Kevin Brown's Sublime Numbers

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June 21, 2025

Abstract

A *perfect number* P is a number such that $\sigma(P) = 2P$. Let $M = 2^m - 1$ be a Mersenne prime, where m is a necessarily prime Mersenne exponent. The *Euclid–Euler Theorem* states that an even number P is perfect if and only if it is of the form $P = 2^{m-1}M$. This paper assumes all perfect numbers are even. Suppose that m, M, and $\bar{M} = 2^M - 1$ are all prime. A *sublime number* S is a number such that both its number of divisors $\tau(S) = 2^{m-1}M$ and sum of divisors $\sigma(S) = 2^{M-1}\bar{M}$ are perfect numbers. It is shown that $S = 2^J M_{m_1} \cdots M_{m_j}$, where $M_{m_1}, \ldots, M_{m_j}$ are distinct Mersenne primes, is a sublime number if and only if $J = \sum m_i$, $J = M - 1$, and $j = m - 1$. The author regards this construction of sublime numbers as one of the most beautiful theorems in Elementary Number Theory.

1 Brief historical time line

Euclid [16]	300 B.C.	$P = 2^{p-1}(2^p - 1)$ is perfect
Euler [16]	1747 A.D.	$P = 2^{p-1}(2^p - 1)$ if and only if even perfect
GIMPS [3]	1996 A.D.	Great Internet Mersenne Prime Search
Kevin Brown [2]	2012 A.D.	Sublime numbers

Of course, $2^p - 1$ is a Mersenne prime [18].

2 Background on τ and σ

Recall that every positive integer n has a unique factorization, $n = p_1^{k_1} \cdots p_j^{k_j}$, where $p_1, \ldots, p_j$ are distinct primes and $k_1, \ldots k_j$ are positive integers. Further, recall that

$$\sigma_k(n) = \sum_{d|n} d^k, \quad k \text{ nonnegative.} \tag{1}$$

2.1 Definition (Number of divisors [13], [4]): Define $\tau(n)$ to be the number of divisors of n, that is,

$$\tau(n) = \sigma_0(n) = \sum_{d|n} 1. \tag{2}$$

2.2 Theorem (Number of divisors [13]): *Let $\tau(n)$ denote the number of divisors of n. Let p be prime, let $p_1, \ldots, p_j$ be distinct primes, and let $k, k_1, \ldots, k_j$ be positive integers. Then*

$$\tau(p^k) = k+1. \tag{3}$$

Furthermore, τ is multiplicative:

$$\tau(p_1^{k_1} \cdots p_j^{k_j}) = \tau(p_1^{k_1}) \cdots \tau(p_j^{k_j}). \tag{4}$$

2.3 Definition (Sum of divisors [13], [6]): Define $\sigma(n)$ to be the sum of the divisors of n, that is,

$$\sigma(n) = \sigma_1(n) = \sum_{d|n} d. \tag{5}$$

2.4 Theorem (Sum of divisors [13]): *Let $\sigma(n)$ denote the sum of the divisors of n. Let p be prime, let $p_1, \ldots, p_j$ be distinct primes, and let $k, k_1, \ldots, k_j$ be positive integers. Then*

$$\sigma(2^k) = 2^{k+1} - 1, \tag{6}$$

$$\sigma(p^k) = \frac{p^{k+1}-1}{p-1}. \tag{7}$$

Furthermore, σ is multiplicative:

$$\sigma(p_1^{k_1} \cdots p_j^{k_j}) = \sigma(p_1^{k_1}) \cdots \sigma(p_j^{k_j}). \tag{8}$$

For example, observe that $\tau(12) = \tau(2^2 \cdot 3) = (2+1)(1+1) = 6$, and that $\sigma(12) = \sigma(2^2 \cdot 3) = (2^3 - 1)(3^2 - 1)/2 = 28$. Furthermore, observe that both $6 = 2^{2-1}(2^2 - 1)$ and $28 = 2^{3-1}(2^3 - 1)$ are perfect numbers. See Section 4.

3 Perfect Numbers

Any positive integer n falls into one of three classes determined by σ.

3.1 Definition: Let n be a positive integer. Then n is called

1. [14], [9] *deficient*, if $\sigma(n) < 2n$,
2. [15], [7] *perfect*, if $\sigma(n) = 2n$,
3. [17], [10] *abundant*, if $\sigma(n) > 2n$.

