Double Stochastic Quantization:Non perturbative approach to Stochastic Quantization $\lambda\varphi^{2n}$model Quantum Field Theory

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Abstract:The 5th-time stochastic-quantization approach to field theory proposed by Parisi and Wu, is put in a path-integral form in [6]. The procedure of taking the limit $\tau \to \infty$ is analyzed and based on new grounds through the introduction of the vacuum-vacuum generating functional. In this paper non perturbative aprouch related to Parisi and Wu stochastic-quantization of the $\lambda\varphi_d^{2n}, n \geq 2, d \geq 4$ model quantum field theory is considered.

1.Introduction

Parisi and Wu' proposed the following alternative method to get the quantum averages [1]:

(i) Introduce a 5-th time τ, in addition to the usual four space-time t, and postulate the following Langevin equation for the dynamics of the field $\varphi(\tau,x)$ in this extra time τ

$$\begin{gathered}\frac{\partial\varphi(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi]}{\delta\varphi(\tau,x)} + \eta(\tau,x),\\ \langle \eta(\tau,x)\rangle_\eta = 0,\\ \langle \eta(\tau,x)\,\eta(\tau',x')\rangle_\eta = 2\delta(\tau-\tau')(x-x'),\end{gathered} \tag{1.1}$$

where the angular brackets denote connected average with respect to the random variable η.

(ii) Evaluate the stochastic average of fields $\phi_\eta(\tau,x)$ satisfying Eq. (1.1), that means

$$\langle\phi_\eta(\tau_1,x_1)\phi_\eta(\tau_2,x_2)\ldots\phi_\eta(\tau_m,x_m)\rangle_\eta. \tag{1.2}$$

(iii) Put $\tau_1 = \tau_2 = \ldots = \tau_m = \tau$ in (1.2) and take the limit

$$\lim_{\tau\to\infty} \langle \phi_\eta(\tau,x_1)\phi_\eta(\tau,x_2)\ldots\phi_\eta(\tau,x_m)\rangle_\eta = G(x_1,x_2,\ldots,x_m) \tag{1.3}$$

It is possible to prove, at least perturbatively, that [6]

$$G(x_1,x_2,\ldots,x_m) = \frac{\int D[\varphi](\varphi(x_1)\varphi(x_2)\ldots\varphi(x_m))\exp\{-S[\varphi]\}}{\int D[\varphi]\exp\{-S[\varphi]\}}. \tag{1.4}$$

To understand this relation see ref.[6].
In this paper in particular we deal with double stochastic relaxation equations of the form:

$$\begin{gathered}\frac{\partial\varphi(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi]}{\delta\varphi(\tau,x)} + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x),\\ \langle\eta(\tau,x)\rangle_\eta = 0, \langle\widetilde{\eta}(\tau,x)\rangle_{\widetilde{\eta}_1} = 0\\ \langle\eta(\tau,x)\eta(\tau',x')\rangle_\eta = 2\delta(\tau-\tau')(x-x'),\\ \langle\widetilde{\eta}(\tau,x)\widetilde{\eta}(\tau',x')\rangle_{\widetilde{\eta}} = 2\delta(\tau-\tau')(x-x').\end{gathered} \tag{1.5}$$

Here $\eta(\tau,x) = \eta(\tau,x;\omega)$ is a space-time white noice on probability space $\Sigma = (\Omega,\mathcal{S},P)$ and $\widetilde{\eta}_{1,2}(\tau,x) = \widetilde{\eta}(\tau,x;\varpi)$ are space-time white noices on probability space $\widetilde{\Sigma} = (\widetilde{\Omega},\mathcal{S},P)$.
(ii) Evaluate the stochastic average of fields $\phi_\eta(\tau,x)$ satisfying Eq. (1.5), that means

$$\langle\phi_{\eta,\widetilde{\eta}}(\tau_1,x_1;\epsilon)\phi_{\eta,\widetilde{\eta}}(\tau_2,x_2;\epsilon)\ldots\phi_{\eta,\widetilde{\eta}}(\tau_m,x_m;\epsilon)\rangle_{\eta,\widetilde{\eta}}. \tag{1.6}$$

(iii) Put $\tau_1 = \tau_2 = \ldots = \tau_m = \tau$ in (1.2) and take the limit

$$\lim_{\tau\to\infty}\lim_{\epsilon\to0}\langle\phi_{\eta,\widetilde{\eta}}(\tau,x_1;\epsilon)\phi_{\eta,\widetilde{\eta}}(\tau,x_2;\epsilon)\ldots\phi_{\eta,\widetilde{\eta}}(\tau,x_m;\epsilon)\rangle_{\eta,\widetilde{\eta}} = G(x_1,x_2,\ldots,x_m;\epsilon) \tag{1.7}$$

To understand this relation we have to introduce the notion of probability (density) $P(\varphi_\eta,\tau)$, that is, the probability (density) of having the system in the configuration $\varphi_\eta(\tau,x)$ at time τ. There exists for $P(\varphi_\eta,\tau)$ an equation that describes its evolution in the time τ. It is called the Stochastic Fokker-Planck (SFP)) equation and it has been derived in [8]:

$$\begin{gathered}\frac{\partial P[\varphi(\tau,x),\tau]}{\partial\tau} = \int d^4x\frac{\delta}{\delta\varphi(\tau,x)}\left[P[\varphi(\tau,x),\tau]\frac{\delta S^\star[\varphi]}{\delta\varphi(\tau,x)}\right] + \int d^4x\frac{\delta^2 S^\star[\varphi]}{\delta\varphi^2(\tau,x)},\\ S^\star[\varphi] = S[\varphi] - \int[\varphi(\tau,x)\eta(\tau,x)]d^4x.\end{gathered} \tag{1.8}$$

It is possible to rewrite this equation in a Schrodinger-type form:

$$\begin{gathered}\frac{\partial\Psi[\varphi(\tau,x),\tau]}{\partial\tau} = -2\mathbf{H}\Psi[\varphi(\tau,x),\tau],\\ \Psi = P[\varphi(\tau,x),\tau]\exp[S^\star[\varphi(\tau,x)]/2],\end{gathered} \tag{1.9}$$

where

$$\mathbf{H} = -\frac{1}{2}\frac{\delta^2}{\delta\varphi^2} + \frac{1}{8}\left[\frac{\delta S^\star[\varphi]}{\delta\varphi}\right]^2 - \frac{1}{4}\frac{\delta^2 S^\star[\varphi]}{\delta\varphi^2}. \tag{1.10}$$

It is a positive semi-definite operator $H\Psi_n = E_n\Psi_n$ whis a ground state E_0 is

$\Psi_0[\varphi(\tau,x),\tau] = \exp[S^\bigstar[\varphi(\tau,x)]/2]$. The solution of Eq.(1.9) is

$$\Psi[\varphi(\tau,x),\tau] = \sum_{n=0}^{\infty} c_n \Psi_n[\varphi(\tau,x),\tau]\exp(-2E_n\tau), \quad (1.11)$$

where $\{c_n\}_{n=0}^{\infty}$ are normalizing constants. The probability density $P[\varphi(\tau,x),\tau]$ can be written as

$$P[\varphi(\tau,x),\tau] = \exp[S^\bigstar[\varphi(\tau,x)]/2]\sum_{n=0}^{\infty} c_n \Psi_n[\varphi(\tau,x),\tau]\exp(-2E_n\tau). \quad (1.12)$$

In the limit $\tau \to \infty$ the only term that does not disappear in this expression is $\Psi_0[\varphi(\tau,x),\tau]$, so finally we have

$$\begin{aligned}\lim_{\tau\to\infty} P[\varphi(\tau,x),\tau] &= c_0 \exp[-S^\bigstar[\varphi(\tau,x)]/2]\exp[-S^\bigstar[\varphi(\tau,x)]/2] = \\ &= c_0 \exp[-S^\bigstar[\varphi(\tau,x)]].\end{aligned} \quad (1.13)$$

This is the formal reason why Eq.(1.4) holds.

2.Non perturbative approach to Stochastic Quantization $\lambda\varphi^{2n}$ model Quantum Field Theory

2.1.The generating functional

In this paper we deal with a system of the double stochastic relaxation equations of the form:

$$\begin{gathered}
\frac{\partial\varphi_{1,\epsilon}(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi_1,\varphi_2]}{\delta\varphi_1(\tau,x)}\bigg|_{\varphi_1=\varphi_{1,\epsilon},\varphi_2=\varphi_{2,\epsilon}} + \eta(\tau,x) + \epsilon\tilde{\eta}_1(\tau,x), \\
\varphi_{1,\epsilon}(0,x) = 0, x \in \mathbb{R}^4, \tau \in \mathbb{R}_+, \\
\frac{\partial\varphi_{2,\epsilon}(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi_1,\varphi_2]}{\delta\varphi_2(\tau,x)}\bigg|_{\varphi_1=\varphi_{1,\epsilon},\varphi_2=\varphi_{2,\epsilon}} - \eta(\tau,x) + \epsilon\tilde{\eta}_2(\tau,x), \\
\varphi_{2,\epsilon}(0,x) = 0, x \in \mathbb{R}^4, \tau \in \mathbb{R}_+, \\
S[\varphi_1,\varphi_2] = \\
\int_{\mathbb{R}^4} d^4x\Big[\frac{1}{2}(\partial_\mu\varphi_1(\tau,x)\partial_\mu\varphi_1(\tau,x) + m^2\varphi_1(\tau,x)) + P(\varphi_1(\tau,x))\Big] + \\
\int_{\mathbb{R}^4} d^4x\Big[\frac{1}{2}(\partial_\mu\varphi_2(\tau,x)\partial_\mu\varphi_2(\tau,x) + m^2\varphi_2(\tau,x)) + P(\varphi_2(\tau,x))\Big] + \\
+\int_{\mathbb{R}^4} d^4x[\gamma \times \varphi_1(\tau,x)\varphi_2(\tau,x)], \gamma > 0.
\end{gathered} \quad (2.1.1)$$

where $0 < \epsilon \ll 1$, $P(\bullet)$ is a polinomial degree $2k, k \geq 2$

$$P(\varphi_{1,2}) = \quad (2.1.2)$$

and where $\eta(\tau,x;\omega), \tilde{\eta}_1(\tau,x;\varpi)$, $\tilde{\eta}_2(\tau,x;\varpi)$ are Gaussian random variables such that

$$\langle \eta(\tau,x) \rangle_\eta = 0,$$
$$\langle \widetilde{\eta}_1(\tau,x) \rangle_{\widetilde{\eta}_1} = 0, \langle \widetilde{\eta}_2(\tau,x) \rangle_{\widetilde{\eta}_2} = 0, \quad (2.1.3)$$

and for the two-point correlation function associated with the random noises fields

$$\langle \eta(\tau,x)\,\eta(\tau',x') \rangle_\eta = 2\delta(\tau-\tau')(x-x'),$$
$$\langle \widetilde{\eta}_1(\tau,x)\,\widetilde{\eta}_1(\tau',x') \rangle_{\widetilde{\eta}} = 2\delta(\tau-\tau')(x-x'), \quad (2.1.4)$$
$$\langle \widetilde{\eta}_2(\tau,x)\,\widetilde{\eta}_2(\tau',x') \rangle_{\widetilde{\eta}} = 2\delta(\tau-\tau')(x-x').$$

