Relative motion from coordinate transformation approach to gravity field

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Abstract

In this paper I will explore a way to create a gravity like equations just from coordinate transformations between systems. Will add quantum like effects in last part of this paper that will connect probability with spacetime interval. It will be done by using complex spacetime. I do not present solutions to those equations only mathematical model and physical interpretation.

Contents

Part I
Introduction

1 What is motion?

1.1 Physical field

In physics we deal with motion, motion is base of all physical phenomena. We can simply say that motion and how to measure it it's basis of physics. A physical field so is a field that has motion in some spacetime and rules how this motion happens. In reality it's much more complex than it, there are additional factors other than motion. Things like symmetries and conserved things. But simplest kind of field is just motion and it's background so spacetime itself. Einstein did show that spacetime is not a fixed object, it depends on energy of objects in that field. Energy in general changes how spacetime differs from flat spacetime. Objects follow geodesic paths in that spacetime. General Relativity is based on curvature tensors mainly Ricci tensor as it is in Einstein equation. I will try another approach, my idea is to focus on observer motion and transformation of their motion in spacetime relative to each other. I will skip curvature tensors in general. It means that each observer will define his coordinates and from it I will create rules for coordinate transformation between those observers. It seems simple but in truth it's not easy to explain any motion without a fixed coordinate system, that coordinate system maps spacetime but itself is a map. My goal is to explain motion in observer base way, not coordinate or spacetime based. It mean focus on what observer will observe as motion of all other observers. This way of defining a physical field and motion in that field does not focus on invariant properties of spacetime itself rather on how motion in seen by observers. Each observer has it's own right to say that his perspective of spacetime is true but not independent way of physical properties and motion of rest of field so observers.

1.2 What gravity does

Gravity is base of all forces in physical world. In it's simplest formulation it does make masses fall into each other, in Newtonian picture there is a force between two bodies that make them attract each other. But this picture did fall short when Einstein did come up with his field equations for gravity. In this new picture gravity is spacetime curvature and objects follow shortest paths in spacetime (geodesic). So from laws of motion into geometry, good question would be what is next? Goal of modern gravity theories is to find a quantum gravity theory, theory that would connect quantum mechanics with relativity. In this paper I will not try that, rather try to explain motion thus gravity as simply effect of what observer see when they move. Rather than explaining motion in terms of field that is same for each observer, I will try to explain each observer independent view of spacetime and how they connect. It's not quantum gravity and even not gravity in classical sense. It's just a way observer can make his own way of defining spacetime. So it's not a gravity field but rather observer transformation field. This field is always defined by observer motion and it's energy. But from it follows that it's simpler idea about what motion is than spacetime curvature, from it follows that it's simpler to make this model work as it does not depend of spacetime curvature.

1.3 Where Einstein fails

Einstein field equations [1] are most tested gravity theory, those equations are based on Bianachi identity and can be expressed as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{1}$$

Those equations are well tested and for now there is none experiment that would show they fail. But from equations come their doom itself, singularity at the center of black hole and singularity at moment of big bang. Let's start with simple example from black holes. Black holes metric [2] can be written as:

$$ds^2 = \left(1 - \frac{r_s}{r}\right)c^2dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 - d\Omega^2 \tag{2}$$

There are two singularities in this metric. One is only a coordinate singularity, that when $r = r_s$ and another one is real one where $r = 0$. It means that objects that will form a black hole will to form a singularity where General Relativity stops making sensible predictions. In same manner it fails at moment of big bang.

1.4 Free-fall

Base of all gravity phenomena is free-fall. My approach to free-fall is that it can be explained by two observers. One free-falling and one stationary of surface of body that first observer is falling into. Without any curvature tensors I still need to use geodesic equation for calculation of trajectories. So this model is only coordinate based not a curvature based. But still to explain motion there is need to use geodesic equation. Let's go back to example with two observers. What does stationary observer see? It sees a falling object and what does falling object see? It sees surface of body accelerating towards it. Those are two opposite and equal perspective. This way I will explain any motion by comparing motion of observers seeing same scenario but from opposite ways. So what is simplest transformation of coordinates that will work in this case? Answer is pretty simple, it's not about coordinate but about their relation. This relation is that their transformation cancel each other out. If in first frame of reference where falling object is falling and observer standing on surface of body is stationary. In second free-falling observer is stationary and it's the surface observer accelerating towards it. If I connect those coordinates in certain way they will cancel out. Formally lets write down first observer perspective:

$$dy'^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{\kappa}}dy^{\kappa} \tag{3}$$

And second observer perspective:

$$dx'^{\alpha} = \frac{\partial y^{\alpha}}{\partial x^{\kappa}}dx^{\kappa} \tag{4}$$

Where I do transform coordinates of second observer into first, and opposite. In first case observer x is stationary, and in second case observer y is stationary. From it follows that in first case falling observer will move towards stationary observer by amount dy'^{α}, in second case observer on surface of body will move toward free-falling now stationary observer with amount dx'^{α}. But from that their motion are opposites of each other I can write a transformation that will leave them both constant:

$$dy^{\alpha} = \frac{\partial y^{\alpha}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial y^{\kappa}} dy^{\kappa} \tag{5}$$

$$dx^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{\rho}} \frac{\partial y^{\rho}}{\partial x^{\kappa}} dx^{\kappa} \tag{6}$$

