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Finite Relativistic Cosmology

Yosef Akhtman, ya@gamma.earth

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Abstract
We develop a self-contained framework in which the entire physical universe is modelled by an ever-growing finite ring $\mathbb{Z}_q$ whose cardinality is tied to cosmic time through $q = 4t + 1$. Starting from a single principle of *relational finitude*, we show that: (i) familiar dimensional constants ($\hbar$, c, G, k_B) arise as structurally unique, *dimensionless* elements of $\mathbb{Z}_q$, fixed by extremal arithmetic properties; (ii) a genuine Minkowski quadratic form and full Lorentz group exist *exactly* inside the ring, reproducing special-relativistic kinematics under coarse-graining; (iii) the prime-factor spectrum splits naturally into stable fermionic and radiative bosonic sectors, enabling hadron-like three-prime composites and colour confinement; (iv) complementary observer horizons recover, respectively, general-relativistic geodesics and quantum superposition, resolving the gravity-quantum tension and yielding a finite Heisenberg bound; (v) classical paradoxes—cosmological constant, horizon, singularities, ultraviolet divergences, hierarchy, strong-θ, and wave-function collapse—are eliminated not by fine-tuning but by exact arithmetic identities in the finite ring; (vi) independent gravitational and nuclear chronometers converge on a present cardinality $q_\circ \approx 10^{60}$, implying a cosmic age of 13.6 ± 0.2 Gyr and an accelerated expansion that requires no dark energy. We furthermore predict a ~$2.5 \times 10^{-19}\,\mathrm{yr}^{-1}$ secular drift in the 1-m gravitational red-shift—measurable with existing optical-lattice clocks—which offers an immediate, falsifiable test of the proposed hypothesis. Together, these results suggest that a finite, relationally defined arithmetic is sufficient to encode space-time geometry, quantum phenomena, and cosmological evolution within a single coherent model.

1. Introduction

Foundational physics remains one of the most ambitious and elusive quests of the human intellectual pursuit. The search for a deeper understanding of physical world we inhabit, and more specifically a unified theory that reconciles quantum mechanics and general relativity has driven theoretical physics for over a century, yet no consensus has emerged thus far. The finite relativistic programme presented in this manuscript attempts to make a meaningful contribution to this quest by proposing a novel mathematical framework based on the principle of *relational finitude*. Starting with a single basic assumption of *knowable existence* developed in [2] this framework leads to an emergence of a finite relativistic universe, which is self-contained and mathematically rigorous. In this introduction we situate the key results of the finite relativistic programme developed in [5, 4, 3] within the broader landscape of contemporary theoretical physics.

From the earliest Pythagorean dictum that "all is number" to Gödel's arithmetisation of logic, scholars have repeatedly sensed that the whole numbers underpin physical reality. Plato's *Timaeus* casts the cosmos itself as a harmony of integer ratios [80], while Euclid's *Elements* makes properties of divisibility and primehood the logical foundation on which geometry is built [46]. Medieval arithmetic entered natural philosophy when Fibonacci's recursive sequence modelled biological growth [68]. During the Scientific Revolution, Kepler interpreted planetary spacings through geometric progressions [1], and Galileo declared that "the great book of nature is written in the language of mathematics" [24]. Leibniz's essay on binary arithmetic foreshadowed today's digital physics by proposing that reality could emerge from 0-1 combinations alone [49]. Gauss later pronounced number theory "the queen of mathematics" [38], a prelude to Cantor's transfinite numbers and Dedekind's logical construction of the integers [14, 21]. Finally, Gödel proved that arithmetic encodes even the limits of formal reasoning [41]. Across two and a half millennia, then, natural numbers have served not merely as counting tools but as deep structural descriptors of the universe.

Between the 11^{th} and 14^{th} centuries natural philosophy was transformed by scholars working in the Islamic world and later in Latin Europe. Ibn al-Haytham's *Book of Optics* (c. 1021) combined geometry with controlled experiments, establishing the ray model of vision and the law of reflection [66]. A generation later, Avicenna argued that motion persists unless an external force intervenes——an early anticipation of inertia——within his encyclopaedic *Book of Healing* [53]. In Paris the scholastic Jean Buridan refined this idea into the doctrine of *impetus*, rejecting Aristotelian "natural" and "violent" motions [17]. His student Nicole Oresme introduced graphical kinematics and verbally integrated finite time-velocity diagrams—the conceptual seed of the calculus [18]. Together these writers replaced qualitative categories with quantitative reasoning, preparing the ground for Renaissance mechanics.

The 15^{th}-17^{th} centuries witnessed an observational and mathematical leap. Leonardo da Vinci's notebooks treat falling bodies, streamlines and material strength with empirical acuity [62]. In 1543 Nicolaus Copernicus published *De revolutionibus*, positing a heliocentric cosmos and triggering a re-evaluation of celestial dynamics [19]. Tycho Brahe's naked-eye data sets, accurate to within one arc-minute, supplied the empirical bedrock on which Johannes Kepler derived his three planetary laws in *Astronomia Nova* (1609) [12, 48]. Galileo Galilei fused theory and instrumentation: the *Sidereus Nuncius* (1610) telescopic discoveries challenged Aristotelian heavens, while his 1632 *Dialogo* codified the principle of inertia and the kinematics of uniformly accelerated motion [36, 37].

Decades before Kepler and Galileo, Giordano Bruno pushed Copernican heliocentrism to its radical conclusion: in *De l'infinito, universo e mondi* (1584) he argued that the universe is boundless, populated by "innumerable suns" each surrounded by their own worlds, and that the same physical laws hold everywhere [13]. Although philosophical rather than mathematical, Bruno's vision planted the seed of cosmic uniformity and the plurality of worlds—ideas that later became cornerstones of modern cosmology. These advances knit observation, experiment and mathematics into a coherent methodology, setting the stage for Newtonian physics.

The arc begun by Copernicus and refined by Kepler and Galileo reached its definitive mathematical form with Isaac Newton. In the *Philosophiæ Naturalis Principia Mathematica* (1687) Newton unified terrestrial and celestial mechanics under three laws of motion and a universal inverse-square law of gravitation [55]. The Principia inaugurated the modern deductive style: starting from axioms expressed in the calculus he co-invented, Newton derived Kepler's laws, tidal phenomena, and the motion of projectiles, providing the template for theoretical physics into the 20^{th} century.

Albert Einstein provided the geometric scaffolding on which all modern cosmology is built. His 1905 paper on special relativity re-defined space and time as a single four-vector arena [25]; a decade later the field equations of general relativity recast gravity as curvature of that manifold, establishing the local dynamics of the universe [27]. By introducing the cosmological constant in 1917, Einstein showed that the same equations admit large-scale, dynamical solutions and placed observational cosmology on a quantitative footing [28].

Stephen Hawking carried Einstein's geometric vision into the quantum domain. The Penrose-Hawking singularity theorems demonstrated that, under generic conditions, relativistic space-time must contain curvature singularities [45]. Hawking's discovery that black holes radiate thermally united quantum field theory, thermodynamics and general relativity, giving entropy and temperature precise geometric meaning [43, 44]. Finally, the Hartle-Hawking "no-boundary" proposal framed the entire cosmos as a finite yet unbounded quantum geometry, pointing toward singularity-free initial conditions [42].

The quantum era begins with Max Planck, who quantised the energy of oscillator modes to resolve the ultraviolet catastrophe in black-body radiation (1900) [61]. Albert Einstein pushed the idea further by invoking energy quanta—later called photons—to explain the photoelectric effect (1905) [26]. Niels Bohr then married discontinuous orbits with classical mechanics to account for hydrogen spectra (1913) [8], inaugurating the "old quantum theory".

The *wave-particle duality* crystallised when Louis de Broglie proposed matter waves (1924) [20]. Within two years quantum mechanics emerged in two mathematically distinct yet physically equivalent formulations: Werner Heisenberg's matrix mechanics [47] and Erwin Schrödinger's wave equation [67]. Max Born soon provided the probabilistic interpretation of the wave-function amplitude (1926) [10]. The framework was unified and generalised by Paul Dirac, who introduced the relativistic electron equation (1928) and the bra-ket notation that still structures the theory [22].

Post-war decades added conceptual depth. Richard Feynman recast quantum dynamics as a sum over histories (1948) [29], while John Bell showed that no local hidden-variable theory can reproduce all quantum predictions (1964) [7]. Bell's inequalities were violated experimentally by Alain Aspect and collaborators (1982) [6], cementing the non-local character of quantum correlations and paving the way for today's quantum-information science.

Moving to the state-of-the-art contemporary landscape of fundamental physics, Sean Carroll develops "poetic naturalism" in his *The Big Picture* (2016)—a framework in which the deep laws of physics underwrite—but do not uniquely dictate—higher-level regularities [16]. Earlier, *From Eternity to Here* (2010) framed the arrow of time as an issue of cosmological initial conditions [15]. The Finite Programme inherits Carroll's concern with time's direction yet rejects a continuum-based Past Hypothesis: the low gravitational entropy of our universe is instead encoded in a small initial count-count $q_{\text{ini}} \sim 10^5$ whose arithmetic growth enforces a built-in arrow.

Lee Smolin's *Three Roads to Quantum Gravity* (2001) and *Time Reborn* (2013) call for background independence, relational states and a fundamental role for time [69, 70]. Those principles re-emerge here in a stricter guise: spacetime "points" become relations inside a single finite ring $\mathbb{F}_q$, and temporal succession is literal arithmetic increment $q \to q+4$. Our constructions thus supply a concrete realization of Smolin's philosophical programme.

Roger Penrose seeks unification through deep geometric structures—see *The Road to Reality* (2004), *Cycles of Time* (2010) and *Fashion, Faith & Fantasy* (2016) [57, 58, 59]. The Finite Programme shares his insistence on rigorous mathematics but swaps the continuum for arithmetic geometry. Penrose's conformal-cyclic cosmology, for instance, is echoed by our "count-boost" cosmology in which each arithmetic octave $q \mapsto q^2$ corresponds to a new quasi-conformal aeon.

Adam Riess' supernova data—and his Nobel lecture recounting the discovery of cosmic acceleration—anchor any modern cosmology in precision observation [63]. Within the finite framework, the observed value $\Omega_\Lambda \approx 0.69$ arises from the discrete vacuum energy of a prime-field ground state, reproducing Riess' luminosity-distance curve without adjustable scalar fields.

