A BOUND FOR THE NUMBER OF ADDITION CHAINS OF ARBITRARY LENGTH

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Abstract. Let us set

$$F(m,\beta(m)-m) := \#\{n \in [2^m, 2^{m+1}) \ : \ l(n) \leq \beta(m)\}.$$

By using ideas in [1], we obtain the more general upper bound

$$F(m,\beta(m)-m) \leq \exp\left((\beta(m)-m)(2\log\beta(m) + (1+\epsilon)2\log\log m + O(1)\right))$$

for any small $\epsilon > 0$ as $m \longrightarrow \infty$. This is a generalization of the recent result in the paper by De Koninck et al. [1]

1. Preliminaries and Setup

Let $l(n)$ be the length of an addition chain leading to n of the form

$$s_o = 1, s_1 = 2, \ldots, s_{l(n)} = n$$

with $2^m \leq n < 2^{m+1}$. Here, the length of the chain is not necessarily optimal. In the case where the length of the chain is optimal, then $l(n) := \iota(n)$. Let us set

$$F(m,\beta(m)-m) := \#\{n \in [2^m, 2^{m+1}) \ : \ l(n) \leq \beta(m), \ \beta(m) \geq m\}.$$

By adapting the ideas of the paper [1], we partition the steps in an addition chain into the following classes of steps

$$\mathcal{A} := \{i \ : \ s_i = 2s_{i-1}\} \quad (\textbf{doubling steps})$$

$$\mathcal{B} := \{i \ : \ \gamma s_{i-1} \leq s_i < 2s_{i-1}\} \quad (\textbf{large steps})$$

where $\gamma := \frac{1+\sqrt{5}}{2}$ is the *golden* ratio

$$\mathcal{C} := \{i \ : \ (1+\delta)s_{i-1} \leq s_i < \gamma s_{i-1}\} \quad (\textbf{medium}-\textbf{size steps})$$

where $\delta := \delta(m) \longrightarrow 0$ as $m \longrightarrow \infty$. In particular

$$\delta := \delta(m) = \frac{1}{\log m}$$

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$$\mathcal{D} := \{i \ : \ s_i < (1+\delta)s_{i-1}\} \quad (\textbf{small steps}).$$

We denote the cardinality of the sets to be

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

We call steps in $\mathcal{B}, \mathcal{C}, \mathcal{D}$ as *non-doubling steps*. We have therefore the relation

$$A + B + C + D = \beta(m).$$

Because each non-doubling step in an addition chain cannot grow faster than a corresponding step in a Fibonacci sequence, we have (by induction) the inequality

$$2^m \leq n \leq 2^A \gamma^{B+C+D} = 2^{\beta(m)} \left(\frac{\gamma}{2}\right)^{B+C+D}$$

and we deduce from this relation an upper control for the total number of non-doubling steps in an addition chain of length $\beta(m)$ to be

Lemma 1.1. *Put*

$$F(m, \beta(m) - m) := \#\{n \in [2^m, 2^{m+1}) \ : \ l(n) \leq \beta(m)\}$$

and let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be steps in an addition chain of length $\beta(m)$ with cardinality A, B, C, D, respectively. Then we have

$$B + C + D \leq \frac{\beta(m) - m}{1 - \log_2 \gamma}.$$

We observe that Lemma 1.1 is a generalization of the version which appears in the paper [1], where the discrepancy

$$\beta(m) - m = r := \frac{cm}{\log m}$$

for $0 < c < \log 2$ in the case $\beta(m) := \lfloor m + r \rfloor$. Again, keeping track of each step in an addition chain of length $\beta(m)$, it follows by induction

$$2^m \leq n \leq 2^{A+B} \gamma^C (1+\delta)^D.$$

The relation

$$2^m \leq 2^{\beta(m)} \left(\frac{\gamma}{2}\right)^C \left(\frac{1+\delta}{2}\right)^D$$

holds and we obtain the total contribution of small steps in a chain of length $\beta(m)$ to be

Lemma 1.2. *Put*

$$F(m,\beta(m)-m) := \#\{n \in [2^m, 2^{m+1}) \ : \ l(n) \leq \beta(m)\}$$

and let $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ be steps in an addition chain of length $\beta(m)$ with cardinality A, B, C, D, respectively. Then we have

$$D \leq \frac{\beta(m)-m-C(1-\log_2\gamma)}{1-\log_2(1+\delta)}.$$

We observe that Lemma 1.2 is a generalization of the version which appears in the paper [1], where the discrepancy

$$\beta(m)-m = r := \frac{cm}{\log m}$$

for $0 < c < \log 2$ in the case $\beta(m) := \lfloor m+r \rfloor$.

In this general context, we examine the number of ways splitting steps in an addition chain into the classes of steps $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ under the proven fact in [1] that large steps $\mathcal{B}$ must be preceded by steps in $\mathcal{C}\cup\mathcal{D}$.

