The visages of the Lorentz-Einstein Law

Speculative analysis with the extrinsic method

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This document presents an application of the extrinsic method through a generic and pedagogical example that can be applied to the co-variant formulation of the Lorentz law. The unsaid hope of this approach is the discovery of some unexplored or weakly explored links between two theories because the formalism of this law is a natural bridge between the electromagnetism and the gravitation. The analysis is able to propose (i) a new formalism for the (2, 0) version of the electromagnetic fields and (ii) a speculative confrontation with the theory of spinors resulting in a theoretical prediction, precisely: the existence of electromagnetic fields mimicking anti-symmetric variations of the metric tensor; and conversely; ©The visages of the Lorentz-Einstein law - Speculative analysis with the extrinsic method.

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1 Introduction

1.1 Context and motivation

The study of interactions between electromagnetic and gravitational fields is an old preoccupation of scientific communities. This was also the one of A. Einstein all along his life. The quest is going on but, up today, no one was able to connect both types of fields correctly within a four-dimensional approach taking the quantum theories into consideration.

1.2 The history of the Lorentz law

The co-variant version of the Lorentz law is sometimes called the Lorentz-Einstein law (short: LEL). The co-variant part of its formalism is the so-called *gravitational term*. This term is a tensor product (i) acting on the 4-speed of some particle and (ii) being deformed by the symmetric cube $\Gamma(2)$ containing the Christoffel's symbols of the second kind [02; p. 49], [07; §90, p. 256, (90.7)]:

$$m\,.\,|\frac{d^{(4)}\mathbf{u}}{ds} > + \underbrace{\otimes_{|\Gamma(2)}({}^{(4)}\mathbf{u},\ {}^{(4)}\mathbf{p}) >}_{gravitational\,term} = \frac{q}{c^2}\,.\,[F(\uparrow,\downarrow)].\,|^{(4)}\mathbf{u} >$$

The classical version of Lorentz's law is written in a three-dimensional formalism. It is an attempt to describe the motion of point-particles immersed in an electromagnetic field [07; §17, pp. 46-49, (17.5)]. This classical version has a four-dimensional formulation [07; §23, pp. 60-62, (23.4)]. The concept of co-variant differentiation plays already an important role in Einstein's master work [01]. A presentation of this kind of differentiation can be read in [07; §85]. Both, the development of this concept and its application to the description of point-particles moving under the influence of electromagnetic and gravitational fields, explain the co-variant formalism of the Lorentz's law [07; §90, pp. 254-257]. A deep analyze of this formalism has been extensively developed and presented in [08]. A remaining problem accompanying the co-variant version of the Lorentz's law is the notion of point-particle itself because it is not directly compatible with the philosophy promoted by the quantified approach of the reality.

1.3 What you will find in this exploration

A previous document [b] has roughly introduced a method allowing the decomposition of deformed tensor products.

In section 2, this document explains the method with all necessary details and constraints through a specific family of deformed tensor products. Among the technical details, attention is focused: (i) on the necessary coherence that should exist between the successive derivations; (ii) on the existence of a non-trivial decomposition mimicking the simplest one; (iii) on the not-evident choice of the non-degenerated bi-linear form [B] and (iv) on a natural link between the concept of co-variant derivation and the decomposition of a subset of deformed tensor products in which the pedagogical at hand can eventually be included.

In section 3, the document (i) proves that the co-variant formulation of the Lorentz law belongs to the family which has been studied in previous section, (ii) proposes a new formalism for the (2, 0) version of the electromagnetic fields, (iii) sketches the difficulties associated with the choice of a suitable bi-linear form [B] and (iv) dares to develop a speculation predicting the existence of electromagnetic fields mimicking anti-symmetric variations of the metric tensor [G]: the so-called *chameleons fields* when and if one can identify the bi-linear form with the metric tensor ([B] = [G]).

2 The extrinsic method

2.1 The extrinsic method: principles

The extrinsic method has the same purpose than the intrinsic one: the discovery of at least one non-trivial decomposition for a given deformed tensor product. The difference between both methods lies in the fact that the former involves mathematical tools which are absent in the initial formulation of so-called (E) question.

Any intrinsic method is condemned to work with only three ingredients: the deforming cube, the projectile and the target (they are the intrinsic tools) with the hope to discover a non-trivial decomposition which is a pair ([Matrix = main part], vector = residual part).

Since any tensor product which has been deformed by an anti-symmetric cube is a deformed Lie product, these methods can be involved in researches looking for the decomposition of deformed Lie products. This possibility has been used in any three-dimensional space; see [a] and [b]. The development of an intrinsic method in a four-dimensional environment is not achieved; see an incomplete introduction in [c]. This is the reason why one must invest all efforts in the development of the extrinsic method. This method, as its name evokes it, involves mathematical tools which are not implicitly present in the initial formulation of the (E) question; precisely: a non-degenerated bi-linear form and the concept of scalar product built with this form.

2.2 Useful definitions and remarks

Definition 2.1. *Presumed decomposition*

Whilst it has been proved that any deformed tensor product accepts at least one trivial decomposition, the so-called simplest decomposition without residual part, it is not certain that at least one non-trivial decomposition exists. Hence, the existence of this non-trivial decomposition is presumed and one will write it:

$$| \otimes_A (\mathbf{q}_1, \mathbf{q}_2) >= [P] . |\mathbf{q}_2 > + |\mathbf{z} >$$

In the coordinates language, this relation writes:

$$A^{\epsilon}_{\alpha\beta} . q_1^{\alpha} . q_2^{\beta} \, = \, p_{\epsilon\beta} . q_2^{\beta} \, + \, z^{\epsilon}$$

Definition 2.2. *The scalar associated with the projectile*

Per definition, it is:

$$S(\mathbf{q}_1) \, = < \mathbf{q}_1, | \otimes_A (\mathbf{q}_1, \mathbf{q}_2) > - \{[P] . |\mathbf{q}_2 > + |\mathbf{z} >\} >_{[B]}$$

In the coordinates language, this scalar writes:

$$S(\mathbf{q}_1) \, = \, b_{\chi\epsilon} . q_1^{\chi} . \{A^{\epsilon}_{\alpha\beta} . q_1^{\alpha} . q_2^{\beta} \, - \, (p_{\epsilon\beta} . q_2^{\beta} \, + \, z^{\epsilon})\}$$

The scalar associated with the projectile is the scalar product between the projectile and the default of realization of the presumed decomposition of the deformed tensor product at hand. This scalar vanishes when the presumed decomposition is realized.

Definition 2.3. *The scalar associated with the target*

Per definition, it is:

$$S(\mathbf{q}_2) \, = < \mathbf{q}_2, | \otimes_A (\mathbf{q}_1, \mathbf{q}_2) > - \{[P] . |\mathbf{q}_2 > + |\mathbf{z} >\} >_{[B]}$$

In the coordinates language, this scalar writes:

$$S(\mathbf{q}_2) \, = \, b_{\chi\epsilon} . q_2^{\chi} . \{A^{\epsilon}_{\alpha\beta} . q_1^{\alpha} . q_2^{\beta} \, - \, (p_{\epsilon\beta} . q_2^{\beta} \, + \, z^{\epsilon})\}$$

The scalar associated with the target is the scalar product between the target and the default of realization of the presumed decomposition of the deformed tensor product at hand. This scalar vanishes when the presumed decomposition is realized.

Definition 2.4. *The polynomial associated with a small variation of the projectile*

Per definition, it is:

$$P_1(\mathbf{q}_1 + d\mathbf{q}_1)$$

$$=$$

$$P_1(\mathbf{q}_1) + < \mathbf{Grad}_{\mathbf{q}_1} P_1(\mathbf{q}_1),\, d\mathbf{q}_1 >_{Id} + \frac{1}{2} . < d\mathbf{q}_1,\, [Hess_{\mathbf{q}_1,0} P_1(\mathbf{q}_1)] . | d\mathbf{q}_1 >>_{Id}$$

$$+ 0(3)$$

Definition 2.5. *The polynomial associated with a small variation of the target*

Per definition, it is:

$$P_2(\mathbf{q}_2 + ds \, . \, \mathbf{u}_2)$$

$$=$$

$$P_2(\mathbf{q}_2) + < \mathbf{Grad}_{\mathbf{q}_2} P_2(\mathbf{q}_2),\, ds \, . \, \mathbf{u}_2 >_{Id} + \frac{1}{2} . < ds \, . \, \mathbf{u}_2,\, [Hess_{\mathbf{q}_2,0} P_2(\mathbf{q}_2)] . | ds \, . \, \mathbf{u}_2 >>_{Id}$$

$$+ 0(3)$$

Definition 2.6. *Pythagorean table*

As generic example, let observe the matrix:

$$T_2(o)(\mathbf{Grad}_{\mathbf{u}},\, \mathbf{z}) = \begin{pmatrix} \partial_{u^0} z^0 & \partial_{u^1} z^0 & \partial_{u^2} z^0 & \partial_{u^3} z^0 \\ \partial_{u^0} z^1 & \partial_{u^1} z^1 & \partial_{u^2} z^1 & \partial_{u^3} z^1 \\ \partial_{u^0} z^2 & \partial_{u^1} z^2 & \partial_{u^2} z^2 & \partial_{u^3} z^2 \\ \partial_{u^0} z^3 & \partial_{u^1} z^3 & \partial_{u^2} z^3 & \partial_{u^3} z^3 \end{pmatrix}$$

Remark 2.1. *Before starting explaining the extrinsic method*

One should first note that:

- The vanishing of a scalar associated with an argument involved in a given deformed tensor product is not a guaranty for the realization of a decomposition. Indeed:

  $$\forall\, i = 1,\, 2 : | \otimes_A (\mathbf{q}_1,\, \mathbf{q}_2) > = [P] . | \mathbf{q}_2 > + | \mathbf{z} > \Rightarrow S(\mathbf{q}_i) = 0$$

  But conversely:

  $$\forall\, i = 1,\, 2 : S(\mathbf{q}_i) = 0 \nRightarrow | \otimes_A (\mathbf{q}_1,\, \mathbf{q}_2) > = [P] . | \mathbf{q}_2 > + | \mathbf{z} >$$

- The polynomial associated with a small variation of an argument involved in a given deformed tensor product is not automatically coinciding with the scalar associated with the deformed tensor product in which this variation appears.

