Solving Polynomial Equations with Fractional Sequences

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Abstract

This paper reports a discovery that there exist the extended standard forms for polynomial equations, which are composed of three items, contains only one parameter and relates to integer or fractional sequences. Using the parameter and the sequences, a series can be constructed of the solution of the equations. If the series converges, it is a root of the equations.

For the extended standard form is always possible for the equations of degree not more than five, this result provides an effective method for the solution of general polynomial equations under and including five degrees without the need of radicals calculating.

This technique can also be extended to polynomial equations with two or more coefficients or parameters, which would be more complex or difficult and will be a big challenge if it be used to solve polynomial equations with higher degrees.

At the same time, our discovery also provides a technique to produce an unlimited number of integer and/or fractional sequences, real or complex. This will enrich related researches.

Keywords

polynomial equation, normal form, extended standard form, integer sequence, fractional sequence

1. Solving polynomial equations with series

Polynomial equations have a long history. It is well-known that for the degrees higher than four, there are no formula or radical solutions. [1,2] Finding a root of polynomial equations is one of basic problems in mathematics. There are many alternative methods for that, the series expression is seen as a general or universal technique.

The Catalan numbers provides an example to solve a quadratic polynomial equation with an integer sequence, which was published by Catalan in 1838. [3,4] But in fact, Mingantu first discovered them no later than 1730's, more than 100 years before, they may be called Mingantu-Catalan numbers. [5,6]

Observation and research show that the similar method can be generalized to define the extended Mingantu-Catalan numbers, which relate a special form of equations with three items and the only parameter appearing as coefficient of the highest degree item, and the special form has been called standard form [7].

Further analysis finds that the idea of the standard form can be extended to include all the single parameter patterns. For a fixed degree, there are three types and 12 cases, each of which defines one or more sequences, integer or fractional, real or complex, the extended Mingantu-Catalan numbers are only a special case of them.

We calculate all the sequences for degree 2,3,4, and 5, totally 336 sequences, some of them are first reported. The higher is the degree of equations, the bigger is the number of the sequences produced. The sequences provide multiple choices to solve an equation. And in this way, our method also provides a mechanism to produce an unlimited number of sequences.

Our method can be used to find a root for the quintic equation in a series form, which has been sought at least since 1600.

2. Extended standard forms of polynomial equations

A general polynomial equation of degree D is

$$a_D x^D + a_{D-1} x^{D-1} + \ldots + a_1 x^1 + a_0 = 0 \qquad (1)$$

where: $a_D a_0 \neq 0$

We are interested in its simplified, three-item form, if possible,

$$p x^D + q x^k + r = 0 \qquad (2)$$

where: $pr \neq 0,\ 0 < k < D$.

For simplicity, denote the three-item equation as (Dk0) to indicate the three items of degree D, k and zero.

Easy to check, the three-item form Eq. (2) can be further simplified into three types of single parameter form:

$$\begin{aligned} p\mathrm{x}^{\mathrm{D}} \pm \mathrm{x}^{\mathrm{k}} \pm 1 = 0 \\ \mathrm{x}^{\mathrm{D}} \pm \mathrm{q}\mathrm{x}^{\mathrm{k}} \pm 1 = 0 \\ \mathrm{x}^{\mathrm{D}} \pm \mathrm{x}^{\mathrm{k}} \pm \mathrm{r} = 0 \end{aligned} \tag{3}$$

And each type has four cases with positive or negative sign in it. Therefore, for fixed D and k, there are 12 cases. The total number of three-item forms of degree 2,3,4 and 5 is 10, which are:

(210)
(310) (320)
(410) (420) (430)
(510) (520) (530) (540)

Definition 1: The extended standard form
The single parameter form (Dk0) in Eq. (3) is called extended standard form. For fixed D and k, it has 3 types and 12 cases.

Proposition 1: The solution of three-item equation (Dk0) in Eq. (3) can be formally expressed by the single parameter's power series as

$$\mathrm{x} = \sum_{\mathrm{n}\geq 0} G_{\mathrm{n}}^{\mathrm{D,k}} \lambda^{\mathrm{n}} \tag{4}$$

where: $\lambda = p, q, r$ the single parameter; $G_{\mathrm{n}}^{\mathrm{D,k}}$, the sequence numbers associated with the extended standard form, which will be given in the following.

The extended standard form here is generalized compared with that in [7], where $G_{\mathrm{n}}^{\mathrm{D,1}}$ for (D10) equation $p\mathrm{x}^{\mathrm{D}} - \mathrm{x} + 1 = 0$ is called the extended Mingantu-Catalan numbers. For example,$G_{\mathrm{n}}^{2,1}$for (210) equation $p\mathrm{x}^2 - \mathrm{x} + 1 = 0$ is the Mingantu-Catalan numbers. [3,4,5,6]

3. Numbers sequences

For an extended standard form (Dk0) in Eq. (3), suppose the solution is Eq. (4), the sequence $G_{\mathrm{n}}^{\mathrm{D,k}}$ can be found. Here all the $G_{\mathrm{n}}^{\mathrm{D,k}}$'s in (Dk0), D=2,3,4,5, k<D are listed in order in Table 1. For known sequences, their OEIS codes like AXXXXXX and/or their formula are indicated.[8]

Table 1: The sequences $G_n^{D,k}$ of (Dk0), D=2,3,4,5, k<D.

(210)		
No.	equation	sequences
1	px^2+x+1	$\{-1,-1,-2,-5,-14,...\}$;
2	px^2+x-1	$\{1,-1,2,-5,14,...\}$;
3	px^2-x+1	$\{1,1,2,5,14,...\}$;
4	px^2-x-1	$\{-1,1,-2,5,-14,...\}$;
5	x^2+qx+1	$\{I,-\frac{1}{2},-\frac{1}{8}I,0,-\frac{1}{128}I,...\};\{-I,-\frac{1}{2},\frac{1}{8}I,0,\frac{1}{128}I,...\}$;
6	x^2+qx-1	$\{1,-\frac{1}{2},\frac{1}{8},0,-\frac{1}{128},...\};\{-1,-\frac{1}{2},-\frac{1}{8},0,\frac{1}{128},...\}$;
7	x^2-qx+1	$\{I,\frac{1}{2},-\frac{1}{8}I,0,-\frac{1}{128}I,...\};\{-I,\frac{1}{2},\frac{1}{8}I,0,\frac{1}{128}I,...\}$;
8	x^2-qx-1	$\{1,\frac{1}{2},\frac{1}{8},0,-\frac{1}{128},...\};\{-1,\frac{1}{2},-\frac{1}{8},0,\frac{1}{128},...\}$;
9	x^2+x+r	$\{-1,1,1,2,5,14,...\};\{0,-1,-1,-2,-5,-14,...\}$;
10	x^2+x-r	$\{-1,-1,1,-2,5,-14,...\};\{0,1,-1,2,-5,14,...\}$;
11	x^2-x+r	$\{0,1,1,2,5,14,...\};\{1,-1,-1,-2,-5,-14,...\}$;
12	x^2-x-r	$\{0,-1,1,-2,5,-14,...\};\{1,1,-1,2,-5,14,...\}$;