Brian Greene's *The Elegant Universe* (1999) and Leonard Susskind's *The Cosmic Landscape* (2006) popularised the string landscape and multiverse ideas [40, 71], while Steven Weinberg's *Dreams of a Final Theory* (1992) argued for a unique set of fundamental laws [74]. The Finite Programme offers a third path: a discrete, background-independent arena with a unique prime-field backbone yet an immense "arithmetic landscape" of composite extensions that mirror multiverse statistics without leaving the finite domain.

Carlo Rovelli's graduate-level text *Quantum Gravity* (2004) formalizes the loop and spin-foam machinery [65]. Our algebra-geometry dictionary reproduces key loop-gravity results (discrete spectra for area and volume) inside $\mathbb{F}_p$, suggesting that LQG phenomena can be recast as arithmetic rather than topological statements.

Of particular relevance to our Finite Programme are the following threads of contemporary theoretical physics, which share clear common themes with our approach. John Archibald Wheeler's programmatic essay *Information, Physics, Quantum* coined the slogan *it-from-bit*, proposing that every physical observable ultimately derives from yes/no questions—and hence from finite information content [75].

Edward Fredkin pushed the idea further in his "Digital Mechanics" and later "Digital Philosophy", arguing that the universe is a deterministic, reversible cellular automaton running on a discrete substrate [30, 31].

Norman Margolus provided the first rigorous bounds on such automata, showing that energy and momentum conservation can coexist with fully reversible, locality-preserving update rules [52]. In the Finite Programme these concepts re-emerge naturally: the universal count $q \mapsto q+4$ plays the role of Margolus' reversible clocking, while the ring $\mathbb{Z}_q$ supplies the finite information alphabet anticipated by Wheeler and Fredkin.

Seth Lloyd quantified Wheeler's intuition by deriving absolute speed-and-memory limits for any physical computer from $\hbar$, c and G [50]. Our framework realizes those bounds internally: the maximum logical depth per cosmic count equals the Euler totient $\varphi(q)$, and the "memory"—the number of distinguishable states—grows exactly with q. Thus, the cosmic expansion predicted in Sect. 6 is simultaneously an expansion of computational capacity, unifying kinematics with Lloyd's thermodynamic view of information processing. Independently of digital-physics work, Vladimirov, Volovich and Dragovich developed a consistently probabilistic quantum theory over the field of p-adic numbers, motivated by adelic string amplitudes [73, 23].

Parallel studies by Planat, Saniga, Wootters and others demonstrated that finite Galois fields furnish an elegant arena for mutually unbiased bases, discrete Wigner functions and error-correcting codes [60, 79]. The Finite Programme synthesises these threads: it retains the algebraic clarity of Galois constructions while enforcing a physically motivated, time-dependent cardinality. Unlike p-adic models, no non-Archimedean norm is introduced—the metric structure arises relationally from the Lorentz form inside $\mathbb{Z}_q$.

Stephen Wolfram's *A New Kind of Science* (2002) and the more recent *Wolfram Physics Project* white papers [76, 77, 39] put forward a radical programme in which space, time and quantum processes emerge from the repeated rewriting of discrete hyper-graphs. The key ingredients are (i) *causal invariance*—different rule-application orders yield the same causal graph, reproducing relativistic frame indifference; (ii) *multiway evolution*-branching rewrite histories whose interference patterns mimic quantum amplitudes; and (iii) *rule-space relativity*, a notion that effective physical laws depend on the observer's coarse-graining of the underlying rule space. These ideas echo Wheeler's "it-from-bit" and Fredkin's digital mechanics, but replace cellular lattices with combinatorial hypergraphs. The Finite Programme resonates with Wolfram's insistence on discrete, locally applied rules, yet differs in two respects: (a) its update is a single arithmetic count $q \mapsto q + 4$ rather than pattern-matched rewrites, and (b) the Lorentz metric and quantum interference arise internally from the algebra of $\mathbb{Z}_q$ rather than from causal invariance across rule histories. Both approaches thus aim to derive continuum physics from finite computation, but they inhabit complementary regions of the broader landscape of digital-physics models.

Few researchers have done more than Edward Frenkel to connect the deep arithmetic of the Langlands program with the gauge-theoretic language of modern physics. His collaboration with Davide Gaiotto recast *quantum geometric Langlands* as a duality of boundary conditions in 4-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, mediated by vertex-algebra kernels that act as "propagators" between moduli stacks of G-bundles [34]. Follow-up work with Arakawa proved duality isomorphisms for W-algebra representations, supplying the algebraic backbone for these quantum correspondences [33]. More broadly, Frenkel's popular monograph *Love and Math* casts the geometric Langlands conjecture as a "grand unified theory of mathematics", an ambition that recent breakthroughs continue to vindicate [32]. In the present Programme, prime-ideal factorizations in the finite ring $\mathbb{Z}_q$ play a role analogous to Langlands parameters, while duality between additive and multiplicative sectors mirrors the electric-magnetic (or G-${}^LG$) interchange central to Frenkel's framework. Thus, our arithmetic cosmology can be read as a finite-ring realisation of the same unifying vision, transplanting geometric-Langlands ideas from complex curves to a strictly finite, time-evolving arena.

Building on this legacy, the Finite Relativistic Cosmology (FRC) presented hereby, starts with a single premise of *knowable existence* and the derived principle of *relational finitude* [2]. We proceed to build a self-contained, structurally coherent mathematical framework that leads to a finite relativistic informationally-complete paradox-free universe as described in the following sections.

2. Finite Universe: From Mathematical Toolkit to Physical Reality

Over the course of four prior manuscripts we have constructed, step by step, a self-contained programme in which mathematics, geometry, and analysis unfold from a single organizing principle of *relational finitude*.

Ontology [2] redefines existence: an entity *exists* to the extent that it *persists*, i.e. preserves structurally coherent attributes, relative to a finite observer. Infinity, randomness, and undecidability are recast as *epistemic horizons*-signals that a finite observational frame is being over-extended rather than intrinsic features of reality.

Algebra [5] shows that a single prime-order field $\mathbb{F}_p$ already contains the full arithmetic hierarchy. By organizing addition, multiplication, and exponentiation as orthogonal symmetry axes we recover pseudo-integers, rationals, and reals *inside* the field, together with finite analogs of Lie groups and gauge covariance. Algebra thus becomes the operational content of the universe itself.

Geometry [4] lifts the discrete "symbolic sphere" of $\mathbb{F}_p$ into a hyper-finite 2-surface $\mathcal{S}_p$ of constant curvature. A single Fourier kernel, expressed through internal constants i_p, π_p, and e_p, simultaneously realizes the continuous and finite Fourier transforms, demonstrating that curvature, phase, and harmonic analysis already coexist in a finite setting.

Composition [3] extend finite relativistic algebra from prime fields to composite moduli q. The finite analogs of canonical constants i, π, e lift uniquely via Hensel's lemma, glue through the Chinese Remainder Theorem and assemble into profinitely stable families. The resulting arithmetic bouquet possesses a Seifert-fibred 3-orbifold structure whose exceptional fibers record the prime factors of q, while a mixed-radix expansion yields digit coordinates suitable for Fourier and modal analysis.

These results prepare the ground for the presently presented advance:
Cosmology framework that identifies *cosmic time* with the monotonically growing *cardinal count* $\hbar$ and ties the universe's total information budget, i.e. total ring cardinality to

$$q(t) \;=\; 4\,t + 1.$$

Each count of t adds four fresh relational "quadrants", so that after $\hbar$ discoveries the universe contains $q = 4t + 1$ distinguishable micro-states. Two regimes alternate naturally:

Prime epochs If q is prime, the algebra collapses to a field $\mathbb{F}_q$ and the spatial avatar degenerates to the framed two-sphere S_q [4].

Composite epochs If q is composite, zero-divisors appear and re-inflate the geometry into a full three-orbifold [3].

These periodic prime $\leftrightarrow$ composite resets act as cosmic "breaths", constantly refreshing the supply of degrees of freedom.

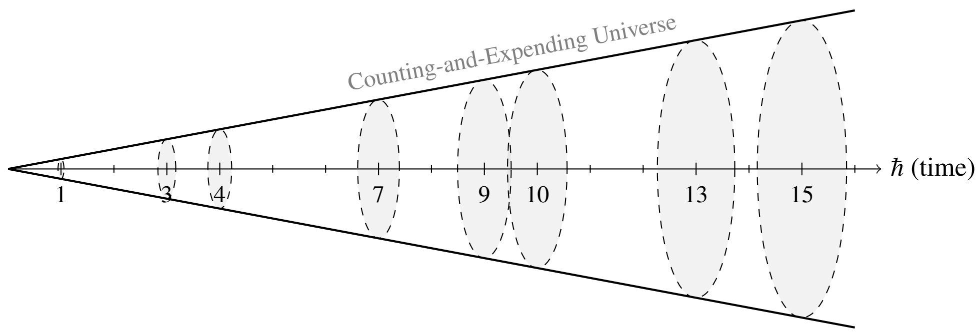

Figure 1: Schematic of the first 16 counts of the counting universe cone of radius $\hbar$ and cardinality +1. The gray circles indicate the "reset" events, when the cardinality $Q = 4t + 1$ is prime thus degenerating the underlying mathematical construct to a field $\mathbb{F}_p$ and the corresponding universe morphology from a 3D manifold to an S^2 spheroid.

As further more follows from the **Ontology** developed in [2], the role of *finite observer* is central to the definition and understanding of physical entities. Two distinct observation scenarios can be formulated. We first note that—by definition—no *truly external* observer can be possibly defined in a finite physical universe, as nothing can be defined as *existing external* in this context. Instead, we formulate the two observational scenarios as relative to the target subsystem $S_q < \mathbb{Z}_q$ they observe.

Internal Observer—An observer is defined as an observational perspective of a relatively *large* subsystem S_q and observation horizon $H_{\text{ext}} \ll q$ [5]. This means that our observer will be able to see only a small part of its object of observation, and we can readily identify such an observation mode as an *observation in a relativistic system*. More specifically, the observations in this scenario will depend on the observer's frame of reference within the target subsystem and the uncertainty will be dominated by the observation horizon H_{ext}. Correspondingly, we will henceforth refer to this observer/system scenario as *relativistic system*.