2.3 Let start with a pedagogical example

Let now consider for the pedagogy a generic deformed tensor product and its presumed decomposition - here, k is an invariant scalar:

$$|\otimes_A (\mathbf{u}_1,\, k\,.\,\mathbf{u}_2) >= [Q]\,.\,|k\,.\,\mathbf{u}_2 > +|\mathbf{Z} >$$

The scalar associated with the projectile of this deformed tensor product is:

$$S(\mathbf{u}_1) \,=< \mathbf{u}_1,\, |\otimes_A (\mathbf{u}_1,\, k\,.\,\mathbf{u}_2) > -\{[Q]\,.\,|k\,.\,\mathbf{u}_2 > +|\mathbf{Z} >\} >_{[B]}$$

And the scalar associated with the target of this deformed tensor product is:

$$S(k\,.\,\mathbf{u}_2) \,=< k\,.\,\mathbf{u}_2,\, |\otimes_A (\mathbf{u}_1,\, k\,.\,\mathbf{u}_2) > -\{[Q]\,.\,|k\,.\,\mathbf{u}_2 > +|\mathbf{Z} >\} >_{[B]}$$

Whilst the polynomials of the respective variations are (recall):

$$P_1(\mathbf{u}_1 + d\mathbf{u}_1)$$

$$=$$

$$P_1(\mathbf{u}_1) + < \mathbf{Grad}_{\mathbf{u}_1} P_1(\mathbf{u}_1),\, d\mathbf{u}_1 >_{Id} + \frac{1}{2}\,.\, < d\mathbf{u}_1,\, [Hess_{(\mathbf{u}_1,\,0)} P_1(\mathbf{u}_1)]\,.\,|d\mathbf{u}_1 >>_{Id}$$

$$+\,0(3)$$

And:

$$P_2(k\,.\,(\mathbf{u}_2 + d\mathbf{u}_2))$$

$$=$$

$$P_2(k\,.\,\mathbf{u}_2)$$

$$+ < \mathbf{Grad}_{k\,.\,\mathbf{u}_2} P_2(k\,.\,\mathbf{u}_2),\, k\,.\,\mathbf{u}_2 >_{Id}$$

$$+\frac{1}{2}\,.\, < k\,.\,\mathbf{u}_2,\, [Hess_{(k\,.\,\mathbf{u}_2,\,0)} P_2(k\,.\,\mathbf{u}_2)]\,.\,|k\,.\,\mathbf{u}_2 >>_{Id}$$

$$+\,0(3)$$

2.4 The essence of the extrinsic method

The extrinsic method lies on the belief of situations for which it is possible to write that, up to terms of degree three:

$$S(\mathbf{u}_1) \,=\, P_1(\mathbf{u}_1 + d\mathbf{u}_1) \,-\, P_1(\mathbf{u}_1) \,-\, 0(3)$$

And:

$$S(k\,.\,\mathbf{u}_2) \,=\, P_2(k\,.\,\mathbf{u}_2 + k\,.\,d\mathbf{u}_2) \,-\, P_2(k\,.\,\mathbf{u}_2) \,-\, 0(3)$$

2.5 Consequences for the variations of the projectile

Concerning the projectile, these situations are such that:

$$< d\mathbf{u}_1,\, | \otimes_A (d\mathbf{u}_1,\, k\,.\,\mathbf{u}_2) > - \{[Q]\,.\,|k\,.\,\mathbf{u}_2 > + |\mathbf{Z} >\} >_{[B]}$$

$$=$$

$$< \mathbf{Grad}_{\mathbf{u}_1} P_1(\mathbf{u}_1),\, d\mathbf{u}_1 >_{Id} + \frac{1}{2}\,.\, < d\mathbf{u}_1,\, [Hess_{(\mathbf{u}_1,\,0)} P_1(\mathbf{u}_1)]\,.\,|d\mathbf{u}_1 >>_{Id}$$

They imply:

$$|\mathbf{Grad}_{\mathbf{u}_1} P_1(\mathbf{u}_1) >= -[B]\,.\,\{[Q]\,.\,|k\,.\,\mathbf{u}_2 > + |\mathbf{Z} >\}$$

And:

$$k\,.\,b_{\chi\epsilon}\,.\,A^{\epsilon}_{\alpha\beta}\,.\,u_2^{\beta} = \frac{1}{2}\,.\,\frac{\partial^2 P_1(\mathbf{u}_1)}{\partial u_1^{\chi} \partial u_1^{\alpha}}$$

A condensed writing of this second relation is strongly depending on the properties of P_1, [B] and A; therefore, it will be left under that formalism. But if in particular the cube A is anti-symmetric, then one may write:

$$k\,.\,[B]\,.\,{}_A\Phi(\mathbf{u}_2) = \frac{1}{2}\,.\,[Hess_{(\mathbf{u}_1,\,0)} P_1(\mathbf{u}_1)]$$

One can also remark the existence of a subset of situations related to the vanishing of k:

$$k = 0 \Rightarrow [Hess_{(\mathbf{u}_1,\,0)} P_1(\mathbf{u}_1)] = [0]$$

... and also:

$$\mathbf{Grad}_{\mathbf{u}_1} P_1(\mathbf{u}_1) = [B]\,.\,|\mathbf{Z} >= \mathbf{constant}\,(\textit{not depending on}\,\mathbf{u}_1)$$

2.6 Consequences for the variations of the target

Concerning the target, these situations are such that:

$$< k\,.\,\mathbf{u}_2,\, | \otimes_A (\mathbf{u}_1,\, k\,.\,\mathbf{u}_2) > - \{[Q]\,.\,|k\,.\,\mathbf{u}_2 > + |\mathbf{Z} >\} >_{[B]}$$

$$=$$

$$< \mathbf{Grad}_{k\,.\,\mathbf{u}_2} P_2(k\,.\,\mathbf{u}_2),\, k\,.\,\mathbf{u}_2 >_{Id} + \frac{1}{2}\,.\, < k\,.\,\mathbf{u}_2,\, [Hess_{(k\,.\,\mathbf{u}_2,\,0)} P_2(k\,.\,\mathbf{u}_2)]\,.\,|k\,.\,\mathbf{u}_2 >>_{Id}$$

They imply:

$$|\mathbf{Grad}_{k\,.\,\mathbf{u}_2} P_2(k\,.\,\mathbf{u}_2) >= -[B]\,.\,|\mathbf{Z} >$$

And:

$$\frac{1}{2}\,.\,[Hess_{(k\,.\,\mathbf{u}_2,\,0)} P_2(k\,.\,\mathbf{u}_2)] = [B]\,.\,\{{}_A\Phi(\mathbf{u}_1) - [Q]\}$$

Here, one starts getting interesting results if (i) the polynomial P_2 is known, (ii) the bi-linear form [B] is not degenerated and (iii) k $\neq$ 0; more precisely:

$$[Q] = {}_A\Phi(\mathbf{u}_1) - \frac{1}{2}\,.\,[B]^{-1}\,.\,[Hess_{(k\,.\,\mathbf{u}_2,\,0)} P_2(k\,.\,\mathbf{u}_2)]$$

And:

$$|\mathbf{Z} >= -[B]^{-1}\,.\,|\mathbf{Grad}_{k\,.\,\mathbf{u}_2} P_2(k\,.\,\mathbf{u}_2) >$$

But this procedure would be incomplete if one would not add the veiled constraints accompanying it.

2.7 The veiled constraints

The veiled constraints concern the link between a gradient and the Hessian that can be obtained with it; concretely, let denote the presumed decomposition with:

$$|\mathbf{D}>= [Q]\,.\,|k\,.\,\mathbf{u}_2> \,+|\mathbf{Z}>$$

1. Concerning the polynomial P_1 depending on the projectile; let consider the components of the gradient:

$$\frac{\partial P_1(\mathbf{u}_1)}{\partial u_1^\alpha} \,=\, -b_{\alpha\epsilon}\,.\,D^\epsilon$$

Let calculate their partial derivations by respect for the components of the projectile:

$$\frac{\partial^2 P_1(\mathbf{u}_1)}{\partial u_1^\chi \partial u_1^\alpha} \,=\, -\frac{\partial b_{\alpha\epsilon}}{\partial u_1^\chi}\,.\,D^\epsilon \,-\, b_{\alpha\epsilon}\,.\,\frac{\partial D^\epsilon}{\partial u_1^\chi}$$

Let confront this relation with the entries of the Hessian arising from the extrinsic method and get the first veiled constraint:

$$2\,.\,k\,.\,b_{\chi\epsilon}\,.\,A^\epsilon_{\alpha\beta}\,.\,u_2^\beta \,+\, \frac{\partial b_{\alpha\epsilon}}{\partial u_1^\chi}\,.\,D^\epsilon \,+\, b_{\alpha\epsilon}\,.\,\frac{\partial D^\epsilon}{\partial u_1^\chi} \,=\, 0$$

This constraint is a set of differential equations depending on the components of the presumed decomposition and one might formally solve them. Nevertheless, even if one would effectively find the components of the presumed decomposition in following that way, an incertitude would remain on the possible pairs ([Q], **Z**).

2. Concerning the polynomial P_2 depending on the target; let consider the components of the gradient:

$$\frac{\partial P_2(k\,.\,\mathbf{u}_2)}{\partial(\,.\,u_2^\alpha)} \,=\, -b_{\alpha\epsilon}\,.\,Z^\epsilon$$

Following the same vein as previously, let calculate their partial derivations by respect for the components of the target:

$$\frac{\partial^2 P_2(k\,.\,\mathbf{u}_2)}{\partial(k\,.\,u_2^\chi)\partial(k\,.\,u_2^\alpha)} \,=\, -\frac{\partial b_{\alpha\epsilon}}{\partial(k\,.\,u_2^\chi)}\,.\,Z^\epsilon \,-\, b_{\alpha\epsilon}\,.\,\frac{\partial Z^\epsilon}{\partial(k\,.\,u_2^\chi)}$$

Let confront this relation with the entries of the Hessian arising from the extrinsic method and get the second veiled constraint:

$$2\,.\,b_{\alpha\epsilon}\,.\,(A^\epsilon_{\beta\chi}\,.\,u_1^\beta \,-\, Q_{\epsilon\chi}) \,+\, \frac{\partial b_{\alpha\epsilon}}{\partial(k\,.\,u_2^\chi)}\,.\,Z^\epsilon \,+\, b_{\alpha\epsilon}\,.\,\frac{\partial Z^\epsilon}{\partial(k\,.\,u_2^\chi)} \,=\, 0$$

Example 2.1. *When the bi-linear form represented by the matrix [B] does not depend on $(k.\boldsymbol{u}_2)$.*

In that case, the veiled constraint on the target is:

$$b_{\alpha\epsilon} \,.\, \{2 \,.\, (A^{\epsilon}_{\beta\chi} \,.\, u_1^{\beta} \,-\, Q_{\epsilon\chi}) \,+\, \frac{\partial Z^{\epsilon}}{\partial(k \,.\, u_2^{\chi})}\} \,=\, 0$$

Let multiply by $(\mathrm{k.du}^{\chi}{}_2)$:

$$k \,.\, \{b_{\alpha\epsilon} \,.\, \{2 \,.\, (A^{\epsilon}_{\beta\chi} \,.\, u_1^{\beta} \,-\, Q_{\epsilon\chi}) \,+\, \frac{\partial Z^{\epsilon}}{\partial(k \,.\, u_2^{\chi})}\}\} \,.\, du_2^{\chi} \,=\, 0$$