3.2 Examples: Any prime is deficient. The first perfect number is 6, $\sigma(6) = 1 + 2 + 3 + 6 = 12$, and 12 is the first abundant number, $\sigma(12) = \sigma(2^2 \cdot 3) = (2^3 - 1)(3^2 - 1)/2 = 28$. Note that $\sigma(28) = \sigma(2^2 \cdot 7) = (2^3 - 1)(7 + 1) = 56$ so that 28 is the second perfect number. Further, note that $\tau(12) = \tau(2^2 \cdot 3) = (2+1)(1+1) = 6$ so that 12 has a perfect number of divisors and a perfect sum of divisors. The number 12 is therefore called *sublime*. See Section 4.

The following theorem characterizes even perfect numbers.

3.3 Theorem (Euclid–Euler [16]): *Let $M = 2^m - 1$ be a Mersenne prime. Then P is an even perfect number if and only if $P = 2^{m-1}M$.*

Proof. Let $P = 2^{m-1}(2^m - 1)$, where $2^m - 1$ is prime and m is necessarily prime. Then, since σ is multiplicative, we have

$$\begin{aligned}\sigma(P) &= \sigma(2^{m-1}(2^m - 1))\\ &= \sigma(2^{m-1})\sigma(2^m - 1)\\ &= (2^m - 1)(2^m)\\ &= 2 \cdot 2^{m-1}(2^m - 1)\\ \sigma(P) &= 2P,\end{aligned}$$

and so P is perfect.

Now suppose that P is an even perfect number, and suppose that $P = 2^k Q$, where Q is odd. Thus, since P is perfect, the condition $\sigma(P) = 2P$ becomes

$$(2^{k+1} - 1)\sigma(Q) = 2^{k+1}Q \tag{9}$$

The odd factor $q = 2^{k+1} - 1$ on the left side of (9) is at least 3, and it must divide Q, the only odd factor on the right side, so $R = Q/q$ is a proper divisor of Q. Dividing both sides of (9) by q we obtain

$$\begin{aligned}\sigma(Q) &= 2^{k+1}R\\ Q + R + Q' &= 2^{k+1}R,\end{aligned}$$

where Q' is the sum of all divisors of Q other than Q and R. Thus,

$$\begin{aligned}Q + Q' &= (2^{k+1} - 1)R\\ Q + Q' &= qR\\ Q + Q' &= Q\\ Q' &= 0,\end{aligned}$$

and so there are no other divisors of Q other than Q and R. Therefore, $R = 1$, and $Q = 2^{k+1} - 1$ is prime, i.e., Q is a Mersenne prime. ■

For the rest of this document all perfect numbers will be assumed to be even.

4 Sublime Numbers

4.1 Definition (Sublime number [20], [2]): A number S is called *sublime* if its number of divisors $\tau(S)$ and its sum of divisors $\sigma(S)$ are both perfect numbers.

The first two perfect numbers are 6 and 28 and $S_1 = 12$ since $\tau(12) = 6$ and $\sigma(12) = 28$. Are there any other sublime numbers? Yes, but it takes a bit of work to construct one. The following beautiful construction, one of the most beautiful in Elementary Number Theory, is due to Kevin Brown [1], [2].

4.2 Theorem ([2]): *If $S = 2^J K$ is sublime, with K odd, then $J+1$ is a Mersenne exponent and K is a squarefree product of distinct Mersenne primes.*

Proof. Since σ is multiplicative, $\sigma(S) = \sigma(2^J)\sigma(K) = (2^{J+1} - 1)\sigma(K)$. Thus, if $\sigma(S)$ is to be perfect, $2^{J+1} - 1$ must be a Mersenne prime. Furthermore, since $2^{J+1} - 1$ is necessarily the only odd factor of $\sigma(S)$, and all powers of two in $\sigma(S)$ must come from $\sigma(K)$. Let p^s be a prime factor of K. Then

$$\sigma(p^s) = 1 + p + p^2 + p^3 + \cdots + p^{s-1} + p^s,$$

Note that s must be odd so that the sum has an even number of odd terms.