Remark 2.1.1. Here $\eta(\tau,x) = \eta(\tau,x;\omega)$ is a space-time white noice on probability space $\Sigma = (\Omega,\mathcal{S},P)$ and $\widetilde{\eta}_{1,2}(\tau,x) = \widetilde{\eta}_{1,2}(\tau,x;\varpi)$ are space-time white noices on probability space $\widetilde{\Sigma} = (\widetilde{\Omega},\mathcal{S},P)$. The angular brackets denote connected average with respect to the random variables $\eta,\widetilde{\eta}_{1,2}$.
We want to build a generating functional $Z_\epsilon[J_1,J_2]$ from which the correlations

$$\langle \varphi_{1,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{1,\eta}(\tau_l,x_l;\omega)\varphi_{2,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{2,\eta}(\tau_l,x_r;\omega)\rangle_\eta$$
$$\frac{\partial\varphi_{1,\eta}(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi_1,\varphi_2]}{\delta\varphi_1(\tau,x)}\bigg|_{\varphi_1=\varphi_{1,\eta},\varphi_2=\varphi_{2,\eta}} + \eta(\tau,x),$$
$$\frac{\partial\varphi_{2,\eta}(\tau,x)}{\partial\tau} = -\frac{\delta S[\varphi_2,\varphi_2]}{\delta\varphi_2(\tau,x)}\bigg|_{\varphi_1=\varphi_{1,\eta},\varphi_2=\varphi_{2,\eta}} + \eta(\tau,x)$$

can be derived by the following fashion:

$$\langle \varphi_{1,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{1,\eta}(\tau_l,x_l;\omega)\varphi_{2,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{2,\eta}(\tau_l,x_r;\omega)\rangle_\eta =$$
$$\lim_{\epsilon\to0}\langle \varphi_{1,\eta,\epsilon\widetilde{\eta}_1}(\tau_1,x_1;\omega,\varpi)\times\ldots\times\varphi_{1,\eta,\epsilon\widetilde{\eta}_1}(\tau_l,x_l;\omega,\varpi)\varphi_{2,\eta,\epsilon\widetilde{\eta}_2}(\tau_1,x_1;\omega,\varpi)\times\ldots$$
$$\ldots\times\varphi_{2,\eta,\epsilon\widetilde{\eta}_2}(\tau_l,x_r;\omega,\varpi)\rangle_\eta = \quad (2.1.5)$$
$$=\lim_{\epsilon\to0}\frac{\delta^l Z_\epsilon[J_1,J_2;\omega]}{\delta J_1(\tau_1,x_1)\ldots\delta J_1(\tau_l,x_l)\delta J_2(\tau_1,x_1)\ldots\delta J_2(\tau_r,x_r)}\bigg|_{J_1=0,J_2=0}$$

By canonical definition [6] one obtains

$$Z_\epsilon[J_1,J_2;\omega] =$$
$$N\int D[\eta(\tau',x;\omega)]\Big(\Big\{\int D[\varphi_1(\tau',x;\omega)]D[\varphi_2(\tau',x;\omega)]D[\widetilde{\eta}_1(\tau',x;\varpi)]D[\widetilde{\eta}_2(\tau',x;\varpi)]$$
$$\delta(\varphi_1(0,x;\omega))\delta(\varphi_2(0,x;\omega))\delta(\varphi_1(\tau',x;\omega)-\varphi_{1,\eta,\epsilon\widetilde{\eta}_1,\epsilon\widetilde{\eta}_2})\delta(\varphi_2(\tau',x;\omega)-\varphi_{2,\eta,\epsilon\widetilde{\eta}_1,\epsilon\widetilde{\eta}_2})\times$$
$$\times\exp\Big[-\int_0^\tau\int_{\mathbb{R}^4}J_1(\tau',x)\varphi_1(\tau',x;\omega)d^4x d\tau'\Big]\exp\Big[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\widetilde{\eta}_1^2(\tau',x;\varpi)d^4x d\tau'\Big]\times \quad (2.1.6)$$
$$\times\exp\Big[-\int_0^\tau\int_{\mathbb{R}^4}J_2(\tau',x)\varphi_2(\tau',x;\omega)d^4x d\tau'\Big]\exp\Big[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\widetilde{\eta}_2^2(\tau',x;\varpi)d^4x d\tau'\Big]\Big\}\times$$
$$\exp\Big[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4x d\tau'\Big]\Big),$$

where $\varphi_{1,2,\eta,\widetilde{\eta}} = \varphi_{1,2}(\tau',x;\omega,\varpi)$ that appears in Eq.(2.1.1)-Eq.(2.1.2) is the solution of the double stochastic Langevin equations (2.1.1), solved with zero initial condidion: $\varphi_{1,2,\eta,\widetilde{\eta}}(0,x;\omega,) \equiv 0$ and N is a normalizing constant and

$$D[\varphi_{1,2};\omega] = \lim_{M\to\infty} \prod_{i=0}^{M} D[\varphi_{1,2,\tau_i}(x;\omega)] \tag{2.1.7}$$

where $\varphi_{1,2,\tau_i}(x;\omega)$, are the field configurations at the time τ_i, having sliced the interval 0 to τ in M infinitesimal parts ε with $\tau_i = i\varepsilon$ and $D[\varphi_{1,2,\tau_i}(x;\omega)]$ is path-integral random measure.

Abbreviation 2.1.1. $D[\varphi_{1,2};\omega] \triangleq D[\varphi_1;\omega]D[\varphi_2;\omega], D\left[\widetilde{\eta}_{1,2}(\tau',x;\omega)\right] \triangleq D[\widetilde{\eta}_1(\tau',x;\omega)] \times D[\widetilde{\eta}_2(\tau',x;\omega)], \delta\left(\varphi_{1,2} - \varphi_{1,2,\eta,\epsilon\widetilde{\eta}_{1,2}}\right) \triangleq \delta(\varphi_1 - \varphi_{1,\eta,\epsilon\widetilde{\eta}_1})\delta(\varphi_2 - \varphi_{2,\eta,\epsilon\widetilde{\eta}_2})$, $J_{1,2}(\tau',x)\varphi_{1,2}(\tau',x;\omega) = J_1(\tau',x)\varphi_1(\tau',x;\omega) + J_2(\tau',x)\varphi_2(\tau',x;\omega)$, etc.

The delta function $\delta(\varphi_{1,2} - \varphi_{1,2,\eta,\widetilde{\eta}})$ in Eq.(2.1.6) we can write as

$$\delta(\varphi_{1,2} - \varphi_{1,2,\eta,\widetilde{\eta}}) = \delta\left[\frac{\partial\varphi_{1,2,\epsilon}}{\partial\tau} + \frac{\delta\widetilde{S}}{\delta\varphi_{1,2}}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} - \epsilon\widetilde{\eta}_{1,2}\right]\left\|\frac{\delta(\epsilon\widetilde{\eta})_{1,2}}{\delta\varphi_{1,2,\epsilon}}\right\|, \tag{2.1.8}$$

where $\widetilde{S} = S - \int_{\mathbb{R}^4}(\varphi_{1,2}\times\eta)d^4x$ and where $\|\delta\widetilde{\eta}_{1,2}/\delta\varphi_{1,2,\epsilon}\|$ is the Jacobian matrix of the transformation $\widetilde{\eta}_{1,2} \to \varphi_{1,2,\epsilon}$, that is

$$\left\|\frac{\delta\left(\epsilon\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}}\right\| = N_\epsilon\left\|\frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}}\right\|,$$

$$\det\left\|\frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}}\right\| = N_\epsilon^{-1}\det\left[\left[\partial_\tau + \frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\delta(\tau-\tau')\right], \tag{2.1.9}$$

$$N_\epsilon = \prod_{i=1}^{\infty} x_i, x_i = \epsilon, i = 1,2,\ldots$$

From Eq.(2.1.9) by canonical calculation we get

$$\det\left\|\frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}}\right\| =$$

$$= N_\epsilon^{-1}\exp\left[\mathbf{tr}\ln\left[\left[\partial_\tau + \frac{\partial^2\widetilde{S}}{\partial\varphi_{1,2}(\tau)\partial\varphi_{1,2}(\tau')}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\delta(\tau-\tau')\right]\right] = \tag{2.1.10}$$

$$= N_\epsilon^{-1}\exp\left[\mathbf{tr}\ln\partial_\tau\left[\delta(\tau-\tau') + \partial_\tau^{-1}\left(\frac{\partial^2\widetilde{S}}{\partial\varphi_{1,2}(\tau)\partial\varphi_{1,2}(\tau')}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right)\right]\right],$$

where ∂_τ^{-1} indicate the Geen's function $G(\tau-\tau')$ that satisfies

$$\partial_\tau G(\tau-\tau') = \delta(\tau-\tau'). \tag{2.1.11}$$

The solutions of the Eq.(2.1.11) are: (i) if we choose propagation forward in time

$$G(\tau-\tau') = \theta(\tau-\tau') \tag{2.1.12}$$

(ii) if we choose propagation backward in time

$$G(\tau-\tau') = -\theta(\tau-\tau') \tag{2.1.13}$$

In case, propagation forward in time, we get

$$
\det\left\| \frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| =
$$

$$
N_{\epsilon}^{-1}\exp\left\{\mathbf{tr}\left[\ln\partial_{\tau} + \ln\left[\delta(\tau-\tau') + \theta(\tau-\tau')\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\right]\right\} \quad (2.1.14)
$$

$$
= N_{\epsilon}^{-1}\exp(\mathbf{tr}\ln\partial_{\tau})\exp\left[\mathbf{tr}\ln\left[\delta(\tau-\tau') + \theta(\tau-\tau')\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\right].
$$

The term $\exp(\mathbf{tr}\ln\partial_{\tau})$ can be dropped, as it cancels with the same term in the denominator of (2.1.6), once we normalize $\widetilde{Z}_{\epsilon}[J_{1,2};\omega] = Z_{\epsilon}[J_{1,2};\omega]/Z_{\epsilon}[0,0;\omega]$.