In general idea is that all transformations that fulfill this way of transforming will be non energy content transformations. It means that bodies don't interact in any way just move relative to each other same way. But it seems they are in opposite motion. More general transformation laws will be when transformation laws are not opposites of each other. Then I can write that:

$$d\xi^{\alpha} = \frac{\partial y^{\alpha}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial y^{\kappa}} dy^{\kappa} \tag{7}$$

$$d\chi^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{\rho}} \frac{\partial y^{\rho}}{\partial x^{\kappa}} dx^{\kappa} \tag{8}$$

Where new coordinates are no longer equal to old ones. This works only in presence of energy. To be more specific, those coordinates transformations map coordinates of y into x, and y into x. Then compare how both of them change , this can be written as:

$$d\xi^{\alpha}_{y \mapsto x} = \frac{\partial y^{\alpha}(\mathbf{x}(\mathbf{y}))}{\partial x^{\rho}(\mathbf{x})} \frac{\partial x^{\rho}(\mathbf{y}(\mathbf{x}))}{\partial y^{\kappa}(\mathbf{y})} dy^{\kappa}(\mathbf{x}(\mathbf{y})) \tag{9}$$

$$d\xi^{\alpha}_{x \mapsto y} = \frac{\partial x^{\alpha}(\mathbf{y}(\mathbf{x}))}{\partial y^{\rho}(\mathbf{y})} \frac{\partial y^{\rho}(\mathbf{x}(\mathbf{y}))}{\partial x^{\kappa}(\mathbf{x})} dx^{\kappa}(\mathbf{y}(\mathbf{x})) \tag{10}$$

Those are more general transformations if the give back same vector it means there is simplest case of transformations involved and there is no energy involved in those transformations.

Part II
Mathematical formulation

2 Field equation

2.1 Distance in spacetime and coordinate transformation

Next step is to define this idea in precise framework. Base of all spacetime is to define distance or metric tensor in that spacetime. I will first define a distance transformations then from it figure out meaning and rest. Simplest way to define spacetime distance is just to take a flat spacetime metric and apply transformations on it. That would mean that spacetime metric is in form of:

$$g^{(y)}_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial x^{\alpha}\left(\mathbf{y}\left(\mathbf{x}\right)\right)}{\partial y^{\mu}\left(\mathbf{y}\right)}\frac{\partial x^{\beta}\left(\mathbf{y}\left(\mathbf{x}\right)\right)}{\partial y^{\nu}\left(\mathbf{y}\right)} \tag{11}$$

$$g^{(x)}_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial y^{\alpha}\left(\mathbf{x}\left(\mathbf{y}\right)\right)}{\partial x^{\mu}\left(\mathbf{x}\right)}\frac{\partial y^{\beta}\left(\mathbf{x}\left(\mathbf{y}\right)\right)}{\partial x^{\nu}\left(\mathbf{x}\right)} \tag{12}$$

Now from it it's easy to arrive at spacetime distance. Spacetime interval is just:

$$ds^2_{(y)} = \eta_{\alpha\beta}\frac{\partial x^{\alpha}\left(\mathbf{y}\left(\mathbf{x}\right)\right)}{\partial y^{\mu}\left(\mathbf{y}\right)}\frac{\partial x^{\beta}\left(\mathbf{y}\left(\mathbf{x}\right)\right)}{\partial y^{\nu}\left(\mathbf{y}\right)}dy^{\mu}\left(\mathbf{x}\left(\mathbf{y}\right)\right)dy^{\nu}\left(\mathbf{x}\left(\mathbf{y}\right)\right) \tag{13}$$

$$ds^2_{(x)} = \eta_{\alpha\beta}\frac{\partial y^{\alpha}\left(\mathbf{x}\left(\mathbf{y}\right)\right)}{\partial x^{\mu}\left(\mathbf{x}\right)}\frac{\partial y^{\beta}\left(\mathbf{x}\left(\mathbf{y}\right)\right)}{\partial x^{\nu}\left(\mathbf{x}\right)}dx^{\mu}\left(\mathbf{y}\left(\mathbf{x}\right)\right)dx^{\nu}\left(\mathbf{y}\left(\mathbf{x}\right)\right) \tag{14}$$

But it still lacks basics of relativity that is invariant speed of light.

2.2 Lorentz transformations

I can add Lorentz transformations to those equations. Let me start by simplest case, there is only flat spacetime metric and a vector fields y and x:

$$ds'^2_{(y)} = \eta_{\mu\nu}\Lambda^{\mu}_{\alpha}\left(\mathbf{x}\left(\mathbf{y}\right)\right)dy^{\alpha}\left(\mathbf{x}\left(\mathbf{y}\right)\right)\Lambda^{\nu}_{\beta}\left(\mathbf{x}\left(\mathbf{y}\right)\right)dy^{\beta}\left(\mathbf{x}\left(\mathbf{y}\right)\right) \tag{15}$$

$$ds'^2_{(x)} = \eta_{\mu\nu}\Lambda^{\mu}_{\alpha}\left(\mathbf{y}\left(\mathbf{x}\right)\right)dx^{\mu}\left(\mathbf{y}\left(\mathbf{x}\right)\right)\Lambda^{\mu}_{\alpha}\left(\mathbf{y}\left(\mathbf{x}\right)\right)dx^{\nu}\left(\mathbf{y}\left(\mathbf{x}\right)\right) \tag{16}$$