... and then sum over χ to get:

$$[B] \,.\, \{2 \,.\, k \,.\, \{{}_{A}\Phi(\mathbf{u}_1) \,-\, [Q]\} \,.\, |d\mathbf{u}_2> + |d\mathbf{Z}>\} \,=\, |\mathbf{0}>$$

The condition is trivially true whatever the bi-linear form [B] is when:

$$2 \,.\, k \,.\, \{{}_{A}\Phi(\mathbf{u}_1) \,-\, [Q]\} \,.\, |d\mathbf{u}_2> + |d\mathbf{Z}> = |\mathbf{0}>$$

2.8 Formalism of a presumed non-trivial decomposition obtained with the extrinsic method

When the extrinsic method can be applied in a coherent manner, one can write the presumed decomposition as:

$$| \otimes_A (\mathbf{u}_1, \, k \,.\, \mathbf{u}_2) >$$

$$=$$

$${}_{A}\Phi(\mathbf{u}_1) \,.\, |k \,.\, \mathbf{u}_2>$$

$$- \, [B]^{-1} \,.\, \{\frac{1}{2} \,.\, [Hess_{(k \,.\, \mathbf{u}_2, 0)} P_2(k \,.\, \mathbf{u}_2)] \,.\, |k \,.\, \mathbf{u}_2> + |\mathbf{Grad}_{k \,.\, \mathbf{u}_2} P_2(k \,.\, \mathbf{u}_2)>\}$$

This formulation:

1. is only useful when the polynomial P_2 and the non-degenerated bi-linear form [B] are known. If the former is not known, one might prefer another formulation involving the pair ([B], $\mathbf{Z}$) and its variations; see below for more technical details.

2. differs obviously from the simplest decomposition without residual part. Indeed, in absence of constraints, any deformed tensor product can be decomposed in a simple way without residual part as:

$$| \otimes_A (\mathbf{u}_1, \, k \,.\, \mathbf{u}_2) > = {}_{A}\Phi(\mathbf{u}_1) \,.\, |k \,.\, \mathbf{u}_2>$$

   Therefore, the extrinsic method gives rise to a difference:

$$- \, [B]^{-1} \,.\, \{\frac{1}{2} \,.\, [Hess_{(k \,.\, \mathbf{u}_2, 0)} P_2(k \,.\, \mathbf{u}_2)] \,.\, |k \,.\, \mathbf{u}_2> + |\mathbf{Grad}_{k \,.\, \mathbf{u}_2} P_2(k \,.\, \mathbf{u}_2)>\}$$

The essence of the extrinsic method lies in the belief that external circumstances (for examples physical circumstances) command how some mathematical operations must be realized. The whole theory of the (E) question - and this document in particular- applies this principle to tensor products which have been deformed by some cube. This choice should be understood as a pedagogical example.

Within this context, one may ask - at least at a formal level- if there exists a link between, on one side the deformation induced by the cube A and, on the other side, the polynomials P_1 and P_2. For a given deformed tensor product of the type which is studied here, the previous result indicates that one gets a non-trivial decomposition in involving the pair (P_2, [B]). Conversely, this result does not say if the nature acts in the same way than the extrinsic method does.

2.9 Characteristics of a non-trivial decomposition equivalent to the simplest decomposition without residual part

This subsection is motivated by an underlying question: "Does a non-trivial decomposition effectively exist? If yes: when?" And this question is itself justified by the fact that there is apparently nothing more natural than the simplest decomposition.

Definition 2.7. *Non-trivial decomposition equivalent to the simplest decomposition without residual part.*

A non-trivial decomposition equivalent to the simplest decomposition without residual part is characterized by the vanishing of the difference between both types of decomposition.

Remark 2.2. *Sufficient condition characterizing a non-trivial decomposition equivalent to the simplest decomposition without residual part.*

For a non-trivial decomposition to be equivalent to the the simplest decomposition without residual part, it is sufficient to verify the relation:

$$\frac{1}{2} \, . \, [Hess_{(k\,.\,\mathbf{u}_2,0)} P_2(k\,.\,\mathbf{u}_2)] \, . \, |k\,.\,\mathbf{u}_2> + |\mathbf{Grad}_{k\,.\,\mathbf{u}_2} P_2(k\,.\,\mathbf{u}_2)> = |\mathbf{0}>$$

This condition describes a set of polynomials with specific characteristics.

Proposition 2.1. *The condition insuring that a non-trivial decomposition resembles the simplest decomposition without residual part can be reformulated with the help of ([B],* $\boldsymbol{Z}$*).*

Proof. Here, one is studying the decomposition of:

$$| \otimes_A (\mathbf{u}_1, \, k\,.\,\mathbf{u}_2) >$$

... and the difference is:

$$- [B]^{-1} \, . \, \{ \frac{1}{2} \, . \, [Hess_{(k\,.\,\mathbf{u}_2,0)} P_2(k\,.\,\mathbf{u}_2)] \, . \, |k\,.\,\mathbf{u}_2> + |\mathbf{Grad}_{k\,.\,\mathbf{u}_2} P_2(k\,.\,\mathbf{u}_2)> \}$$

The Hessian at hand, like any Hessian, can be understood as a superposition of gradients. Therefore, the Hessian can be represented by a special type of Pythagorean table:

$$[Hess_{(k\,.\,\mathbf{u}_2,\,0)} P_2(k\,.\,\mathbf{u}_2)]$$
$$=$$
$$T_2(o)(\mathbf{Grad}_{k\,.\,\mathbf{u}_2}, \mathbf{Grad}_{k\,.\,\mathbf{u}_2} P_2(k\,.\,\mathbf{u}_2))$$
$$=$$
$$-T_2(o)(\mathbf{Grad}_{k\,.\,\mathbf{u}_2},\, [B]\,.\,|\mathbf{Z}>)$$

And the condition writes now:

$$\frac{1}{2}\,.\,[B]^{-1}\,.\,T_2(o)(\mathbf{Grad}_{k\,.\,\mathbf{u}_2},\, [B]\,.\,|\mathbf{Z}>)\,.\,|k\,.\,\mathbf{u}_2> \;+ |\mathbf{Z}>= \;|\mathbf{0}>$$

□

Diverse situations must now be envisaged to go further.

Remark 2.3. *When [B] does not depend on (k.$\boldsymbol{u}_2$).*

When, per hypothesis:

$$\forall\,\beta:\;\frac{\partial[B]}{\partial(k\,.\,u_2^\beta)} \;=\; 0$$

Then, the condition has a particular formalism:

$$\frac{1}{2}\,.\,[B]^{-1}\,.\,T_2(o)(\mathbf{Grad}_{k\,.\,\mathbf{u}_2},\, [B]\,.\,|\mathbf{Z}>)\,.\,|k\,.\,\mathbf{u}_2> \;+ |\mathbf{Z}>= \;|\mathbf{0}>$$
$$\downarrow$$
$$\frac{k}{2}\,.\,b^{\alpha\chi}\,.\,\partial_{(k\,.\,u_2^\beta)}(b_{\chi\psi}\,.\,Z^\psi)\,.\,u_2^\beta \;+\; Z^\alpha \;=\; 0$$
$$\downarrow$$
$$\frac{k}{2}\,.\,b^{\alpha\chi}\,.\,b_{\chi\psi}\,.\,\partial_{(k\,.\,u_2^\beta)} Z^\psi\,.\,u_2^\beta \;+\; Z^\alpha \;=\; 0$$
$$\downarrow$$
$$\frac{k}{2}\,.\,\delta^\alpha_\psi\,.\,\partial_{(k\,.\,u_2^\beta)} Z^\psi\,.\,u_2^\beta \;+\; Z^\alpha \;=\; 0$$
$$\downarrow$$
$$\frac{k}{2}\,.\,\partial_{(k\,.\,u_2^\beta)} Z^\alpha\,.\,u_2^\beta \;+\; Z^\alpha \;=\; 0$$
$$\downarrow$$
$$\frac{1}{2}\,.\,T_2(o)(\mathbf{Grad}_{(k\,.\,\mathbf{u}_2)},\, \mathbf{Z})\,.\,|k\,.\,\mathbf{u}_2> \;+ |\mathbf{Z}>= \;|\mathbf{0}>$$

Whatever the solutions of this relation are, the latter is extremely problematic because it signs a lack of precision which is perfectly symbolized by the fact that it can be rewritten in an infinite numbers of ways ... in re-injecting the expression of $\mathbf{Z}$ into the gradient. Let introduce:

$$\mathbf{U} = k \,.\, \mathbf{u}_2$$

The condition insuring the coincidence between a non-trivial decomposition and the simplest one when [B] doesn't depend on $\mathbf{U}$ is equivalent to:

$$\frac{1}{2} \,.\, T_2(o)(\mathbf{Grad_U},\, \mathbf{Z}) \,.\, |\mathbf{U}> + |\mathbf{Z}> = |\mathbf{0}>$$

Proposition 2.2. *The condition insuring the coincidence between a non-trivial decomposition and the simplest one when [B] doesn't depend on* $\boldsymbol{U}$ *can be reformulated as a set of relations depending on the Hessian of each component of the residual part.*

Proof. Since:

$$Z^\alpha = -\frac{1}{2} \,.\, \sum_\gamma \partial_{U^\gamma} Z^\alpha \,.\, U^\gamma$$

The following calculations can be made:

$$-\frac{1}{2} \,.\, \sum_\beta \partial_{U^\beta} \{ \sum_\gamma \partial_{U^\gamma} Z^\alpha \,.\, U^\gamma \} \,.\, U^\beta + 2.Z^\alpha = 0$$

$$\downarrow$$

$$-\frac{1}{2} \,.\, \sum_\beta \sum_\gamma \partial^2_{U^\beta U^\gamma} Z^\alpha \,.\, U^\gamma \,.\, U^\beta - \frac{1}{2} \,.\, \{ \sum_\beta \sum_\gamma \partial_{U^\gamma} Z^\alpha \,.\, \delta^\gamma_\beta \} \,.\, U^\beta + 2.Z^\beta = 0$$

$$\downarrow$$

$$-\frac{1}{2} \,.\, \sum_\beta \sum_\gamma \partial^2_{U^\beta U^\gamma} z^\alpha \,.\, U^\gamma \,.\, U^\beta - \frac{1}{2} \,.\, \sum_\beta \partial_{U^\beta} Z^\alpha \,.\, U^\beta + 2 \,.\, Z^\alpha = 0$$

$$\downarrow$$

$$Z^\alpha - \frac{1}{6} \,.\, \sum_\beta \sum_\gamma \partial^2_{U^\beta U^\gamma} Z^\alpha \,.\, U^\gamma \,.\, U^\beta = 0$$

$$\downarrow$$

$$Z^\alpha = \frac{1}{6} \,.\, <\mathbf{U}| \,.\, \{ [Hess_{(\mathbf{U},0)} Z^\alpha] \,.\, |\mathbf{U}> \}$$