$$\begin{aligned}
&= (1+p) + p^2(1+p) + \cdots + p^{s-1}(1+p)\\
&= (1+p)(1+p^2+p^4+\cdots+p^{s-1})\\
&= (1+p)(1+p^2+p^4+\cdots+p^{2t}) \quad (s-1=2t)
\end{aligned}$$

Note that t must be odd so that the second factor has an even number of odd terms.

$$\sigma(p^s) = (1+p)(1+p^2)(1+p^4+\cdots+p^{4u}) \quad \text{(repeating the process)}$$

Observe that $1+p$ and $1+p^2$ cannot both be powers of 2 since if $1+p = 2^r$, then $p = 2^r - 1$ and $1+p^2 = 2(2^{2r-1} - 2^r + 1)$. Consequently, $1+p^2$ has an odd factor, and p^s is a prime factor of S only if $s = 1$, that is, K is squarefree. Since $p = 2^r - 1$, p must be a Mersenne prime. ∎

4.3 Theorem (Number of divisors of a sublime number [2]): *Suppose $S = 2^J M_{m_1} \cdots M_{m_j}$ is a sublime number, and let $M = 2^m - 1$ be the Mersenne prime such that $\tau(S) = 2^{m-1}M$, is perfect. Then*

$$J = M - 1, \tag{10a}$$

$$j = m - 1. \tag{10b}$$

Proof. Observe that $\tau(S) = \tau(2^J M_{m_1} \cdots M_{m_j}) = (J+1) \cdot 2^j = 2^j(J+1)$. Imposing $2^j(J+1) = 2^{m-1}M$ yields $J = M-1$ and $j = m-1$. ∎

4.4 Theorem (Sum of divisors of a sublime number [2]): *Suppose* $S = 2^J M_{m_1} \cdots M_{m_j}$ *is a sublime number, and let* $\bar{M} = 2^M - 1$ *be the Mersenne prime such that*

$$\sigma(S) = 2^{M-1}\bar{M} \tag{11a}$$

is perfect. Then

$$J = \sum_{i=1}^{j} m_i, \tag{11b}$$

$$J = M - 1. \tag{11c}$$

Proof. Observe that

$$\begin{aligned} \sigma(S) &= \sigma(2^J M_{m_1} \cdots M_{m_j}) \\ &= (2^{J+1} - 1) \cdot 2^{m_1} \cdots 2^{m_j} \\ &= (2^{J+1} - 1) \cdot 2^k \quad (\text{where } k = \textstyle\sum m_i) \\ 2^{M-1}(2^M - 1) &= 2^k \cdot (2^{J+1} - 1). \end{aligned}$$

Thus, $k = M - 1$ and $J + 1 = M$, whence $J = \sum m_i$ and $J = M - 1$. ■

Combining Theorems 4.4 and 4.3 we have

4.5 Theorem (Sublime numbers [2]): *Suppose* $S = 2^J M_{m_1} \cdots M_{m_j}$ *is sublime, and suppose that* $M = 2^m - 1$ *and* $\bar{M} = 2^M - 1$ *are Mersenne primes such that*

$$\tau(S) = 2^{m-1} M \tag{12a}$$

and

$$\sigma(S) = 2^{M-1}\bar{M} \tag{12b}$$

are both perfect. Then

$$J = \sum_{i=1}^{j} m_i, \tag{12c}$$

$$J = M - 1, \tag{12d}$$

$$j = m - 1. \tag{12e}$$

Thus, the basic ingredient of a sublime number is a pair of double Mersenne primes $M = 2^m - 1$ *and* $\bar{M} = 2^M - 1$ *such that* $J = M - 1$ *is expressible as the sum of* $j = m - 1$ *distinct Mersenne exponents.*

	m	$M = 2^m - 1$
1.	2	3
2.	3	7
3.	5	31
4.	7	127
5.	31	2147483647
6.	127	170141183460469231731687303715884105727

Table 1: The list of known double Mersenne primes M [19], i.e., Mersenne primes M such that $2^M - 1$ is prime.

5 The second sublime number

We will now use Table 1 to investigate if there are other sublime numbers.

1. Suppose $m = 2$, then $M_2 = 3$ and $\bar{M} = M_3 = 7$ is prime, so $J = 3 - 1 = 2$ must be expressible as a sum of $j = 2-1 = 1$ distinct Mersenne exponents. Trivially, $2 = 2$ so we have $S_1 = 2^J M_2 = 2^2 \cdot 3 = 12$ is the first sublime number.

