varphi(X_1)\varphi(X_2)\rangle_\eta^{(1)} = -\frac{\lambda}{3!}\langle\int dX_3 \int dX_4 (G(X_1-X_4)G(X_2-X_3) + \\ G(X_1-X_3)G(X_2-X_4))\eta(X_3)\Big(\int dX_5\, G(X_4-X_5)\eta(X_5)\Big)^3\rangle_\eta. \qquad (2.13)$$

where, for simplicity, we have introduced a compact notation:

$$\int_0^\tau d\tau \int_\Omega d^dx \equiv \int dX, \qquad (2.14)$$

and also $\varphi(\tau,x) \equiv \varphi(X)$ and finally $\eta(\tau,x) \equiv \eta(X)$.

The process can be repeated and therefore the stochastic quantization can be used as an alternative approach to describe scalar quantum fields. We stress here that the stochastic quantization is based in the fact that although one starts with the system out of equilibrium, the Markovian Langevin equation forces it into equilibrium. Moreover, when the thermodynamic equilibrium is reached, the stochastic expectation values will coincide with the Schwinger functions of the Euclidean field theory.

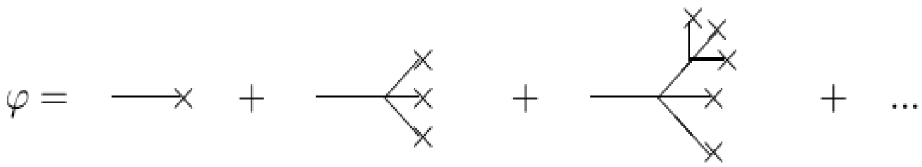

Figure1:Perturbative expansion for the scalar field
where crosses denote noise fields.

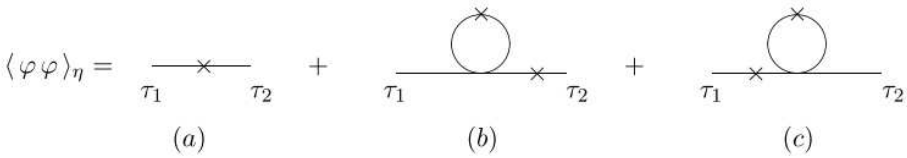

Figure2:The corrections up to one-loop to the
two-point correlation function.

We can represent Eq.(6) graphically as figure (1) (the random noise field is represented by a cross). Using this diagrammatical expansion, it is possible to show that the two-point correlation function up to one-loop level is given by figure (2), where we represent the retarded Green function by a line and the free two-point function by a crossed line. The rules to obtain the algebraic values of the stochastic diagrams are similar to the usual Feynman rules. For instance the two-point function at one-loop level is given by

$$(b) = -\frac{\lambda}{2}\,\delta^d(k_1+k_2)\int d^dk\int_0^{\tau_1} d\tau\, G(k_1;\tau_1-\tau)D(k;\tau,\tau)D(k_2;\tau_2,\tau). \qquad (2.15)$$

$$(c) = -\frac{\lambda}{2}\,\delta^d(k_1+k_2)\int d^dk\int_0^{\tau_2} d\tau\, G(k_2;\tau_2-\tau)D(k;\tau,\tau)D(k_1;\tau_1,\tau). \qquad (2.16)$$

A simple computation shows that we recover the correct equilibrium result at equal asymptotic Markov parameters ($\tau_1 = \tau_2 \to \infty$):

$$(b)|_{\tau_1=\tau_2\to\infty} = -\frac{\lambda}{2}\,\delta^d(k_1+k_2)\frac{1}{(k_2^2+m_0^2)}\,\frac{1}{(k_1^2+k_2^2+2m_0^2)}\int\frac{d^dk}{(k^2+m_0^2)}. \qquad (2.17)$$

Obtaining the Schwinger functions in the asymptotic limit does not guarantee that we gain a finite physical theory. The next step is to implement a suitable regularization scheme. A crucial point to find a satisfactory regularization scheme is to use one that preserves the symmetries of the original model. In the stochastic regularization method the symmetries of the physical theory is maintained. There are in general two different ways to implement the stochastic regularization. The first one is to start from a Langevin equation with a memory kernel. It is known from the literature [9] that this method can at best only remove two degrees of divergence. Another possibility is to smear only the noise field in the probability functional [10]-[11]:

$$\langle F[\varphi]\rangle_\eta = \frac{\int D[\eta]F[\varphi]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\int d\tau'\,\eta(\tau,x)K_\Lambda^{-1}\eta(\tau',x)\Big]}{\int D[\eta]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\int d\tau'\,\eta(\tau,x)K_\Lambda^{-1}\eta(\tau',x)\Big]}, \tag{2.18}$$

where K_Λ is a memory kernel. In this case we change the Einstein relations of the noise

field to:

$$\langle\,\eta(\tau,x)\,\eta(\tau',x')\,\rangle_\eta = 2K_\Lambda(\tau,\tau'^d(x-x'). \tag{2.19}$$

The smearing function should be chosen such that, when $\Lambda\to\infty$:

$$\lim_{\Lambda\to\infty}K_\Lambda(\tau-\tau') = \delta(\tau-\tau'), \tag{2.20}$$

recovering the usual theory.