□

Proposition 2.3. *A first set of solutions for these conditions are quadratic forms depending on the components of $\boldsymbol{U}$.*

Proof. Let suppose that each component of the residual part is a polynomial of degree two with coefficient not depending on $\mathbf{U}$:

$$Z^\alpha(\mathbf{U}) = g^\alpha_{\beta\gamma} . U^\beta . U^\gamma = \sum_\beta g^\alpha_{\beta\beta} . (U^\beta)^2 + \sum_{\beta<\gamma} \sum_\gamma (g^\alpha_{\beta\gamma} + g^\alpha_{\gamma\beta}) . U^\beta . U^\gamma$$

A first partial derivation by respect for U^γ is:

$$\frac{\partial Z^\alpha(\mathbf{U})}{\partial U^\gamma} = 2 . g^\alpha_{\gamma\gamma} . U^\gamma + (g^\alpha_{\beta\gamma} + g^\alpha_{\gamma\beta}) . U^\beta$$

And a second partial derivation by respect for U^β is:

$$\frac{\partial^2 Z^\alpha(\mathbf{U})}{\partial U^\beta \partial U^\gamma} = (g^\alpha_{\beta\gamma} + g^\alpha_{\gamma\beta})$$

With this result one gets:

$$\sum_\beta \sum_\gamma \frac{\partial^2 Z^\alpha(\mathbf{U})}{\partial U^\beta \partial U^\gamma} . U^\beta . U^\gamma = \sum_\beta \sum_\gamma (g^\alpha_{\beta\gamma} + g^\alpha_{\gamma\beta}) . U^\beta . U^\gamma = 2 . Z^\alpha(\mathbf{U})$$

Therefore, the conditions characterizing the residual part of a non-trivial decomposition mimicking the simplest one have the following generic solutions:

$$Z^\alpha(\mathbf{U}) = 3 . g^\alpha_{\beta\gamma} . U^\beta . U^\gamma, \; g^\alpha_{\beta\gamma} = constant$$

They are quadratic forms (one for each component of the residual part) with constant coefficients depending on the components of $\mathbf{U}$. □

Lemma 2.1. *When the bi-linear form represented by the square matrix [B] doesn't depend on $\boldsymbol{U} = k.\boldsymbol{u}_2$ (k is invariant), the deformed tensor product at hand has eventually a non-trivial decomposition but the latter is finally equal to the simplest one when there exists a cube G of which the knots don't depend on the components of $\boldsymbol{U}$ such that:*

$$\mathbf{Z} = 3 . \otimes_G (\mathbf{U}, \mathbf{U})$$

One may remark here the similarity between the formalism of these residual parts and the formalism of the so-called *gravitational term* characterizing the co-variance of the Lorentz law; see below. In these circumstances:

$$| \otimes_A (\mathbf{u}_1, \mathbf{U}) >= {}_A\Phi(\mathbf{u}_1) . |\mathbf{U} >$$

Although the main part of the decomposition is theoretically:

$$[Q] = {}_A\Phi(\mathbf{u}_1) + \frac{1}{2} . T_2(o)(\mathbf{Grad_U}, \mathbf{Z})$$

Furthermore, the veiled constraint which has been obtained in example 2.1 for the same circumstances is automatically verified.

Remark 2.4. *Formalism of an effective non-trivial decomposition obtained with the help of the extrinsic method.*

When the non-trivial decomposition cannot be identified with the simplest one, the extrinsic method suggests to write:

$$| \otimes_A (\mathbf{u}_1, \mathbf{U}) >$$

$$=$$

$$\underbrace{\{_A\Phi(\mathbf{u}_1) + \frac{1}{2} \,.\, [B]^{-1} \,.\, T_2(o)(\mathbf{Grad_U}, [B] \,.\, |\mathbf{Z} >)\}}_{=[Q]} \,.\, |\mathbf{U} > + |\mathbf{Z} >$$

$$=$$

$$|\mathbf{D} >$$

... and there is no reason to think that the non trivial part of the decomposition **D** vanishes. In opposition, one should write in general:

$$\frac{1}{2} \,.\, [B]^{-1} \,.\, T_2(o)(\mathbf{Grad_U}, [B] \,.\, |\mathbf{Z} >) + |\mathbf{Z} > = |\theta >$$

This relation can be transposed in the language of components:

$$\frac{1}{2} \,.\, b^{\alpha\chi} \,.\, \partial_{U^\beta}(b_{\chi\psi} \,.\, Z^\psi) \,.\, U^\beta + Z^\alpha = \theta^\alpha$$

$$\downarrow$$

$$\frac{1}{2} \,.\, b^{\alpha\chi} \,.\, \{\partial_{U^\beta} b_{\chi\psi} \,.\, Z^\psi + b_{\chi\psi} \,.\, \partial_{U^\beta} Z^\psi\} \,.\, U^\beta + Z^\alpha = \theta^\alpha$$

$$\downarrow$$

$$\frac{1}{2} \,.\, b^{\alpha\chi} \,.\, \partial_{U^\beta} b_{\chi\psi} \,.\, Z^\psi \,.\, U^\beta + \frac{1}{2} \,.\, \underbrace{b^{\alpha\chi} \,.\, b_{\chi\psi}}_{=\delta^\alpha_\psi} \,.\, \partial_{U^\beta} Z^\psi \,.\, U^\beta + Z^\alpha = \theta^\alpha$$

$$\downarrow$$

$$\frac{1}{2} \,.\, b^{\alpha\chi} \,.\, \partial_{U^\beta} b_{\chi\psi} \,.\, Z^\psi \,.\, U^\beta + \underbrace{\frac{1}{2} \,.\, \partial_{U^\beta} Z^\alpha \,.\, U^\beta + Z^\alpha}_{=\theta^\alpha_0} = \theta^\alpha$$

The formalism clearly exhibits two parts: (i) a first one, denoted ${\theta^\alpha}_0$, is the component of the non-trivial part of **D** when the bi-linear form [B] does not depend on the components of **U** and (ii) a second one is the contribution to the component of the non-trivial part of **D** related to a modification of [B] by respect for the components of **U**.

2.10 A natural link between a non-trivial decomposition and a co-variant derivation

In a canonical basis Ω, let consider the vector $\mathbf{u}_1 \equiv (\mathrm{u}^0, \mathrm{u}^1, ...)$ and a connection C; [1]. The components of the co-variant derivation of this contra-variant vector in this connection are per definition:

$$\nabla_\alpha u_1^\beta \,=\, \partial_\alpha u_1^\beta \,+\, C^\beta_{\rho\alpha} \,.\, u_1^\rho$$

Let now suppose that this contra-variant vector is the gradient of some function f($\mathbf{q}$) by respect for the components of the vector $\mathbf{q}$; per convention, this fact can be written as:

$$\forall \beta \,:\, u_1^\beta \,=\, \partial_\beta f(\mathbf{q}) \,=\, \frac{\partial f(\mathbf{q})}{\partial q^\beta} \iff \mathbf{u}_1 \,=\, \partial_{\mathbf{q}} f(\mathbf{q}) \,=\, \mathbf{Grad}_{\mathbf{q}} f(\mathbf{q})$$

Let now inject these specific components into the components of the co-variant derivation of this vector:

$$\nabla_\alpha \partial_\beta f(\mathbf{q}) \,=\, \partial_\alpha \partial_\beta f(\mathbf{q}) \,+\, C^\beta_{\rho\alpha} \,.\, \partial_\rho f(\mathbf{q})$$

At this stage, let introduce a non-degenerated bi-linear form [B]:

$$b^{\chi\alpha} \,.\, \nabla_\alpha \partial_\beta f(\mathbf{q}) \,=\, b^{\chi\alpha} \,.\, \partial_\alpha \partial_\beta f\mathbf{q}) \,+\, b^{\chi\alpha} \,.\, C^\beta_{\rho\alpha} \,.\, \partial_\rho f(\mathbf{q})$$

Nothing forbids the definition of a new cube A such that:

$$b^{\chi\alpha} \,.\, C^\beta_{\rho\alpha} \,=\, -A^\chi_{\rho\beta}$$

One can also write:

$$A^\chi_{\rho\beta} \,.\, \partial_\rho f(\mathbf{q}) \,=\, A^\chi_{\rho\beta} \,.\, u_1^\rho \,=\, {}_A\Phi_{\chi\beta}(\mathbf{u}_1) \,=\, {}_A\Phi_{\chi\beta}(\partial_{\mathbf{q}} f(\mathbf{q}))$$

The relations can be condensed into:

$$-[B]^{-1} \,.\, [\nabla_\alpha u_1^\beta] \,=\, {}_A\Phi(\mathbf{Grad}_{\mathbf{q}} f(\mathbf{q})) \,-\, [B]^{-1} \,.\, [Hess_{(\mathbf{q},\,0)} f(\mathbf{q})]$$

The r.h.s is the main part of a non-trivial decomposition which would have been obtained ... for the deformed tensor product:

$$|\otimes_A (\mathbf{u}_1,\, \mathbf{q}) >= \,-[B]^{-1} \,.\, T_2(o)(\nabla_{\mathbf{q}},\, \mathbf{u}_1) \,.\, |\mathbf{q} > \,+ |\mathbf{Z} >$$

... with the help of the extrinsic method when:

$$P_2(\mathbf{q}) \,=\, f(\mathbf{q}),\, \mathbf{u}_1 \,=\, \mathbf{Grad}_{\mathbf{q}} f(\mathbf{q}) \,=\, \mathbf{Grad}_{\mathbf{q}} P_2(\mathbf{q})$$

At this stage, one may remark that:

[1] The letter C has here nothing to do with the notation $\mathbb{C}$ representing the complex numbers.

- The pedagogical example which has been introduced in subsection 2.3 is (recall):

$$k \,.\, |\otimes_A (\mathbf{u}_1,\, \mathbf{u}_2) >= \,k \,.\, [Q] \,.\, |\mathbf{u}_2 > \,+ |\mathbf{Z}>$$

  The previous result can be applied to it when:

$$\mathbf{q} \,=\, k \,.\, \mathbf{u}_2 \,=\, \mathbf{U};\, \mathbf{u}_1 \,=\, \mathbf{Grad_U} P_2(\mathbf{U})$$

  In particular, if it happens that:

$$\mathbf{u}_1 \,=\, \mathbf{U}$$

  Then, the main part of a decomposition is effectively related to the concept of co-variance if one can write:

$$\mathbf{U} \,=\, \mathbf{Grad_U} f(\mathbf{U})$$

  This condition imposes strange constraints to the vector $\mathbf{U}$ because it must be the gradient of some function f depending on its components by respect for these components. For example, in a one-dimensional space, this constraint is equivalent to:

$$U \,=\, \frac{\partial f(U)}{\partial U} \,\Rightarrow\, f(U) \,=\, \frac{1}{2} \,.\, U^2 \,+\, constant$$

  If U represents a speed, this relation is a kind of reminiscence for the kinetic energy per unit of mass of some particle of which the mass (m = 1) does not depend on its speed (the classical case).