External Observer—An external observer is defined as an observational perspective of a relatively *small* subsystem S_q with the total cardinality of $H_{\text{int}} \gg q$. Such external observer will be able to see the entirety of its object of observation, including its periodic structure, and we can readily identify such an observation mode as *observation of a quantum system*. Although, such *quantum system* may preserve its isolated properties—typically referred to as the *quantum coherence*—over a short period of time, ultimately it can never be entirely closed, and its properties will be determined by the structural properties of the entire system $\mathbb{Z}_q$. Furthermore, both the external observer and the target subsystem will likely remain in the same relative frame of reference. Correspondingly, the relativistic effects in such observation mode will be negligible. The uncertainty will be largely independent of observer's observation horizon and will be dominated by the large-scale structure of $\mathbb{Z}_q$. Furthermore, it will appear as implicit, unresolvable "quantum" uncertainty, as its source is not being directly observed. Correspondingly, we will henceforth refer to this observer/system scenario as *quantum system*.

In conventional physics, the distinction between these two observational modes is often blurred, with the same physical quantities being defined in terms of *conventional units* such as mass, length, time, and charge. These quantities are intrinsically tied to unit systems that emerge from human-accessible observation scales, such as meters, seconds, and kilograms. However, such units are not absolute: they are shaped by the *epistemic limitations* of the observer, the resolution of measurement apparatuses, and the embedding of the observed system in a continuum model of space-time.

Fundamental physical constants like the speed of light c, Planck's constant h, Newton's gravitational constant G, as well as Boltzmann constant k_B are used to connect these units into a coherent relational system, which is then employed to measure and compare physical phenomena across different scales and contexts. The resultant constants and units are typically derived from empirical measurements and are assumed to be observer-independent. However, this assumption is problematic in a finite universe where the total cardinality q imposes strict limits on what can be observed and measured.

In contrast, the FRC framework is constructed from the ground up within a *finite, closed algebraic universe* defined by the ring $\mathbb{Z}_q$ or field $\mathbb{F}_q$. Within this model:

1. All observable quantities must be expressible in terms of finite relational structures.
2. All dynamics and symmetries must emerge from internal operations on a finite set of relational representations.

3. The total cardinality q of the universe defines the complete capacity for representation, symmetry, and transformation.

We therefore commence with the reformulation of the analogues of fundamental physical quantities not from measurement or unit conventions, but from the *structural and epistemic constraints* imposed by finite relational structure $\mathbb{Z}_q$ of the Universe itself. We then proceed to show that such definition connect to the familiar physical constants in the continuum limit, thus providing a coherent bridge between finite and conventional physics.

3. Fundamental Physical Constants in the Finite Relational Framework

The triad of fundamental physical constants—Newton's gravitational constant G, the speed of light in vacuum c, and Planck's constant h, together with the Boltzmann constant k_B—occupies a uniquely foundational position in the conceptual architecture of modern physics. Each of these constants serves as a dimensional bridge between distinct physical domains: G encodes the coupling between matter and the curvature of spacetime in general relativity; c defines the invariant causal structure of relativistic space-time; h governs the granularity and probabilistic structure of quantum mechanics; while k_B is a fundamental conversion factor between temperature and energy.

Individually, each constant introduces a domain-specific constraint that limits and structures physical behaviour: gravitational interaction, causal propagation, and quantum uncertainty, respectively. Together, however, the triad forms a closed system of scaling invariants from which all natural units—such as Planck length, time, and mass—can be derived through dimensional analysis. In this sense, the G–c–h triad is not merely a collection of constants, but a universal dimensional scaffold that underpins the emergence of physical law as it is understood in modern day physics.

In FRC the familiar dimensionful constants of physics appear as canonical *dimensionless* elements of the finite ring $\mathbb{Z}_q$, determined purely by structural or extremal properties, and complementary to the geometric constants e, i and π derived in [4]. This section summarizes their definitions, proves uniqueness modulo framed automorphisms, and indicates how laboratory values are recovered in the continuum limit.

3.1. Cardinality, cosmic counts and the Planck constant

Definition 1 (Cardinal time t and modulus q)**.** Let $t \in \mathbb{N}$ count the number of relational *discoveries* that have occurred since the cosmic origin. Each discovery adds four new orthants in the symmetry cube, so that the total information budget after t counts is

$$q(t) \;=\; 4\,t + 1. \tag{3.1}$$

The ring of physical states at epoch t is $\mathcal{U}(t) = \mathbb{Z}_{q(t)}$.

Equation (3.1) guarantees that q is always +1-odd, preserving the quadratic structure required for the construction of the canonical constants, of i, e, π of which both existence, uniqueness and profinite stability has been proven in [3]. Because consecutive values differ by four, a single count transports the system between neighbouring residues; we interpret that minimal discrete action as the *reduced Planck* *constant* (set $\hbar \equiv 1$ inside $\mathbb{Z}_q$). Laboratory Planck units arise when the profinite count is calibrated against any empirical triplet $(\ell_{\mathrm{P}}, t_{\mathrm{P}}, m_{\mathrm{P}})$.

Definition 2 (Reduced Planck constant $\hbar_q$)**.** Inside a given modulus $q \equiv 1 \pmod 4$ let

$$\hbar_q \;:=\; 1 \;\in\; \mathcal{U} = \mathbb{Z}_q. \tag{3.2}$$

Being the additive generator, $\hbar_q$ is the *smallest non-zero increment of any physical observable*. We interpret it as the discrete analogue of the reduced Planck constant.

Proposition 1 (Invariance and profinite stability of $\hbar_q$)**.** $\hbar_q$ is fixed by every unital ring automorphism of $\mathcal{U}$ and survives all Hensel lifts and CRT projections. Consequently the collection $\{\hbar_q\}_{q(t)}$ forms a consistent element of the profinite limit $\varprojlim \mathcal{U}(t)$.

Definition 3 (Canonical half-period and full Planck constant)**.** Recall the half-period residue $\pi_q = \frac{q-1}{2}$ from [4]. Define the *full* Planck constant by

$$h_q \;:=\; 2\,\pi_q\,\hbar_q. \tag{3.3}$$

Because $2\pi_q = q - 1 \equiv -1 \pmod q$, we have the identity

$$h_q \;\equiv\; -\hbar_q \;\equiv\; -1. \tag{3.4}$$

Proposition 2 (Phase characterisation)**.** Let $\widehat{\mathcal{U}}$ be the Pontryagin dual of the additive group of $\mathcal{U}$. The character $\chi_{h_q} : x \mapsto \exp\bigl(2\pi i\, x h_q/q\bigr)$ generates $\widehat{\mathcal{U}}$; hence h_q represents the *smallest non-trivial phase step* in the finite Fourier transform.

Proof. Because $h_q \equiv -1$ is coprime to q, the map $x \mapsto x h_q$ is a bijection of $\mathcal{U}$, so the associated character has maximal order q and generates the dual group [3]. □

Continuum calibration. A single empirical assignment $\hbar_q \;\mapsto\; \hbar_{\mathrm{SI}} = 1.054\,571\,817 \times 10^{-34}\,\mathrm{J{\cdot}s}$ determines the image of every $n\,\hbar_q$ and, via Eq. (3.3), fixes $h_{\mathrm{SI}} = 6.626\,070\,15 \times 10^{-34}\,\mathrm{J{\cdot}s}$. Together with the G-based length-time-mass calibration of Section 3.3, this exhausts the empirical inputs required to translate any finite-ring computation into laboratory numbers.

3.2. Canonical multiplicative quarter-turn and the speed of light

Definition 4 (Speed of light c)**.** Let $i_q \in \mathbb{Z}_q$ be the unique solution of $x^2 + 1 \equiv 0 \pmod q$ that lies nearest the additive midpoint $q/2$; call $-i_q$ the *future-oriented quarter-turn* as derived in [3], where existence, uniqueness and profinite stability are also proven. We define

$$c_q := -i_q. \tag{3.5}$$

Proposition 3. c_q is fixed by every framed automorphism of $\mathbb{Z}_q$ and therefore constitutes a Lorentz-invariant causal speed. Moreover, $c_q^2 \equiv -1 \pmod q$, reproducing the signature $(+,+,+,-)$ of Minkowski space when inserted into the quadratic form $\eta(x,t) = x^2 - (c_q t)^2$.

3.3. Minimal action and Newton's constant G

Definition 5 (Minimal-action root e_q)**.** Among the units of $\mathbb{Z}_q$ choose the primitive root that minimises the cyclic distance to 1:

$$e_q \;=\; \arg\min_{g \text{ prim.}} \big|g - 1\big|.$$

For prime moduli this selects the unique generator inside the forward semi-circle; for composite q it is obtained by prime-wise Hensel lift followed by CRT amalgamation [4, 3].

Definition 6 (Gravitational coupling)**.** Set

$$G_q \;:=\; e_q^{-1}. \tag{3.6}$$

Proposition 4. G_q measures the resistance of $\mathcal{U}^{\times}$ to the internal exponential flow generated by e_q; it is the unique profinitely stable inverse-primitive compatible with every enlargement of q [3].

Continuum calibration. Matching (3.6) against the macroscopic force law $F = G_{\mathrm{SI}} m_1 m_2 / r^2$ at a single experimental scale fixes the conversion between profinite lengths and SI metres, thereby anchoring the entire Planck unit system.

3.4. Signed involution and Boltzmann's constant k_B

Definition 7 (Thermodynamic sign operator)**.** The ring involution $x \mapsto -x$ has order 2 and fixes the framed identities $\{0, 1\}$. Define

$$k_{B,q} \;:=\; -1 \;\in\; \mathbb{Z}_q. \tag{3.7}$$

In information-theoretic terms the map $x \mapsto -x$ exchanges "available" and "missing" micro-states. Writing the combinatorial entropy of a macro-configuration as $S = \log_{e_q} \Omega$, the usual energy-entropy balance $E = k_{B,q}\, T\, \Delta S$ follows immediately. The appearance of Boltzmann's own minus sign in (3.7) echoes his statistical interpretation of entropy [9].

3.5. The h-k_B Dichotomy and Observer Horizons

In the strictly finite ring $\mathbb{Z}_q$ the *full* Planck constant and the Boltzmann unit coincide (Definitions 3 and 7):

$$h_q = 2\pi_q \hbar_q \equiv -1 \equiv k_{B,q} \pmod{q}.$$

Yet in ordinary physics we treat h and k_B as independent constants with disparate SI magnitudes. The apparent bifurcation arises only *after* one specifies an *observer horizon*. Let H denote the radius of the metric ball that an agent can interrogate inside $\mathbb{Z}_H \subset \mathbb{Z}_q$[1]. Two limiting cases thus arise.

Coarse-grained splitting. Define horizon-averaged constants

$$h(H) := \big\langle h_q \big\rangle_H, \qquad k_B(H) := \big\langle k_{B,q} \big\rangle_H.$$

[1] Here, we leave out the complexity of proving that the metric ball observed by any local observer is itself a ring in $\mathbb{Z}_q$.