Remark:

1.Total 20, where: real 16, complex 4; integer 12, fraction 8;

2.I, the imaginary unit;

3.$\{1,1,2,5,14,...\}$, A000108, $\frac{1}{n+1}\binom{2n}{n}$, the Mingantu-Catalan numbers.

(310)		
No.	equation	sequences
1	px^3+x+1	$\{-1,1,-3,12,-55,...\}$;
2	px^3+x-1	$\{1,-1,3,-12,55,...\}$;
3	px^3-x+1	$\{1,1,3,12,55,...\}$;
4	px^3-x-1	$\{-1,-1,-2,-5,-14,...\}$;
5	x^3+qx+1	$\{-1,\frac{1}{3},0,-\frac{1}{81},-\frac{1}{243},...\}$; $\{b,-\frac{1}{3b},0,\frac{1}{81b^2},-\frac{1}{243b^7},...\}$;
6	x^3+qx-1	$\{1,-\frac{1}{3},0,\frac{1}{81},\frac{1}{243},...\}$; $\{c,-\frac{1}{3c},0,\frac{1}{81c^5},-\frac{1}{243c^7},...\}$;
7	x^3-qx+1	$\{-1,-\frac{1}{3},0,\frac{1}{81},-\frac{1}{243},...\}$; $\{b,\frac{1}{3b},0,-\frac{1}{81b^2},\frac{1}{243b^7},...\}$;
8	x^3-qx-1	$\{1,\frac{1}{3},0,-\frac{1}{81},\frac{1}{243},...\}$; $\{c,\frac{1}{3c},0,-\frac{1}{81c^5},\frac{1}{243c^7},...\}$;
9	x^3+x+r	$\{0,-1,0,1,0,-3,0,12,...\}$; $\{I,\frac{1}{2},\frac{3}{8}I,-\frac{1}{2},-\frac{105}{128}I,...\}$; $\{-I,\frac{1}{2},-\frac{3}{8}I,-\frac{1}{2},\frac{105}{128}I,...\}$;
10	x^3+x-r	$\{0,1,0,-1,0,3,0,-12,...\}$; $\{I,-\frac{1}{2},\frac{3}{8}I,\frac{1}{2},-\frac{105}{128}I,...\}$; $\{-I,-\frac{1}{2},-\frac{3}{8}I,\frac{1}{2},\frac{105}{128}I,...\}$;
11	x^3-x+r	$\{0,1,0,1,0,3,0,12,...\}$; $\{1,-\frac{1}{2},-\frac{3}{8},-\frac{1}{2},-\frac{105}{128},...\}$; $\{-1,-\frac{1}{2},\frac{3}{8},-\frac{1}{2},\frac{105}{128},...\}$;
12	x^3-x-r	$\{0,-1,0,-1,0,-3,0,-12,...\}$; $\{1,\frac{1}{2},-\frac{3}{8},\frac{1}{2},-\frac{105}{128},...\}$; $\{-1,\frac{1}{2},\frac{3}{8},\frac{1}{2},\frac{105}{128},...\}$;

Remark：

1. Total 28, where: real 16, complex 12; integer 8, fraction 20;
2. $b=\frac{1}{2}\pm\frac{\sqrt{3}}{2}I$; $c=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}I$; I, imaginary unit;
3. $\{1,1,3,12,...\}$, A001764, $\frac{1}{2n+1}\binom{3n}{n}$, the extended Mingantu-Catalan numbers;
4. $\{\frac{1}{2},\frac{3}{8},\frac{1}{2},\frac{105}{128},...\}=\{\frac{1}{2},\frac{3}{2^3},\frac{16}{2^5},\frac{105}{2^7},...\}$, the numerator $\{1,3,16,105,...\}$, A085614, $\frac{2^n(3n)!!}{(n+1)!n!!}$.

(320)		
No.	equation	sequences
1	px^3+x^2+1	$\{I,\frac{1}{2},-\frac{5}{8}I,-1,\frac{231}{128}I,...\};\{-I,\frac{1}{2},\frac{5}{8}I,-1,-\frac{231}{128}I,...\};$
2	px^3+x^2-1	$\{1,-\frac{1}{2},\frac{5}{8},-1,\frac{231}{128},...\};\{-1,-\frac{1}{2},-\frac{5}{8},-1,-\frac{231}{128},...\};$
3	px^3-x^2+1	$\{-1,\frac{1}{2},-\frac{5}{8},1,-\frac{231}{128},...\};\{1,\frac{1}{2},\frac{5}{8},1,\frac{231}{128},...\};$
4	px^3-x^2-1	$\{I,-\frac{1}{2},-\frac{5}{8}I,1,\frac{231}{128}I,...\};\{-I,-\frac{1}{2},\frac{5}{8}I,1,-\frac{231}{128}I,...\};$
5	x^3+qx^2+1	$\{-1,-\frac{1}{3},-\frac{1}{9},-\frac{2}{81},0,\frac{2}{729},...\};\{b,-\frac{1}{3},\frac{1}{9b},-\frac{1}{81b^2},0,...\};$
6	x^3+qx^2-1	$\{1,-\frac{1}{3},\frac{1}{9},-\frac{2}{81},0,...\};\{c,-\frac{1}{3},\frac{1}{9c},-\frac{2}{81c^2},0,...\};$
7	x^3-qx^2+1	$\{-1,\frac{1}{3},-\frac{1}{9},\frac{2}{81},0,...\};\{b,\frac{1}{3},\frac{1}{9b},\frac{2}{81b^2},0,...\};$
8	x^3-qx^2-1	$\{1,\frac{1}{3},\frac{1}{9},\frac{2}{81},0,...\};\{c,\frac{1}{3},\frac{1}{9c},\frac{2}{81c^2},0,...\};$
9	x^3+x^2+r	$\{-1,-1,2,-7,30,-143,...\};$
10	x^3+x^2-r	$\{-1,1,2,7,30,143,...\};$
11	x^3-x^2+r	$\{1,-1,-2,-7,-30,...\};$
12	x^3-x^2-r	$\{1,1,-2,7,-30,143,...\};$