- Future developments are possible when f is a continuous function in $\mathbf{q}$:

$$-[\nabla_\alpha U_\beta] \,=\, [B] \,.\, {}_A\Phi(\mathbf{U}^*) \,-\, [Hess_{(\mathbf{q},\,0)} f(\mathbf{q})]$$

$$-[\nabla_\alpha T_\beta]^t \,=\, {}_A\Phi(\mathbf{U}^*)^t \,.\, [B]^t \,-\, [Hess_{(\mathbf{q},\,0)} f(\mathbf{q})]^t$$

$$[Hess_{(\mathbf{q},\,0)} f(\mathbf{q})]^t \,=\, [Hess_{(\mathbf{q},\,0)} f(\mathbf{q})]$$

$$[\nabla_\alpha U_\beta]^t \,-\, [\nabla_\alpha U_\beta] \,=\, [B] \,.\, {}_A\Phi(\mathbf{U}^*) \,-\, {}_A\Phi(\mathbf{U}^*)^t \,.\, [B]^t$$

  This kind of relation appears again a little bit later in subsection 3.6.

3 The Lorentz Einstein Law and the extrinsic method

3.1 Why the extrinsic method can be applied to the Lorentz Einstein Law

Up to now, the discussion concerns elements in $\mathrm{V}_4 = \{\mathrm{E}(\mathrm{D}=4, \mathbb{R}), \otimes_{\Gamma(2)}\}$ and attention will be focused on the analysis of the Lorentz-Einstein law for non-mass-less particles (m $\neq$ 0) :

$$|\frac{d\mathbf{u}}{ds} > \,+ |\otimes_{\Gamma(2)} (\mathbf{u},\, \mathbf{u}) >= \,\frac{q}{m \,.\, c^2} \,.\, [F(\uparrow,\, \downarrow)] \,.\, |\mathbf{u}>$$

Proposition 3.1. *The Lorentz-Einstein law is equivalent to a decomposition of the gravitational term and this decomposition is in the family of the generic and pedagogical deformed tensor product (recall):*

$$| \otimes_A (\mathbf{u}_1, k \,.\, \mathbf{u}_2) >= [Q] \,.\, |k \,.\, \mathbf{u}_2 > + |\mathbf{Z} >$$

Proof. When the mass of the particle at hand is not null ($m \neq 0$), the Lorentz-Einstein law can be rewritten as:

$$| \otimes_{\Gamma(2)} (\mathbf{u}, \mathbf{u}) >= \frac{q}{m \,.\, c^2} \,.\, [F(\uparrow, \downarrow)] \,.\, |\mathbf{u} > - |\frac{d\mathbf{u}}{ds} >$$

In multiplying this expression by an invariant k, it is also:

$$| \otimes_{\Gamma(2)} (\mathbf{u}, k \,.\, \mathbf{u}) >= \frac{q}{m \,.\, c^2} \,.\, [F(\uparrow, \downarrow)] \,.\, |k \,.\, \mathbf{u} > - k \,.\, |\frac{d\mathbf{u}}{ds} >$$

Hence, provided one introduces the following identifications:

$$\Gamma(2) = A, \; \mathbf{u} = \mathbf{u}_1 = \mathbf{u}_2, \; [Q] = \frac{q}{m \,.\, c^2} \,.\, [F(\uparrow, \downarrow)], \; \mathbf{Z} = -k \,.\, \frac{d\mathbf{u}}{ds}$$

... one can affirm that the study of k times the co-variant version of the Lorentz law is equivalent to the study of the generic example introduced in subsection 2.3. □

Therefore, the results of previous subsections can now be involved to decompose a vector which is k times the gravitational term. Let comment the definitions:

- The cube A contains all Christoffel's symbols of the second kind [02].
- The parameter "s" is referring to a curvilinear abscissa.
- $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}$ is the four-dimensional speed (u^0, u^1, u^2, u^3) of some event in V_4.

Q is a matrix representing the electromagnetic field.

- **Z** represents minus k times an acceleration. Therefore, within a discussion related to the decomposition of the gravitational term, the vanishing of the residual part **Z** corresponds to an invariant four-speed.

3.2 The Lorentz-Einstein law as non-trivial decomposition of the gravitational term

Considering the subsection 2.8, one may think that the Lorentz-Einstein law is a representation for a decomposition of the gravitational term which has been obtained when a non-degenerated bi-linear form [B] and a polynomial $P_2(k \,.\, \mathbf{u})$ are known. The extrinsic method yields in general:

$$[Q] = {}_A\Phi(\mathbf{u}_1) + \frac{1}{2} \,.\, [B]^{-1} \,.\, T_2(o)(\mathbf{Grad}_{(k \,.\, \mathbf{u}_2)}, [B] \,.\, |\mathbf{Z} >)$$

...

$$|\mathbf{Z}> = -[B]^{-1} . |\mathbf{Grad}_{(k\,.\,\mathbf{u}_2)} P_2(k\,.\,\mathbf{u}_2)>$$

Here, these relations have now a precise visage:

$$[Q]$$

$$=$$

$$_{\Gamma(2)}\Phi(\mathbf{u}) - \frac{1}{2} . [B]^{-1} . [Hess_{(k\,.\,\mathbf{u},\,0)} P_2(k\,.\,\mathbf{u})]$$

$$=$$

$$\frac{q}{m\,.\,c^2} . [F(\uparrow, \downarrow)]$$

$$=$$

$$\frac{q}{m\,.\,c^2} . [G]^{-1} . [F(2,\,0)]$$

And:

$$|\mathbf{Z}> = -[B]^{-1} . |\mathbf{Grad}_{k\,.\,\mathbf{u}} P_2(k\,.\,\mathbf{u})> = -k . \frac{d\mathbf{u}}{ds}$$

They offer a new visage for the (2, 0) representation of the electromagnetic field:

$$\frac{q}{m\,.\,c^2} . [F(2,\,0)] = [G] . _{\Gamma(2)}\Phi(\mathbf{u}) - \frac{1}{2} . [G] . [B]^{-1} . [Hess_{(k\,.\,\mathbf{u},\,0)} P_2(k\,.\,\mathbf{u})]$$

3.3 The Lorentz-Einstein law and the first veiled constraint

At this stage attention has not yet been given to the first veiled constraint and this should be done; recall that:

$$2 . k . b_{\chi\epsilon} . A^{\epsilon}_{\alpha\beta} . u^{\beta}_2 + \frac{\partial b_{\alpha\epsilon}}{\partial u^{\chi}_1} . D^{\epsilon} + b_{\alpha\epsilon} . \frac{\partial D^{\epsilon}}{\partial u^{\chi}_1} = 0$$

Here, more precisely:

$$2 . k . b_{\chi\epsilon} . \Gamma(2)^{\epsilon}_{\alpha\beta} . u^{\beta} + \frac{\partial b_{\alpha\epsilon}}{\partial u^{\chi}} . D^{\epsilon} + b_{\alpha\epsilon} . \frac{\partial D^{\epsilon}}{\partial u^{\chi}} = 0$$

With [02]:

$$\Gamma(2)^{\epsilon}_{\alpha\beta} = \Gamma(2)^{\epsilon}_{\beta\alpha}$$

And:

$$\mathbf{D}$$

$$=$$

$$[Q] . |k\,.\,\mathbf{u}> + |\mathbf{Z}>$$

$$=$$

$$k . \{\frac{q}{m\,.\,c^2} . [F(\uparrow, \downarrow)] . |\mathbf{u}> - |\frac{d\mathbf{u}}{ds}>\}$$

3.4 The Lorentz-Einstein law and the second veiled constraint

At this stage attention has not yet been given to the second veiled constraint and this should be done; recall that:

$$2 \,.\, b_{\alpha\epsilon} \,.\, (A^{\epsilon}_{\beta\chi} \,.\, u_1^{\beta} - Q_{\epsilon\chi}) + \frac{\partial b_{\alpha\epsilon}}{\partial(k \,.\, u_2^{\chi})} \,.\, Z^{\epsilon} + b_{\alpha\epsilon} \,.\, \frac{\partial Z^{\epsilon}}{\partial(k \,.\, u_2^{\chi})} = 0$$

Here:

$$2 \,.\, b_{\alpha\epsilon} \,.\, (\Gamma(2)^{\epsilon}_{\beta\chi} \,.\, u^{\beta} - Q_{\epsilon\chi}) - k \,.\, \frac{\partial b_{\alpha\epsilon}}{\partial(k \,.\, u^{\chi})} \,.\, \frac{du^{\epsilon}}{ds} - k \,.\, b_{\alpha\epsilon} \,.\, \frac{\partial \frac{du^{\epsilon}}{ds}}{\partial(k \,.\, u^{\chi})} = 0$$

3.5 The problematic choice of a non-degenerated bi-linear form for the Lorentz-Einstein law

There is only a small number of indications concerning the choice of [B]:

1. The first one is resulting from the well-accepted fact that the Lorentz transformations $[\Lambda]$ modify the electromagnetic fields:

   $$[F'(0,\, 2)](\mathbf{x}') = [\Lambda] \,.\, [F(0,\, 2)](\mathbf{x}) \,.\, [\Lambda]^t$$

   It is also known that the Lorentz transformations preserve the metric:

   $$[\hat{\Lambda}] \,.\, [G] \,.\, [\Lambda] = [G]$$

   Here, the extrinsic method suggests the formalism:

   $$\frac{q}{m \,.\, c^2} \,.\, [F(2,\, 0)] = [G] \,.\, {}_{\Gamma(2)}\Phi(\mathbf{u}) - \frac{1}{2} \,.\, [G] \,.\, [B]^{-1} \,.\, [Hess_{(k \,.\, \mathbf{u},\, 0)} P_2(k \,.\, \mathbf{u})]$$

   In another frame, one should have similarly:

   $$\frac{q}{m \,.\, c^2} \,.\, [F(2,\, 0)]' = [G]' \,.\, {}_{\Gamma(2)'}\Phi(\mathbf{u}') - \frac{1}{2} \,.\, [G]' \,.\, [B']^{-1} \,.\, [Hess_{(k' \,.\, \mathbf{u}',\, 0)} P_2'(k' \,.\, \mathbf{u}')]$$

   These facts should help finding [B].