Since the Langevin equation is unaffected by the stochastic regularization, the physical field is the same as in the regularized case. However, the zeroth-order two-point correlation function is given by:

$$\begin{aligned} D(k;\tau,\tau') &= 2\delta^d(k+k')\int_0^\tau ds\int_0^{\tau'}ds'\,G(k;\tau-s)\,G(k;\tau'-s')K_\Lambda(s-s') = \\ &2\delta^d(k+k')\int_0^\tau ds\int_0^{\tau'}ds'\exp[-(\tau+\tau'-s-s'^2+m_0^2)]K_\Lambda(s-s'). \end{aligned} \tag{2.21}$$

It is possible to prove that a necessary condition that the regularization function K_Λ should satisfy in order to render the divergent loops finite is $K_\Lambda(\tau)\ \mid_{\tau=0}\ = 0$. The following series of kernels obeying this condition were proposed:

$$K_\Lambda^{(n)}(\tau) = \frac{1}{2n!}\Lambda^2(\Lambda^2\ \mid\ \tau\ \mid)^n\exp[-\Lambda^2\ \mid\ \tau\ \mid]. \tag{2.22}$$

For the case $n=0$ we obtain, for the free two-point correlation function:

$$\lim_{\tau\to\infty}D(k;\tau,\tau) = \frac{\delta^d(k+k')}{(k^2+m_0^2)}\,\frac{\Lambda^2}{(\Lambda^2+k^2+m_0^2)}. \tag{2.23}$$

Since the stochastic diagrams contains crossed lines in its loops, we have that the ultraviolet divergences can be regularized choosing an appropriate n. Note that it is possible to use a different regulator of the type $K_\sigma(\tau) = \frac{1}{2}\,\sigma\,\tau^{\sigma-1}$.

We now return to φ_4^4 theory. To first order in λ, and in momentum space, the two point function in Euclidean space is

$$G(p,-p) = \frac{1}{p^2+m^2} + \frac{-\lambda}{2}\left(\int\frac{d^4q}{(2\pi)}\,\frac{1}{p^2+m^2}\right)\left(\frac{1}{p^2+m^2}\right) + O(\lambda^2). \tag{2.24}$$

Let us define now by $\mu_{\mathbf{ren}}^2$ the effective or renormalized mass (squared) such that

$$\frac{1}{p^2+\mu_{\mathbf{ren}}^2} = \frac{1}{p^2+m^2}\left\{1-\frac{\lambda}{2}\left(\int\frac{d^4q}{(2\pi)}\,\frac{1}{p^2+m^2}\right)\left(\frac{1}{p^2+m^2}\right)+O(\lambda^2)\right\}. \tag{2.25}$$

Again, to first order in λ, we can write the equivalent equation

$$G(p,-p) = \frac{1}{p^2+m^2+\frac{\lambda}{2}\int\frac{d^4q}{(2\pi)}\,\frac{1}{p^2+m^2}} + O(\lambda^2). \tag{2.26}$$

This equation leads us to define $\mu_{\mathbf{ren}}^2$ by

$$\mu^2_{\mathbf{ren}} = m^2 + \frac{\lambda}{2}\int \frac{d^4q}{(2\pi)}\frac{1}{p^2+m^2} = m^2 + \delta m^2, \tag{2.27}$$

and the quantity $\mu_{\mathbf{ren}}$ represents the physical (or renormalized) mass of the particle. Thus, the interaction changes the mass of the particle. However $\delta m^2 = \infty$ since that

$$\int \frac{d^4q}{(2\pi)}\frac{1}{p^2+m^2} \simeq \int_0^\infty \frac{q^3dq}{q^2} = \int_0^\infty qdq = \infty. \tag{2.28}$$

This is an example of a typical “ultraviolet” divergence in canonical perturbative expansion in quantum field theory.

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Apendix. 3.Generalized Banach fixed-point theorem.

Theorem 3.1. [15] Let (X,d) be a complete metric space and $T : X \to X$ be a contraction
mapping with the Lipschitz constant $\alpha \in [0,1)$. Then:

(1) T has a unique fixed point $\bar{x}$ in X.

(2) For an <u>arbitrary</u> point $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$generated by the Picard iteration process (defined by $x_{n+1} = T(x_n), n \in \mathbb{N} \cup \{0\}$) converges to $\bar{x}$.