Abbreviation 2.1.2.$Z_{\epsilon}[J_{1,2};\omega] \triangleq Z_{\epsilon}[J_1,J_2;\omega]$

So in Eq.(2.1.14) we are left with

$$
\det\left\| \frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| =
$$

$$
N_{\epsilon}^{-1}\exp\left[\mathbf{tr}\ln\left[\delta(\tau-\tau') + \theta(\tau-\tau')\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}}\right]\right]. \quad (2.1.15)
$$

By using the canonical expansion for the $\ln$, we obtain

$$
\det\left\| \frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| =
$$

$$
N_{\epsilon}^{-1}\exp\left[\mathbf{tr}\left[\theta(\tau-\tau')\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} + \right.\right.
$$

$$
\left.\left.\theta(\tau-\tau')\theta(\tau'-\tau)\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau')\delta\varphi_{1,2}(\tau)}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} + \ldots\right]\right] = \quad (2.1.16)
$$

$$
= N_{\epsilon}^{-1}\exp\left[\int d\tau\int_{\mathbb{R}^4} d^4x\,\theta(0)\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}^2(\tau)}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} + \right.
$$

$$
\left.\int d\tau'\int_{\mathbb{R}^4} d^4x\,\theta(\tau-\tau')\theta(\tau'-\tau)\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2}(\tau')\delta\varphi_{1,2}(\tau)}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\epsilon}} + \ldots\right]
$$

The second term in this expression is zero because $\theta(\tau-\tau')\theta(\tau'-\tau) = 0$ and the

same for all the subsequent terins. The only one left is the first term and choosing $\theta(0) = 1/2$ we get

$$
\det\left\| \frac{\delta\left(\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2,\epsilon}} \right\| = N_{\epsilon}^{-1}\exp\left[\frac{1}{2}\int d\tau'\frac{\delta^2\widetilde{S}}{\delta\varphi_{1,2,\epsilon}^2}\right]. \quad (2.1.17)
$$

Inserting Eq.(2.1.17) and Eq.(2.1.8) into Eq.(2.1.6) and performing the $\widetilde{\eta}$ integration, we get

$$
\begin{aligned}
Z_\epsilon[J_1,J_2;\omega] &= N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\times\\
&\times\exp\left\{-\int_0^\tau\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau}+\frac{\delta\widetilde{S}}{\delta\varphi_1(\tau',x;\omega)}\right]^2-\frac{1}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_1^2(\tau',x;\omega)}\right]d^4xd\tau'\right\}\\
&\times\exp\left\{-\int_0^\tau\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau}+\frac{\delta\widetilde{S}}{\delta\varphi_2(\tau',x;\omega)}\right]^2-\frac{1}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_2^2(\tau',x;\omega)}\right]d^4xd\tau'\right\}\\
&\times\exp\left\{-\int_0^\tau\int_{\mathbb{R}^4}J_{1,2}(\tau',x)\varphi_{1,2}(\tau',x;\omega)d^4xd\tau'\right\}\exp\left[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right].
\end{aligned}
\tag{2.1.18}
$$

From Eq.(2.1.18) finally we obtain

$$
\begin{aligned}
Z_\epsilon[J_{1,2};\omega] &= N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\times\\
&\times\exp\left\{-\int_0^\tau\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta\right]^2-\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_1^2}\right]d^4xd\tau'\right.\\
&-\int_0^\tau\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta\right]^2-\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_2^2}\right]d^4xd\tau'-\\
&\left.-\int_0^\tau\int_{\mathbb{R}^4}J_{1,2}(\tau',x)\varphi_{1,2}(\tau',x;\omega)d^4xd\tau'\right\}\times\exp\left[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right].
\end{aligned}
\tag{2.1.19}
$$

If we want also to specify that we are interested only in the correlations at the same 5-th time τ_1, we have just to choose $J_{1,2}(x,\tau')$ of the form $J_{1,2}(x,\tau')=\bar{J}(x)\delta(\tau'-\tau_1),\ \tau_1<\tau$ and Eq.(2.1.19) then becomes

$$
\begin{aligned}
Z_\epsilon[J_{1,2};\omega] &= N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\times\\
&\times\exp\left\{-\int_0^\tau\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta\widetilde{S}}{\delta\varphi_1(\tau',x;\omega)}\right]^2-\frac{1}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_1^2(\tau',x;\omega)}\right]d^4xd\tau'\right.\\
&-\int_0^\tau\int_{\mathbb{R}^4}\left[\frac{1}{4\epsilon^2}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta\widetilde{S}}{\delta\varphi_2(\tau',x;\omega)}\right]^2-\frac{1}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_2^2(\tau',x;\omega)}\right]d^4xd\tau'-\\
&\left.-\int_{\mathbb{R}^4}J_{1,2}(\tau_1,x)\varphi_{1,2}(\tau_1,x;\omega)d^4x\right\}\times\exp\left[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right].
\end{aligned}
\tag{2.1.20}
$$

Remark 2.1.2. In all this we have to remember, of course, that once we set $\tau_1\to\infty$ we have also to extend the interval of integration from $[0,\tau]$ to $[0,\infty]$.
From Eq.(2.1.21) with $\epsilon\ll 1$ for two-point correlation function $\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle$ defined by

$$
\langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2)\rangle=\lim_{\epsilon\to 0}\left[\frac{\delta^l Z_\epsilon[J_1,J_2;\omega]}{\delta J_1(\tau_1,x_1)\delta J_2(\tau_2,x_2)}\right]
\tag{2.1.21$'$}
$$

for mutually two-point correlation function $\langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2)\rangle_\eta$ we get

$$
\begin{gathered}
\langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2)\rangle_\eta \asymp \langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2);\epsilon\rangle \triangleq \\
\triangleq N_\epsilon^{-1}\int D[\eta(\tau',x;\omega)]\exp\Big[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\Big]\times \\
\Big(\int D[\varphi_{1,2}(\tau',x;\omega)](\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega))\times \\
\delta(\varphi_{1,2}(0,x;\omega))\times \\
\times\exp\Bigg\{-\frac{1}{4\epsilon^2}\Bigg[\int_0^\tau\int_{\mathbb{R}^4}\Bigg[\frac{\partial\varphi(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta(\tau',x;\omega)\Bigg]^2- \\
\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_1^2(\tau',x;\omega)}\Bigg]d^4xd\tau'\Bigg\} \\
\times\exp\Bigg\{-\frac{1}{4\epsilon^2}\int_0^\tau\int_{\mathbb{R}^4}\Bigg[\frac{\partial\varphi(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta(\tau',x;\omega)\Bigg]^2- \\
\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_2^2(\tau',x;\omega)}\Bigg]d^4xd\tau'\Bigg\}\Bigg).
\end{gathered}
\tag{2.1.22}
$$

Performing the $\varphi_{1,2}(\tau',x;\omega)$ integration in Eq.(2.1.22) by using saddle point approximation we get

$$
\begin{gathered}
\langle\varphi_{1,\eta}(\tau_1,x_1)\varphi_{2,\eta}(\tau_2,x_2)\rangle_\eta \asymp \\
\det\Bigg\|\frac{\delta(\varphi_1)}{\delta\eta}\Bigg\|\det\Bigg\|\frac{\delta(\varphi_2)}{\delta\eta}\Bigg\|\int D[\eta(\tau',x;\omega)](\varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_2,x_2;\omega))\times \\
\times\exp\Bigg\{-\int_0^\tau\Bigg[-\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_1^2(\tau',x;\omega)}\Bigg|_{\varphi_{1,2}=\varphi_{1,2,\eta}}\Bigg]d^4xd\tau'\Bigg\}\times \\
\exp\Bigg\{-\int_0^\tau\Bigg[-\frac{1}{2}\frac{\delta^2 S}{\delta\varphi_2^2(\tau',x;\omega)}\Bigg|_{\varphi_{1,2}=\varphi_{1,2,\eta}}\Bigg]d^4xd\tau'\Bigg\} \\
\exp\Big[-\frac{1}{4}\int_0^\tau\eta^2(\tau',x;\omega)d^4xd\tau'\Big]= \\
\int D[\eta(\tau',x;\omega)](\varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_2,x_2;\omega))\exp\Big[-\frac{1}{4}\int_0^\tau\eta^2(\tau',x;\omega)d^4xd\tau'\Big]\asymp \\
\asymp\langle\varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_1,x_2;\omega)\rangle_\eta
\end{gathered}
\tag{2.1.23}
$$

$\varphi_{1,2,\eta}(\tau_1,x_1;\omega)$ that appears in Eq.(2.1.23) is the solution of the Langevin equation (2.1.26), solved with zero initial condition. From Eq.(2.1.22)-Eq.(2.1.23) we get

$$
\begin{gathered}
\langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega)\rangle_\eta \simeq \\
N\int D[\varphi_{1,2}(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega)) \\
\Big(\int D[\eta(\tau',x;\omega)](\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega))\exp\Big[-\frac{1}{4}\int_0^\tau\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\Big]\Big)\times \\
\times\exp\Bigg\{-\int_0^\tau\Bigg[\frac{1}{4\epsilon^2}\Bigg[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta(\tau',x;\omega)\Bigg]^2\Bigg]d^4xd\tau'\Bigg\} \\
\times\exp\Bigg\{-\int_0^\tau\Bigg[\frac{1}{4\epsilon^2}\Bigg[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta(\tau',x;\omega)\Bigg]^2\Bigg]d^4xd\tau'\Bigg\}.
\end{gathered}
\tag{2.1.24}
$$

From Eq.(2.1.24) finally we get

$$\langle\varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_2,x_2;\omega)\rangle_\eta \simeq \langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2);\epsilon\rangle \triangleq$$
$$\triangleq N_\epsilon^{-1}\int D[\varphi_{1,2}(\tau',x;\omega)]\langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega)\rangle_\eta \times$$
$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_1}\int_{\mathbb{R}^4}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2 d^4xd\tau'\right\}$$
$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_2}\int_{\mathbb{R}^4}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2 d^4xd\tau'\right\} \quad (2.1.25)$$

where

$$\langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega)\rangle_\eta \triangleq$$
$$\int D[\eta(\tau',x;\omega)](\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega))\exp\left[-\frac{1}{4}\int_0^{\tau}\int_{\mathbb{R}^4}\eta^2(\tau',x;\omega)d^4xd\tau'\right]$$

Proposition 2.1.1. It follows from Eq.(2.1.22)-Eq.(2.1.23) that in Eq.(2.1.22) we can interchange integration on variable $\eta(\tau',x;\omega)$ and integration on variables $\varphi_{1,2}(\tau',x;\omega)$ in Eq.(1.24)-Eq.(2.1.25)