Remark：

1. Total 24, where: real 12, complex 12; integer 4, fraction 20;

2. $b=\frac{1}{2}\pm\frac{\sqrt{3}}{2}I$; $c=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}I$; $d=\pm\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}I$; I, imaginary unit;

3.$\{1,2,7,30,143,...\}$, A006013, $\frac{1}{n+1}\binom{3n+1}{n}$;

4. $\{1,\frac{1}{2},\frac{5}{8},1,\frac{231}{128},...\}=\{1,\frac{1}{2},\frac{5}{2^3},\frac{32}{2^5},\frac{231}{2^7},...\}$, the numerator $\{1,1,5,32,231,...\}$ is a new sequence.

(410)		
No.	equation	sequences
1	px^4+x+1	$\{-1,-1,-4,-22,...\}$;
2	px^4+x-1	$\{1,-1,4,-22,...\}$;
3	px^4-x+1	$\{1,1,4,22,...\}$;
4	px^4-x-1	$\{-1,1,-4,22,...\}$;
5	x^4+qx+1	$\{d,-\frac{1}{4d^2},-\frac{1}{32d^5},0,...\}$;
6	x^4+qx-1	$\{1,-\frac{1}{4},-\frac{1}{32},0,...\};\{-1,-\frac{1}{4},\frac{1}{32},0,...\};$ $\{I,\frac{1}{4},\frac{1}{32}I,0,...\};\{-I,\frac{1}{4},-\frac{1}{32}I,0,...\}$
7	x^4-qx+1	$\{d,\frac{1}{4d^2},-\frac{1}{32d^5},0,...\}$;
8	x^4-qx-1	$\{1,\frac{1}{4},-\frac{1}{32},0,...\};\{-1,\frac{1}{4},\frac{1}{32},0,...\};$ $\{I,-\frac{1}{4},\frac{1}{32}I,0,...\};\{-I,-\frac{1}{4},-\frac{1}{32}I,0,...\};$
9	x^4+x+r	$\{-1,-\frac{1}{3},\frac{2}{9},\frac{20}{81},...\};\{0,-1,0,0,-1,0,0,-4,...\};$ $\{b,-\frac{1}{4b^3+1},-\frac{6b^2}{(4b^3+1)^3},...\};$
10	x^4+x-r	$\{0,1,0,0,-1,0,0,4,...\};\{-1,-\frac{1}{3},\frac{2}{9},-\frac{20}{81},...\};$ $\{b,\frac{1}{4b^3+1},-\frac{6b^2}{(4b^3+1)^3},...\};$
11	x^4-x+r	$\{1,-\frac{1}{3},-\frac{2}{9},-\frac{20}{81},...\};\{0,1,0,0,1,0,0,4,...\};$ $\{c,-\frac{1}{4c^3+1},-\frac{6c^2}{(4c^3+1)^3},...\};$
12	x^4-x-r	$\{0,-1,0,0,1,0,0,-4,...\};\{1,\frac{1}{3},-\frac{2}{9},\frac{20}{81},...\};$ $\{c,\frac{1}{4c^3+1},-\frac{6c^2}{(4c^3+1)^3},...\};$

Remark：

1. Total 36, where: real 16, complex 20; integer 8, fraction 28;

2. $b=\frac{1}{2}\pm\frac{\sqrt{3}}{2}I$; $c=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}I$; $d=\pm\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}I$; I, imaginary unit;

3. $\{1,1,4,22,...\}$, A002293, $\frac{1}{3n+1}\binom{4n}{n}$, the extended Mingantu-Catalan numbers;

4. $\{1,\frac{1}{3},\frac{2}{9},\frac{20}{81},...\}$ = $\{1,\frac{1}{3},\frac{2}{3^2},\frac{20}{3^4},...\}$, the numerator $\{1,1,2,20,243,...\}$ is a new sequence.