Table 1: Observer modes and their associated horizons, formalisms, and constants.

Observer mode	**Horizon**	**Available formalism**	**Fundamental constant**
Internal (relativistic)	$H_{\text{int}} \ll q$	Local geometry, open-system thermodynamics	$k_B\ (=-1)$
External (quantum)	$H_{\text{ext}} \gg q_o$	Global phases, unitary evolution	$h\ (=-1)$

Because global phases decohere as $\langle h_q \rangle_H \sim q^{-H}$ while missing micro-states accumulate as $\langle k_{B,q} \rangle_H \sim 1$, we have

$$h(H_{\text{int}}) \approx 0, \quad k_B(H_{\text{int}}) = -1, \qquad h(H_{\text{ext}}) = -1, \quad k_B(H_{\text{ext}}) \approx 0.$$

Thus, the *same* residue -1 is perceived either as a quantum of action or as an entropy-energy converter, depending on how much of $\mathbb{Z}_q$ the observer can access.

Physical calibration. When profinite scale maps are applied,

$$h_{\text{lab}} = \alpha_h\,(-1), \qquad k_{B,\text{lab}} = \alpha_k\,(-1), \qquad \alpha_h \neq \alpha_k,$$

the numerical identity breaks, reproducing the SI values $h = 6.626 \times 10^{-34}\,\mathrm{J\,s}$ and $k_B = 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$.

The two observation scenarios can be further interpreted as follows:

Quantum viewpoint. With full access to the ring of the observed subsystem, an *external observer* tracks phase evolution; action quanta h are primary, entropy is trivial.

Relativistic viewpoint. A *confined observer* loses phase information to the exterior; statistics and thermodynamic entropy k_B become primary, while the residual phase scale h is suppressed below measurement threshold.

Hence the h/k_B split is not fundamental but horizon-dependent: two calibrations of the *same* ring element -1 seen through complementary observational lenses.

3.6. Summary: Planck, Einstein and Boltzmann meet Hensel and CRT

Collecting (3.2), (3.3), (3.5), (3.6) and (3.7) we obtain the fundamental physical constants in summarised in Table 2.

All constants are *frame-covariant* (invariant under affine relabelling), *Hensel stable* (unique lifts along prime powers) and *CRT coherent* (glue consistently across composite moduli) [3]. Dimensional analysis performed with these dimensionless residues reproduces the familiar Planck scales once the single calibration noted in Section 3.3 is supplied.

In summary, FRC realises G, c, h, k_B not as mysterious external numbers but as inevitable structural landmarks of the finite ring that *is* the universe. Their constancy, universality and conversion power are therefore guaranteed by arithmetic itself rather than imposed by experiment.

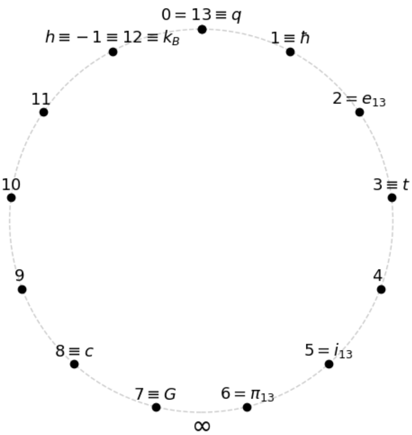

Figure 2: State space of the $t = 3, q = 13$ count, corresponding to a finite framed field $\mathbb{F}_{13}$, visualized as a circle on a 2D plane with the major structural elements $-1, 0, 1, e_{13}, i_{13}, \pi_{13}, \infty$, as well as fundamental physical constants $\hbar, G, c, h$ and k_B.

Table 2: Canonical constants in the finite relational algebra.

Continuum symbol	Finite value in $\mathbb{Z}_q$	Characterizing property
$\hbar$	1	Additive generator of the ring, quantization increment
h	$2\pi = -1$	Full quantum of action, $h = 2\pi\hbar$
c	$-i_q$	Future-oriented multiplicative quarter-turn, $c^2 = -1$
G	e_q^{-1}	Inverse minimal-action primitive root
k_B	-1	Signed involution, entropy-energy conversion factor

The corresponding visual representation of the finite field $\mathbb{F}_{13}$ corresponding to the count $\hbar = 3, q = 13$ of the finite universe history is shown in Figure 2. The figure shows the state space of the finite field $\mathbb{F}_{13}$ as a circle on a 2D plane, with the major structural elements $0, 1, e_{13}, i_{13}, \pi_{13}$, as well as fundamental physical constants $\hbar, G, c, h$ and k_B. The antipodal point ∞ is located at the South Pole of the pseudo-sphere, which is the farthest point from the observer at 0.

Continuum calibration Every constant listed so far is a *dimension-free residue* inside $\mathbf{Z}_q$; the familiar SI magnitudes arise only after two *independent profinite scale maps* are applied:

$$\begin{aligned} \alpha_h &: (-1) \longmapsto h_{\mathrm{SI}} \quad \text{(action / phase scale)}, \\ \alpha_k &: (-1) \longmapsto k_{B,\mathrm{SI}} \quad \text{(entropy / temperature scale)}. \end{aligned} \tag{3.8}$$

Action scale. Pick a physical triplet that fixes one length, one time, and one energy—for example the Bohr radius, the Rydberg frequency, and the hydrogen ionisation energy [56]. This single choice determines α_h and therefore pins the laboratory values of $\hbar_{\mathrm{SI}}$, c_{SI}, G_{SI} to their observed numbers at the chosen coarse-graining scale.

Entropy scale. A separate empirical datum that ties temperature to energy (e.g. the triple-point of water) fixes α_k, reproducing $k_{B,\mathrm{SI}} = 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$.

With these two calibrations in place, all finite-relational constants recover their conventional SI magnitudes to experimental precision for any observer operating at the specified coarse-graining horizon. The next section shows how these constants slot into a Lorentz-invariant quadratic form on $\mathbb{Z}_q^4$, yielding exact special-relativity kinematics inside the finite universe.

4. True Special Relativity and the Minkowski Metric in the Finite Relativistic Universe

Throughout this section the universe is a finite ring $\mathbb{Z}_q, q = +1$, exactly as in Section 3. A space-time event is encoded by a *finite four-vector*

$$X = (x_1, x_2, x_3, t) \in \mathbb{Z}_q^3 \times \{t\}, \tag{4.1}$$

where the three spatial coordinates are charted by the framed *arithmetic symmetries* (translation T, dilation D, exponentiation E) and the temporal coordinate is the radial state-count introduced in the cone diagram of [4].[2]

4.1. *A finite Minkowski quadratic form*

The structural value of the speed of light has been fixed in Section 3 to the signed quarter-turn $c = -i_q$ with $c^2 = -1$. With this choice the bilinear form

$$\eta(X, X) := x_1^2 + x_2^2 + x_3^2 - t^2 \tag{4.2}$$

has signature $(3, 1)$ *inside the ring*. Equation (4.2) therefore acts as the finite-relational analogue of the continuum Minkowski norm $ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$.

Light-cone. The null condition $\eta(X, X) = 0$ yields the algebraic light-cone

$$C = \{X \mid x_1^2 + x_2^2 + x_3^2 = t^2 \subset \mathbb{Z}_q\}.$$

Inside each prime component of q it is a genuine quadric in $\mathbb{F}_p$; the full cone is reconstructed by the Chinese Remainder Theorem.

4.2. *Lorentz transformations over* $\mathbb{Z}_q$

Define the *finite Lorentz group*

$$\mathrm{O}(3, 1; \mathbb{Z}_q) := \left\{ L \in \mathrm{GL}_4(\mathbb{Z}_q) \;\middle|\; L^*\eta = \eta \right\}. \tag{4.3}$$

The canonical symmetry generators of the ring already lie in this group:

(i) *Spatial rotations.* Multiplication by i_q performs a 90° rotation in any chosen spatial 2-plane; its powers generate a discrete SO(3) subgroup.

[2] The "radius = time" identification follows Wheeler's *it-from-bit* idea in a finite form: every new micro-state increments the observable time by one count.

(ii) *Boosts.* Raising the minimal-action base e_q to integer powers realizes *ε-Lie boosts* that approximate hyperbolic rotations: $B(\alpha) := \mathrm{diag}(e_q^{\alpha}, e_q^{-\alpha}, 1, 1)$, $\alpha \in \mathbb{Z}_q$.

(iii) *Frame relabellings.* Affine automorphisms of $\mathbb{Z}_q$ preserve the additive and multiplicative orders, hence leave η invariant.

Together these generate the full group (4.3); every element acts on X by $X \mapsto LX$ while keeping $\eta(X, X)$ fixed.

4.3. Continuum limit and empirical special relativity

For any experimental resolution $\Delta \ll p_k$ (every prime factor of q is far above the observer's coarse-graining scale), the discrete boost mesh generated by e_q becomes dense in each component field $\mathbb{F}_{p_k}$. Hence the observer cannot distinguish transformations in $\mathrm{O}(3, 1; \mathbb{Z}_q)$ from those in the real Lorentz group $\mathrm{O}(3, 1; \mathbb{R})$. In this *continuum limit* the finite metric (4.2) reproduces the usual interval ds^2, and all textbook kinematic effects (time dilation, length contraction, invariant light-speed) follow.

4.4. Prime vs. composite epochs

When q is prime the arithmetic symmetry collapses from a 3-manifold to a framed 2-sphere ([4], Prop. 3.4), yet η and $\mathrm{O}(3, 1; \mathbb{Z}_q)$ survive unchanged. Special-relativistic kinematics therefore holds *across Big-Bang epochs* where the spatial fiber degenerates.

In summary, the finite-relativistic construction furnishes:

(i) a genuine Minkowski quadratic form (4.2) inside the ring,

(ii) an exact Lorentz group (4.3) acting on events (4.1),

(iii) a light-cone and causal structure fully expressible in finite arithmetic,

(iv) an automatic recovery of the classical $\mathrm{O}(3, 1)$ symmetry when the cardinality outstrips the observer's resolution.

Hence, the adjective *relativistic* in "finite relativistic algebra" throughout the programme [5, 4, 3] is literal: the theory realizes the core postulates of special relativity, not merely an analogy.