Remark 2.1.3.Note that for any fixed values of parameters τ_1,x_1,τ_2,x_2 and $\gamma \simeq 0$ we get

$$\langle[\varphi_1(\tau_1,x_1;\omega)-a][\varphi_2(\tau_2,x_2;\omega)+a];\epsilon\rangle_\eta \simeq \langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega);\epsilon\rangle_\eta - a^2, \quad (2.1.26)$$

where by translation invariance

$$\langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega);\epsilon\rangle_\eta - a^2 \simeq 0 \Rightarrow a \simeq a(\tau_1,\tau_2,x_1-x_2). \quad (2.1.27)$$

From Eq.(2.1.25) by the replacement

$$\begin{aligned}
\varphi_1(\tau,x;\omega)-\theta(\tau)a &= v_-(\tau,x;\omega),\\
\varphi_2(\tau,x;\omega)+\theta(\tau)a &= v_+(\tau,x;\omega),\\
\varphi_1(\tau,x;\omega) &= v_-(\tau,x;\omega)+\theta(\tau)a,\\
\varphi_2(\tau,x;\omega) &= v_+(\tau,x;\omega)-\theta(\tau)a,\\
\frac{\partial\varphi_1(\tau,x;\omega)}{\partial\tau} &= \frac{\partial v_-(\tau,x;\omega)}{\partial\tau}+\delta(\tau)a,\\
\frac{\partial\varphi_2(\tau,x;\omega)}{\partial\tau} &= \frac{\partial v_+(\tau,x;\omega)}{\partial\tau}-\delta(\tau)a,
\end{aligned} \quad (2.1.28)$$

we obtain

$$\Omega(\tau_1,\tau_2,x_1-x_2,a;\epsilon)\triangleq$$

$$N_\epsilon^{-1}\int D[\varphi_{1,2}(\tau',x;\omega)]\langle[\varphi_1(\tau_1,x_1;\omega)-a][\varphi_2(\tau_2,x_2;\omega)+a]\rangle_\eta\times$$

$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_1}\int_{\mathbb{R}^4}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2 d^4x d\tau'\right\}\times$$

$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_2}\int_{\mathbb{R}^4}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2 d^4x d\tau'\right\}=$$

$$=N_\epsilon^{-1}\int D[v_-(\tau',x;\omega)]D[v_+(\tau',x;\omega)]\langle v_-(\tau_1,x_1)v_+(\tau_1,x_1)\rangle_\eta\times \tag{2.1.29}$$

$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_1}\int_{\mathbb{R}^4}\left[\frac{\partial v_-(\tau',x;\omega)}{\partial\tau'}+\delta(\tau)a+\frac{\delta S}{\delta\varphi_1(\tau',x;\omega)}\right|_{\varphi_1=v_-(\tau',x;\omega)+a_-}$$

$$-\eta(\tau',x;\omega)d^4xd\tau']^2\Big\}\times$$

$$\times\exp\left\{-\frac{1}{4\epsilon^2}\int_0^{\tau_2}\int_{\mathbb{R}^4}\left[\frac{\partial v_+(\tau',x;\omega)}{\partial\tau'}-\delta(\tau)a+\frac{\delta S}{\delta\varphi_2(\tau',x;\omega)}\right|_{\varphi_2=v_+(\tau_1,x_1;\omega)-a}$$

$$-\eta(\tau',x;\omega)d^4xd\tau'\Big\}.$$

Remark 2.1.4. Note that

$$\lim_{\epsilon\to0}\Omega(\tau_1,\tau_2,x_1-x_2,a;\epsilon)=0\Rightarrow\lim_{\epsilon\to0}\langle\varphi_1(\tau_1,x_1)\varphi_2(\tau_2,x_2);\epsilon\rangle-a^2=0. \tag{2.1.30}$$

Definition 2.1.1. Let $v_\mp(\tau,x;\omega)$ be the solution of the Langevin equations (2.1.31)

$$\begin{aligned}\frac{\partial v_-(\tau,x;\omega)}{\partial\tau}&=-\delta(\tau)a-\frac{\delta S[\varphi_{1,2}]}{\delta\varphi_1(\tau,x;\omega)}\Bigg|_{\substack{\varphi_1=v_-(\tau',x;\omega)+a_-\\ \varphi_2=v_+(\tau_1,x_1;\omega)-a}}+\eta(\tau,x;\omega),\\ \frac{\partial v_+(\tau,x;\omega)}{\partial\tau}&=\delta(\tau)a-\frac{\delta S[\varphi_{1,2}]}{\delta\varphi_2(\tau,x;\omega)}\Bigg|_{\substack{\varphi_1=v_-(\tau',x;\omega)+a_-\\ \varphi_2=v_+(\tau_1,x_1;\omega)-a}}+\eta(\tau,x;\omega),\\ v_\mp(0,x;\omega)&=0,\end{aligned} \tag{2.1.31}$$

Linear stochastic differental *master equation* corresponding to the Langevin equations (2.1.31) reads

$$\begin{aligned}\frac{\partial v_-(\tau,x,a;\omega)}{\partial\tau}&=-\delta(\tau)a-\mathcal{L}\left\{\frac{\delta S[\varphi_{1,2}]}{\delta\varphi_1(\tau,x;\omega)}\Bigg|_{\varphi_1=v_-(\tau',x;\omega)+a_-}\right\}+\eta(\tau,x;\omega),\\ \frac{\partial v_+(\tau,x,a;\omega)}{\partial\tau}&=\delta(\tau)a-\mathcal{L}\left\{\frac{\delta S[\varphi_{1,2}]}{\delta\varphi_2(\tau,x;\omega)}\Bigg|_{\varphi_2=v_+(\tau_1,x_1;\omega)-a}\right\}+\eta(\tau,x;\omega),\\ v_\mp(0,x,a;\omega)&=0,\end{aligned} \tag{2.1.32}$$

where

$$\mathcal{L}\left\{\delta S[\varphi_{1,2}]/\delta\varphi_1(\tau,x;\omega)|_{\substack{\varphi_1=v_-(\tau',x;\omega)+a_-\\ \varphi_2=v_+(\tau_1,x_1;\omega)-a}}\right\} \tag{2.1.33}$$

is a linear part of variational derivative

$$\delta S[\varphi_{1,2}]/\delta\varphi_1(\tau,x;\omega)|_{\substack{\varphi_1=v_-(\tau',x;\omega)+a_-\\ \varphi_2=v_+(\tau_1,x_1;\omega)-a}} \tag{2.1.34}$$

and where

$$\mathcal{L}\left\{\delta S[\varphi_{1,2}]/\delta\varphi_2(\tau,x;\omega)|_{\substack{\varphi_1=v_-(\tau',x;\omega)+a_-\\ \varphi_2=v_+(\tau_1,x_1;\omega)-a}}\right\}. \quad (2.1.35)$$

is a linear part of variational derivative

$$\delta S[\varphi_{1,2}]/\delta\varphi_2(\tau,x;\omega)|_{\substack{\varphi_1=v_-(\tau',x;\omega)+a_-\\ \varphi_2=v_+(\tau_1,x_1;\omega)-a}} \quad (2.1.36)$$

2.2.Transcendental master equation corresponding to two-point Green function $G(x_1,x_2,\lambda)$.

Definition 2.2.1.Let $v_{\mp}(\tau,x,a;\omega)$ be the solution of the *stochastic differental master equations* (2.1.32). Transcendental *master equation* corresponding to the stochastic Langevin equation (2.1.31) reads

$$\langle v_-(\tau_1,x_1,a;\omega)v_+(\tau_2,x_2,a;\omega)\rangle_\eta = 0. \quad (2.2.1)$$

Theorem 2.2.1.Let $a_{1,2}(\bar{\tau},\bar{x})$ be an solution of the equation (2.2.1) at fixed point $(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2) \in (\mathbb{R}_+\times\mathbb{R}^4)\times(\mathbb{R}_+\times\mathbb{R}^4)$ i.e.,

$$\langle v_-(\bar{\tau}_1,\bar{x}_1,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\omega)v_+(\bar{\tau}_2,\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\omega)\rangle_\eta = 0. \quad (2.2.2)$$

Let $\Delta(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2)$ be a set such that

$$\begin{gathered}a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2) \in \Delta(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2) \Leftrightarrow\\ \Leftrightarrow \langle v_-(\bar{\tau}_1,\bar{x}_1,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\omega)v_+(\bar{\tau}_2,\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\omega)\rangle_\eta = 0,\end{gathered} \quad (2.2.3)$$

and let $\widetilde{\Omega}(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2))$ be a set such that

$$\begin{gathered}a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2) \in \widetilde{\Omega}(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2)) \Leftrightarrow\\ \Leftrightarrow \underline{\lim}_{\epsilon\to 0}\Omega(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\epsilon) = 0,\end{gathered} \quad (2.2.4)$$

where the quantity $\Omega(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2);\epsilon)$ defined by Eq.(2.1.29). Then

$$\widetilde{\Omega}(\bar{\tau}_1,\bar{\tau}_2,\bar{x}_1-\bar{x}_2,a(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2)) \subseteq \Delta(\bar{\tau}_1,\bar{x}_1;\bar{\tau}_2,\bar{x}_2). \quad (2.2.5)$$

2.3. Double Stochastic Quantization the Free Scalar Fields

For a scalar field theory governed by the action in terms of the Euclidean spacetime is given by

$$S_E = \int d^4x\left[\frac{1}{2}(\partial\varphi(x))^2 + \frac{1}{2}(m\varphi(x))^2\right] \quad (2.3.1)$$

differental *master equations* corresponding to the Langevin equations (2.1.31) reads

$$\begin{gathered}\frac{\partial v_-(x,\tau;\omega,\varpi)}{\partial\tau} = (\partial^2-m^2)v_-(x,\tau;\omega,\varpi)+\eta(x,\tau;\omega),\\ \frac{\partial v_+(x,\tau;\omega,\varpi)}{\partial\tau} = (\partial^2-m^2)v_+(x,\tau;\omega,\varpi)+\eta(x,\tau;\omega),\\ v_{\mp}(x,0;\omega,\varpi) = 0.\end{gathered} \quad (2.3.2)$$

Fourier transformed stochastic differential equations (2.3.2) in k and τ given as

$$\frac{\partial}{\partial\tau}\widehat{v}_{-}(k,\tau) =$$
$$-(k^2+m^2)\widehat{v}_{-}(k,\tau)-(2\pi)^4 a\delta(\tau)\delta^4(k)-(2\pi)^4 m^2 a\delta^4(k)+\widehat{\eta}(k,\tau;\omega),$$
$$\frac{\partial}{\partial\tau}\widehat{v}_{+}(k,\tau) =$$
$$-(k^2+m^2)\widehat{v}_{+}(k,\tau)-(2\pi)^4 a\delta(\tau)\delta^4(k)+(2\pi)^4 m^2 a\delta^4(k)+\widehat{\eta}(k,\tau;\omega). \qquad (2.3.3)$$