(420)		
No.	equation	sequences
1	$p\mathrm{x}^4+\mathrm{x}^2+1$	$\{\mathrm{I},\frac{1}{2}\mathrm{I},\frac{7}{8}\mathrm{I},\frac{33}{16}\mathrm{I},...\};\{-\mathrm{I},-\frac{1}{2}\mathrm{I},-\frac{7}{8}\mathrm{I},-\frac{33}{16}\mathrm{I},...\};$
2	$p\mathrm{x}^4+\mathrm{x}^2-1$	$\{1,-\frac{1}{2},\frac{7}{8},-\frac{33}{16},...\};\{-1,\frac{1}{2},-\frac{7}{8},\frac{33}{16},...\};$
3	$p\mathrm{x}^4-\mathrm{x}^2+1$	$\{1,\frac{1}{2},\frac{7}{8},\frac{33}{16},...\};\{-1,-\frac{1}{2},-\frac{7}{8},-\frac{33}{16},...\};$
4	$p\mathrm{x}^4-\mathrm{x}^2-1$	$\{\mathrm{I},-\frac{1}{2}\mathrm{I},\frac{7}{8}\mathrm{I},-\frac{33}{16}\mathrm{I},...\};\{-\mathrm{I},\frac{1}{2}\mathrm{I},-\frac{7}{8}\mathrm{I},\frac{33}{16}\mathrm{I},...\};$
5	$\mathrm{x}^4+\mathrm{qx}^2+1$	$\{\mathrm{d},-\frac{1}{4\mathrm{d}},\frac{1}{32\mathrm{d}^3},\frac{1}{128\mathrm{d}^5},...\};$
6	$\mathrm{x}^4+\mathrm{qx}^2-1$	$\{1,-\frac{1}{4},\frac{1}{32},\frac{1}{128},...\};\{-1,\frac{1}{4},-\frac{1}{32},-\frac{1}{128},...\};$ $\{\mathrm{I},\frac{1}{4}\mathrm{I},\frac{1}{32}\mathrm{I},-\frac{1}{128}\mathrm{I},...\};\{-\mathrm{I},-\frac{1}{4}\mathrm{I},-\frac{1}{32}\mathrm{I},\frac{1}{128}\mathrm{I},...\};$
7	$\mathrm{x}^4-\mathrm{qx}^2+1$	$\{\mathrm{d},\frac{1}{4\mathrm{d}},\frac{1}{32\mathrm{d}^3},-\frac{1}{128\mathrm{d}^5},...\};$
8	$\mathrm{x}^4-\mathrm{qx}^2-1$	$\{1,\frac{1}{4},\frac{1}{32},-\frac{1}{128},...\};\{-1,-\frac{1}{4},-\frac{1}{32},\frac{1}{128},...\};$ $\{\mathrm{I},-\frac{1}{4}\mathrm{I},\frac{1}{32}\mathrm{I},\frac{1}{128}\mathrm{I},...\};\{-\mathrm{I},\frac{1}{4}\mathrm{I},-\frac{1}{32}\mathrm{I},-\frac{1}{128}\mathrm{I},...\};$
9	$\mathrm{x}^4+\mathrm{x}^2+\mathrm{r}$	$\{\mathrm{I},-\frac{1}{2}\mathrm{I},-\frac{5}{8}\mathrm{I},-\frac{21}{16}\mathrm{I},...\};\{-\mathrm{I},\frac{1}{2}\mathrm{I},\frac{5}{8}\mathrm{I},\frac{21}{16}\mathrm{I},...\};$
10	$\mathrm{x}^4+\mathrm{x}^2-\mathrm{r}$	$\{\mathrm{I},\frac{1}{2}\mathrm{I},-\frac{5}{8}\mathrm{I},\frac{21}{16}\mathrm{I},...\};\{-\mathrm{I},-\frac{1}{2}\mathrm{I},\frac{5}{8}\mathrm{I},-\frac{21}{16}\mathrm{I},...\};$
11	$\mathrm{x}^4-\mathrm{x}^2+\mathrm{r}$	$\{1,-\frac{1}{2},-\frac{5}{8},-\frac{21}{16},...\};\{-1,\frac{1}{2},\frac{5}{8},\frac{21}{16},...\};$
12	$\mathrm{x}^4-\mathrm{x}^2-\mathrm{r}$	$\{1,\frac{1}{2},-\frac{5}{8},\frac{21}{16},...\};\{-1,-\frac{1}{2},\frac{5}{8},-\frac{21}{16},..\};$

Remark：

1. Total 32, where: real 12, complex 20; integer 0, fraction 32;
2. $\mathrm{d}=\pm\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}\mathrm{I}$; I, imaginary unit;
3. $\{\frac{1}{4},\frac{1}{32},\frac{1}{128},...\}=\{\frac{1}{2^2},\frac{1}{2^5},\frac{2}{2^8},\frac{5}{2^{11}},...\}$, the numerator $\{1,1,2,5,...\}$ is the Mingantu-Catalan numbers, A000108, $\frac{1}{n+1}\binom{2n}{n}$;

3.$\{\frac{1}{2},\frac{5}{8},\frac{21}{16},\frac{429}{128},...\}=\{\frac{1}{2},\frac{10}{2^4},\frac{84}{2^6},\frac{858}{2^8},...\}$, the numerator is A024491, $\frac{1}{4n-1}\binom{4n}{2n}$;

4. $\{1,\frac{1}{2},\frac{7}{8},\frac{33}{16},...\}=\{1,\frac{1}{2},\frac{7}{8},\frac{33}{16},...\}=\{1,\frac{2}{2^2},\frac{14}{2^4},\frac{132}{2^6},...\}$, the numerator is A028990, $\frac{1}{2n+1}\binom{4n}{2n}$.

(430)		
No.	equation	sequences
1	px^4+x^3+1	$\{-1,-\frac{1}{3},-\frac{1}{3},-\frac{35}{81},...\};\{b,-\frac{1}{3}b^2,\frac{1}{3}b^3,-\frac{35}{81}b^4,...\};$
2	px^4+x^3-1	$\{1,-\frac{1}{3},\frac{1}{3},-\frac{35}{81},...\};\{c,-\frac{1}{3}c^2,\frac{1}{3}c^3,-\frac{35}{81}c^4,...\};$
3	px^4-x^3+1	$\{1,\frac{1}{3},\frac{1}{3},\frac{35}{81},...\};\{c,\frac{1}{3}c^2,\frac{1}{3}c^3,\frac{35}{81}c^4,...\};$
4	px^4-x^3-1	$\{-1,\frac{1}{3},-\frac{1}{3},\frac{35}{81},...\};\{b,\frac{1}{3}b^2,\frac{1}{3}b^3,\frac{35}{81}b^4,...\};$
5	x^4+qx^3+1	$\{d,-\frac{1}{4},\frac{3}{32d},-\frac{1}{32d^2},...\};$
6	x^4+qx^3-1	$\{1,-\frac{1}{4},\frac{3}{32},-\frac{1}{32},...\};\{-1,-\frac{1}{4},-\frac{3}{32},-\frac{1}{32},...\};$ $\{I,-\frac{1}{4},-\frac{3}{32}I,\frac{1}{32},...\};\{-I,-\frac{1}{4},\frac{3}{32}I,\frac{1}{32},...\};$
7	x^4-qx^3+1	$\{d,\frac{1}{4},\frac{3}{32d},\frac{1}{32d^2},...\};$
8	x^4-qx^3-1	$\{1,\frac{1}{4},\frac{3}{32},\frac{1}{32},...\};\{-1,\frac{1}{4},-\frac{3}{32},\frac{1}{32},...\}$ $\{I,\frac{1}{4},-\frac{3}{32}I,-\frac{1}{32},...\};\{-I,\frac{1}{4},\frac{3}{32}I,-\frac{1}{32},...\};$
9	x^4+x^3+r	$\{-1,1,3,15,91,...\};$
10	x^4+x^3-r	$\{-1,-1,3,-15,91,...\};$
11	x^4-x^3+r	$\{1,-1,-3,-15,-91,...\};$
12	x^4-x^3-r	$\{1,1,3,15,91,...\};$

Remark：

1. Total 32, where: real 12, complex 20; integer 4, fraction 28;

2. $a=\pm\frac{1}{2}\pm\frac{\sqrt{3}}{2}I$; $b=\frac{1}{2}\pm\frac{\sqrt{3}}{2}I$; $c=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}I$; $d=\pm\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}I$; I, imaginary unit;

3. $\{1,3,15,91,...\}$, A006632, $\frac{3}{4n-1}\binom{4n-1}{n-1}$;

4. $\{\frac{1}{3},\frac{1}{3},\frac{35}{81},...\}$ = $\{\frac{1}{3},\frac{3}{3^2},\frac{35}{3^4},...\}$, the numerator $\{1,3,35,462,6561,...\}$ is a new sequence.