2. Suppose $m = 3$, then $M = 7$ and $\bar{M} = M_7 = 127$ is prime, so $J = 7 - 1 = 6$ must be expressible as a sum of $j = 3 - 1 = 2$ distinct Mersenne exponents. Clearly, this is impossible.

3. Suppose $m = 5$, then $M = 31$ and $\bar{M} = M_{31}$ is prime, so $J = 31 - 1 = 30$ must be expressible as a sum of $j = 5 - 1 = 4$ distinct Mersenne exponents. Clearly, this is impossible.

4. Suppose $m = 7$, then $M = 127$ and $\bar{M} = M_{127}$ is prime, so $J = 127 - 1 = 126$ must be expressible as a sum of $j = 7 - 1 = 6$ distinct Mersenne exponents. Recall that the list of Mersenne exponents [5] less than 127 is

$$2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107.$$

Here we strike gold [2], since

$$126 = 61 + 31 + 19 + 7 + 5 + 3, \tag{13}$$

we have

$$S_2 = 2^{126}(2^{61} - 1)(2^{31} - 1)(2^{19} - 1)(2^7 - 1)(2^5 - 1)(2^3 - 1) \tag{14}$$

as the second even sublime number. Note that

$$\tau(S_2) = (127)(2^6) = 2^6(2^7 - 1), \tag{15}$$

$$\sigma(S_2) = (2^{127} - 1)2^{126} = 2^{126}(2^{127} - 1), \tag{16}$$

are both perfect. The explicit values are

$$S_2 = 6\,086\,555\,670\,238\,378\,989\,670\,371\,734\,243\,169\,622\,657\,830\,773\,351\,885\,970\\ 528\,324\,860\,512\,791\,691\,264,$$
$$\tau(S_2) = 8128,$$
$$\sigma(S_2) = 14\,474\,011\,154\,664\,524\,427\,946\,373\,126\,085\,988\,481\,573\,677\,491\,474\,835\,889\\ 066\,354\,349\,131\,199\,152\,128.$$

5.1 Remark: The author is grateful to `https://tex.stackexchange.com/` for assistance in typesetting long numbers. The `siunitx` package is critical. The crucial command is

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\parbox[t]{the width}{\num[group-separator={\ \linebreak[1]}]{the number}}
```

enclosed in a tabular environment.

Whether or not there are other even sublime numbers depends on the existence of further double Mersenne pairs M, $\bar{M}$. For example, if $m = 61$ and $M = 2^{61} - 1 \approx 2.306 \times 10^{18}$, then

$$\bar{M} = 2^M - 1 \approx 1.695 \times 10^{694127911065419641},$$

is far too large for any currently known primality test [11]. Furthermore, we would have to express $J = 2305843009213693950$ as a sum of $j = 60$ distinct Mersenne exponents, but there are only 52 Mersenne primes known as of June 2025, with $m = 136{,}279{,}841$, so it is unlikely that further progress will be made in discovering any more sublime numbers. We should be grateful that we have two of them!

6 Sequences

The definition of each OEIS sequence has been edited to conform to the requirements of this document.

6.1 OEIS (A000005): Number of divisors: $\tau(n) = \sigma_0(n)$ is the number of divisors of n.

6.2 OEIS (A000043): Mersenne exponents: Primes p such that $2^p - 1$ is prime.

6.3 OEIS (A000203): Sum of divisors: $\sigma(n) = \sigma_1(n)$ is the sum of the divisors of n.

6.4 OEIS (A000396): Perfect numbers: Numbers n such that $\sigma(n) = 2n$.

6.5 OEIS (A000668): Mersenne prime: Primes of the form $2^p - 1$, where p is necessarily prime.

6.6 OEIS (A005100): Deficient numbers: Numbers n such that $\sigma(n) < 2n$.

6.7 OEIS (A005101): Abundant numbers: Numbers n such that $\sigma(n) > 2n$.

6.8 OEIS (A077586): Double Mersenne prime: Primes of the form $2^p - 1$, where p is itself a Mersenne prime.

6.9 OEIS (A081357): Sublime number: a number such that the number of divisors and the sum of the divisors of the number are both perfect.

The references have been formatted as “raggedright” so that the urls are not broken.

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