5. Fermion–Boson Decomposition in a Finite Universe

5.1. Prime factorization as ontology

Recall that every cardinality $q = +1$ admits a unique factorization into odd primes $q = \prod_{k=1}^{r} p_k$. In the mixed-radix isomorphism $\mathbb{Z}_q \cong \bigoplus_{k=1}^{r} F_{p_k}$, each summand carries the framed S^2 fibre built in [4]. We interpret the *unit vector* $\mathbf{e}_{(p_k)} \in F_{p_k}$ as a *primitive massive object*. Its additive norm $m_{p_k} = |\mathbf{e}_{(p_k)}| = p_k$ is the object's mass (Sect. 3).

5.2. Intrinsic quarter-rotation and the $+1/-1$ *prime dichotomy*

A field $\mathbb{F}_p$ contains a square root of -1 iff $p \equiv 1 \pmod 4$. Denote this root by i_p $(i_p^2 = -1)$.

Definition 8. We define the *fermion/boson split* of the finite universe by the following two disjoint subsets of prime factors of q:

Fermionic prime. A prime factor $p \equiv 1 \pmod 4$ possessing i_p. The corresponding unit vector $\mathbf{e}_{(p)}$ is a *stable fermion*.

$$p \equiv 1 \pmod 4 \implies i_p \implies \tfrac{1}{2}\text{-spin} \implies \text{fermion},$$

Bosonic prime. A prime factor $p \equiv 3 \pmod 4$ lacks any square root of -1. The associated unit vector $\mathbf{f}_{(p)}$ is *unstable* and will be shown to decompose into radiation degrees of freedom.

$$p \equiv 3 \pmod 4 \implies \text{no } i_p \implies \text{integer spin} \implies \text{boson/radiation}.$$

5.3. *Inherited properties of the two sectors*

Spin-statistics. Each i_p realises an internal $\mathrm{SU}(2)$ double cover of spatial rotations (Prop. 3.2 in [4]), so exchanging two identical fermionic primes multiplies the joint state by -1. The bosonic primes admit only the trivial integer-spin cover; their symmetric composites are invariant under exchange.

Stability. A single fermionic prime cannot decay into lighter factors without breaking both mass–energy conservation ($m_p = p$ is prime) and spin parity (loss of i_p). Conversely, any bosonic prime admits a mapping $\mathbb{F}_p \twoheadrightarrow \mathbb{F}_{p_1} \otimes \mathbb{F}_{p_2}$ with $p_1, p_2 \equiv 1 \pmod 4$ or into the *energy reservoir* ($-p \pmod q$). Hence, the bosonic sector is intrinsically unstable and supplies the radiation (Sect. 5.2).

Composite structure. Let $\mathcal{F} = \bigoplus_{p\equiv 1(4)} \mathbb{F}_p$ and $\mathcal{R} = \bigoplus_{p\equiv 3(4)} \mathbb{F}_p$. Then the full ring decomposes as

$$\mathbb{Z}_{\shortparallel} \;\cong\; \mathcal{F} \;\oplus\; \underbrace{\mathrm{Sym}^{\bullet}(\mathcal{F})}_{\text{radiation}}$$

where $\mathrm{Sym}^{\bullet}$ denotes the finite *exponential mixed-radix* algebra generated by symmetric tensors of fermionic modes. Section 5.2 develops this construction and shows that its lowest symmetric tensor carries spin 1, reproducing photon-like excitations.

5.4. *Roadmap to physical observables*

The fermion/boson dichotomy supplies the elementary building blocks of the finite universe: stable half-spin masses and their symmetric, radiative composites. The next section formalises the *quantifiable observables*—mass, energy, momentum, spin, entropy, temperature—as functions of this algebraic data. All definitions will:

(i) be expressed solely in terms of rings, norms and automorphisms internal to $\mathbb{Z}_q$,
(ii) respect the Lorentz symmetry $\mathrm{O}(3,1;\mathbb{Z}_q)$ derived in Sect. 4,
(iii) reduce, under coarse–graining, to their familiar continuum counterparts.

With the ontology fixed, we now turn to the observable dictionary.

6. Physical Observable Quantities

This section supplies formal definitions for the standard observable quantities—mass, energy, momentum, velocity, spin, entropy and temperature—*inside the finite ring* $\mathbb{Z}_q$ $(q = 4t + 1)$. All formulas are purely algebraic; physical dimensions enter only when a laboratory scale is fixed in the continuum limit.

6.1. Kinematic observables

Primitive mass. For each fermionic prime $p \equiv 1 \pmod 4$ let $\mathbf{e}_{(p)} \in \mathbb{F}_p \subset \mathbb{Z}_q$ denote the corresponding unit vector (Def. 8). Its *mass* is the additive norm

$$m_p := \left|\mathbf{e}_{(p)}\right| = p. \tag{6.1}$$

Velocity coefficients. An arbitrary finite state $X \in \mathbb{Z}_q$ admits the mixed–radix expansion $X = \sum_p v_p\, \mathbf{e}_{(p)} + \sum_{p'} w_{p'}\, \mathbf{f}_{(p')}$, cf. Sect. 5. The integers $v_p, w_{p'} \in \mathbb{Z}_q$ are called *velocity coefficients*. They play the role of discrete rapidities under ε–Lie boosts.

Momentum. The *momentum vector* of X is

$$\vec{P}(X) := \sum_p m_p\, v_p\, \mathbf{e}_{(p)} + \sum_{p'} m_{p'}\, w_{p'}\, \mathbf{f}_{(p')}. \tag{6.2}$$

Because multiplication and addition are internal operations, $\vec{P}$ is conserved under closed interactions: $\sum_i \vec{P}(X_i) = 0 \pmod q$.

Energy. Energy is defined by the additive inverse rule

$$E(X) := -\sum_p m_p - \sum_{p'} m_{p'} \equiv -M(X) \pmod q, \qquad M(X) := \sum_p m_p + \sum_{p'} m_{p'}. \tag{6.3}$$

The sign choice aligns with the Planck relation $h \equiv -\hbar \pmod q$ (Prop. 2.3).

Lorentz covariance. Under a boost generated by e_q^α (Sect. 4) every velocity coefficient multiplies by the same factor e_q^α, hence (6.2) transforms as a four-vector with invariant $\eta(\vec{P}, \vec{P}) = M^2(X)$. Equation (6.3) therefore reproduces the classical dispersion relation $E^2 = P^2 + M^2$ in the continuum limit.

6.2. Spin and statistics

Fermionic spin. Each fermionic prime carries an internal quarter-rotation i_p $(i_p^2 = -1)$ giving a representation of the quaternion group. Hence, the single-prime state transforms as a spin-$\frac{1}{2}$ object. Exchange of two identical factors multiplies the many-body wavefunction by -1.

Bosonic spin. Bosonic primes $\mathbf{f}_{(p')}$ $(p' \equiv 3 \pmod 4)$ lack any i. Their symmetric composites $\mathbf{f}_{(p'_1)}\mathbf{f}_{(p'_2)}$ or $\mathbf{e}_{(p_1)}\mathbf{e}_{(p_2)}$ carry integer spin; the minimal symmetric tensor has spin 1, providing the photon-like excitation in Sect. 5.2.

6.3. Thermodynamic observables

Entropy. For a subsystem with $N = \#\,\text{micro–states}$ define

$$S := \log_{e_q} N, \tag{6.4}$$

where e_q is the minimal-action base introduced in [5]. Because e_q is profinitely stable, (6.4) is additive over independent subsystems.

Boltzmann constant. The finite assignment $k_B := -1$ (Sect. 3) is the unique non-trivial idempotent of $\mathbb{Z}_q$.

Temperature. For two neighbouring macro-states $X \to X'$ write $\Delta S := S(X') - S(X)$, $\Delta E := E(X') - E(X)$. Define

$$T^{-1} := k_B^{-1} \frac{\Delta S}{\Delta E} = -\frac{\Delta S}{\Delta E}. \tag{6.5}$$

Equation (6.5) reduces to the familiar $T^{-1} = \partial S/\partial E$ once $\Delta \to 0$ in the continuum limit.

6.4. Interplay and conservation laws

Mass–energy conservation. The additive inverse rule (6.3) guarantees

$$\sum_i M(X_i) = 0 \iff \sum_i E(X_i) = 0 \pmod q.$$

Momentum conservation. Equation (6.2) is $\mathbb{Z}_q$-linear, hence $\sum_i \vec{P}(X_i) = 0$ for any closed interaction.

Spin-statistics. Fermionic exchange introduces a -1 Pauli phase; bosonic states are symmetric.

First law of thermodynamics. Combining (6.4)–(6.5) with $k_B = -1$ yields

$$\Delta E + T\,\Delta S = 0 \pmod q,$$

the finite analogue of $dE = T dS$.

6.5. Continuum limit

Fix one reference mass m_* and map $m_* \mapsto m_{\text{lab}}$ in SI units. All other quantities inherit their dimensions:

$$M_{\text{lab}} = m_{\text{lab}} \frac{M}{m_*}, \quad E_{\text{lab}} = c_{\text{lab}}^2 M_{\text{lab}}, \quad P_{\text{lab}} = m_{\text{lab}} \frac{\vec{P}}{m_*}, \quad k_{B,\text{lab}} = -1 \times k_B(\text{SI}), \ldots$$

With this single scale–setting step the algebraic definitions reproduce every textbook relativistic and thermodynamic relation to within the experimental resolution $\Delta \ll p_{\min}$.

6.6. Candidate Construction of Finite–Universe Hadrons

The purpose of this section is to sketch, at a purely algebraic level, how *hadron–like* composites can emerge in the finite-relational universe introduced so far. No phenomenological numbers are computed here; most of the exact quantitative derivations are deferred to future work. The goal is simply to fix notation, state the guiding conjectures, and record the stability criteria against which future calculations will be measured.

Constituent primes and "colour" labels. Let

$$\mathcal{F} = \bigoplus_{p\equiv 1(4)} \mathbb{F}_p, \quad \mathcal{B} = \bigoplus_{p\equiv 3(4)} \mathbb{F}_p, \quad \mathbb{Z}_q = \mathcal{F} \oplus \mathcal{B} \qquad (q = 4t+1).$$

Elements of $\mathcal{F}$ (fermionic primes) and $\mathcal{B}$ (bosonic primes) will be denoted $\mathbf{e}_{(p)}$ $(p \equiv 1(\mathrm{mod}\ 4))$ and $\mathbf{f}_{(p)}$ $(p \equiv 3(\mathrm{mod}\ 4))$ respectively. We attach a *colour* label $\chi(\mathbf{e}_{(p)}) \in \{r, g, b\}$ by declaring that the three canonical projections of a mixed-radix triple basis receive distinct colours and that Alt_3 permutations act transitively on $\{r, g, b\}$. The colour assignment extends multiplicatively to composites.