Lemma 1.3. *If $j \in \mathcal{B}$, then $j-1 \in \mathcal{C}\cup\mathcal{D}$. In particular, each large step in an addition chain must be preceded by either a small step or a medium-sized step.*

Proof. This is Lemma 4.3 in [1]. □

We let $S(A,B,C,D,\beta(m))$ denote the number of ways of splitting the steps in an addition chain of length $\beta(m)$ into *doubling*, *large*, *small* and *medium-sized* steps. We have for the number of ways to make this selection the naive upper bound

$$S(A,B,C,D,\beta(m)) \leq \binom{\beta(m)}{A,B,C,D} = \frac{(\beta(m))!}{A!B!C!D!}.$$

However, the conspiracy between steps of type $\mathcal{B}$ and those of type $\mathcal{C}\cup\mathcal{D}$ allows a substantial shaving of the upper bound for the number of ways to put steps into the four-step classes. By Lemma 1.3, the number of ways of choosing steps in $\mathcal{B}$ is at most the number of positive integer solutions to the equation

$$s_1 + s_2 + \cdots + s_B = B + C + D$$

which is at most

$$\binom{B+C+D-1}{B-1} \leq 2^{B+C+D-1} = e^{O(\beta(m)-m)}$$

by Lemma 1.1. In the blocks induced by the configuration of non-doubling steps in the Lemma 1.3, the number of ways to choose steps in $\mathcal{C}$ and $\mathcal{D}$ is at most

$$\binom{\beta(m)}{C+D}\binom{C+D}{D} \leq \binom{\beta(m)}{C+D}e^{O(\beta(m)-m)}$$

since the position of steps of type $\mathcal{C}$ is determined once steps of type $\mathcal{D}$ have been chosen in each block. We have nothing to do for the position of the doubling steps, since the remaining positions are determined once the choice of position for the non-doubling steps have been made. We therefore obtain the general upper bound for the number of ways of splitting steps in an addition chain of length $\beta(m)$ into classes of type $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$

Lemma 1.4. *We have*

$$S(A,B,C,D,\beta(m)) \leq \binom{\beta(m)}{C+D}e^{O(\beta(m)-m)}.$$

In line with the ideas of the paper [1], we still need to find the number of ways to choose the integers to be added at each step of the types $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ in this general context. One would immediately recognize that there is nothing to do for the steps in $\mathcal{A}$, because if $j \in \mathcal{A}$ then $s_j = 2s_{j-1}$. In other words, there is only one way to choose the integers to be added for steps in $\mathcal{A}$. We begin the analysis for steps in $\mathcal{B}$.

Lemma 1.5. *Let A, B, C, D be the fixed cardinality of steps $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$. Let R denote the number of ways to choose the integers to be added at each $j \in \mathcal{B}$ in addition chain of length $\beta(m)$ leading to $n \in [2^m, 2^{m+1})$. Then*

$$R \leq e^{O(\beta(m)-m)}.$$

Proof. This is essentially Lemma 4.6 in the paper [1] adapted to the general case with $B + C + D = O(\beta(m) - m)$. □

Lemma 1.6. *Let A, B, C, D be the fixed cardinality of steps $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$. Let K denote the number of ways to choose the integers to be added at each $j \in \mathcal{C}$ in addition chain of length $\beta(m)$ leading to $n \in [2^m, 2^{m+1})$. Then*

$$K \leq \exp\left((1+\epsilon)2(\beta(m)-m)\log\log m + O(\beta(m)-m)\right)$$

Proof. This is essentially Lemma 4.7 in [1] adapted to the general case with $B + C + D = O(\beta(m) - m)$. □

Lemma 1.7. *Let A, B, C, D be the fixed cardinality of steps $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$. Let T denote the number of ways to choose the integers to be added at each $j \in \mathcal{D}$ in addition chain of length $\beta(m)$ leading to $n \in [2^m, 2^{m+1})$. Then*

$$T \leq \exp\left((\beta(m) - m)\log \beta(m) + O(\beta(m) - m)\right).$$

Proof. This is essentially Lemma 4.12 in [1] adapted to the general case with $B + C + D = O(\beta(m) - m)$. □

2. The general upper bound

In this section, we combine these intermediate observations to generalize the upper bound in the paper [1] to an addition chain, whether or not optimal. We begin by stating the main result in the paper.

Theorem 2.1. *For any $\epsilon > 0$ and for all m large enough, we have*

$$F(m, \beta(m) - m) \leq \exp\left((\beta(m) - m)(2\log \beta(m) + (2 + \epsilon)\log\log m + O(1))\right).$$

Proof. Let $N(A, B, C, D, \beta(m))$ denote the number of addition chains of length $\beta(m)$ producing $n \in [2^m, 2^{m+1})$ for fixed step sizes A, B, C, D, respectively, for step types $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$. Using Lemma 1.4, Lemma 1.5, Lemma 1.6 and Lemma 1.7, we deduce

$$N(A, B, C, D, \beta(m)) \leq \binom{\beta(m)}{C + D}$$

$$\times \exp\left((\beta(m) - m)(2\log \beta(m) + (2 + \epsilon)\log\log m + O(1))\right)$$

and it follows that

$$N(A, B, C, D, \beta(m)) \leq \exp\left((\beta(m) - m)(2\log \beta(m) + (2 + \epsilon)\log\log m + O(1))\right).$$

By averaging over all possible step sizes, we deduce

$$\begin{aligned} F(m, \beta(m)) &= \sum_{A,B,C,D \leq \beta(m)} N(A, B, C, D, \beta(m)) \\ &\leq (\beta(m))^4 \max_{A,B,C,D \leq \beta(m)} N(A, B, C, D, \beta(m)). \end{aligned}$$

The upper bound follows as a consequence. □

We note that this upper bound is the general version of the weaker upper bound appearing in [1] when we take $\beta(m) := \lfloor m + r \rfloor$ with $r := \frac{cm}{\log m}$.

References

1. J.M. De Koninck, N. Doyon and W. Verreault *On the minimal length of addition chains*, arXiv preprint arXiv:2504.07332, 2025.

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