2. The second indication helping choosing [B] comes from Cartan's work on metrics related to Hessian matrices [12].

3.6 The simplest decomposition without residual part of the gravitational term and the bi-vectors "à la Cartan"

Warning: as long as the matrix [B] has not been precisely discovered, what follows is pure speculation or a simple exercise.

Proposition 3.2. *Provided two conditions are realized:*

- *The non-degenerated bi-linear form [B] coincides with the metric [G]:*

  $$[B] = [G]$$

- *The polynomial P_2 is smooth and continuous for each speed* $\mathbf{u}$*:*

$$[Hess_{(k\,.\,\mathbf{u},0)}P_2(k\,.\,\mathbf{u})] \,=\, [Hess_{(k\,.\,\mathbf{u},0)}P_2(k\,.\,\mathbf{u})]^t$$

... the treatment of the Lorentz-Einstein law with the extrinsic method gives a specific formalism for the electromagnetic field:

$$\frac{q}{c^2}\,.\,[F_{\alpha\beta}] \,=\, \frac{1}{2}\,.\,\{[G]\,.\,{}_{\Gamma(2)}\Phi(\mathbf{p}) \,-\, {}_{\Gamma(2)}\Phi^t(\mathbf{p})\,.\,[G]^t\}$$

... and there exist mathematical configurations for which the formalism resulting from the treatment of the Lorentz-Einstein law with the extrinsic method is an infinitesimal variation of the metric tensor [03; §172, pp. 145-146]:

$$\delta[G] \,=\, \frac{1}{2}\,.\,\{[G]\,.\,{}_{\Gamma(2)}\Phi(\mathbf{p}) \,-\, {}_{\Gamma(2)}\Phi(\mathbf{p})\,.\,[G]\}$$

Remark 3.1. *Preliminaries*

A comparison between the expected formalism and the formalism obtained with the extrinsic method yields:

$$\delta[G] \,=\, 2q\,.\,[F_{\alpha\beta}] \,+\, \{{}_{\Gamma(2)}\Phi^t(\mathbf{p})\,.\,[G]^t \,-\, {}_{\Gamma(2)}\Phi(\mathbf{p})\,.\,[G]\}$$

Therefore, the proposition can only be validated when:

- Either the simplest decomposition without residual part and the metric are symmetric:
$$ {}_{\Gamma(2)}\Phi(\mathbf{p}) \,=\, {}_{\Gamma(2)}\Phi^t(\mathbf{p}),\; [G] \,=\, [G]^t$$

- Or the simplest decomposition without residual part and the metric are anti-symmetric:
$$ {}_{\Gamma(2)}\Phi(\mathbf{p}) \,=\, -{}_{\Gamma(2)}\Phi^t(\mathbf{p}),\; [G] \,=\, -[G]^t$$

Proof. Within the theory of spinors [03], each element ${}^{(5)}\mathbf{X}$ in $\mathrm{E}(5,\mathbb{R})$ can be represented in $\mathrm{M}(4,\mathbb{R})$ [03 ; §93, pp. 81-82]:

$$\mathbf{X}_\alpha \;:\; (x^0_\alpha,\, x^1_\alpha,\, x^2_\alpha,\, x'^1_\alpha,\, x'^2_\alpha) \,\in\, E(5,\, R) \;:\; [C(\mathbf{X}_\alpha)] \,=\, \begin{bmatrix} x^0_\alpha & x^1_\alpha & x^2_\alpha & 0 \\ x'^1_\alpha & -x^0_\alpha & 0 & x^2_\alpha \\ x'^2_\alpha & 0 & -x^0_\alpha & -x^1_\alpha \\ 0 & x'^2_\alpha & -x'^1_\alpha & x^0_\alpha \end{bmatrix}$$

And each pair $(\mathbf{X}_1,\, \mathbf{X}_2)$ of elements in $\mathrm{E}(5,\mathbb{R})$ has a representation [03 ; §95, p. 83]:

$$[C(\mathbf{X}_1,\, \mathbf{X}_2)] \,=\, \frac{1}{2}\,.\,\{[C(\mathbf{X}_1)]\,.\,[C(\mathbf{X}_2)] \,-\, [C(\mathbf{X}_2)]\,.\,[C(\mathbf{X}_1)]\}$$

The approach proposed in [03] can be applied without reduction of the generality to elements in $\mathrm{E}(4,\mathbb{R})$; for that purpose, it is enough to write $\mathrm{x}^0 = 0$.

In that context, one can easily verify that:

$$[C(\mathbf{X}_1)]\,.\,[C(\mathbf{X}_2)]$$

$$=$$

$$\begin{bmatrix} (x_1^1\,.\,x_2'^1 \,+\, x_1^2\,.\,x_2'^2) & 0 & 0 & (x_1^1\,.\,x_2^2 \,-\, x_1^2\,.\,x_2^1) \\ 0 & (x_1'^1\,.\,x_2^1 \,+\, x_1^2\,.\,x_2'^2) & (x_1'^1\,.\,x_2^2 \,-\, x_1^2\,.\,x_2'^1) & 0 \\ 0 & (x_1'^2\,.\,x_2^1 \,-\, x_1^1\,.\,x_2'^2) & (x_1^1\,.\,x_2'^1 \,+\, x_1'^2\,.\,x_2^2) & 0 \\ (x_1'^2\,.\,x_2'^1 \,-\, x_1'^1\,.\,x_2'^2) & 0 & 0 & (x_1'^1\,.\,x_2^1 \,+\, x_1'^2\,.\,x_2^2) \end{bmatrix}$$

And, in inverting the subscripts 1 and 2, that:

$$[C(\mathbf{X}_2)]\,.\,[C(\mathbf{X}_1)]$$

$$=$$

$$\begin{bmatrix} (x_2^1\,.\,x_1'^1 \,+\, x_2^2\,.\,x_1'^2) & 0 & 0 & (x_2^1\,.\,x_1^2 \,-\, x_2^2\,.\,x_1^1) \\ 0 & (x_2'^1\,.\,x_1^1 \,+\, x_2^2\,.\,x_1'^2) & (x_2'^1\,.\,x_1^2 \,-\, x_2^2\,.\,x_1'^1) & 0 \\ 0 & (x_2'^2\,.\,x_1^1 \,-\, x_2^1\,.\,x_1'^2) & (x_2^1\,.\,x_1'^1 \,+\, x_2'^2\,.\,x_1^2) & 0 \\ (x_2'^2\,.\,x_1'^1 \,-\, x_2'^1\,.\,x_1'^2) & 0 & 0 & (x_2'^1\,.\,x_1^1 \,+\, x_2'^2\,.\,x_1^2) \end{bmatrix}$$

If one wants to prove the proposition in following the approach developed in [03; §172, pp. 145-146], one must discover realistic circumstances for which the simplest decomposition without residual part is a bi-vector:

$$[C(\mathbf{X}_1)]\,.\,[C(\mathbf{X}_2)] \;=\; {}_{\Gamma(2)}\Phi(\mathbf{p})$$

In details:

$$\begin{bmatrix} (x_1^1\,.\,x_2'^1 \,+\, x_1^2\,.\,x_2'^2) & 0 & 0 & (x_1^1\,.\,x_2^2 \,-\, x_1^2\,.\,x_2^1) \\ 0 & (x_1'^1\,.\,x_2^1 \,+\, x_1^2\,.\,x_2'^2) & (x_1'^1\,.\,x_2^2 \,-\, x_1^2\,.\,x_2'^1) & 0 \\ 0 & (x_1'^2\,.\,x_2^1 \,-\, x_1^1\,.\,x_2'^2) & (x_1^1\,.\,x_2'^1 \,+\, x_1'^2\,.\,x_2^2) & 0 \\ (x_1'^2\,.\,x_2'^1 \,-\, x_1'^1\,.\,x_2'^2) & 0 & 0 & (x_1'^1\,.\,x_2^1 \,+\, x_1'^2\,.\,x_2^2) \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \Gamma^0_{\mu 0}\cdot p^\mu & \Gamma^0_{\mu 1}\cdot p^\mu & \Gamma^0_{\mu 2}\cdot p^\mu & \Gamma^0_{\mu 3}\cdot p^\mu \\ \Gamma^1_{\mu 0}\cdot p^\mu & \Gamma^1_{\mu 1}\cdot p^\mu & \Gamma^1_{\mu 2}\cdot p^\mu & \Gamma^1_{\mu 3}\cdot p^\mu \\ \Gamma^2_{\mu 0}\cdot p^\mu & \Gamma^2_{\mu 1}\cdot p^\mu & \Gamma^2_{\mu 2}\cdot p^\mu & \Gamma^2_{\mu 3}\cdot p^\mu \\ \Gamma^3_{\mu 0}\cdot p^\mu & \Gamma^3_{\mu 1}\cdot p^\mu & \Gamma^3_{\mu 2}\cdot p^\mu & \Gamma^3_{\mu 3}\cdot p^\mu \end{bmatrix}$$

If this equality would be true, then one would automatically remark that:

$$\Gamma^0_{\mu 1}\cdot p^\mu \;=\; \Gamma^0_{\mu 2}\cdot p^\mu \;=\; \Gamma^1_{\mu 3}\cdot p^\mu \;=\; \Gamma^2_{\mu 3}\cdot p^\mu$$

$$=$$

$$\Gamma^3_{\mu 2}\cdot p^\mu \;=\; \Gamma^3_{\mu 1}\cdot p^\mu \;=\; \Gamma^2_{\mu 0}\cdot p^\mu \;=\; \Gamma^1_{\mu 0}\cdot p^\mu \;=\; 0$$

Let adopt the conventional writings:

$$\begin{aligned} a &= x_1^1\,.\,x_2'^1 \,+\, x_1^2\,.\,x_2'^2 \\ b &= x_1'^1\,.\,x_2^1 \,+\, x_1'^2\,.\,x_2^2 \\ c &= x_1^1\,.\,x_2'^1 \,+\, x_1'^2\,.\,x_2^2 \\ d &= x_1'^1\,.\,x_2^1 \,+\, x_1^2\,.\,x_2'^2 \end{aligned}$$

These conventions allow:

- ... A rewriting of the non-vanishing entries in the simplest decomposition:

$$\Gamma^0_{\mu 0} \, . \, p^\mu \;=\; -\Gamma^3_{\mu 3} \, . \, p^\mu \;=\; \frac{1}{2} \, . \, (a \,-\, b)$$

$$\Gamma^1_{\mu 1} \, . \, p^\mu \;=\; -\Gamma^2_{\mu 2} \, . \, p^\mu \;=\; \frac{1}{2} \, . \, (d \,-\, c)$$

$$\Gamma^0_{\mu 3} \, . \, p^\mu \;=\; (x^1_2 \, . \, x^2_1 \,-\, x^2_2 \, . \, x^1_1)$$

$$\Gamma^1_{\mu 2} \, . \, p^\mu \;=\; (x'^1_2 \, . \, x^2_1 \,-\, x^2_2 \, . \, x'^1_1)$$

$$\Gamma^2_{\mu 1} \, . \, p^\mu \;=\; (x'^2_2 \, . \, x^1_1 \,-\, x^1_2 \, . \, x'^2_1)$$

$$\Gamma^3_{\mu 0} \, . \, p^\mu \;=\; (x'^2_2 \, . \, x'^1_1 \,-\, x'^1_2 \, . \, x'^2_1)$$