(3) $d(x_n,\bar{x}) \leq \frac{\alpha^n}{1-\alpha}\, d(x_0,x_1)$ for all $n \in \mathbb{N}$.

Definition 3.1. [15] We say that $\bar{x} \in X$ is a contractive fixed point (abbr. CFP) of T if $\bar{x} = T(\bar{x})$ and the Picard iterates $T_n(x)$ converge to $\bar{x}$ as $n \to \infty$ for all $x \in X$.

The operator T from Definition 3.1 became known as a Picard operator (shortly PO),

Definition 3.2. We say that $\bar{x} \in X$ is a quasy contractive fixed point (abbr. QCFP) of T if $\bar{x} = T(\bar{x})$ and there is an $x_0 \in X$ such that the Picard iterates $T_n(x_0)$ converge to $\bar{x}$ as $n \to \infty$.

Definition 3.3.The operator T from Definition 3.2 became known as a quasy Picard operator (abbr. QPO)

Theorem 3.2.(Generalized Banach fixed-point theorem). Let (X,d) be a complete metric
space and $T : X \to X$ is continuous.Assume that there is an $y_0 \in X$ such that the inequalities are satisfied

$$\begin{gathered} d(y_1,y_0) = d(T(y_0),y_0) \leq \alpha \times const, \\ d(y_2,y_1) = d(T(y_1),T(y_0)) \leq \alpha d(y_1,y_0), \\ d(y_3,y_2) = d(T(y_2),T(y_1)) \leq \alpha d(y_2,y_1), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ d(y_{n+1},y_n) = d(T(y_n),T(y_{n-1})) \leq \alpha d(y_n,y_{n-1}), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{gathered} \tag{3.1}$$

where $\alpha < 1$, $y_n = T(y_{n-1})$. Ten there is a quasy contractive fixed point $\bar{x}$ of T, such that the Picard iterates $T_n(y_0)$ converge to $\bar{y} \in X$ as $n \to \infty$.

Proof. From inequalities (3.1) we obtain

$$d(y_1,y_0) = d(T(y_0),y_0) \leq \alpha \times const,$$
$$d(y_2,y_1) = d(T(y_1),T(y_0)) \leq \alpha d(y_1,y_0),$$
$$d(y_3,y_2) = d(T(y_2),T(y_1)) \leq \alpha d(y_2,y_1) \leq \alpha^2 d(y_1,y_0), \quad (3.2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$d(y_{n+1},y_n) = d(T(y_n),T(y_{n-1})) \leq \alpha d(y_n,y_{n-1}) \leq \alpha^n d(y_1,y_0),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

From inequalities (3.2) by using triangle inequality we obtain for some $\varepsilon \ll 1$

$$d(y_n,y_{n+m}) \leq d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) +\ldots+ d(y_{n+m-1},y_{n+m}) \leq$$
$$\leq (\alpha^n + \alpha^{n+1} +\ldots+ \alpha^{n+m-1}) d(y_1,y_0) = \alpha^n \frac{1-\alpha^{n+m}}{1-\alpha} d(y_1,y_0) < \quad (3.3)$$
$$< \frac{\alpha^n}{1-\alpha} d(y_1,y_0) < \varepsilon.$$

Thus a sequence $\{y_n\}_{n=0}^{\infty} \subset X$ is a fundamental sequence in X and there is exists $\lim_{n\to\infty} y_n = \bar{y} \in X$. We assume now that $T(\bar{y}) = \bar{\bar{y}}$, by using triangle inequality we obtain

$$d(\bar{y},\bar{\bar{y}}) \leq d(\bar{y},y_n) + d(y_n,y_{n+1}) + d(y_{n+1},\bar{\bar{y}}). \quad (3.4)$$

For any $\varepsilon > 0$ we can choose $N = N(\varepsilon)$ such that for any $n \geq N$
(i) $d(\bar{y},y_n) < \varepsilon/3$, since that is $\bar{y} = \lim_{n\to\infty} y_n$;
(ii) $d(y_n,y_{n+1}) < \varepsilon/3$, since that $\{y_n\}_{n=0}^{\infty} \subset X$ is a fundamental sequence;
(iii) $d(y_{n+1},\bar{\bar{y}}) = d(T(y_n),T(\bar{y})) \leq \varepsilon/3$, since that is $\lim_{n\to\infty} T(y_n) = T(\lim_{n\to\infty} y_n) = T(\bar{y})$.
Therefore for any ϵ such that $0 < \epsilon < \varepsilon : d(\bar{y},\bar{\bar{y}}) \leq \epsilon$ it follows that $d(\bar{y},\bar{\bar{y}}) = 0$ and consequently $\bar{\bar{y}} = \bar{y}$.

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