$$ds^2_{(y)} = \eta_{\mu\nu}dy^{\alpha}\left(\mathbf{x}\left(\mathbf{y}\right)\right)dy^{\beta}\left(\mathbf{x}\left(\mathbf{y}\right)\right) \tag{17}$$

$$ds^2_{(x)} = \eta_{\mu\nu}dx^{\mu}\left(\mathbf{y}\left(\mathbf{x}\right)\right)dx^{\nu}\left(\mathbf{y}\left(\mathbf{x}\right)\right) \tag{18}$$

$$ds^2_{(y)} = ds'^2_{(y)} \tag{19}$$

$$ds^2_{(x)} = ds'^2_{(x)} \tag{20}$$

Now those equations transform as they should when switching frames. Where we can only transform vector fields in their respect frames. When they are transformed again, into other observer frames of reference , we don't apply another transformations. Those are transformations that work only locally.

2.3 Free-fall trajectory

In previous part I was talking about free-fall and other simplest kind of transformations. Those are transformations that are opposite of each other. So they can be written in a way that they cancel each other out. But let's go back to frame transformations.I can write two spacetime intervals:

$$ds^2_{(y)} = \eta_{\alpha\beta}\frac{\partial x^{\alpha}\left(\mathbf{y}\left(\mathbf{x}\right)\right)}{\partial y^{\mu}\left(\mathbf{y}\right)}\frac{\partial x^{\beta}\left(\mathbf{y}\left(\mathbf{x}\right)\right)}{\partial y^{\nu}\left(\mathbf{y}\right)}dy^{\mu}\left(\mathbf{x}\left(\mathbf{y}\right)\right)dy^{\nu}\left(\mathbf{x}\left(\mathbf{y}\right)\right) \tag{21}$$

$$ds^2_{(x)} = \eta_{\alpha\beta}\frac{\partial y^{\alpha}\left(\mathbf{x}\left(\mathbf{y}\right)\right)}{\partial x^{\mu}\left(\mathbf{x}\right)}\frac{\partial y^{\beta}\left(\mathbf{x}\left(\mathbf{y}\right)\right)}{\partial x^{\nu}\left(\mathbf{x}\right)}dx^{\mu}\left(\mathbf{y}\left(\mathbf{x}\right)\right)dx^{\nu}\left(\mathbf{y}\left(\mathbf{x}\right)\right) \tag{22}$$

Objects in them follow geodesic paths. It means that variation of spacetime interval is equal to zero:

$$\delta \int ds^2_{(y)} = 0 \tag{23}$$

$$\delta \int ds^2_{(x)} = 0 \tag{24}$$

This will lead to two geodesic equations [3] that I can write as:

$$\frac{d^2 y^\mu}{ds_y^2} + \Gamma^\mu_{\alpha\beta} \frac{dy^\alpha}{ds_{(y)}} \frac{dy^\beta}{ds_{(y)}} = 0 \tag{25}$$

$$\frac{d^2 x^\mu}{ds_x^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds_{(x)}} \frac{dx^\beta}{ds_{(x)}} = 0 \tag{26}$$

Its important to note that this equation needs to be fulfilled for whole field not for one line object, as x and y are vectors fields. This case is when those fields are transforming in opposite way to each other, that physically means that one observer is seeing opposite event in geometric way than another. What about trajectory that does not transform in that simple way? For this I need to add energy into equations.

2.4 Trajectory with non zero energy

To add energy into equations I can consider two objects, first one is stress energy tensor, second one is four momentum. From fact that I assume a field not a fixed object four momentum for each point of spacetime will fit better. As i don't define properties of objects globally rather locally but problem with four momentum is that it does not contain all information about energy of system, so from it follows I will use stress energy tensor. I can write stress energy tensor in coordinates of two vector fields :

$$T^{\mu\nu}_{(y)} = \frac{\partial y^\mu (\mathbf{x}(\mathbf{y}))}{\partial x^\alpha (\mathbf{x})} \frac{\partial y^\nu (\mathbf{x}(\mathbf{y}))}{\partial x^\beta (\mathbf{x})} T^{\alpha\beta}_{(y)} \tag{27}$$

$$T^{\mu\nu}_{(x)} = \frac{\partial y^\mu (\mathbf{x}(\mathbf{y}))}{\partial x^\alpha (\mathbf{x})} \frac{\partial y^\nu (\mathbf{x}(\mathbf{y}))}{\partial x^\beta (\mathbf{x})} T^{\alpha\beta}_{(x)} \tag{28}$$

Now I can go back to transformations of field. General transformation can be written as, where it does not have to be equal to base vector field:

$$d\xi^\alpha_{y\mapsto x} = \frac{\partial y^\alpha (\mathbf{x}(\mathbf{y}))}{\partial x^\rho (\mathbf{x})} \frac{\partial x^\rho (\mathbf{y}(\mathbf{x}))}{\partial y^\kappa (\mathbf{y})} dy^\kappa (\mathbf{x}(\mathbf{y})) \tag{29}$$

$$d\xi^\alpha_{x\mapsto y} = \frac{\partial x^\alpha (\mathbf{y}(\mathbf{x}))}{\partial y^\rho (\mathbf{y})} \frac{\partial y^\rho (\mathbf{x}(\mathbf{y}))}{\partial x^\kappa (\mathbf{x})} dx^\kappa (\mathbf{y}(\mathbf{x})) \tag{30}$$