- and, as consequence, a condensed formulation of the matrix $[\mathrm{C}(\mathbf{X}_1)] \, . \, [\mathrm{C}(\mathbf{X}_2)]$:

$$[C(\mathbf{X}_1)] \, , . \, [C(\mathbf{X}_2)]$$

$$=$$

$$\begin{bmatrix} \frac{1}{2} \, . \, (a \,-\, b) & 0 & 0 & (x^1_2 \, . \, x^2_1 \,-\, x^2_2 \, . \, x^1_1) \\ 0 & \frac{1}{2} \, . \, (d \,-\, c) & (x'^1_2 \, . \, x^2_1 \,-\, x^2_2 \, . \, x'^1_1) & 0 \\ 0 & (x'^2_2 \, . \, x^1_1 \,-\, x^1_2 \, . \, x'^2_1) & \frac{1}{2} \, . \, (c \,-\, d) & 0 \\ (x'^2_2 \, . \, x'^1_1 \,-\, x'^1_2 \, . \, x'^2_1) & 0 & 0 & \frac{1}{2} \, . \, (b \,-\, a) \end{bmatrix}$$

Remark 3.2. *A useful identity*

$$a \, . \, b$$

$$=$$

$$(x^1_1 \, . \, x'^1_2 \,+\, x^2_1 \, . \, x'^2_2) \, . \, (x'^1_1 \, . \, x^1_2 \,+\, x'^2_1 \, . \, x^2_2)$$

$$=$$

$$x^1_1 \, . \, x'^1_2 \, . \, x'^1_1 \, . \, x^1_2 \,+\, x^1_1 \, . \, x'^1_2 \, . \, x'^2_1 \, . \, x^2_2 \,+\, x^2_1 \, . \, x'^2_2 \, . \, x'^1_1 \, . \, x^1_2 \,+\, x^2_1 \, . \, x'^2_2 \, . \, x'^2_1 \, . \, x^2_2$$

On the same vein:

$$c \, . \, d$$

$$=$$

$$(x^1_1 \, . \, x'^1_2 \,+\, x'^2_1 \, . \, x^2_2) \, . \, (x'^1_1 \, . \, x^1_2 \,+\, x^2_1 \, . \, x'^2_2)$$

$$=$$

$$x^1_1 \, . \, x'^1_2 \, . \, x'^1_1 \, . \, x^1_2 \,+\, x^1_1 \, . \, x'^1_2 \, . \, x^2_1 \, . \, x'^2_2 \,+\, x'^2_1 \, . \, x^2_2 \, . \, x'^1_1 \, . \, x^1_2 \,+\, x'^2_1 \, . \, x^2_2 \, . \, x^2_1 \, . \, x'^2_2$$

The previous results are now yielding:

$$a \, . \, b \,-\, c \, . \, d$$

$$=$$

$$x^1_1 \, . \, x'^1_2 \, . \, x'^2_1 \, . \, x^2_2 \,-\, x^1_1 \, . \, x'^1_2 \, . \, x^2_1 \, . \, x'^2_2 \,+\, x^2_1 \, . \, x'^2_2 \, . \, x'^1_1 \, . \, x^1_2 \,-\, x'^2_1 \, . \, x^2_2 \, . \, x'^1_1 \, . \, x^1_2$$

$$=$$

$$x_1^1 \,.\, x_2'^1 \,.\, (x_1'^2 \,.\, x_2^2 \,-\, x_1^2 \,.\, x_2'^2) \,+\, (x_1^2 \,.\, x_2'^2 \,-\, x_1'^2 \,.\, x_2^2) \,.\, x_1'^1 \,.\, x_2^1$$

$$=$$

$$(x_1^1 \,.\, x_2'^1 \,-\, x_1'^1 \,.\, x_2^1) \,.\, (x_1'^2 \,.\, x_2^2 \,-\, x_1^2 \,.\, x_2'^2)$$

Let state that:

$$\Gamma^0_{\mu 3} \,.\, \Gamma^3_{\nu 0} \,.\, p^\mu \,.\, p^\nu$$

$$=$$

$$(x_2^1 \,.\, x_1^2 \,-\, x_2^2 \,.\, x_1^1) \,.\, (x_2'^2 \,.\, x_1'^1 \,-\, x_2'^1 \,.\, x_1'^2)$$

$$=$$

$$x_2^1 \,.\, x_1^2 \,.\, x_2'^2 \,.\, x_1'^1 \,-\, x_2^1 \,.\, x_1^2 \,.\, x_2'^1 \,.\, x_1'^2 \,-\, x_2^2 \,.\, x_1^1 \,.\, x_2'^2 \,.\, x_1'^1 \,+\, x_2^2 \,.\, x_1^1 \,.\, x_2'^1 \,.\, x_1'^2$$

And that:

$$\Gamma^1_{\mu 2} \,.\, \Gamma^2_{\nu 1} \,.\, p^\mu \,.\, p^\nu$$

$$=$$

$$(x_2'^1 \,.\, x_1^2 \,-\, x_2^2 \,.\, x_1'^1) \,.\, (x_2'^2 \,.\, x_1^1 \,-\, x_2^1 \,.\, x_1'^2)$$

$$=$$

$$x_2'^1 \,.\, x_1^2 \,.\, x_2'^2 \,.\, x_1^1 \,-\, x_2'^1 \,.\, x_1^2 \,.\, x_2^1 \,.\, x_1'^2 \,-\, x_2^2 \,.\, x_1'^1 \,.\, x_2'^2 \,.\, x_1^1 \,+\, x_2^2 \,.\, x_1'^1 \,.\, x_2^1 \,.\, x_1'^2$$

At the end of the day:

$$(\Gamma^0_{\mu 3} \,.\, \Gamma^3_{\nu 0} \,-\, \Gamma^1_{\mu 2} \,.\, \Gamma^2_{\nu 1}) \,.\, p^\mu \,.\, p^\nu$$

$$=$$

$$x_2^1 \,.\, x_1^2 \,.\, x_2'^2 \,.\, x_1'^1 \,-\, x_2'^1 \,.\, x_1^2 \,.\, x_2'^2 \,.\, x_1^1 \,+\, x_2^2 \,.\, x_1^1 \,.\, x_2'^1 \,.\, x_1'^2 \,-\, x_2^2 \,.\, x_1'^1 \,.\, x_2^1 \,.\, x_1'^2$$

$$=$$

$$x_2^1 \,.\, x_1'^1 \,.\, (x_1^2 \,.\, x_2'^2 \,-\, x_2^2 \,.\, x_1'^2) \,-\, x_1^1 \,.\, x_2'^1 \,.\, (x_1^2 \,.\, x_2'^2 \,-\, x_2^2 \,.\, x_1'^2)$$

$$=$$

$$(x_1^1 \,.\, x_2'^1 \,-\, x_1'^1 \,.\, x_2^1) \,.\, (x_1'^2 \,.\, x_2^2 \,-\, x_1^2 \,.\, x_2'^2)$$

$$=$$

$$a \,.\, b \,-\, c \,.\, d$$

This is a remarkable result.

Remark 3.3. *Admissible matrices*

Very important indications concerning the formalism of the matrix representation for a bi-vector can be read in [03 ; §125, pp. 109-110]. They allow the definition of two families:

1. **First family**:
$$\Gamma^0_{\mu 3}\,.\,p^\mu \,=\, (x^1_2\,.\,x^2_1 \,-\, x^2_2\,.\,x^1_1) \,=\, 0$$
$$\Gamma^1_{\mu 2}\,.\,p^\mu \,=\, (x'^1_2\,.\,x^2_1 \,-\, x^2_2\,.\,x'^1_1) \,=\, 0$$
$$\Gamma^2_{\mu 1}\,.\,p^\mu \,=\, (x'^2_2\,.\,x^1_1 \,-\, x^1_2\,.\,x'^2_1) \,=\, 0$$
$$\Gamma^3_{\mu 0}\,.\,p^\mu \,=\, (x'^2_2\,.\,x'^1_1 \,-\, x'^1_2\,.\,x'^2_1) \,=\, 0$$
When neither $\mathbf{X}_1$ nor $\mathbf{X}_2$ have vanishing components:
$$\frac{x^2_2}{x^2_1} \,=\, \frac{x'^1_2}{x'^1_1} \,=\, \frac{x'^2_2}{x'^2_1} \,=\, \frac{x^1_2}{x^1_1} \,=\, k,\, \forall\, k \,\neq\, 0$$
It is equivalent to write:
$$\mathbf{X}_2 \,=\, k\,.\,\mathbf{X}_2$$
In that case:
$$a \,=\, b \,=\, c \,=\, d \,=\, k\,.\,(x^1_1\,.\,x'^1_1 \,+\, x^2_1\,.\,x'^2_1)$$
And:
$$[C(\mathbf{X}_1)]\,.\,[C(\mathbf{X}_2)] \,=\, [0]$$
Elements in the first family are identified with the null matrix. This situation coincides with any invariant geometry.