(510)		
No.	equation	sequences
1	px^5+x+1	$\{-1,1,-5,35,...\}$
2	px^5+x-1	$\{1,-1,5,-35,...\}$;
3	px^5-x+1	$\{1,1,5,35,...\}$;
4	px^5-x-1	$\{-1,-1,-5,-35,...\}$;
5	x^5+qx+1	$\{-1,\frac{1}{5},\frac{1}{25},\frac{1}{125},...\}$; $\{g,-\frac{1}{5g^3},-\frac{1}{25g^7},-\frac{1}{125g^{11}},...\}$;
6	x^5+qx-1	$\{1,-\frac{1}{5},-\frac{1}{25},-\frac{1}{125},...\}$; $\{h,-\frac{1}{5h},-\frac{1}{25h^7},-\frac{1}{125h^{11}},...\}$;
7	x^5-qx+1	$\{-1,-\frac{1}{5},\frac{1}{25},-\frac{1}{125},...\}$; $\{g,\frac{1}{5g^3},-\frac{1}{25g^7},\frac{1}{125g^{11}},...\}$;
8	x^5-qx-1	$\{1,\frac{1}{5},-\frac{1}{25},\frac{1}{125},...\}$; $\{h,\frac{1}{5h},-\frac{1}{25h^7},\frac{1}{125h^{11}},...\}$;
9	x^5+x+r	$\{0,-1,0,0,0,1,0,0,0,-5,...\}$; $\{d,-\frac{1}{5d^4+1},-\frac{5}{(5d^4+1)^3},-\frac{10d^2(15d^4-1)}{(5d^4+1)^5},...\}$;
10	x^5+x-r	$\{0,1,0,0,0,-1,0,0,0,5,...\}$; $\{d,\frac{1}{5d^4+1},-\frac{5}{(5d^4+1)^3},\frac{10d^2(15d^4-1)}{(5d^4+1)^5},...\}$;
11	x^5-x+r	$\{0,1,0,0,0,1,0,0,0,5,...\}$; $\{1,-\frac{1}{4},-\frac{5}{32},-\frac{5}{32},...\}$; $\{-1,-\frac{1}{4},\frac{5}{32},-\frac{5}{32},...\}$; $\{I,-\frac{1}{4},\frac{5}{32}I,\frac{5}{32},...\}$; $\{-I,-\frac{1}{4},-\frac{5}{32}I,\frac{5}{32},...\}$;
12	x^5-x-r	$\{0,-1,0,0,0,-1,0,0,0,-5,...\}$; $\{1,\frac{1}{4},-\frac{5}{32},\frac{5}{32},...\}$; $\{-1,\frac{1}{4},\frac{5}{32},\frac{5}{32},...\}$; $\{I,\frac{1}{4},\frac{5}{32}I,-\frac{5}{32},...\}$; $\{-I,\frac{1}{4},-\frac{5}{32}I,-\frac{5}{32},...\}$;

Remark:

1. Totally 44, where: real 16, complex 28; integer 8, fraction 36;

2. $g=\pm\frac{\sqrt{5}}{4}+\frac{1}{4}\pm\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}I$; $h=\pm\frac{\sqrt{5}}{4}-\frac{1}{4}\pm\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}I$; $d=\pm\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}I$; I, imaginary unit;

3. $\{1,1,5,35,285,...\}$, A002294, $\frac{1}{4n+1}\binom{5n}{n}$, the extended Mingantu-Catalan numbers;

4. $\{\frac{1}{4},\frac{5}{32},\frac{5}{32},\frac{385}{2048}...\}=\{\frac{1}{2^2},\frac{5}{2^5},\frac{40}{2^8},\frac{385}{2^{11}},...\}$, the numerator $\{1,5,40,385,4096,...\}$ is a new sequence.

(520)		
No.	equation	sequences
1	$px^5 + x^2 + 1$	$\{I, -\frac{1}{2}, -\frac{9}{8}I, \frac{7}{2}, ...\}; \{-I, -\frac{1}{2}, \frac{9}{8}I, \frac{7}{2}, ...\};$
2	$px^5 + x^2 - 1$	$\{1, -\frac{1}{2}, \frac{9}{8}, -\frac{7}{2}, ...\}; \{-1, -\frac{1}{2}, -\frac{9}{8}, -\frac{7}{2}, ...\};$
3	$px^5 - x^2 + 1$	$\{1, \frac{1}{2}, \frac{9}{8}, \frac{7}{2}, ...\}; \{-1, \frac{1}{2}, -\frac{9}{8}, \frac{7}{2}, ...\};$
4	$px^5 - x^2 - 1$	$\{I, \frac{1}{2}, -\frac{9}{8}I, -\frac{7}{2}, ...\}; \{-I, \frac{1}{2}, \frac{9}{8}I, -\frac{7}{2}, ...\};$
5	$x^5 + qx^2 + 1$	$\{-1, -\frac{1}{5}, 0, \frac{1}{125}, -\frac{1}{625}, ...\};$ $\{\alpha, -\frac{1}{5\alpha^2}, 0, \frac{1}{125\alpha^8}, \frac{1}{625\alpha^{11}}, ...\};$
6	$x^5 + qx^2 - 1$	$\{1, -\frac{1}{5}, 0, \frac{1}{125}, \frac{1}{625}, ...\};$ $\{\beta, -\frac{1}{5\beta^2}, 0, \frac{1}{125\beta^8}, \frac{1}{625\beta^{11}}, ...\};$
7	$x^5 - qx^2 + 1$	$\{-1, \frac{1}{5}, 0, -\frac{1}{125}, -\frac{1}{625}, ...\};$ $\{\alpha, \frac{1}{5\alpha^2}, 0, -\frac{1}{125\alpha^8}, \frac{1}{625\alpha^{11}}, ...\};$
8	$x^5 - qx^2 - 1$	$\{1, \frac{1}{5}, 0, -\frac{1}{125}, \frac{1}{625}, ...\};$ $\{\beta, \frac{1}{5\beta^2}, 0, -\frac{1}{125\beta^8}, \frac{1}{625\beta^{11}}, ...\};$
9	$x^5 + x^2 + r$	$\{-1, -\frac{1}{3}, \frac{1}{3}, -\frac{44}{81}, ...\};$ $\{b, -\frac{1}{b(5b^3+2)}, -\frac{10b^3+1}{b^3(5b^3+2)^3}, -\frac{2(75b^6+10b^3+1)}{b^5(5b^3+2)^5}, ...\};$
10	$x^5 + x^2 - r$	$\{-1, \frac{1}{3}, \frac{1}{3}, \frac{44}{81}, ...\};$ $\{b, \frac{1}{b(5b^3+2)}, -\frac{10b^3+1}{b^3(5b^3+2)^3}, \frac{2(75b^6+10b^3+1)}{b^5(5b^3+2)^5}, ...\};$
11	$x^5 - x^2 + r$	$\{1, -\frac{1}{3}, -\frac{1}{3}, -\frac{44}{81}, ...\};$ $\{c, -\frac{1}{c(5c^3-2)}, -\frac{10c^3-1}{c^3(5c^3-2)^3}, -\frac{2(75c^6-10c^3+1)}{c^5(5c^3-2)^5}, ...\};$
12	$x^5 - x^2 - r$	$\{1, \frac{1}{3}, -\frac{1}{3}, \frac{44}{81}, ...\};$ $\{c, \frac{1}{c(5c^3-2)}, -\frac{10c^3-1}{c^3(5c^3-2)^3}, \frac{2(75c^6-10c^3+1)}{c^5(5c^3-2)^5}, ...\};$