Let us consider ODE

$$\dot{x}\,(\tau,\lambda)+\lambda x(\tau,\lambda) = g(\tau,\lambda), x(0) = 0. \qquad (2.3.4)$$

The corresponding solution $x(\tau,\lambda)$ is

$$x(\tau,\lambda) = e^{-\lambda\tau}\int_0^{\tau} e^{\lambda\tau_1} g(\tau_1,\lambda)d\tau_1. \qquad (2.3.5)$$

From Eq.(2.3.3)-Eq.(2.3.5) one obtains

$$\widehat{v}_{-}(k,\tau,a) =$$
$$e^{-(k^2+m^2)\tau}\int_0^{\tau} e^{(k^2+m^2)\tau_1}\left[-(2\pi)^4 a\delta(\tau_1)\delta^4(k)-(2\pi)^4 m^2 a\delta^4(k)+\widehat{\eta}(k,\tau_1;\omega)\right]d\tau_1 =$$
$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau}-(2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\int_0^{\tau} e^{(k^2+m^2)\tau_1}d\tau_1 +$$
$$+e^{-(k^2+m^2)\tau}\int_0^{\tau} e^{(k^2+m^2)\tau_1}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$
$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau}-(2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\int_0^{\tau} e^{(k^2+m^2)\tau_1}d\tau_1 +$$
$$+\int_0^{\tau} e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$
$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau}-$$
$$(2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\left[\frac{e^{(k^2+m^2)\tau}}{k^2+m^2}-\frac{1}{k^2+m^2}\right]+$$
$$+\int_0^{\tau} e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$
$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau}-(2\pi)^4\frac{m^2 a\delta^4(k)}{k^2+m^2}\left[1-e^{-(k^2+m^2)\tau}\right]+$$
$$+\int_0^{\tau} e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 =$$
$$-(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m^2)\tau}+\frac{m^2}{k^2+m^2}\left[1-e^{-(k^2+m^2)\tau}\right]\right]+$$
$$+\int_0^{\tau} e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1. \qquad (2.3.6)$$

and

$$
\begin{gathered}
\widehat{v}_{+}(k,\tau,a) = \\
e^{-(k^2+m^2)\tau}\int\limits_0^\tau e^{(k^2+m^2)\tau_1}\left[-(2\pi)^4 a\delta(\tau_1)\delta^4(k)+(2\pi)^4 m^2 a\delta^4(k)+\widehat{\eta}(k,\tau_1;\omega)\right]d\tau_1 = \\
-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau}-(2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\int\limits_0^\tau e^{(k^2+m^2)\tau_1}d\tau_1 + \\
+e^{-(k^2+m^2)\tau}\int\limits_0^\tau e^{(k^2+m^2)\tau_1}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau}+(2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\int\limits_0^\tau e^{(k^2+m^2)\tau_1}d\tau_1 + \\
+\int\limits_0^\tau e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau} - \\
(2\pi)^4 m^2 a\delta^4(k)e^{-(k^2+m^2)\tau}\left[\frac{e^{(k^2+m^2)\tau}}{k^2+m^2}-\frac{1}{k^2+m^2}\right]+ \\
+\int\limits_0^\tau e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
-(2\pi)^4 a\delta^4(k)e^{-(k^2+m^2)\tau}-(2\pi)^4\frac{m^2 a\delta^4(k)}{k^2+m^2}\left[1-e^{-(k^2+m^2)\tau}\right]+ \\
+\int\limits_0^\tau e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1 = \\
-(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m^2)\tau}+\frac{m^2}{k^2+m^2}\left[1-e^{-(k^2+m^2)\tau}\right]\right]+ \\
+\int\limits_0^\tau e^{-(k^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k,\tau_1;\omega)d\tau_1.
\end{gathered} \tag{2.3.7}
$$

From Eq.(2.3.6)-Eq.(2.3.7) one obtains

$$
\begin{gathered}
\widehat{v}_{-}(k_1,\tau,a)\widehat{v}_{+}(k_2,\tau',a') = \\
\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m^2)\tau}+\frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right]+\right. \\
\left.+\int\limits_0^\tau e^{-(k_1^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right\}\times \\
\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m^2)\tau'}+\frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_2^2+m^2)\tau'}\right]\right]+\right. \\
\left.+\int\limits_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_1)}\widehat{\eta}(k_2,\tau_1;\omega)d\tau_1\right\}
\end{gathered} \tag{2.3.8}
$$

From Eq.(3.2.11) one obtains

$$
\begin{gathered}
\langle \widehat{v}_-(k_1,\tau,a)\widehat{v}_+(k_2,\tau',a')\rangle_\eta = \\
\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right]\right\} \times \\
\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m^2)\tau'} + \frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_2^2+m^2)\tau'}\right]\right] + \right. \\
+\left\langle\left(\int_0^\tau e^{-(k_1^2+m^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right)\left(\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_2)}\widehat{\eta}(k_2,\tau_2;\omega)d\tau_2\right)\right\rangle_\eta = \\
-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right] \times \\
\left[e^{-(k_2^2+m^2)\tau'} + \frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_2^2+m^2)\tau'}\right]\right] + \\
+\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_2)}\int_0^\tau e^{-(k_1^2+m^2)(\tau-\tau_1)}\langle\widehat{\eta}(k_1,\tau_1;\omega)\widehat{\eta}(k_2,\tau_2;\omega)\rangle_\eta d\tau_1 d\tau_2
\end{gathered}
\tag{2.3.9}
$$

We set now $\tau' = \tau, a' = a$. Note that

$$
\begin{gathered}
-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right] \times \\
\left.\left[e^{-(k_2^2+m^2)\tau'} + \frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau'}\right]\right]\right|_{\tau'=\tau} = \\
-(2\pi)^8 a^2\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_1^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right] \times \\
\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2}{k_2^2+m^2}\left[1-e^{-(k_1^2+m^2)\tau}\right]\right]
\end{gathered}
\tag{2.3.10}
$$

and

$$
\begin{gathered}
\left.\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_2)}\int_0^\tau e^{-(k_1^2+m^2)(\tau-\tau_1)}\langle\widehat{\eta}(k_1,\tau_1;\omega)\widehat{\eta}(k_2,\tau_2;\omega)\rangle_\eta d\tau_1 d\tau_2\right|_{\tau'=\tau} = \\
\delta(k_1+k_2)\int_0^\tau e^{-(k_2^2+m^2)(\tau-\tau_2)}\int_0^\tau e^{-(k_1^2+m^2)(\tau-\tau_1)}\delta(\tau_1-\tau_2)d\tau_1 d\tau_2 = \\
2(2\pi)^4\delta(k_1+k_2)\int_0^\tau e^{-(k_1^2\dot{+}k_2^2+2m^2)(\tau-\tau_1)}d\tau_1 = \\
2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2\dot{+}k_2^2+2m^2)\tau}\int_0^\tau e^{(k_1^2\dot{+}k_2^2+2m^2)\tau_1}d\tau_1 = \\
2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2\dot{+}k_2^2+2m^2)\tau}\left[\frac{1}{(k_1^2\dot{+}k_2^2+2m^2)}e^{(k_1^2\dot{+}k_2^2+m^2)\tau} - \frac{1}{(k_1^2\dot{+}k_2^2+2m^2)}\right] = \\
= \frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2\dot{+}k_2^2+2m^2} - 2(2\pi)^4\delta(k_1+k_2)\frac{e^{-(k_1^2\dot{+}k_2^2+2m^2)\tau}}{(k_1^2\dot{+}k_2^2+2m^2)}
\end{gathered}
\tag{2.3.11}
$$

From Eq.(2.3.9)-Eq.(2.3.11) we get

$$\lim_{\tau\to\infty}\langle\hat{v}(k_1,\tau,a)\hat{v}(k_2,\tau,a)\rangle_\eta = (2\pi)^8 a^2\delta^4(k_1)\delta^4(k_2)\frac{m^4}{(k_1^2+m^2)(k_2^2+m^2)} - \\ -\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m^2}. \tag{2.3.12}$$

Therefore

$$\lim_{\tau\to\infty}\langle\hat{v}(x_1,\tau,a)\hat{v}(x_2,\tau,a)\rangle_\eta = \\ \times a^2(2\pi)^{-8}\times\int d^4k_1 e^{ik_1x_1}\int d^4k_2 e^{ik_2x_2}\delta^4(k_1)\delta^4(k_2)\frac{m^4}{(k_1^2+m^2)\times(k_2^2+m^2)} - \\ -(2\pi)^{-8}\times\int d^4k_1 e^{ik_1x_1}\int d^4k_2 e^{ik_2x_2}\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m^2} = \\ = \left[\frac{a^2m^4}{m^2\times m^2}+(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2}\right] = a^2-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2} \tag{2.3.13}$$

Two point function $G(x_1,x_2)$ of euclidean QFT corresponding to the action (2.3.1) is

$$G(x_1,x_2) = \lim_{\tau\to\infty}\langle\varphi(x_1,\tau;\omega)\varphi(x_2,\tau;\omega)\rangle_\eta \tag{2.3.14}$$

Master equation corresponding to two-point function $G(x_1,x_2)$ reads

$$\lim_{\tau\to\infty}\langle\varphi(x_1,\tau;\omega)\varphi(x_2,\tau;\omega)\rangle - a^2 = \lim_{\tau\to\infty}\langle\hat{v}(k_1,\tau,a)\hat{v}(k_2,\tau,a)\rangle_\eta = 0. \tag{2.3.15}$$

From (2.3.15) we get

$$\lim_{\tau\to\infty}\langle\varphi(x_1,\tau;\omega)\varphi(x_2,\tau;\omega)\rangle_\eta = a^2. \tag{2.3.16}$$

From Eq.(2.3.13)-Eq.(2.3.16) we get

$$\lim_{\tau\to\infty}\langle\hat{v}(k_1,\tau,a)\hat{v}(k_2,\tau,a')\rangle_\eta = 0 \Leftrightarrow a^2-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2} = 0 \tag{2.3.17}$$

and therefore

$$a^2 = (2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2}. \tag{2.3.18}$$

From Eq.(2.3.16) and Eq.(2.3.21) finally we get desired result

$$G(x_1,x_2) = \lim_{\tau\to\infty}\langle\varphi(x_1,\tau;\omega)\varphi(x_2,\tau;\omega)\rangle_\eta = \\ (2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2} = (2\pi)^{-2}\left(\frac{m}{|x|}\right)K_1(m|x_1-x_2|), \tag{2.3.19}$$

where K_1 is the modified Bessel functions of the second kind, integer order 1.