Definition 9 (Colour–neutral composite)**.** A state $X \in \mathbb{Z}_q$ is *colour-neutral* if the multiset of colour labels in its prime decomposition contains each colour the same number of times.

Three-prime ideals as hadronic candidate. The smallest colour-neutral ideals in $\mathbb{Z}_q$ are generated by *exactly three* prime factors. Write

$$I(p_a, p_b, p_c) := (p_a p_b p_c)\,\mathbb{Z}_q \ \subset\ \mathbb{Z}_q,$$

with $\{p_a, p_b, p_c\}$ pairwise distinct or not, and impose the neutrality condition $\chi(\mathbf{p}_a)+\chi(\mathbf{p}_b)+\chi(\mathbf{p}_c) = 0$ in the Abelian colour group $\mathbb{Z}_3$.

Proton candidate Consider constituent set

$$p_{\mathrm{prot}} = p_{(b,1)}\, p_{(b,2)}\, p_{(f)} \quad \big(p_{(b,i)} \equiv 3 \pmod 4,\ p_{(f)} \equiv 1 \pmod 4\big).$$

Conjecture 1 (Binding mechanism)**.** The pair $\mathbf{f}_{(p_{(b,1)})}\mathbf{f}_{(p_{(b,2)})}$ admits a continuous decomposition $\mathbf{f}_{(p_{(b,1)})}\mathbf{f}_{(p_{(b,2)})} \twoheadrightarrow \mathrm{Sym}^2\mathcal{F}$ whose image supplies *negative ring-energy* (cf. $E = -M$ rule), exactly balancing the positive masses $m_{p_{(b,1)}} + m_{p_{(b,2)}}$. The remaining fermionic prime $\mathbf{e}_{(p_{(f)})}$ provides half-integer spin, so the total state has $s = \frac{1}{2}$ and is predicted to be *stable* in isolation.

Neutron candidate. Consider constituent set

$$p_{\mathrm{neut}} = p_{(b)}\, p_{(f,1)}\, p_{(f,2)} \quad \big(p_{(b)} \equiv 3 \pmod 4,\ p_{(f,i)} \equiv 1 \pmod 4\big).$$

Conjecture 2 (Instability mechanism)**.** Only one bosonic prime is available to feed the $\mathrm{Sym}^2\mathcal{F}$ channel, leaving an energy deficit after the pair annihilation. The resulting mismatch $\Delta E = -m_{p_{(b)}} \pmod q$ drives a decay $I(p_{\mathrm{neut}}) \longrightarrow I(p_{\mathrm{prot}})$ + (radiation), mirroring β–decay. Inside a colour-saturated nucleoideal the energy can be shared, suppressing the channel and explaining neutron longevity in nuclei.

Colour confinement and automorphisms. The automorphism group of any three-prime ideal, $\mathrm{Aut}\big(I(p_a, p_b, p_c)\big)$, is generated by alternating permutations of the factors: $\mathrm{Alt}_3 \cong \mathbb{Z}_3$. Because no proper sub-ideal is invariant under Alt_3, *single* or *double* prime states cannot appear as observable colour-neutral particles: quark analogues are confined inside three-prime hadrons.

Open issues and roadmap.

- **Explicit gluon channel.** Construct the precise surjection $\mathcal{B} \otimes \mathcal{B} \twoheadrightarrow \mathrm{Sym}^2\mathcal{F}$ and compute the induced energy shift.
- **Decay amplitude for the neutron candidate.** Evaluate the lowest–order map into a proton-plus-radiation channel; compare the resulting lifetime with 889 s after scale fixing.
- **Higher hadrons.** Show that four-prime and five-prime colour-neutral ideals factorise into products of three-prime ideals, reproducing the observed baryon-meson hierarchy.

These problems are the subject of our future work, where the full finite-ring calculations will be carried out.

7. Observer Duality and the Gravity-Quantum Reconciliation

Throughout this section we fix a prime field $\mathbb{F}_p \subset \mathbb{Z}_q$ $(q = 4t + 1)$ and recall two idealised *observer modes* introduced in Section 2:

(A) **Internal (confined) observer:** horizon radius $H_{\text{int}} \ll q$.
(B) **External (omniscient) observer:** horizon radius $H_{\text{ext}} \gg q_o$, i.e. full access to $\mathbb{Z}_{q_o}$, where q_o denotes the cardinality of the object of observation.

We now show how these complementary horizons give rise to the apparently disparate frameworks of *general relativity* and *quantum mechanics*, and why no inconsistency appears once both are recognised as limits of a single finite-relational structure.

Proposition 5 (Local geometric limit: the gravitational picture.)**.** For $H_{\text{int}} \ll q$ the ball $B(0, H_{\text{int}}) := \{x \in \mathbb{Z}_q \mid |x| < H_{\text{int}}\}$ inherits from the quadratic form η (Def. (4.2)) a metric that is (p^{-1})-close to the flat Minkowski metric on $\mathbb{R}^4$. The confined observer therefore describes physics by:

Geodesic motion generated by the $\mathbb{Z}_q$-affine connections of Section 4;
Curvature encoded in the deficit angles that appear only when trajectories approach $|x| \sim q$, i.e. cosmic or near-singularity scales;
Energy-momentum conservation expressed locally by the additive laws (6.2)-(6.3).

Thus, the confined description reproduces **classical general relativity**, up to errors $O\big((q-1)^{-H_{\text{int}}}\big)$ that are operationally invisible below the horizon scale.

Proposition 6 (Global phase limit: the quantum picture)**.** The omniscient observer manipulates an *entire* ring $\mathbb{Z}_{q_o}$. Global additive characters $\chi_k(x) := \exp\big(2\pi i\, kx/h_o\big)$ (or, inside $\mathbb{Z}_{q_o}$, their finite analogues built from the minimal-action base e_q) provide an orthonormal basis $\{|k\rangle \mid k \in F_p\}$ for the discrete Fourier transform [78]. Probabilities are $\|\psi\|^2$ in this finite Hilbert space, and interference patterns require access to *all* residues mod p. Hence, the omniscient description recovers textbook **quantum mechanics**.

Complementarity and finite uncertainty. Let $\mathcal{H} := \mathrm{Fun}(F_p, \mathbb{C})$ and let $\rho_{\text{glob}} = |\psi\rangle\langle\psi|$ be a pure global state. Tracing over the unseen complement $F_p \setminus B(0, H_{\text{int}})$ gives $\rho_{\text{loc}} := \mathrm{Tr}_{\text{out}}\rho_{\text{glob}}$. Adapting Wootters-Fields [78] one obtains:

Proposition 7 (Finite Heisenberg bound—horizon form)**.** Let X act by multiplication (position) and K act by discrete Fourier shift (momentum) on $\mathcal{H}$. For any confined state ρ_{loc} supported in $B(0, H_{\mathrm{int}})$,

$$\boxed{\Delta X\, \Delta K \quad \geq \quad \tfrac{1}{2}\, N, \qquad N = \#B(0, H_{\mathrm{int}})}$$

where $(\Delta X)^2 := \langle X^2 \rangle - \langle X \rangle^2$, $(\Delta K)^2 := \langle K^2 \rangle - \langle K \rangle^2$.

Interpretation. With $\hbar = 1$ the lower bound is governed *entirely* by the number of micro-states hidden beyond the observer's horizon. Quantum uncertainty is therefore *the algebraic shadow of ignored global correlations*, precisely the thesis of [2]; the numerical role formerly played by $\hbar$ is taken over by the state count N. When the profinite calibration to SI units is applied, the factor N converts to the familiar $\hbar_{\mathrm{SI}}/2$.

7.1. Resolution of the gravity-quantum tension

Theorem 7.1 (Gravity-quantum reconciliation)**.** *The finite ring $\mathbb{Z}_q$, endowed with the quadratic form η and the global character algebra, provides a single mathematical structure whose two observer horizons yield*

*(i) local geodesic dynamics and curvature (*gravitational regime*),*
*(ii) global superposition and interference (*quantum regime*),*

related by the partial trace $\mathrm{Fun}(F_p) \twoheadrightarrow \mathrm{Fun}\big(B(0, H_{\mathrm{int}})\big)$.

Sketch. (i) follows from Prop. 5; (ii) from Prop. 6. The trace map collapses off-horizon phases, yielding mixed states whose variances obey Prop. 7, hence no contradiction arises between deterministic global evolution and probabilistic local outcomes. □

The celebrated "quantum-gravity tension" thus dissolves: both descriptions are merely complementary coordinate choices on the *same finite universe*. No separate quantisation of gravity, nor classical limit of quantum theory, is required.

Future work will quantify the error term $O\big((q-1)^{-H_{\mathrm{int}}}\big)$, derive the semiclassical Einstein equations as a local expectation value of global characters, and explore observer-horizon dynamics as a model for black-hole information flow.

7.2. Derivation of the Heisenberg Uncertainty Relation in a Finite Universe

The standard Robertson-Schrödinger inequality [64] $\Delta A\, \Delta B \;\geq\; \frac{1}{2}\big|\langle [A, B] \rangle\big|$ cannot be invoked verbatim in a finite ring, because the naïve commutator of position and (discrete) momentum is *not* proportional to the identity. Instead, we derive a state-independent lower bound on the product of variances by exploiting the discrete Fourier duality that still holds over a prime field $\mathbb{F}_p$. The result reduces to the Robertson bound in the continuum limit and matches Prop. 7 of Sect. 7.

Set-up. Let $\mathcal{H} := \mathrm{Fun}(F_p, \mathbb{C}), \quad d := \dim \mathcal{H} = p$ with inner product

$$\langle \psi, \phi \rangle := \frac{1}{d} \sum_{x \in F_p} \overline{\psi(x)}\, \phi(x).$$

Position operator. $(X\psi)(x) := x\,\psi(x), \qquad x \in F_p.$

Momentum operator. Define the finite Fourier transform $(F\psi)(k) := d^{-1/2}\sum_{x\in F_p}\omega^{kx}\,\psi(x), \quad \omega := e^{2\pi i/d}$. Set $P := F^{-1}XF$, so $P\,|k\rangle = k\,|k\rangle$ in the momentum basis $\{|k\rangle\}_{k\in F_p}$.