Remark:

1. Totally 40, where: real 12, complex 28; integer 0, fraction 40;
2. $\alpha = g_1, h_2;\ \beta = g_2,\ h_1;\ b = \frac{1}{2} \pm \frac{\sqrt{3}}{2}I; c = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}I;$ I, imaginary unit;

$$g_1 = \frac{\sqrt{5}}{4} + \frac{1}{4} \pm \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}I;\ g_2 = -\frac{\sqrt{5}}{4} - \frac{1}{4} \pm \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}I;$$

$$h_1 = \frac{\sqrt{5}}{4} - \frac{1}{4} \pm \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}I;\ h_2 = -\frac{\sqrt{5}}{4} + \frac{1}{4} \pm \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}I;$$

3. $\{\frac{1}{3}, \frac{1}{3}, \frac{44}{81}, \ldots\} = \{\frac{1}{3}, \frac{3}{3^2}, \frac{44}{3^4}, \ldots\}$, the numerator $\{1,3,44,780,\ldots\}$ is a new sequence;

4. $\{1, \frac{1}{2}, \frac{9}{8}, \frac{7}{2}, \ldots\} = \{1, \frac{1}{2}, \frac{9}{2^3}, \frac{112}{2^5}, \ldots\}$, the numerator $\{1,1,9,112,\ldots\}$ is a new sequence.

(530)		
No.	equation	sequences
1	px^5+x^3+1	$\{-1,\frac{1}{3},-\frac{4}{9},\frac{65}{81},...\};\{b,-\frac{1}{3}b^3,\frac{4}{9}b^5,-\frac{65}{81}b^7,...\};$
2	px^5+x^3-1	$\{1,-\frac{1}{3},\frac{4}{9},-\frac{65}{81},...\};\{c,-\frac{1}{3}c^3,\frac{4}{9}c^5,-\frac{65}{81}c^7,...\};$
3	px^5-x^3+1	$\{1,\frac{1}{3},\frac{4}{9},\frac{65}{81},...\};\{c,\frac{1}{3}c^3,\frac{4}{9}c^5,\frac{65}{81}c^7,...\};$
4	px^5-x^3-1	$\{-1,-\frac{1}{3},-\frac{4}{9},-\frac{65}{81},...\};\{b,\frac{1}{3}b^3,\frac{4}{9}b^5,\frac{65}{81}b^7,...\};$
5	x^5+qx^3+1	$\{-1,\frac{1}{5},-\frac{1}{25},0,\frac{2}{625},...\};$ $\{\alpha,-\frac{1}{5\alpha},\frac{1}{25\alpha^3},0,-\frac{2}{625\alpha^7},...\};$
6	x^5+qx^3-1	$\{1,-\frac{1}{5},\frac{1}{25},0,-\frac{2}{625},...\};$ $\{\beta,-\frac{1}{5\beta},\frac{1}{25\beta^3},0,-\frac{2}{625\beta^7},...\};$
7	x^5-qx^3+1	$\{-1,-\frac{1}{5},-\frac{1}{25},0,\frac{2}{625},...\};$ $\{\alpha,\frac{1}{5\alpha},\frac{1}{25\alpha^3},0,-\frac{2}{625\alpha^7},...\};$
8	x^5-qx^3-1	$\{1,\frac{1}{5},\frac{1}{25},0,-\frac{2}{625},...\};$ $\{\gamma,\frac{1}{5\gamma},\frac{1}{25\gamma^3},0,-\frac{2}{625\gamma^7},...\};$
9	x^5+x^3+r	$\{I,-\frac{1}{2},\frac{7}{8}I,\frac{5}{2},...\};\{-I,-\frac{1}{2},-\frac{7}{8}I,\frac{5}{2},...\};$
10	x^5+x^3-r	$\{I,\frac{1}{2},\frac{7}{8}I,-\frac{5}{2},...\};\{-I,\frac{1}{2},-\frac{7}{8}I,-\frac{5}{2},...\};$
11	x^5-x^3+r	$\{1,-\frac{1}{2},-\frac{7}{8},-\frac{5}{2},...\};\{-1,-\frac{1}{2},\frac{7}{8},-\frac{5}{2},...\};$
12	x^5-x^3-r	$\{1,\frac{1}{2},-\frac{7}{8},\frac{5}{2},...\};\{-1,\frac{1}{2},\frac{7}{8},\frac{5}{2},...\};$

Remark:

1. Totally 40, where: real 12, complex 28; integer 0, fraction 40;
2. $\alpha = g_1, h_2;\ \beta = g_2,\ h_1;\ \gamma = g_3, h_4$

$$g_1 = \frac{\sqrt{5}}{4} + \frac{1}{4} \pm \frac{\sqrt{2}\ \sqrt{5-\sqrt{5}}}{4} I;\ g_2 = -\frac{\sqrt{5}}{4} - \frac{1}{4} \pm \frac{\sqrt{2}\ \sqrt{5-\sqrt{5}}}{4} I;$$

$$g_3 = \frac{\sqrt{5}}{4} - \frac{1}{4} \pm \frac{\sqrt{2}\ \sqrt{5-\sqrt{5}}}{4} I;\ h_4 = -\frac{\sqrt{5}}{4}\ - \frac{1}{4} \pm \frac{\sqrt{2}\ \sqrt{5+\sqrt{5}}}{4} I;$$

$$h_1 = \frac{\sqrt{5}}{4} - \frac{1}{4} \pm \frac{\sqrt{2}\ \sqrt{5+\sqrt{5}}}{4} I;\ \ h_2 = -\frac{\sqrt{5}}{4}\ + \frac{1}{4} \pm \frac{\sqrt{2}\ \sqrt{5+\sqrt{5}}}{4} I;$$

$b = \frac{1}{2} \pm \frac{\sqrt{3}}{2} I;\ c = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} I;\ I$, imaginary unit;

3. $\{1, \frac{1}{3}, \frac{4}{9}, \frac{65}{81}, ...\} = \{1, \frac{1}{3}, \frac{4}{3^2}, \frac{65}{3^4}, ...\}$, the numerator $\{1,1,4,65,...\}$ is a new sequence;

4. $\{1, \frac{1}{2}, \frac{7}{8}, \frac{5}{2}, ...\} = \{1, \frac{1}{2}, \frac{7}{2^3}, \frac{80}{2^5}, ...\}$, the numerator $\{1,1,7,80,...\}$ is a new sequence.

(540)		
No.	equation	sequences
1	$px^5 + x^4 + 1$	$\{d, -\frac{d^2}{4}, \frac{7d^3}{32}, -\frac{d^4}{4}, ...\};$
2	$px^5 + x^4 - 1$	$\{1, -\frac{1}{4}, \frac{7}{32}, \frac{1}{4}, ...\}; \{-1, -\frac{1}{4}, -\frac{7}{32}, -\frac{1}{4}, ...\};$ $\{I, \frac{1}{4}, -\frac{7}{32}I, -\frac{1}{4}, ...\}; \{-I, \frac{1}{4}, \frac{7}{32}I, -\frac{1}{4}, ...\};$
3	$px^5 - x^4 + 1$	$\{1, \frac{1}{4}, \frac{7}{32}, \frac{1}{4}, ...\}; \{-1, \frac{1}{4}, -\frac{7}{32}, \frac{1}{4}, ...\};$ $\{I, -\frac{1}{4}, -\frac{7}{32}I, \frac{1}{4}, ...\}; \{-I, -\frac{1}{4}, \frac{7}{32}I, \frac{1}{4}, ...\};$
4	$px^5 - x^4 - 1$	$\{d, \frac{d^2}{4}, \frac{7d^3}{32}, \frac{d^4}{4}, ...\};$
5	$x^5 + qx^4 + 1$	$\{1, -\frac{1}{5}, \frac{2}{25}, -\frac{4}{125}, ...\}; \{\alpha, -\frac{1}{5}, \frac{2}{25\alpha}, -\frac{4}{125\beta^2}, ...\};$
6	$x^5 + qx^4 - 1$	$\{-1, -\frac{1}{5}, -\frac{2}{25}, -\frac{4}{125}, ...\}; \{\gamma, -\frac{1}{5}, \frac{2}{25\gamma}, -\frac{4}{125\gamma^2}, ...\};$
7	$x^5 - qx^4 + 1$	$\{-1, \frac{1}{5}, -\frac{2}{25}, \frac{4}{125}, ...\}; \{\alpha, \frac{1}{5}, \frac{2}{25\alpha}, \frac{4}{125\beta^2}, ...\};$
8	$x^5 - qx^4 - 1$	$\{1, \frac{1}{5}, \frac{2}{25}, \frac{4}{125}, ...\}; \{\gamma, \frac{1}{5}, \frac{2}{25\gamma}, \frac{4}{125\gamma^2}, ...\};$
9	$x^5 + x^4 + r$	$\{-1, -1, 4, -26, ...\};$
10	$x^5 + x^4 - r$	$\{-1, 1, 4, 26, ...\};$
11	$x^5 - x^4 + r$	$\{1, -1, -4, -26, ...\};$
12	$x^5 - x^4 - r$	$\{-1, 1, -4, 26, ...\};$

Remark:

1. Totally 40, where: real 12, complex 28; integer 4, fraction 36;
2. $\alpha = g_1, h_2; \gamma = g_3, h_4;$

$$g_1 = \frac{\sqrt{5}}{4} + \frac{1}{4} \pm \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}I;\ h_2 = -\frac{\sqrt{5}}{4} + \frac{1}{4} \pm \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}I;$$

$$g_3 = \frac{\sqrt{5}}{4} - \frac{1}{4} \pm \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}I;\ h_4 = -\frac{\sqrt{5}}{4} - \frac{1}{4} \pm \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}I;$$

$$d = \pm\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}I;$$ I, imaginary unit;

3. $\{1, \frac{1}{5}, \frac{2}{25}, \frac{4}{125}, ...\}$, the numerator $\{1,2,4,7, ...\}$ is part of A302938; the denominator, $\{1,5,25,125, ...\}$ is part of A069030;
4. $\{1,4,26, ...\}$, A118971, $\frac{1}{n+1}\binom{5n+3}{n}$;
5. $\{\frac{1}{4}, \frac{7}{32}, \frac{1}{4}, ...\} = \{\frac{1}{2^2}, \frac{7}{2^5}, \frac{64}{2^8}, ...\}$, the numerator $\{1,7,64,663, ...\}$ is a new sequence.