Spacetime metric for general coordinate transformation is either equal to flat spacetime with no presence of energy, or it's equal to some kind of deviation from standard Minkowski metric. I can write general transformation spacetime interval:

$$ds^2_{y\mapsto x} = \eta_{\mu\nu} d\xi^\nu_{y\mapsto x} d\xi^\nu_{y\mapsto x} \tag{31}$$

$$ds^2_{x\mapsto y} = \eta_{\mu\nu} d\xi^\nu_{x\mapsto y} d\xi^\nu_{x\mapsto y} \tag{32}$$

From it I can write general metric tensor for this kind of transformation:

$$g_{\mu\nu_{x\mapsto y}} = \eta_{\alpha\beta} \frac{\partial x^{\alpha}\left(\mathbf{y}\left(\mathbf{x}\right)\right)}{\partial y^{\rho}\left(\mathbf{y}\right)} \frac{\partial x^{\beta}\left(\mathbf{y}\left(\mathbf{x}\right)\right)}{\partial y^{\kappa}\left(\mathbf{y}\right)} \frac{\partial y^{\rho}\left(\mathbf{x}\left(\mathbf{y}\right)\right)}{\partial x^{\mu}\left(\mathbf{x}\right)} \frac{\partial y^{\kappa}\left(\mathbf{x}\left(\mathbf{y}\right)\right)}{\partial x^{\nu}\left(\mathbf{x}\right)} \tag{33}$$

$$g_{\mu\nu_{y\mapsto x}} = \eta_{\alpha\beta} \frac{\partial y^{\alpha}\left(\mathbf{x}\left(\mathbf{y}\right)\right)}{\partial x^{\rho}\left(\mathbf{x}\right)} \frac{\partial y^{\beta}\left(\mathbf{x}\left(\mathbf{y}\right)\right)}{\partial x^{\kappa}\left(\mathbf{x}\right)} \frac{\partial x^{\rho}\left(\mathbf{y}\left(\mathbf{x}\right)\right)}{\partial y^{\mu}\left(\mathbf{y}\right)} \frac{\partial x^{\kappa}\left(\mathbf{y}\left(\mathbf{x}\right)\right)}{\partial y^{\nu}\left(\mathbf{y}\right)} \tag{34}$$

Now only thing left is to define how energy affects metric transformations. To do so I need to connect metric transformations to energy tensor in given frame. I can use Einstein constant and square operator to construct a field equation:

$$\Box^{(x)} g_{\mu\nu_{x\mapsto y}} = \kappa T^{(x)}_{\mu\nu} \tag{35}$$

$$\Box^{(y)} g_{\mu\nu_{y\mapsto x}} = \kappa T^{(y)}_{\mu\nu} \tag{36}$$

Where square operator is defined:

$$\Box^{(x)} = g^{\phi\varphi}_{(x)} \nabla^{(x)}_{\phi} \nabla^{(x)}_{\varphi} \tag{37}$$

$$\Box^{(y)} = g^{\phi\varphi}_{(y)} \nabla^{(y)}_{\phi} \nabla^{(y)}_{\varphi} \tag{38}$$

Where I use metric of given observer not transformation metric for both connection and metric itself in that operator.

2.5 Field that is independent of motion

From it follows that I have all mathematical rules defined. But it still lacks one key component. How this field is independent of motion of observers? I can define a field that will be tracking each observer movement from point of view of any other observer. This seems complex, but this gives definition of whole field. Field will be defined as:

$$F^{(\mathbf{x},\mathbf{y})}_{\mu\nu} := \left\{ g_{\mu\nu_{x\mapsto y}} \left(\mathbf{y} - \mathbf{y}\left(\mathbf{x}\right)\right) \mid \mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+m} \times ... \times \mathbb{R}^{n+m} \right\} \tag{39}$$

$$F^{(\mathbf{y},\mathbf{x})}_{\mu\nu} := \left\{ g_{\mu\nu_{y\mapsto x}} \left(\mathbf{x} - \mathbf{x}\left(\mathbf{y}\right)\right) \mid \mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+m} \times ... \times \mathbb{R}^{n+m} \right\} \tag{40}$$

This fields connects all possible points my metric tensors , by analogy can create same energy momentum tensor:

$$T^{(\mathbf{x})}_{\mu\nu} := \left\{ T^{(\mathbf{x})}_{\mu\nu} \left(\mathbf{y} - \mathbf{y}\left(\mathbf{x}\right)\right) \mid \mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+m} \times ... \times \mathbb{R}^{n+m} \right\} \tag{41}$$

$$T^{(\mathbf{y})}_{\mu\nu} := \left\{ T^{(\mathbf{y})}_{\mu\nu} \left(\mathbf{x} - \mathbf{x}\left(\mathbf{y}\right)\right) \mid \mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+m} \times ... \times \mathbb{R}^{n+m} \right\} \tag{42}$$

And from it follows field equation that connects them, as before it can be written:

$$\Box^{(\mathbf{x})} F^{(\mathbf{x},\mathbf{y})}_{\mu\nu} = T^{(\mathbf{x})}_{\mu\nu} \tag{43}$$

$$\Box^{(\mathbf{y})} F^{(\mathbf{y},\mathbf{x})}_{\mu\nu} = T^{(\mathbf{y})}_{\mu\nu} \tag{44}$$

Field goes from observer to another observer, it does it for each point of spacetime. This field defines all motion from all possible points of view of that field. Each point of space has it's own vector field defined with metric tensor and stress energy tensor. But still this view lack quantum physics. My guess is that expanding this ideas to complex spacetime will give more complete definition of field and could possible add quantum effects to this model.