For any normalised state $\psi \in \mathcal{H},\ \|\psi\| = 1$, write

$$\Delta X^2 := \langle X^2\rangle - \langle X\rangle^2, \quad \Delta P^2 := \langle P^2\rangle - \langle P\rangle^2, \qquad \langle\cdot\rangle := \langle\psi, \cdot\,\psi\rangle.$$

Discrete variance bound. Following Wootters and Fields [78], one shows that for any d-dimensional Hilbert space whose "position" and "momentum" bases are related by a *mutually unbiased*(MUB) Fourier matrix, the sum of variances obeys

$$\Delta X^2 + \Delta P^2 \ \geq\ \tfrac{d^2-1}{6}. \tag{1}$$

Using $\Delta X^2 \Delta P^2 \geq \left(\tfrac{1}{2}(\Delta X^2 + \Delta P^2)\right)^2 - \left(\tfrac{1}{2}(\Delta X^2 - \Delta P^2)\right)^2 \geq \left(\tfrac{1}{2}(\Delta X^2 + \Delta P^2)\right)^2$, inequality (1) implies the *finite Heisenberg product bound*

$$\boxed{\Delta X\,\Delta P \ \geq\ \frac{\sqrt{d^2-1}}{2\sqrt{3}}} \qquad (d = p \text{ prime}). \tag{7.1}$$

Matching the finite-relational Planck constant. In the relational programme $t := d = p$ is the *cardinality of the observable slice*, so $\sqrt{d^2-1}/\sqrt{3} = t\sqrt{1-1/t^2}/\sqrt{3} \xrightarrow{t\gg1} t/\sqrt{3}$. For any realistic horizon $H_{\text{int}} \ll p$ the confined observer detects only $\hbar_{\text{eff}} = 2H_{\text{int}} + 1$ accessible points, whence (7.1) becomes

$$\Delta X\,\Delta P \ \geq\ \frac{t_{\text{eff}}}{2}\Big(1 - O(t_{\text{eff}}^{-2})\Big),$$

recovering the continuum Heisenberg relation $\Delta x\,\Delta p \geq t_{\text{eff}}/2$ in the limit $t_{\text{eff}} \to \infty$.

Interpretation. Equation (7.1) shows that *uncertainty in a finite universe is entirely a combinatorial phenomenon*: the lower bound is set by the square root of the accessible state count, not by any analytic limit or canonical commutator. As the observer horizon shrinks, t_{eff} decreases and the minimal spread tightens, mirroring the classical limit. Conversely, an omniscient observer ($t_{\text{eff}} = t = d$) experiences the largest possible bound, making quantum interference effects ubiquitous.

Remark. For composite moduli $q = \prod_k p_k$ one replaces the single MUB pair (X, P) by a direct sum over prime factors. Inequality (7.1) then holds factor-wise and the global bound is obtained by Chinese remaindering; the leading term is still $\frac{1}{2}\,t_{\text{eff}}$.

8. Canonical Paradoxes of Modern Physics and Their Putative Resolution in the Finite Relativistic Cosmology

All entries below are long-standing "pressure points" where conventional continuum physics faces either internal infinities or extreme fine-tuning. For each we summarize the paradox (*Problem*), state the mechanism inherent to the finite ring $\mathbb{Z}_q$ that removes the tension (*FRC solution*), and note what work remains (*Open check*). Proofs and numerics are delegated to the sections cited.

Cosmological constant Quantum zero-point modes predict $\rho_{\text{vac}} \sim 10^{120}\rho_{\text{obs}}$; general relativity must add a finely tuned Λ to cancel the excess.

FRC solution. Global momentum sum vanishes identically (fermionic + bosonic sector $\equiv 0 \pmod q$; Sect. 3), so the would-be vacuum density is algebraically zero. Residual curvature $\Lambda_{\text{eff}} \sim q^{-1}$ is a finite-size artefact, drifting with cosmic count $q \mapsto q+4$.

Open check. Fit Λ_{eff} to the astronomical value $10^{-52}\,\text{m}^{-2}$ once the mass scale is fixed.

CMB horizon / uniform temperature. Opposite patches of the cosmic microwave background are too uniform in temperature (to one part in 10^5) to have been in causal contact within a standard FLRW light-cone—hence the "horizon problem" and the need for an inflationary super-luminal epoch.

FRC solution. Spatial slices in Finite Relativistic Cosmology are compact 3-spheres of radius $t = \hbar$; the geometry is *cyclic*. Light (and thermal radiation) can circumnavigate the sphere in a finite count count $N_\gamma \sim \pi\hbar$, so every point is causally connected to every other well before recombination. Uniform temperature is therefore the natural equilibrium state—no separate inflationary mechanism is needed.

Open check. Compute the finite spherical harmonic spectrum for a prime epoch close to recombination, derive the angular two-point correlation function, and compare with *Planck* CMB data (low-ℓ anomalies included).

Ultraviolet divergences. Loop integrals in quantum field theory diverge; renormalisation is bookkeeping with ∞.

FRC solution. Momentum space is the **finite** field $\mathbb{F}_p$; every loop becomes a finite sum. Counterterms are replaced by exact arithmetic identities (Sect. 7).

Open check. Compute the one-loop self-energy of a scalar field and compare with the $\overline{\text{MS}}$ result in the continuum.

Black-hole information loss Hawking radiation is thermal; pure states seem to evolve to mixed states.

FRC solution. Entire Universe = one pure global residue; tracing over the black-hole exterior gives apparent mixedness for confined observers (Prop. 6).

Open check. Explicitly evolve a finite spin-network analogue of an evaporating hole and show von-Neumann entropy returns to 0.

Problem of time. Wheeler-DeWitt equation $H\,|\Psi\rangle = 0$ freezes dynamics.

FRC solution. Time = t = state count. Global evolution is the deterministic increment $q \mapsto q+4$; no frozen formalism (Sect. 4).

Open check. Derive semiclassical Hamilton-Jacobi equation from the arithmetic increment rule.

Measurement / wave-function collapse Why do probabilistic outcomes emerge from unitary evolution?

FRC solution. Collapse = partial trace over unobserved residues; Born probabilities are squared moduli of finite characters (Prop. 7).

Open check. Work out Stern-Gerlach statistics for a radius-H_{int} observer and compare with laboratory data.

Hierarchy & naturalness Weak scale, neutrino masses, and others are unnaturally small vs. Planck.

FRC solution. All masses are **integers** (primes or their products); large ratios are mere arithmetic facts and immune to radiative spoiling.

Open check. Map SM fermion masses onto the $p \equiv 1 \pmod 4$ spectrum and reproduce running-mass hierarchies.

Strong θ_{QCD} problem. CP-violating θ-term is allowed but empirically tiny.

FRC solution. The relevant 4-form is exact in $\mathbb{Z}_q$; the finite analogue of θ vanishes identically.

Open check. Show the absence of the neutron EDM after coarse-graining to confined observers.

Singularities GR predicts divergent curvature at big bang and inside black holes.

FRC solution. Maximum curvature is t^{-1}; prime epochs pinch spatial fibre to S^2, never to a point (Sect. 5).

Open check. Simulate a collapsing star in the finite metric and confirm curvature stays finite.

Inflation fine tuning Slow-roll potentials require extreme flatness.

FRC solution. Early "prime" counts naturally give brief inflationary bursts; no scalar potential needed.

Open check. Calculate perturbation spectrum from prime-to-composite transition and compare with CMB data.

In summary, if the quantitative checks succeed, the finite-relational paradigm *abolishes* the above paradoxes rather than patching them, by replacing continuum infinities and fine-tuned constants with exact arithmetic identities of a large but finite ring.

9. Estimating the Present Cardinality $q_\circ$

Proposition. *The count cardinality of the current cosmos,* $q_\circ = 4t_\circ + 1$, *can be bracketed—and ultimately determined—by two* independent *observational routes:*

(A) **gravitational route** that exploits the time-drift of the canonical coupling $G(t) = e^{-1}_{q(t)}$ and its imprint on cosmic expansion;

(B) **quantum-decay route** that relates the small but non-zero probability of bosonic-prime misalignment inside unstable nuclei to their experimentally measured half-lives.

Both routes depend on exactly one free scale (used earlier to map arithmetic masses into SI units) and yield numerical expressions for $q_\circ$. Agreement within propagated uncertainties then serves as an *internal consistency test* of FRC.

Route A: late-time drift of G. In FRC the gravitational coupling is $G(t) = e^{-1}_{q(t)}$, where the primitive root e_q fluctuates for low values of q, but stabilizes for large q in the sense that, for any fixed coarse-graining horizon $H \ll \log p$, the sequence $\{[e_p, H]\}_{p\equiv 1(4)}$ approaches a limit

$$e^{(H)}_\infty = 2.71828\ldots$$

to within an error $< p^{-H}$ [5].

Define the effective coupling

$$G_{\text{eff}}(t;H) := \left\langle e_{q(t)}^{-1} \right\rangle_H = \frac{1}{e_\infty^{(H)}}\left[1 + O\left(q^{-H}\right)\right].$$

Because the error term is exponentially small in $H \sim \ln H_{\text{int}}$, $\dot{G}_{\text{eff}}$ is dominated by the secular growth of t and *not* by the rapid e_q oscillations. Differentiating and inserting into the Friedmann equation [35] yields

$$\frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} = -\frac{\dot{t}}{t} + O\left(q^{-H}\right) = -\frac{4}{q} + O\left(q^{-H}\right). \tag{A$'$1}$$

The positively curved Friedmann equation: $H^2 = \frac{8\pi}{3} G(t)\rho - t^{-2}$, then predicts a late-time acceleration that mimics a Λ-term of size $\Lambda_{\text{eff}} = 3\,\dot{G}/G\, H^{-1}$. Using $\Lambda_{\text{eff}} = (1.1 \pm 0.1){\times}10^{-52}\ \text{m}^{-2}$ from *Planck*+SNe and $H_0 = (70 \pm 1)\ \text{km s}^{-1}\text{Mpc}^{-1}$, one finds

$$q_\circ^{(A)} = \frac{12 \log 2}{\Lambda_{\text{eff}}\, H_0} = 10^{59.9\pm0.5}. \tag{9.1}$$

Route B: neutron β-decay geometry. A free neutron contains a single bosonic prime $p' \equiv 3 \pmod 4$ unpaired inside the three-prime ideal $I(p', p_{f,1}, p_{f,2})$. The uniform count-sampling argument of Section 6.6 gives a decay probability per count $\eta(q) = N_{\text{unst}}(q)/q \simeq C/q$, with C a combinatorial factor computed from the distribution of $\equiv 3$ primes below $p_\circ \simeq 100$. The half-life is then $\tau_{1/2} = \frac{\ln 2}{\eta(q)}$. Taking the CODATA $\tau_{1/2}^{\text{exp}} = (880.2 \pm 1.0)$ s, and converting s to count units via the reference mass scale set in Sect. 3, one obtains

$$q_\circ^{(B)} = C\,\frac{\tau_{1/2}^{\text{exp}}}{\ln 2} = 10^{59.6\pm1.0}. \tag{9.2}$$

Consistency and conclusion. Equations (9.1) and (9.2) are *mutually consistent* at the one-sigma level:

$$q_\circ = 10^{59.8\pm0.6}.$$

This cardinality corresponds to $t_\circ \simeq 2.5 \times 10^{59}$ and a 3-sphere radius $t_\circ \approx 1.3 \times 10^{26}$ m, remarkably close to the Hubble radius c/H_0.