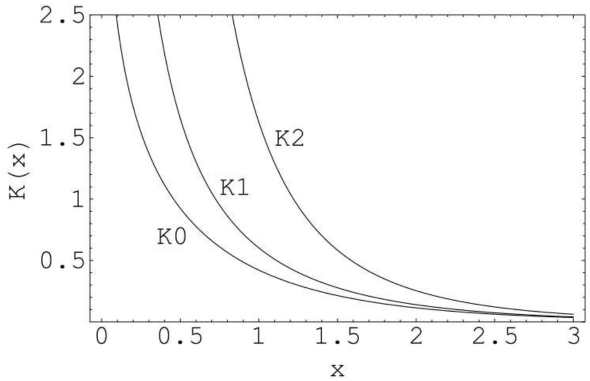

Figure 2.3.1.Plot of the modified Bessel functions of the second kind,integer order 1.

2.4.Double stochastic quantization the $\lambda\phi_d^4$ theory.

In this section we consider a neutral scalar field with a $\frac{\lambda}{4!}\phi_d^4$ $, d \geq 4,$ self-interaction, defined in a d-dimensional Minkowski spacetime. The vacuum persistence functional is the generating functional of all vacuum expectation value of time-ordered products of the theory. The Euclidean field theory can be obtained by analytic continuation to imaginary time supported by the positive energy condition for the relativistic field theory. In the Euclidean field theory, we have the Euclidean counterpart for the vacuum persistence functional, that is, the generating functional of complete Schwinger functions. Actually, the $(\lambda\varphi^4)_d$ Euclidean theory is defined by these Euclidean Green's functions. The Euclidean generating functional $Z[h]$ is formally defined by the following functional integral:

$$Z[h] = \int [d\varphi] \ \exp\Big(-S_0 - S_I + \int d^d x\, h(x)\varphi(x)\Big), \tag{2.4.1}$$

where the action that usually describes a free scalar field is

$$S_0[\varphi] = \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m_0^2\,\varphi^2(x)\right), \tag{2.4.2}$$

and the interacting part, defined by the non-Gaussian contribution, is

$$S_I[\varphi] = \int d^d x \,\frac{\lambda}{4!}\,\varphi^4(x). \tag{2.4.3}$$

In Eq.(2.4.1), $[d\varphi]$ is a translational invariant measure, formally given by $[d\varphi] = \prod_{x\in\mathbb{R}^d} d\varphi(x)$. The terms λ and m_0^2 are respectively the bare coupling constant and the squared mass of the model. Finally, $h(x)$ is a smooth function that we introduce to generate the Schwinger functions of the theory by functional derivatives. In the weak-coupling perturbative expansion, which is the conventional procedure, we perform a formal perturbative expansion with respect to the non-Gaussian terms of the action. As a consequence of this formal expansion, all the n-point unrenormalized Schwinger functions are expressed in a powers series of the bare coupling constant λ.

The aim of this section is to discuss the double stochastic quantization of a free scalar field. It can be shown that it is equivalent to the usual path integral quantization. The starting point of the stochastic quantization to obtain the Euclidean field theory is a Markovian Langevin equation. Assume an Euclidean d-dimensional manifold, where we are choosing periodic boundary conditions for a scalar field and also a random noise. In other words, they are defined in a d-torus $\Omega \equiv T^d$. To implement the stochastic quantization we supplement the scalar field $\varphi(x)$ and the random noises $\eta(x)$ and $\hat{\eta}(\tau,x)$ with an extra coordinate τ, the Markov parameter, such that $\varphi(x) \to \varphi(\tau,x)$ and $\eta(x) \to \eta(\tau,x)$.
Therefore, the fields and the random noises $\eta(\tau,x)$ and $\widetilde{\eta}(\tau,x)$ are defined in a domain: $T^d \times R^{(+)}$. Let us consider that this dynamical system is out of equilibrium, being described by the following equation of evolution:

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -\frac{\delta S_0[\varphi]}{\delta\varphi(x)}\Big|_{\varphi(x)=\varphi(\tau,x)} + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x), \qquad (2.4.4)$$

where τ is a Markov parameter, $\eta(\tau,x)$ is a random noise field and S_0 is the usual free action defined in Eq.(2.4.2). For a free scalar field, the double stochastic Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -(-\Delta + m_0^2)\varphi(\tau,x) + \eta(\tau,x) + \epsilon\hat{\eta}(\tau,x), \qquad (2.4.5)$$

where Δ is the d-dimensional Laplace operator. The Eq.(2.4.5) describes a Ornstein-Uhlenbeck process and we are assuming the Einstein relations, that is:

$$\begin{aligned}\langle\eta(\tau,x)\rangle_\eta &= 0,\\ \langle\widetilde{\eta}(\tau,x)\rangle_\eta &= 0,\end{aligned} \qquad (2.4.6)$$

and for the two-point correlation function associated with the random noise fields

$$\begin{aligned}\langle\eta(\tau,x)\eta(\tau',x')\rangle_\eta &= 2\delta(\tau-\tau')(x-x'),\\ \langle\widetilde{\eta}(\tau,x)\widetilde{\eta}(\tau',x')\rangle_{\hat{\eta}} &= 2\delta(\tau-\tau')(x-x'),\end{aligned} \qquad (2.4.7)$$

where $\langle\ldots\rangle_\eta$ means stochastic averages. In a generic way, the stochastic average for any functional of φ given by $F[\varphi]$ is defined by

$$\langle F[\varphi]\rangle_{\eta,\hat{\eta}} = \frac{\int D[\eta]D[\hat{\eta}]F[\varphi]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\eta^2(\tau,x)\Big]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\widetilde{\eta}^2(\tau,x)\Big]}{\Big(\int D[\eta]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\eta^2(\tau,x)\Big]\Big)\Big(\int D[\widetilde{\eta}]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\widetilde{\eta}^2(\tau,x)\Big]\Big)}. \qquad (2.4.8)$$

Let us define the retarded Green function for the diffusion problem that we call $G(\tau-\tau',x-x')$. The retarded Green function satisfies $G(\tau-\tau',x-x') = 0$ if $\tau-\tau' < 0$ and also

$$\Big[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2)\Big]G(\tau-\tau',x-x') = \delta^d(x-x')\delta^d(\tau-\tau'). \qquad (2.4.9)$$

Using the retarded Green function and the initial condition $\varphi(\tau,x)|_{\tau=0} = 0$, the solution for Eq.(3.5.5) reads

$$\varphi(\tau,x) = \int_0^\tau d\tau'\int_\Omega d^dx'\,G(\tau-\tau',x-x')[\eta(\tau',x') + \epsilon\hat{\eta}(\tau',x)]. \qquad (2.4.10)$$

In the following we are interested in calculating the quantity $\langle\varphi(\tau,x)\varphi(\tau',x')\rangle_{\eta,\hat{\eta}}$. Using

Eq.(2.4.6), Eq.(2.4.7) and Eq.(2.4.10), we have

$$\langle \varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle_{\eta,\hat{\eta}} = 2\int_0^{\min(\tau_1,\tau_2)} d\tau' \int_\Omega d^dx' G(\tau_1-\tau',x_1-x')\,G(\tau_2-\tau',x_2-x'), \tag{2.4.11}$$

where $\min(\tau_1,\tau_2)$ means the minimum of τ_1 and τ_2. Using a Fourier representation, the two-point correlation function $\langle \varphi(\tau,x)\varphi(\tau',x')\rangle_\eta \equiv D(\tau,x;\tau',x')$ is given by

$$D(\tau,x;\tau',x') = \frac{1}{(2\pi)^d}\int d^dp\,\frac{e^{-ip(x-x')}}{(p^2+m_0^2)}\,e^{-(p^2+m_0^2)(\tau-\tau')}. \tag{2.4.12}$$

It is not difficult to show that Eq.(2.4.12) can be written as:

$$D(\tau,x;\tau',x') = \frac{1}{(2\pi)^d}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(1-n)n!}(\tau-\tau')^n\Big(\frac{m_0}{r}\Big)^{\frac{d}{2}+n-1}K_{\frac{d}{2}+n-1}(m_0 r). \tag{2.4.13}$$

where $r = |\ x-x'\ |$ and K_ν is the modified Bessel function of order ν.
We can use the Fourier analysis to show that when the Markov parameters τ and τ' go to infinity we recover the standard Euclidean free field theory. Therefore let us define the Fourier transforms for the field and the noises given by $\varphi(\tau,k)$ and $\eta(\tau,k)$. We have respectively

$$\varphi(\tau,k) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int d^dx\,e^{-ikx}\,\varphi(\tau,x), \tag{2.4.14}$$

and

$$\begin{aligned}\hat{\eta}(\tau,k) &= \frac{1}{(2\pi)^{d/2}}\int e^{-ikx}\,\eta(\tau,x),\\ \hat{\hat{\eta}}(\tau,k) &= \frac{1}{(2\pi)^{d/2}}\int e^{-ikx}\,\hat{\hat{\eta}}(\tau,x)\end{aligned} \tag{2.4.15}$$

Substituting Eq.(2.4.14) in Eq.(2.4.2), the free action for the scalar field in the $(d+1)$-dimensional space writing in terms of the Fourier coefficients reads

$$S_0[\varphi(k)]|_{\varphi(k)=\varphi(\tau,k)} = \frac{1}{2}\int d^dk\,\varphi(\tau,k)(k^2+m_0^2)\varphi(\tau,k). \tag{2.4.16}$$

Substituting Eq.(2.4.14) and Eq.(2.4.15) in Eq.(2.4.5) we have that each Fourier coefficient satisfies a Langevin equation given by

$$\frac{\partial}{\partial\tau}\varphi(\tau,k) = -(k^2+m_0^2)\varphi(\tau,k) + \eta(\tau,k) + \epsilon\hat{\eta}(\tau,k). \tag{2.4.17}$$

The solution for this equation reads

$$\varphi(\tau,k) = \exp(-(k^2+m_0^2)\tau)\varphi(0,k) + \int_0^\tau d\tau'\exp(-(k^2+m_0^2)(\tau-\tau'))[\eta(\tau',k)+\epsilon\hat{\eta}(\tau',k)]. \tag{2.4.18}$$

Using the Einstein relation, we get that the Fourier coefficients for the random noise satisfies

$$\begin{aligned}\langle \eta(\tau,k)\rangle_\eta &= 0,\\ \langle \hat{\eta}(\tau,k)\rangle_\eta &= 0\end{aligned} \tag{2.4.19}$$

and

$$\langle \eta(\tau,k)\eta(\tau',k')\rangle_\eta = 2(2\pi)^d\delta^d(\tau-\tau')(k+k'),$$
$$\langle \widehat{\widetilde{\eta}}(\tau,k)\widehat{\widetilde{\eta}}(\tau',k')\rangle_\eta = 2(2\pi)^d\delta^d(\tau-\tau')(k+k') \quad (2.4.20)$$