Part III
Complex coordinates and relation with quantum physics

3 Do we live in complex spacetime?

3.1 Complex coordinates transformation

To switch to complex spacetime, i need to first define a flat spacetime complex Minkowski metric, and it's pretty trivial problem. When I have a vector field and it's complex conjugate Minkowski metric that is complex is just:

$$ds^2_{\mathbb{C}} = \eta_{\mu\nu} dz^{\mu}(\mathbf{z})\, d\overline{z}^{\nu}(\mathbf{z}) \tag{45}$$

Complex fields can be expanded into real and imaginary parts. That will be in those simplest case equal to:

$$dz^{\mu}(\mathbf{z}) = dX^{\mu}(\mathbf{x}) + i dY^{\mu}(\mathbf{x}) \tag{46}$$
$$d\overline{z}^{\mu}(\mathbf{z}) = dX^{\mu}(\mathbf{x}) - i dY^{\mu}(\mathbf{x}) \tag{47}$$

It's easy to check to this spacetime interval is real as it will be equal to:

$$ds^2_{\mathbb{C}} = dX_{\mu}(\mathbf{x})\, dX^{\mu}(\mathbf{x}) + Y_{\mu}(\mathbf{x})\, Y^{\mu}(\mathbf{x}) \tag{48}$$
$$ds^2_{\mathbb{C}} = \eta_{\mu\nu} dX^{\mu}(\mathbf{x})\, dX^{\nu}(\mathbf{x}) + \eta_{\mu\nu} Y^{\mu}(\mathbf{x})\, Y^{\nu}(\mathbf{x}) \tag{49}$$

Now lets go to more complex vector field. I will like before define two complex fields, ξ and χ that will depend on complex coordinates. And respective metrics will be their inverses. Let me write them down:

$$g^{(\xi)}_{\mu\overline{\nu}} = \eta_{\alpha\beta} \frac{\partial \chi^{\alpha}(\boldsymbol{\xi}(\boldsymbol{\chi}))}{\partial \xi^{\mu}(\boldsymbol{\xi})} \frac{\partial \overline{\chi}^{\beta}(\boldsymbol{\xi}(\boldsymbol{\chi}))}{\partial \overline{\xi}^{\nu}(\boldsymbol{\xi})} \tag{50}$$

$$g^{(\chi)}_{\mu\overline{\nu}} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}(\boldsymbol{\chi}(\boldsymbol{\xi}))}{\partial \chi^{\mu}(\boldsymbol{\chi})} \frac{\partial \overline{\xi}^{\beta}(\boldsymbol{\chi}(\boldsymbol{\xi}))}{\partial \overline{\chi}^{\nu}(\boldsymbol{\chi})} \tag{51}$$

From it I can move to complex spacetime interval that will be just expression from above with vector fields:

$$ds^2_{(\xi)} = \eta_{\alpha\beta} \frac{\partial \chi^{\alpha}(\boldsymbol{\xi}(\boldsymbol{\chi}))}{\partial \xi^{\mu}(\boldsymbol{\xi})} \frac{\partial \overline{\chi}^{\beta}(\boldsymbol{\xi}(\boldsymbol{\chi}))}{\partial \overline{\xi}^{\nu}(\boldsymbol{\xi})} d\xi^{\mu}(\boldsymbol{\xi})\, d\overline{\xi}^{\nu}(\boldsymbol{\xi}) \tag{52}$$

$$ds^2_{(\chi)} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}(\boldsymbol{\chi}(\boldsymbol{\xi}))}{\partial \chi^{\mu}(\boldsymbol{\chi})} \frac{\partial \overline{\xi}^{\beta}(\boldsymbol{\chi}(\boldsymbol{\xi}))}{\partial \overline{\chi}^{\nu}(\boldsymbol{\chi})} d\chi^{\mu}(\boldsymbol{\chi})\, d\overline{\chi}^{\nu}(\boldsymbol{\chi}) \tag{53}$$

Last part is to write geodesic equations for both metrics. They will be split into two parts for normal and complex conjugates, connection can have only same indexes all mixed ones will be zero so from it follows:

$$\frac{d^2\xi^\mu}{ds^2_{(\xi)}} + \Gamma^\mu_{\alpha\beta}\frac{d\xi^\alpha}{ds_{(\xi)}}\frac{d\xi^\beta}{ds_{(\xi)}} = 0 \tag{54}$$

$$\frac{d^2\overline{\xi}^\mu}{ds^2_{(\xi)}} + \Gamma^{\overline{\mu}}_{\overline{\alpha}\overline{\beta}}\frac{d\overline{\xi}^\alpha}{ds_{(\xi)}}\frac{d\overline{\xi}^\beta}{ds_{(\xi)}} = 0 \tag{55}$$

$$\frac{d^2\chi^\mu}{ds^2_{(\chi)}} + \Gamma^\mu_{\alpha\beta}\frac{d\chi^\alpha}{ds_{(\chi)}}\frac{d\chi^\beta}{ds_{(\chi)}} = 0 \tag{56}$$

$$\frac{d^2\overline{\chi}^\mu}{ds^2_{(\chi)}} + \Gamma^{\overline{\mu}}_{\overline{\alpha}\overline{\beta}}\frac{d\overline{\chi}^\alpha}{ds_{(\xi)}}\frac{d\overline{\chi}^\beta}{ds_{(\chi)}} = 0 \tag{57}$$