Table 2: Summary of the sequences of (Dk0), D=2,3,4,5, k<D.

three-items	sequences	real s.	complex s.	integer s.	fractional s.	new s.
(210)	20	16	4	12	8	0
(310)	28	16	12	8	20	0
(320)	24	12	12	4	20	1
(410)	36	16	20	8	28	1
(420)	32	12	20	0	32	0
(430)	32	12	20	4	28	1
(510)	44	16	28	8	36	1
(520)	40	12	28	0	40	2
(530)	40	12	28	0	40	2
(540)	40	12	28	4	36	1
total	336	136	200	48	288	9

5. Solving polynomial equations with fractional sequences

As discussed in section 2, if a polynomial equation of degree D is given in three-item form like in Eq. (2), it can have one of three types of the extended standard form in Eq. (3). The three types can be transformed each other, which may sometimes mean to use a complex number. If a complex root is expected, the complex parameter is recommended.

For all the (Dk0) discussed here in this paper, the first type of (D10) is a special one, see the Table 1.

$$p\mathrm{x}^{\mathrm{D}} \pm \mathrm{x} \pm 1 = 0 \qquad (5)$$

which always produces the integer sequences, defines the extended Mingantu-Catalan numbers and the convergent radii are obvious. Therefor it is convenient to use.[7]

But for a general polynomial equation in the three-item form, (D10) may not be always satisfied with the convergent condition, and polynomial transforms can be employed to guarantee the convergence. Some examples are given in [7] to show the possibility, though the calculating sometimes is complicated.

Here we would like to show another way, that is, the fractional sequences may be used to do the same thing.

Example 1: $x^2 - 3x + 64 = 0$
The two roots are $1.500000000 \pm 7.858116825\ \mathrm{I}$.
First it is transformed to

$$\mathrm{py}^2 - \mathrm{y} + 1 = 0$$

where: $\mathrm{p} = 7.111111111 > \frac{1}{4}$, out of the convergent radius.

Now we can change it into the second type, say

$$\mathrm{z}^2 + \mathrm{qz} + 1 = 0$$

where: $\mathrm{q} =- 0.375, \mathrm{x} = 8\ \mathrm{z}$;

And use the correspondent fractional sequence $\{\mathrm{I}, -\frac{1}{2}, -\frac{1}{8}\mathrm{I}, 0, -\frac{1}{128}\mathrm{I}, 0, -\frac{1}{1024}\mathrm{I}, \ldots\}$, get $\mathrm{z} = 0.1875000000 + 0.9822646641$ and $\mathrm{x} = 1.500000000 + 7.858117313\ \mathrm{I}$.

Example 2: $x^3 - 2x - 5 = 0$

The roots are $2.094551482,\ -1.047275741 \pm 1.135939890\ \mathrm{I}$.

The first type parameter $p = \frac{25}{8} > \frac{4}{27}$, out of convergent interval. We choose the second type

$$\mathrm{z}^3 + \mathrm{qz} + 1 = 0$$

where: $\mathrm{q} = 0.3419951894 + 0.5923530436\ \mathrm{I}, \mathrm{x} = \mathrm{k}\ \mathrm{z}, \mathrm{k} = \sqrt[3]{-5}$;

use the correspondent fractional sequence $\{-1, \frac{1}{3}, 0, -\frac{1}{81}, -\frac{1}{243}, 0, \frac{4}{6561}, \ldots\}$, get

$\mathrm{z} =- 0.8815381927 + 0.1982310679,\ \mathrm{x} =- 1.047261494 - 1.135969400\ \mathrm{I}$.

Example 3: $x^4 - 2x - 5 = 0$

The complex roots are $-0.2233844114 \pm 1.512577973$.

The first type parameter $p =- 7.8125$, $|p| > \frac{27}{256}$, out of convergent interval. We choose the second type

$$\mathrm{z}^4 + \mathrm{qz} + 1 = 0$$

where: $\mathrm{q} = 0.4229485052 + 0.4229485052\ \mathrm{I}, \mathrm{x} = \mathrm{k}\ \mathrm{z}, \mathrm{k} = \sqrt[4]{-5}$;

use the correspondent fractional sequence $\{\mathrm{d}, -\frac{1}{4\mathrm{d}^2}, -\frac{1}{32\mathrm{d}^5}, 0, \ldots\},\ \mathrm{d} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{I}$, get

$\mathrm{z} = 0.6096899510 + 0.8209527296\ \mathrm{I},\ \mathrm{x} =- 0.2233831910 + 1.512720458\ \mathrm{I}$.

Example 4: $x^5 - 2x - 5 = 0$

The two complex roots are $-1.075770544 \pm 0.6489646940\ \mathrm{I}$.

The first type parameter $p =- 19.53125000$, $|p| > \frac{256}{3125}$, out of convergent interval. We choose the second type

$$\mathrm{z}^5 + \mathrm{qz} + 1 = 0$$

where: $\mathrm{q} = 0.4464898976 + 0.3243938989\ \mathrm{I}, \mathrm{x} = \mathrm{k}\ \mathrm{z}, \mathrm{k} = \sqrt[5]{-5}$;

use the correspondent fractional sequence $\{\mathrm{d}, -\frac{1}{4\mathrm{d}^2}, -\frac{1}{32\mathrm{d}^5}, 0, \ldots\},\ \mathrm{d} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{I}$, get

$\mathrm{z} =- 0.9072610715 + 0.07776141398\ \mathrm{I},\ \mathrm{x} =- 1.075770578 - 0.6489756763\ \mathrm{I}$.

All the examples above are calculated in the 10 digits precision, which shows that the fractional sequences of both real and complex are also useful in the solution of polynomial equations and they occupy the most part of the whole sequences, providing more choices to guarantee the convergence, though their calculation is usually a little complex.

6. Discussions and conclusions

We have produced totally 336 numbers series with the single-parameter equations of degree 2,3,4, and 5. Among them, there are real 136, complex 200, integer 48, fractional 288, and 9 sequences which are first reported.

Among the 336 sequences, nearly 60% are complex, more than 85% are fractional, their potentiality should be further developed.

More sequences will be found if the degree becomes higher and the number of parameters is increasing, that will be more complex and more difficult, and a big challenge.

The solution of polynomial equations in terms of series provides a direct expression. Most of the sequences have no explicit formula, and not easy to specify the convergent radii. From a viewpoint of applications, finding the formula or fixing the convergent radii of the sequences is important. Most difficult is for that kind of fractional sequences, in which one or more zeros are contained.

If all the problems and the difficulty above are solved, we will be in the position where we can find the root of general polynomial equations of any degree.

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