Before investigate the interacting field theory, let us calculate the Fourier representation for the two-point correlation function, i.e., $\langle \varphi(\tau,k)\varphi(\tau',k')\rangle_\eta$. Using Eq.(2.4.18), we obtain three contributions to the scalar two-point correlation function. The first one is given by

$$\exp(-(k^2+m_0^2)\,\tau + (k'^2+m_0^2)\,\tau')\varphi(0,k)\varphi(0,k'), \quad (2.4.21)$$

and decay to zero at long time. Let us assume that $\varphi(\tau,k)|_{\tau=0} = 0$. There are also two crossed terms, each first order in the noise Fourier component given by

$$2\,\varphi(0,k)\exp(-(k^2+m_0^2)\tau)\int_0^{\tau'} ds\,\exp(-(k'^2+m_0^2)(\tau'-s))\Big[\eta(s,k')+\epsilon\widehat{\widetilde{\eta}}(\sigma,k')\Big]. \quad (2.4.22)$$

Since we are assuming the Einstein relations, i.e., $\langle \eta(\tau,x)\rangle_\eta = 0, \langle \widetilde{\eta}(\tau,x)\rangle_\eta = 0$ on averaging on noise, these cross terms vanish. The final term is second-order in the noise Fourier component. Again, the solution subject to the initial condition $\varphi(\tau,k)|_{\tau=0} = 0$ can be used to give

$$\left\{\int_0^{\tau} ds\,\exp(-(k^2+m_0^2)(\tau-s))\Big[\eta(s,k)+\epsilon\widehat{\widetilde{\eta}}(s,k)\Big]\right\}\times$$
$$\left\{\int_0^{\tau'} d\sigma\,\exp(-(k'^2+m_0^2)(\tau'-\sigma))\Big[\eta(\sigma,k')+\epsilon\widehat{\widetilde{\eta}}(\sigma,k')\Big]\right\}. \quad (2.4.23)$$

Again averaging on noises and using the Einstein relation given by Eq.(2.4.20) we have
that this term becomes

$$2\delta^d(k+k'_0)\int_0^{\min(\tau,\tau')} ds\,\exp(-(k^2+m_0)(\tau+\tau'-2s)). \quad (2.4.24)$$

Assuming that $\tau = \tau'$ and using $\langle \varphi(\tau,k)\varphi(\tau',k')\rangle_\eta|_{\tau=\tau'} \equiv D(k,k';\tau,\tau')$ we have

$$D(k;\tau,\tau) = (2\pi)^d\delta^d(k+k')\frac{1}{(k^2+m_0^2)}(1-\exp(-2\tau(k^2+m_0^2))). \quad (2.4.25)$$

In the following, we are redefining the two-point correlation function as $D(k;\tau,\tau) \to (2\pi)^d D(k;\tau,\tau)$. In the limit when $\tau\to\infty$ we recover the standard two-point function of the Euclidean free field theory. Before going to the next section, we would like to mention the existence of more general Markovian Langevin equations. We can introduce a kernel defined in the d-torus. The kerneled Langevin equation reads:

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -\int d^dy\,K(x,y)\frac{\delta S_0}{\delta\varphi(y)}|_{\varphi(y)=\varphi(\tau,y)} + \eta(\tau,x) + \epsilon\widehat{\eta}(\tau,x). \quad (2.4.26)$$

The second moment of the noise fields will be modified to:

$$\langle \eta(\tau,x)\,\eta(\tau',x')\rangle_\eta = 2\delta(\tau-\tau')\,K(x,x'),$$
$$\langle \widetilde{\eta}(\tau,x)\,\widetilde{\eta}(\tau',x')\rangle_\eta = 2\delta(\tau-\tau')\,K(x,x'). \quad (2.4.27)$$

Choosing an appropriate kernel, it can be shown that all the above conclusions remain unchanged.

The double stochastic Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi_\epsilon(\tau,x) = (\Delta - m_0^2)\varphi_\epsilon(\tau,x) - \frac{\lambda}{3!}\varphi_\epsilon^3(\tau,x) + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x). \quad (2.4.28)$$

By the replacements

$$\begin{aligned} \varphi_\epsilon(x,\tau;\omega,\varpi) &= v_{\epsilon-}(x,\tau;\omega,\varpi) + \theta(\tau)a, \\ \varphi_\epsilon(x,\tau;\omega,\varpi) &= v_{\epsilon+}(x,\tau;\omega,\varpi) - \theta(\tau)a, \\ \theta(\tau) &= \begin{cases} 0 & \text{if } \tau \le 1 \\ 1 & \text{if } \tau > 1 \end{cases} \end{aligned} \quad (2.4.29)$$

we obtain from Eqs.(2.4.28)

$$\begin{gathered} \frac{\partial[v_{\epsilon-}(x,\tau;\omega,\varpi) + \theta(\tau)a]}{\partial\tau} = \frac{\partial v_{\epsilon-}(x,\tau;\omega,\varpi)}{\partial\tau} + a\delta(\tau) = \\ = (\partial^2 - m^2)[v_{\epsilon-}(x,\tau;\omega,\varpi) + \theta(\tau)a] - \frac{\lambda}{3!}[v_{\epsilon-}(x,\tau;\omega,\varpi) + a]^3 + \\ +\eta(x,\tau;\omega) + \epsilon\widetilde{\eta}(x,\tau;\varpi) = \\ (\partial^2 - m^2)[v_{\epsilon-}(x,\tau;\omega,\varpi) + \theta(\tau)a] - \frac{\lambda}{3!}(v_{\epsilon-}^3 + 3av_{\epsilon-}^2 + 3a^2v_{\epsilon-} + a^3) + \\ +\eta(x,\tau;\omega) + \epsilon\widetilde{\eta}(x,\tau;\varpi) \\ \text{and} \\ \frac{\partial[v_{\epsilon+}(x,\tau;\omega,\varpi) - \theta(\tau)a]}{\partial\tau} = \frac{\partial v_{\epsilon+}(x,\tau;\omega,\varpi)}{\partial\tau} - a\delta(\tau) = \\ (\partial^2 - m^2)[v_{\epsilon+}(x,\tau;\omega,\varpi) - \theta(\tau)a] - \frac{\lambda}{3!}(v_{\epsilon+}^3 - 3av_{\epsilon+}^2 + 3a^2v_{\epsilon+} - a^3) + \\ +\eta(x,\tau;\omega) + \epsilon\widetilde{\eta}(x,\tau;\varpi) \end{gathered} \quad (2.4.30)$$

Differential master equations corresponding to double stochastic Langevin equations (2.4.30) reads

$$\begin{gathered} \frac{\partial v_-(x,\tau;\omega)}{\partial\tau} = -a\delta(\tau) + [\partial^2 - (m^2 + 0.5\lambda a^2))]v_-(x,\tau;\omega) - \\ -\frac{\lambda}{6}a^3 - m^2a + \eta(x,\tau;\omega) = \\ -a\delta(\tau) + [\partial^2 - m_1^2]v_-(x,\tau;\omega) - \left(\frac{\lambda}{6}a^3 + m^2a\right) + \eta(x,\tau;\omega) \\ \text{and} \\ \frac{\partial v_+(x,\tau;\omega)}{\partial\tau} = a\delta(\tau) + [\partial^2 - (m^2 + 0.5\lambda a^2))]v_+(x,\tau;\omega) + \\ +\frac{\lambda}{6}a^3 + m^2a + \eta(x,\tau;\omega) = \\ a\delta(\tau) + [\partial^2 - m_1^2]v_+(x,\tau;\omega) + \left(\frac{\lambda}{6}a^3 + m^2a\right) + \eta(x,\tau;\omega) \\ m_1^2 = m^2 + 0.5\lambda a^2. \end{gathered} \quad (2.4.31)$$

Consider the Fourier transformed stochastic differential equation (2.3.6) in k and τ given as

$$\begin{gathered}
\frac{\partial}{\partial\tau}\widehat{v}_{-}(k,\tau) = \\
-(k^2+m_1^2)\widehat{v}_{-}(k,\tau)-(2\pi)^4 a\delta(\tau)\delta^4(k)-(2\pi)^4\left(m^2a+\frac{\lambda}{6}a^3\right)\delta^4(k)+\widehat{\eta}(k,\tau;\omega) \\
\text{and} \\
\frac{\partial}{\partial\tau}\widehat{v}_{+}(k,\tau) = \\
-(k^2+m_1^2)\widehat{v}_{-}(k,\tau)+(2\pi)^4 a\delta(\tau)\delta^4(k)+(2\pi)^4\left(m^2a+\frac{\lambda}{6}a^3\right)\delta^4(k)+\widehat{\eta}(k,\tau;\omega)
\end{gathered} \tag{2.4.32}$$

Let us consider ODE

$$\dot{x}\,(\tau,\lambda)+\lambda x(\tau,\lambda) = g(\tau,\lambda), x(0) = 0. \tag{2.4.33}$$

The corresponding solution $x(t,\lambda)$ reads

$$x(\tau,\lambda) = e^{-\lambda\tau}\int_0^\tau e^{\lambda\tau_1}g(\tau_1,\lambda)d\tau_1. \tag{2.4.34}$$

From Eq.(2.4.32)-Eq.(2.4.34) one obtains

$$\hat{v}_{-}(k,\tau,a) = e^{-(k^2+m_1^2)\tau} \times$$

$$\times \int_0^\tau e^{(k^2+m_1^2)\tau_1}\left[-(2\pi)^4 a\delta(\tau_1)\delta^4(k) - (2\pi)^4\left(m^2 a + \frac{\lambda}{6}a^3\right)\delta^4(k) + \hat{\eta}(k,\tau_1;\omega)\right]d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4\left(m^2 a + \frac{\lambda}{6}a^3\right)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}\hat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4 a\left(m^2 + \frac{\lambda}{6}a^2\right)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\hat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} -$$

$$-(2\pi)^4 a\left(m^2 + \frac{\lambda}{6}a^2\right)\delta^4(k)e^{-(k^2+m^2)\tau}\left[\frac{e^{(k^2+m^2)\tau}}{k^2+m_1^2} - \frac{1}{k^2+m_1^2}\right] + \qquad (2.4.35)$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\hat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4\frac{\left(m^2 a + \frac{\lambda}{6}a^3\right)\delta^4(k)}{k^2+m_1^2}\left[1 - e^{-(k^2+m_1^2)\tau}\right] +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\hat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$-(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m_1^2)\tau} + \frac{\left(m^2 + \frac{\lambda}{6}a^2\right)}{k^2+m_1^2}\left[1 - e^{-(k^2+m_1^2)\tau}\right]\right] +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\hat{\eta}(k,\tau_1;\omega)d\tau_1,$$