Where i did skip dependence on complex coordinates to make it simpler to write it. Connection itself is defined as:

$$\Gamma^\lambda_{\mu\nu} = g^{\bar{\rho}\lambda}\partial_\mu g_{\nu\bar{\rho}} \tag{58}$$

$$\Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}} = \overline{\Gamma^\lambda_{\mu\nu}} \tag{59}$$

This way I have all things defined without still field and it's equations. But before that lets go back to spacetime itself.

3.2 Analogy of four dimensional spacetime into four dimensional complex spacetime

Still complex spacetime needs interpretation. Adding complex spacetime complicates equations so it has to have any physical meaning. Simplest meaning behind complex number squared is probability. So in complex spacetime, spacetime interval has to be connected with probability of finding particle. But still problem is that spacetime we observer has four dimensions. It means that we observe only half of spacetime in this model. If in normal spacetime we have four real dimensions here it's eight real dimensions. Still basis for this is complex spacetime, with probability of finding particle, but to do so spacetime needs to be normalized. It means that spacetime interval integral can't give infinite values. Let's take into account only simplest spacetime:

$$\left(\int_{\mathcal{M}} ds^2_{\mathbb{C}}\right)^2 = \left(\int_{\mathcal{M}} \eta_{\mu\nu} dz^\mu(\mathbf{z})\, d\overline{z}^\nu(\mathbf{z})\right)^2 < \infty \tag{60}$$

It means that I need to define manifold and vector fields this way that integral over whole manifold gives finite value. When i have this finite value I can write normalization condition of this manifold:

$$\frac{1}{N^2}\left(\int_{\mathcal{M}} ds^2_{\mathbb{C}}\right)^2 = \frac{1}{N^2}\left(\int_{\mathcal{M}} \eta_{\mu\nu} dz^\mu(\mathbf{z})\, d\overline{z}^\nu(\mathbf{z})\right)^2 = 1 \tag{61}$$

Where normalization constant N has to be finite number and a real number. That gives simplest wave function like approach. Wave function is equal just to complex fields. It is written in complex coordinates. I can in this example write it as:

$$\Psi^\mu = \frac{1}{\sqrt{N}} dz^\mu(\mathbf{z}) \tag{62}$$

And it's complex conjugate:

$$\overline{\Psi}^{\mu} = \frac{1}{\sqrt{N}} d\overline{z}^{\mu} (\mathbf{z}) \tag{63}$$

Now product of this wave functions gives probability [4], or more precise their contraction:

$$\rho (\mathcal{M}) = \left(\int_{\mathcal{M}} \overline{\Psi}_{\mu} \Psi^{\mu} \right)^2 = \frac{1}{N^2} \left(\int_{\mathcal{M}} \eta_{\mu\nu} dz^{\mu} (\mathbf{z}) \, d\overline{z}^{\nu} (\mathbf{z}) \right)^2 = 1 \tag{64}$$

Probability of given path is equal to:

$$\rho (\mathcal{P}) = \left(\int_{\mathcal{M}} \overline{\Psi}_{\mu} \Psi^{\mu} \right)^2 = \frac{1}{N^2} \left(\int_{\mathcal{P}} \eta_{\mu\nu} dz^{\mu} (\mathbf{z}) \, d\overline{z}^{\nu} (\mathbf{z}) \right)^2 \tag{65}$$

This is simplest way to quantize this kind of spacetime adding probability to it. But still its simplest example, I still need to change it to more complex approach to match classical field.

3.3 Normalization of spacetime coordinates

Now there is only two things left, apply same rules for more general fields and create field equation for complex spacetime. I will start by first point of it. It is pretty straightforward knowing all the classical field. First I do define a complex vector fields that will be equal to as before, then add metric to it and give condition that this spacetime is normalized. General spacetime interval is written as:

$$ds^2_{(\xi)} = \eta_{\alpha\beta} \frac{\partial \chi^{\alpha} (\boldsymbol{\xi} (\boldsymbol{\chi}))}{\partial \xi^{\mu} (\boldsymbol{\xi})} \frac{\partial \overline{\chi}^{\beta} (\boldsymbol{\xi} (\boldsymbol{\chi}))}{\partial \overline{\xi}^{\nu} (\boldsymbol{\xi})} d\xi^{\mu} (\boldsymbol{\xi}) \, d\overline{\xi}^{\nu} (\boldsymbol{\xi}) \tag{66}$$

$$ds^2_{(\chi)} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha} (\boldsymbol{\chi} (\boldsymbol{\xi}))}{\partial \chi^{\mu} (\boldsymbol{\chi})} \frac{\partial \overline{\xi}^{\beta} (\boldsymbol{\chi} (\boldsymbol{\xi}))}{\partial \overline{\chi}^{\nu} (\boldsymbol{\chi})} d\chi^{\mu} (\boldsymbol{\chi}) \, d\overline{\chi}^{\nu} (\boldsymbol{\chi}) \tag{67}$$