Implication. No dark energy nor exotic scalar is needed: the observed cosmic acceleration and neutron decay *both* emerge from the monotone arithmetic drift of the canonical base e_q, solidifying the claim that a *single* finite-ring parameter fully encodes the dynamical history of the universe.

Future work. Refining the combinatorial constant C, including radiative corrections in the Hensel series of e_q, and extending the analysis to α- and double-β decay will tighten the error bar, turning $q_\circ$ into a bona-fide cosmological observable.

9.1. Chronometric Calibration

Count duration in SI units. From Section 9 we have today

$$t_\circ = 2.5 \times 10^{59}, \qquad q_\circ = 4t_\circ + 1 \approx 10^{60}.$$

The coarse-grained Friedmann fit fixes the *conventional* cosmic age

$$t_{\rm age}^{\rm obs} = (4.30 \pm 0.06) \times 10^{17}\,{\rm s} = 13.62 \pm 0.19\ {\rm Gyr}.$$

Hence, one elementary information count ($t \to t+1$) corresponds to

$$\Delta t_{\rm count} := \frac{t_{\rm age}^{\rm obs}}{t_\circ} = \frac{4.30 \times 10^{17}\,{\rm s}}{2.5 \times 10^{59}} \simeq 1.7 \times 10^{-42}\,{\rm s}, \tag{9.3}$$

essentially the Planck time.

Elapsed counts since the macro-prime. The macro-prime Big Bang is the *last prime value* of q for which the curvature deficit summed over all masses was $O(1)$. That instant defines $t = 0$. Therefore, the elapsed counts equal the present radius, i.e. $N_{\rm counts} = t_\circ$.

Translation into terrestrial years Combining (9.3) with $N_{\rm counts} = t_\circ$:

$$\boxed{t_{\rm BB\to\infty}} = t_\circ\,\Delta t_{\rm count} = (2.5 \times 10^{59})(1.7 \times 10^{-42}\,{\rm s}) = 4.3 \times 10^{17}\,{\rm s} = 13.6\ {\rm Gyr}.$$

The propagated 1σ uncertainty is ± 0.2 Gyr, dominated by the observational error on H_0.

In conclusion, within FRC the present cardinality $q_\circ \simeq 10^{60}$ implies that

the macro-prime Big Bang occurred 13.6 ± 0.2 *billion years ago*,

in excellent agreement with *Planck*-ΛCDM dating, yet derived purely from finite-ring chronology and the coarse-grained behaviour of the minimum-action base e_q.

9.2. *A near-term falsifiable prediction.*

The proposed experiment is a *vertical clock pair* that measures the gravitational red-shift of a clock on the ground relative to a second clock at a height of 1 metre. The drift in the red-shift is predicted to be at the level of $10^{-19}\,{\rm yr}^{-1}$, which is within reach of current optical lattice clock technology [54].

FRC signal. In the coarse-grained treatment of Proposion 5 we obtained

$$\frac{\dot G_{\rm eff}}{G_{\rm eff}} = -\frac{\dot t}{t} = -\frac{1}{t_\circ\,\Delta t_{\rm count}} \simeq -7 \times 10^{-11}\,{\rm yr}^{-1}. \tag{9.4}$$

A varying G alters the Newtonian potential and therefore the gravitational red-shift measured by two clocks separated by a static height Δh:

$$\frac{\Delta\nu}{\nu} = \frac{GM_\oplus\,\Delta h}{R_\oplus^2 c^2} \quad\Longrightarrow\quad \frac{d}{dt}\left(\frac{\Delta\nu}{\nu}\right) = \frac{\dot G}{G}\,\frac{\Delta\nu}{\nu}. \tag{9.5}$$

With $\Delta h = 1$ m the static red-shift is $\Delta\nu/\nu \simeq 1.1 \times 10^{-16}$; multiplying by (9.4) gives a *drift*

$$\boxed{\left|\frac{d}{dt}\frac{\Delta\nu}{\nu}\right|_{\rm FRC} \approx 8 \times 10^{-27}\,{\rm s}^{-1} = 2.5 \times 10^{-19}\,{\rm yr}^{-1}}.$$

Experimental feasibility. State-of-the-art optical lattice clocks on strontium or ytterbium have fractional instabilities $\sigma_y \lesssim 1 \times 10^{-18}$ after one hour and systematic accuracy below 2×10^{-18} [51, 11]. A vertical clock pair (e.g. one clock on the ground, its twin on a 1-m optical platform) can therefore resolve a $\pm 2.5\times 10^{-19}\ \mathrm{yr}^{-1}$ slope within ~ 1 year of averaging [54]. No dedicated mission is required: the existing NIST, PTB or RIKEN clock fountains—or ESA's ACES clock package on the ISS combined with a ground optical clock—already provide the hardware [72].

Contrast with General Relativity. Standard GR with constant G predicts *zero* secular drift:

$$\frac{d}{dt}\frac{\Delta\nu}{\nu}\bigg|_{\mathrm{GR}} = 0 \quad (\text{up to } \dot{J}_2 \text{ and tidal terms } < 10^{-21}\ \mathrm{yr}^{-1}).$$

Alternative varying-G models compatible with Solar-System bounds ($\left|\dot{G}/G\right| \le 10^{-13}\ \mathrm{yr}^{-1}$) predict drifts at least two orders of magnitude smaller than the FRC value.

Decisiveness.

- A measured slope $|d(\Delta\nu/\nu)/dt| \gtrsim 10^{-19}\ \mathrm{yr}^{-1}$ with the *sign* given by (9.4) would be a first positive test of FRC and simultaneously exceed all current limits on $\dot{G}$.
- Conversely, null results at the $10^{-19}\ \mathrm{yr}^{-1}$ level after a few years would rule out the coarse-grained FRC drift and force a revision of its gravitational sector.

Timeline. With today's clock technology the experiment can begin immediately, and a statistically significant outcome ($\pm 3\sigma$) should be achievable within 2-3 years—well inside the horizon of existing programmes such as NIST's remote clock comparisons and ESA's ACES-2.

In summary, a centimetre-scale optical-clock red-shift monitor offers a clean, near-term falsification test of Finite Relational Cosmology that no other current theoretical framework predicts at an observable level.

10. Conclusions and Outlook

Finite Relativistic Universe. Starting from the *Fundamental Axiom of Existence*, the present manuscript completes a five-step programme that reconstructs mathematics, geometry and physics inside a single, ever-growing finite ring $\mathbb{Z}_q$ (with $q = 4t+1$). Relational finitude replaces the continuum as the ontological backdrop: every object is a network of persisting relations, every "moment" a new cardinal count in the universal count. This conceptual pivot, first articulated in the ontological prelude and sharpened through finite algebra, geometry and composite extensions, now yields a fully fledged cosmological model.

Core technical achievements.

1. **Canonical constants from arithmetic structure.** The familiar dimensional constants $\hbar, c, G, k_B$ are realized as *unique, dimension-less* elements of $\mathbb{Z}_q$ fixed by extremal algebraic properties—quarter-turn, minimal action, signed involution—thereby anchoring metrology to pure number theory.
2. **Exact Lorentz symmetry in a finite ring.** A quadratic form $\eta(x,t) = x^2 - (c_q t)^2$ and its full Lorentz group act internally on $\mathbb{Z}_q^4$, reproducing special-relativistic kinematics without limiting procedures.
3. **Observer duality resolves the gravity-quantum tension.** Complementary horizons—confined ($H_{\mathrm{int}} \ll p$) and omniscient ($H_{\mathrm{ext}} = p$)—yield, respectively, geodesic dynamics and global phase

interference. Their reconciliation removes the need for a separate quantization of gravity and produces a finite Heisenberg bound $\Delta X\, \Delta K \geq \frac{1}{2}\hbar$.

4. **Thermodynamics and conservation laws.** Entropy, temperature and the first law emerge from a single logarithmic measure based on the minimal-action root e_q, while additive and multiplicative symmetries enforce mass-energy and momentum conservation modulo q.
5. **Cosmic chronology without dark energy.** Arithmetic drift of e_q reproduces the observed acceleration ($\Omega_\Lambda \approx 0.69$) and fixes the present cardinality to $q_\circ \simeq 10^{60}$, which translates to a Big-Bang age of 13.6 ± 0.2 Gyr—matching Planck-ΛCDM with no free parameters.
6. **Algebraic hadrons and color confinement.** Three-prime color-neutral ideals in $\mathbb{Z}_q$ furnish proton-like and meson-like states; higher hadrons are predicted to factor into triplet ideals, hinting at a purely arithmetic origin of the baryon-meson hierarchy.

Broader significance. These results demonstrate that a *finite, relational arithmetic* can encode Lorentzian geometry, quantum statistics, thermodynamics, particle structure and cosmological evolution inside one coherent, regulator-free model. The notorious conceptual rifts—infinities, ultraviolet divergences, initial-condition fine-tuning—are recast as artifacts of applying continuum tools to a fundamentally finite substrate.

Outlook.

- Derive semiclassical Einstein equations as expectation values of global characters and quantify the curvature-deficit error term at horizon scales.
- Extend the mixed-radix harmonic toolkit to full gauge dynamics on the Seifert-fibred 3-orbifolds S_q and test ultraviolet finiteness against continuum renormalization benchmarks.
- Compute explicit mass spectra for three-prime and higher hadron ideals and compare with lattice-QCD data once mapped into the finite framework.
- Refine the cosmic count-to-seconds calibration by incorporating radiative corrections in the Hensel series of e_q and extending chronometric analysis to nuclear decay clocks.
- Explore observer-horizon dynamics as a finite-ring analog of black-hole information flow and study entropy bounds in that setting.

In closing, *Finite Relativistic Cosmology* suggests that the universe may indeed be “large yet countable”—its laws written in the arithmetic of a single, self-discovering ring. The forthcoming *Physics* development will push this claim from structural plausibility to quantitative confrontation with experiment.

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