and

$$\hat{v}_+(k,\tau,a) = e^{-(k^2+m_1^2)\tau} \times$$

$$\times\int_0^\tau e^{(k^2+m_1^2)\tau_1}\Big[(2\pi)^4 a\delta(\tau_1)\delta^4(k) - (2\pi)^4\Big(-m^2a - \frac{\lambda}{6}a^3\Big)\delta^4(k) \pm \hat{\eta}(k,\tau_1;\omega)\Big]d\tau_1 =$$

$$(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} - (2\pi)^4\Big(-m^2a - \frac{\lambda}{6}a^3\Big)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}\hat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4 a\Big(m^2 + \frac{\lambda}{6}a^3\Big)\delta^4(k)e^{-(k^2+m_1^2)\tau}\int_0^\tau e^{(k^2+m_1^2)\tau_1}d\tau_1 +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\hat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} +$$

$$+(2\pi)^4 a\Big(m^2 + \frac{\lambda}{6}a^2\Big)\delta^4(k)e^{-(k^2+m^2)\tau}\Big[\frac{e^{(k^2+m^2)\tau}}{k^2+m_1^2} - \frac{1}{k^2+m_1^2}\Big] + \qquad (2.4.35')$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\hat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k)e^{-(k^2+m_1^2)\tau} + (2\pi)^4\frac{\Big(m^2 + \frac{\lambda}{6}a^3\Big)\delta^4(k)}{k^2+m_1^2}\Big[1 - e^{-(k^2+m_1^2)\tau}\Big] +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\hat{\eta}(k,\tau_1;\omega)d\tau_1 =$$

$$+(2\pi)^4 a\delta^4(k)\left[e^{-(k^2+m_1^2)\tau} + \frac{\Big(m^2 + \frac{\lambda}{6}a^3\Big)}{k^2+m_1^2}\Big[1 - e^{-(k^2+m_1^2)\tau}\Big]\right] +$$

$$+\int_0^\tau e^{-(k^2+m_1^2)(\tau-\tau_1)}\hat{\eta}(k,\tau_1;\omega)d\tau_1.$$

From Eq.(2.4.35)-Eq.(2.4.35′) one obtains

$$
\begin{aligned}
&\widehat{v}_{-}(k_1,\tau,a)\widehat{v}_{+}(k_2,\tau',a') = \\
&\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m_1^2)\tau} + \frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right] + \right. \\
&\left. + \int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1 \right\} \times \\
&\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right] + \right. \\
&\left. + \int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_1)}\widehat{\eta}(k_2,\tau_1;\omega)d\tau_1 \right\}
\end{aligned}
\tag{2.4.36}
$$

From Eq.(2.4.36) one obtains

$$
\begin{aligned}
&\langle \widehat{v}_{-}(k_1,\tau,a)\widehat{v}_{+}(k_2,\tau',a')\rangle_{\eta} = \\
&\left\{-(2\pi)^4 a\delta^4(k_1)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\right\} \times \\
&\left\{(2\pi)^4 a'\delta^4(k_2)\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right]\right\} - \\
&-\left\langle\left(\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\widehat{\eta}(k_1,\tau_1;\omega)d\tau_1\right)\left(\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_2)}\widehat{\eta}(k_2,\tau_2;\omega)d\tau_2\right)\right\rangle_{\eta} = \\
&-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m_1^2)\tau} + \frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right] \times \\
&\left[e^{-(k_2^2+m_1^2)\tau'} + \frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_2^2+m_1^2)\tau'}\right]\right] - \\
&-\int_0^{\tau'} e^{-(k_2^2+m_1^2)(\tau'-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\langle\widehat{\eta}(k_1,\tau_1;\omega)\widehat{\eta}(k_2,\tau_2;\omega)\rangle_{\eta}d\tau_1 d\tau_2
\end{aligned}
\tag{2.4.37}
$$

Note that

$$-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times$$
$$\left[e^{-(k_2^2+m^2)\tau'} + \frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau'}\right]\right]\Bigg|_{\tau'=\tau,a'=a} =$$
$$-(2\pi)^8 aa'\delta^4(k_1)\delta^4(k_2)\left[e^{-(k_1^2+m_1^2)\tau} + \frac{m^2+\frac{\lambda}{6}a^2}{k_1^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right]\times$$
$$\times\left[e^{-(k_1^2+m^2)\tau} + \frac{m^2+\frac{\lambda}{6}a'^2}{k_2^2+m_1^2}\left[1-e^{-(k_1^2+m_1^2)\tau}\right]\right] \tag{2.4.38}$$

and note that

$$\int_0^{\tau'} e^{-(k_2^2+m^2)(\tau'-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m^2)(\tau-\tau_1)}\langle\widehat{\eta}(k_1,\tau_1;\omega)\widehat{\eta}(k_2,\tau_2;\omega)\rangle_\eta d\tau_1 d\tau_2\Bigg|_{\tau'=\tau,a'=a} =$$
$$\delta(k_1+k_2)\int_0^{\tau} e^{-(k_2^2+m_1^2)(\tau-\tau_2)}\int_0^{\tau} e^{-(k_1^2+m_1^2)(\tau-\tau_1)}\delta(\tau_1-\tau_2)d\tau_1 d\tau_2 =$$
$$2(2\pi)^4\delta(k_1+k_2)\int_0^{\tau} e^{-(k_1^2+k_2^2+2m_1^2)(\tau-\tau_1)}d\tau_1 = \tag{2.4.39}$$
$$2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m_1^2)\tau}\int_0^{\tau} e^{(k_1^2+k_2^2+2m_1^2)\tau_1}d\tau_1 =$$
$$2(2\pi)^4\delta(k_1+k_2)e^{-(k_1^2+k_2^2+2m_1^2)\tau}\left[\frac{1}{(k_1^2+k_2^2+2m_1^2)}e^{(k_1^2+k_2^2+m_1^2)\tau} - \frac{1}{(k_1^2+k_2^2+2m_1^2)}\right] =$$
$$= \frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m_1^2} - 2(2\pi)^4\delta(k_1+k_2)\frac{e^{-(k_1^2+k_2^2+2m_1^2)\tau}}{(k_1^2+k_2^2+2m_1^2)}$$

From Eq.(2.4.38)-Eq.(2.4.39) we get

$$\lim_{\tau\to\infty}\langle\widehat{v}_-(k_1,\tau,a)\widehat{v}_+(k_2,\tau,a)\rangle_\eta = (2\pi)^8 a^2\delta^4(k_1)\delta^4(k_2)\frac{\left(m^2+\frac{\lambda}{6}a^2\right)^2}{(k_1^2+m_1^2)(k_2^2+m_1^2)} -$$
$$-\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m_1^2} \tag{2.4.40}$$
$$m_1^2 = m^2+0.5\lambda a^2$$

Therefore

$$\lim_{\tau\to\infty}\langle\hat{v}_{-}(x_1,\tau,a)\hat{v}_{+}(x_2,\tau,a)\rangle_\eta =$$

$$-a^2(2\pi)^{-8}\times\int d^4k_1 e^{ik_1x_1}\int d^4k_2 e^{ik_2x_2}\delta^4(k_1)\delta^4(k_2)\frac{\left(m^2+\frac{\lambda}{6}a^2\right)^2}{(k_1^2+m_1^2)\times(k_2^2+m_1^2)}+$$

$$(2\pi)^{-8}\times\int d^4k_1 e^{ik_1x_1}\int d^4k_2 e^{ik_2x_2}\frac{2(2\pi)^4\delta(k_1+k_2)}{k_1^2+k_2^2+2m_1^2} =$$

$$=\left[\frac{a^2\left(m^2+\frac{\lambda}{6}a^2\right)^2}{m_1^2\times m_1^2}-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m_1^2}\right]= \qquad (2.4.41)$$

$$\frac{a^2\left(m^2+\frac{\lambda}{6}a^2\right)^2}{m_1^2\times m_1^2}-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m_1^2}$$

$$m_1^2 = m^2+0.5\lambda a^2$$

Transcendental master equation corresponding to two-point Green function $G(x_1,x_2,\lambda)$ reads

$$\frac{\left[a(x_1-x_2)^2\right]\left(m^2+\frac{\lambda}{6}a(x_1-x_2)^2\right)^2}{\left(m^2+0.5\lambda a(x_1-x_2)^2\right)^2}-$$

$$-(2\pi)^{-4}\int d^4k\frac{e^{ik(x_1-x_2)}}{k^2+m^2+0.5\lambda a(x_1-x_2)^2} = 0. \qquad (2.4.41)$$

3.Weak coupling. Nonperturbative result.

3.1.Weak coupling.The $\lambda\phi_d^4$ theory

We assume now that

$$\varepsilon = \frac{\lambda\theta_\delta(|x|)a(x_1-x_2)^2}{m^2} \ll 1, \qquad (3.1.1)$$

where $\theta_\delta(|x|) = \theta_\delta(|x_1-x_2|) = \theta(|x_1-x_2|-\delta), x = |x_1-x_2|$.
From Eq.(2.4.41) and Eq.(3.1.1) we get

$$\theta_\delta(|x|)a^2(x)\frac{\left(1+\frac{\lambda a^2(x)}{6m^2}\right)^2}{\left(1+\frac{0.5\lambda a^2(x)}{m^2}\right)^2}-(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)\left[1+\frac{0.5\lambda a^2(x)}{m^2+k^2}\right]}. \qquad (3.1.2)$$

and

$$\theta_\delta(|x|)a^2(x) = (2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}\left[1-\frac{0.5\lambda a^2(x)}{m^2+k^2}+\left(\frac{0.5\lambda a^2(x)}{m^2+k^2}\right)^2+\ldots\right]=$$

$$(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)}-0.5\lambda(2\pi)^{-4}a^2(x)\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^2}+\ldots \qquad (3.1.3)$$

Therefore under condition (3.1.41) we get

$$a^2(x)\left[1+0.5\lambda(2\pi)^{-4}\int\frac{d^4ke^{ikx}\theta_\delta(|x|)}{(m^2+k^2)^2}\right]=(2\pi)^{-4}\int\frac{\theta_\delta(|x|)d^4ke^{ikx}}{(m^2+k^2)}+o(\varepsilon), \quad (3.1.4)$$

and thus for two-point for ϕ_4^4 theory in the Euclidean QFT we obtain nonperturbative result

$$\begin{gathered}\theta_\delta(|x_1-x_2|)G(x_1-x_2)=\theta_\delta(|x_1-x_2|)a^2(x_1-x_2)=\\(2\pi)^{-4}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)}\times\\\left[1+0.5\lambda(2\pi)^{-4}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)^2}\right]^{-1}=\\=(2\pi)^{-4}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)}-\\-0.5\lambda(2\pi)^{-8}\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)^2}\times\\\left(\int\frac{d^4ke^{ik(x_1-x_2)}\theta_\delta(|x_1-x_2|)}{(m^2+k^2)}\right)+o(\varepsilon).\end{gathered} \quad (3.1.\ 5)$$

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