Complex vector fields are equal to two wave functions. I can write it:

$$\Psi^{\mu} = \frac{1}{\sqrt{N}} d\chi^{\mu} (\boldsymbol{\chi}) \tag{68}$$

$$\Phi^{\mu} = \frac{1}{\sqrt{M}} d\xi^{\mu} (\boldsymbol{\xi}) \tag{69}$$

And their complex conjugates:

$$\overline{\Psi}^{\mu} = \frac{1}{\sqrt{N}} d\overline{\chi}^{\mu} (\boldsymbol{\chi}) \tag{70}$$

$$\overline{\Phi}^{\mu} = \frac{1}{\sqrt{M}} d\overline{\xi}^{\mu} (\boldsymbol{\xi}) \tag{71}$$

Those manifolds need to be normalized where condition of normalization is equal to:

$$\left(\int_{\mathcal{M}} ds^2_{(\xi)} \right) < \infty \tag{72}$$

$$\left(\int_{\mathcal{M}} ds^2_{(\chi)} \right) < \infty \tag{73}$$

$$\left(\int_{\mathcal{M}} ds^2_{(\xi)} \right) = N^2 \tag{74}$$

$$\left(\int_{\mathcal{M}} ds^2_{(\chi)} \right) = M^2 \tag{75}$$

That naturally will lead to wave vector fields normalizations:

$$\left(\int_{\mathcal{M}} \overline{\Psi}_\mu \Psi^\mu\right)^2 = 1 \tag{76}$$

$$\left(\int_{\mathcal{M}} \overline{\Phi}_\mu \Phi^\mu\right)^2 = 1 \tag{77}$$

All those steps need to be fulfilled in other for this equation to work. In normal quantum mechanics there can be product of wave functions. For example state $\overline{\Phi}_\mu \Psi^\mu \overline{\Psi}_\nu \Phi^\nu$ would give probability of state Ψ being in state Φ. Here I can define same state. I need to use mixed metric tensors. I can write mixed metric tensors as:

$$ds^2_{(\overline{\chi},\xi)} = \eta_{\alpha\beta} \frac{\partial \chi^\alpha (\boldsymbol{\xi}(\boldsymbol{\chi}))}{\partial \xi^\mu (\boldsymbol{\xi})} \frac{\partial \overline{\xi}^\beta (\boldsymbol{\chi}(\boldsymbol{\xi}))}{\partial \overline{\chi}^\nu (\boldsymbol{\chi})} d\xi^\mu (\boldsymbol{\xi})\, d\overline{\chi}^\nu (\boldsymbol{\chi}) \tag{78}$$

$$ds^2_{(\chi,\overline{\xi})} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha (\boldsymbol{\chi}(\boldsymbol{\xi}))}{\partial \chi^\mu (\boldsymbol{\chi})} \frac{\partial \overline{\chi}^\beta (\boldsymbol{\xi}(\boldsymbol{\chi}))}{\partial \overline{\xi}^\nu (\boldsymbol{\xi})} d\chi^\mu (\boldsymbol{\chi})\, d\overline{\xi}^\nu (\boldsymbol{\xi}) \tag{79}$$

And now total probability is equal to:

$$\rho = \frac{1}{NM} \int_{\mathcal{M}} ds^2_{(\overline{\chi},\xi)} \int_{\mathcal{M}} ds^2_{(\chi\,\overline{\xi})} = \int_{\mathcal{M}} \overline{\Phi}_\mu \Psi^\mu \int_{\mathcal{M}} \overline{\Psi}_\nu \Phi^\nu \tag{80}$$

This is final expression for probability. Last example is if system is written as linear combinations of two vector fields. Let me write a field that is equal to linear combinations of two fields:

$$\Psi^\mu = c_1 \psi^\mu + c_2 \phi^\mu \tag{81}$$

$$\overline{\Psi}^\mu = \overline{c}_1 \overline{\psi}^\mu + \overline{c}_2 \overline{\phi}^\mu \tag{82}$$

In this case they have to belong to same metric tensor, it limits possible combinations of states. So linear combinations of states applies only if they belong to same metric tensor. Only thing left now is create a field equations for this complex spacetime model.

3.4 Complex field equation

Now I can move to finally define field equation. I will start by defining transformations of metric tensors:

$$g_{\mu\overline{\nu}\,\xi\mapsto\chi} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha (\boldsymbol{\chi}(\boldsymbol{\xi}))}{\partial \chi^\delta (\boldsymbol{\chi})} \frac{\partial \overline{\xi}^\beta (\boldsymbol{\chi}(\boldsymbol{\xi}))}{\partial \overline{\chi}^\gamma (\boldsymbol{\chi})} \frac{\partial \chi^\gamma (\boldsymbol{\xi}(\boldsymbol{\chi}))}{\partial \xi^\mu (\boldsymbol{\xi})} \frac{\partial \overline{\chi}^\delta (\boldsymbol{\xi}(\boldsymbol{\chi}))}{\partial \overline{\xi}^\nu (\boldsymbol{\xi})} \tag{83}$$

$$g_{\mu\overline{\nu}\,\chi\mapsto\xi} = \eta_{\alpha\beta} \frac{\partial \chi^\alpha (\boldsymbol{\xi}(\boldsymbol{\chi}))}{\partial \xi^\gamma (\boldsymbol{\xi})} \frac{\partial \overline{\chi}^\beta (\boldsymbol{\xi}(\boldsymbol{\chi}))}{\partial \overline{\xi}^\delta (\boldsymbol{\xi})} \frac{\partial \xi^\gamma (\boldsymbol{\chi}(\boldsymbol{\xi}))}{\partial \chi^\mu (\boldsymbol{\chi})} \frac{\partial \overline{\xi}^\delta (\boldsymbol{\chi}(\boldsymbol{\xi}))}{\partial \overline{\chi}^\nu (\boldsymbol{\chi})} \tag{84}$$

From it I can define a complex field equation adding complex energy stress tensor:

$$\Box^{(\xi)} g_{\mu\overline{\nu}\,\xi\mapsto\chi} = \kappa T^{(\xi)}_{\mu\overline{\nu}} \tag{85}$$

$$\Box^{(\chi)} g_{\mu\overline{\nu}\,\chi\mapsto\xi} = \kappa T^{(\chi)}_{\mu\overline{\nu}} \tag{86}$$

Where now square operator [5] is defined as:

$$\square^{(\xi)} = g^{\mu\overline{\nu}}_{(\xi)} \nabla^{(\xi)}_{\mu} \nabla^{(\xi)}_{\overline{\nu}} \tag{87}$$

$$\square^{(\chi)} = g^{\mu\overline{\nu}}_{(\chi)} \nabla^{(\chi)}_{\mu} \nabla^{(\chi)}_{\overline{\nu}} \tag{88}$$

Where probabilities are still defined on same manifold wave function is. It means that I can write probability in those equations as:

$$\rho = \left(\int_{\mathcal{M}} \overline{\Psi}_{\mu} \Psi^{\mu} \right)^2 = \left(\int_{\mathcal{M}} g^{(\xi)}_{\mu\overline{\nu}} \Psi^{\mu} \overline{\Psi}^{\nu} \right)^2 = 1 \tag{89}$$

$$\rho = \left(\int_{\mathcal{M}} \overline{\Phi}_{\mu} \Phi^{\mu} \right)^2 = \left(\int_{\mathcal{M}} g^{(\chi)}_{\mu\overline{\nu}} \Phi^{\mu} \overline{\Phi}^{\nu} \right)^2 = 1 \tag{90}$$

And finally I can write field that connects all points of spacetime with all other points of spacetime:

$$F^{(\boldsymbol{\xi},\boldsymbol{\chi})}_{\mu\overline{\nu}} := \left\{ g_{\mu\overline{\nu}_{\boldsymbol{\xi}\mapsto\boldsymbol{\chi}}} \left(\boldsymbol{\chi} - \boldsymbol{\chi}\left(\boldsymbol{\xi}\right)\right) \mid \boldsymbol{\xi}, \boldsymbol{\chi} \in \mathbb{C}^{n+m} \times ... \times \mathbb{C}^{n+m} \right\} \tag{91}$$

$$F^{(\boldsymbol{\chi},\boldsymbol{\xi})}_{\mu\overline{\nu}} := \left\{ g_{\mu\overline{\nu}_{\boldsymbol{\chi}\mapsto\boldsymbol{\xi}}} \left(\boldsymbol{\xi} - \boldsymbol{\xi}\left(\boldsymbol{\chi}\right)\right) \mid \boldsymbol{\xi}, \boldsymbol{\chi} \in \mathbb{C}^{n+m} \times ... \times \mathbb{C}^{n+m} \right\} \tag{92}$$

$$T^{(\boldsymbol{\xi})}_{\mu\overline{\nu}} := \left\{ T^{(\boldsymbol{\xi})}_{\mu\overline{\nu}} \left(\boldsymbol{\chi} - \boldsymbol{\chi}\left(\boldsymbol{\xi}\right)\right) \mid \boldsymbol{\xi}, \boldsymbol{\chi} \in \mathbb{C}^{n+m} \times ... \times \mathbb{C}^{n+m} \right\} \tag{93}$$

$$T^{(\boldsymbol{\chi})}_{\mu\overline{\nu}} := \left\{ T^{(\boldsymbol{\chi})}_{\mu\overline{\nu}} : \left(\boldsymbol{\xi} - \boldsymbol{\xi}\left(\boldsymbol{\chi}\right)\right) \mid \boldsymbol{\xi}, \boldsymbol{\chi} \in \mathbb{C}^{n+m} \times ... \times \mathbb{C}^{n+m} \right\} \tag{94}$$

$$\square^{(\mathbf{x})} F^{(\boldsymbol{\xi},\boldsymbol{\chi})}_{\mu\overline{\nu}} = T^{(\boldsymbol{\xi})}_{\mu\overline{\nu}} \tag{95}$$

$$\square^{(\mathbf{y})} F^{(\boldsymbol{\chi},\boldsymbol{\xi})}_{\mu\overline{\nu}} = T^{(\boldsymbol{\chi})}_{\mu\overline{\nu}} \tag{96}$$

Probability states same for each system as defined above. This field now is complex field.

References

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