2. **Second family**
$$\Gamma^0_{\mu 0}\,.\,p^\mu \,=\, -\Gamma^3_{\mu 3}\,.\,p^\mu \,=\, \frac{1}{2}\,.\,(a \,-\, b) \,=\, 0$$
$$\Gamma^1_{\mu 1}\,.\,p^\mu \,=\, -\Gamma^2_{\mu 2}\,.\,p^\mu \,=\, \frac{1}{2}\,.\,(d \,-\, c) \,=\, 0$$
Remark 3.4. *A fist characteristic of the second family*

Because of [10 ; p. 89, (17.5), D]:
$$\forall\,\alpha \,\in\, I_4 \,=\, \{0,\, 1,\, 2,\, 3\} \,:\, \Gamma^\alpha_{\mu\alpha}\,.\,p^\mu \,=\, \frac{\partial log\sqrt{|g|}}{\partial x^\mu}\,.\,p^\mu \,=\, 0$$
A sum on μ when one accepts the classical definition $\mathbf{p} = \mathrm{m}.\mathbf{u}$ for the kinetic momentum yields:
$$m\,.\,\frac{dlog\sqrt{|g|}}{ds} \,=\, 0$$
This characteristic is trivially true in two sets of circumstances:

- The particle at hand is mass-less: $\mathrm{m} = 0$.
- $$\forall\, m,\, s \,:\, log\sqrt{|g|} \,=\, constant$$

Remark 3.5. *A second characteristic of the second family*

All matrices in the second family have only a small number of entries and they all lie in a line orthogonal to the diagonal:

$$\Gamma^0 = \Gamma^0_{\mu 3} . p^\mu ; \Gamma^1 = \Gamma^1_{\mu 2} . p^\mu ; \Gamma^2 = \Gamma^2_{\mu 1} . p^\mu ; \Gamma^3 = \Gamma^3_{\mu 0} . p^\mu$$

$$_{\Gamma(2)}\Phi(\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & \Gamma^0 \\ 0 & 0 & \Gamma^1 & 0 \\ 0 & \Gamma^2 & 0 & 0 \\ \Gamma^3 & 0 & 0 & 0 \end{bmatrix}$$

These entries appear in the useful identity which has been discovered in remark 2.2.:

$$\Gamma^0 . \Gamma^3 - \Gamma^1 . \Gamma^2 = (\Gamma^0_{\mu 3} . \Gamma^3_{\nu 0} - \Gamma^1_{\mu 2} . \Gamma^2_{\nu 1}) . p^\mu . p^\nu = a . b - c . d$$

But here, a = b and c = d (look at the top of this remark); hence:

$$\Gamma^0 . \Gamma^3 - \Gamma^1 . \Gamma^2 = a^2 - c^2$$

(a) **Sub-family 2.1** : The simplest decomposition without residual part is entirely symmetric.
Since :

$$\Gamma^0 = \Gamma^3 ; \Gamma^1 = \Gamma^2$$

The useful identity writes:

$$(\Gamma^0)^2 - (\Gamma^1)^2 = a^2 - c^2$$

One can define four configurations:

i. The configuration (+, +) :

$$\Gamma^0 = a ; \Gamma^1 = c$$

$$_{\Gamma(2)}\Phi(\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & c & 0 \\ 0 & c & 0 & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$$

ii. The configuration (-, +) :

$$\Gamma^0 = -a ; \Gamma^1 = c$$

$$_{\Gamma(2)}\Phi(\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & c & 0 \\ 0 & c & 0 & 0 \\ -a & 0 & 0 & 0 \end{bmatrix}$$

...

iii. The configuration (+, -) :

$$\Gamma^0 = a\,;\,\Gamma^1 = -c$$

$$_{\Gamma(2)}\Phi(\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -c & 0 \\ 0 & -c & 0 & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$$

iv. The configuration (-, -) :

$$\Gamma^0 = -a\,;\,\Gamma^1 = -c$$

$$_{\Gamma(2)}\Phi(\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & -c & 0 \\ 0 & -c & 0 & 0 \\ -a & 0 & 0 & 0 \end{bmatrix}$$

(b) **Sub-family 2.2** : The simplest decomposition without residual part is entirely anti-symmetric.
Since :

$$\Gamma^0 = -\Gamma^3\,;\,\Gamma^1 = -\Gamma^2$$

Here, the useful identity writes:

$$(\Gamma^1)^2 - (\Gamma^0)^2 = a^2 - c^2$$

One can define four configurations:

i. The configuration (+, +) :

$$\Gamma^1 = a\,;\,\Gamma^0 = c$$

$$_{\Gamma(2)}\Phi(\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & c \\ 0 & 0 & a & 0 \\ 0 & -a & 0 & 0 \\ -c & 0 & 0 & 0 \end{bmatrix}$$

ii. The configuration (-, +) :

$$\Gamma^1 = -a\,;\,\Gamma^0 = c$$

$$_{\Gamma(2)}\Phi(\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & c \\ 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -c & 0 & 0 & 0 \end{bmatrix}$$

iii. The configuration (+, -) :

$$\Gamma^1 = a\,;\,\Gamma^0 = -c$$

$$_{\Gamma(2)}\Phi(\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & -c \\ 0 & 0 & a & 0 \\ 0 & -a & 0 & 0 \\ c & 0 & 0 & 0 \end{bmatrix}$$

iv. The configuration (-, -) :

$$\Gamma^1 = -a\,;\,\Gamma^0 = -c$$

$$_{\Gamma(2)}\Phi(\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & -c \\ 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ c & 0 & 0 & 0 \end{bmatrix}$$

Remark 3.6. *A third characteristic of the second family*

Some elements in the second family are Dirac's matrices [09 ; §2.13, pp.29-32].

□

3.7 The chameleons fields

Definition 3.1. *What is a chameleon field?*

Per convention, a chameleon field is an electromagnetic field:

1. resulting from the treatment of the Lorentz-Einstein law with the extrinsic method,
2. when the simplest decomposition without residual term of the gravitational term is a bi-vector "à la Cartan",
3. allowing the relation:
$$\delta[G] = [F(2,\,0)]$$

The justification of this semantic is clear: a chameleon field is an electromagnetic field resembling an anti-symmetric variation of the metric.

Remark 3.7. *When simplest decomposition without residual term of the gravitational term and the underlying metric are symmetric matrices*

Here, one works with:

$$\forall\,[G] = [G]^t = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{bmatrix}$$

... and matrices in the sub-family 2.1 which will be denoted:

$$_{\Gamma(2)}\Phi(\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & \chi \\ 0 & 0 & \Upsilon & 0 \\ 0 & \Upsilon & 0 & 0 \\ \chi & 0 & 0 & 0 \end{bmatrix}$$

In this context:

$$[G]\,.\,_{\Gamma(2)}\Phi(\mathbf{p})$$

...

$$=$$

$$\begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{bmatrix} . \begin{bmatrix} 0 & 0 & 0 & \chi \\ 0 & 0 & \Upsilon & 0 \\ 0 & \Upsilon & 0 & 0 \\ \chi & 0 & 0 & 0 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \chi \cdot g_{03} & \Upsilon \cdot g_{02} & \Upsilon \cdot g_{01} & \chi \cdot g_{00} \\ \chi \cdot g_{13} & \Upsilon \cdot g_{12} & \Upsilon \cdot g_{11} & \chi \cdot g_{01} \\ \chi \cdot g_{23} & \Upsilon \cdot g_{22} & \Upsilon \cdot g_{12} & \chi \cdot g_{02} \\ \chi \cdot g_{33} & \Upsilon \cdot g_{23} & \Upsilon \cdot g_{13} & \chi \cdot g_{03} \end{bmatrix}$$

And:

$$_{\Gamma(2)}\Phi(\mathbf{p}) \, . \, [G]$$

$$=$$

$$\begin{bmatrix} 0 & 0 & 0 & \chi \\ 0 & 0 & \Upsilon & 0 \\ 0 & \Upsilon & 0 & 0 \\ \chi & 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \chi \cdot g_{03} & \chi \cdot g_{13} & \chi \cdot g_{23} & \chi \cdot g_{33} \\ \Upsilon \cdot g_{02} & \Upsilon \cdot g_{12} & \Upsilon \cdot g_{22} & \Upsilon \cdot g_{23} \\ \Upsilon \cdot g_{01} & \Upsilon \cdot g_{11} & \Upsilon \cdot g_{12} & \Upsilon \cdot g_{13} \\ \chi \cdot g_{00} & \chi \cdot g_{01} & \chi \cdot g_{02} & \chi \cdot g_{03} \end{bmatrix}$$

Hence:

$$[G] \, . \, _{\Gamma(2)}\Phi(\mathbf{p}) \; - \; _{\Gamma(2)}\Phi(\mathbf{p}) \, . \, [G]$$

$$=$$

$$\begin{bmatrix} 0 & \Upsilon \cdot g_{02} - \chi \cdot g_{13} & \Upsilon \cdot g_{01} - \chi \cdot g_{23} & \chi \cdot (g_{00} - g_{33}) \\ \chi \cdot g_{13} - \Upsilon \cdot g_{02} & 0 & \Upsilon \cdot (g_{11} - g_{22}) & \chi \cdot g_{01} - \Upsilon \cdot g_{23} \\ \chi \cdot g_{23} - \Upsilon \cdot g_{01} & \Upsilon \cdot (g_{22} - g_{11}) & 0 & \chi \cdot g_{02} - \Upsilon \cdot g_{13} \\ \chi \cdot (g_{33} - g_{00}) & \Upsilon \cdot g_{23} - \chi \cdot g_{01} & \Upsilon \cdot g_{13} - \chi \cdot g_{02} & 0 \end{bmatrix}$$

Let recall that [07]:

$$[F_{\alpha\beta}] = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{bmatrix} ; \, [F^{\alpha\beta}] = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{bmatrix}$$

Hence, as expected, the matrix which has been calculated can formally be identified with a matrix mimicking an electromagnetic field (equivalently: can formally be identified with a chameleon field) in writing:

$$q \, . \, E_x = \frac{1}{2} \cdot \{\Upsilon \, . \, g_{02} - \chi \cdot g_{13}\}$$

$$q \, . \, E_y = \frac{1}{2} \cdot \{\Upsilon \, . \, g_{01} - \chi \cdot g_{23}\}$$

$$q\,.\,E_z \;=\; \frac{1}{2}\,.\,\{\chi\,.\,(g_{00} \;-\; g_{33})\}$$
$$q\,.\,H_x \;=\; \frac{1}{2}\,.\,\{\Upsilon\,.\,g_{13} \;-\; \chi\,.\,g_{02}\}$$
$$q\,.\,H_y \;=\; \frac{1}{2}\,.\,\{\chi\,.\,g_{01} \;-\; \Upsilon\,.\,g_{23}\}$$
$$q\,.\,H_z \;=\; \frac{1}{2}\,.\,\{\Upsilon\,.\,(g_{22} \;-\; g_{11})\}$$

And this chameleon field is equivalent to an infinitesimal anti-symmetric variation of the metric such that:

$$\delta g_{00} \;=\; \delta g_{11} \;=\; \delta g_{22} \;=\; \delta g_{33} \;=\; 0$$
$$\delta g_{01} \;=\; \frac{1}{2}\,.\,\{\Upsilon\,.\,g_{02} \;-\; \chi\,.\,g_{13}\} \;=\; -\delta g_{10}$$
$$\delta g_{02} \;=\; \frac{1}{2}\,.\,\{\Upsilon\,.\,g_{01} \;-\; \chi\,.\,g_{23}\} \;=\; -\delta g_{20}$$
$$\delta g_{03} \;=\; \frac{1}{2}\,.\,\{\chi\,.\,(g_{00} \;-\; g_{33})\} \;=\; -\delta g_{30}$$
$$\delta g_{32} \;=\; \frac{1}{2}\,.\,\{\Upsilon\,.\,g_{13} \;-\; \chi\,.\,g_{02}\} \;=\; -\delta g_{23}$$
$$\delta g_{13} \;=\; \frac{1}{2}\,.\,\{\chi\,.\,g_{01} \;-\; \Upsilon\,.\,g_{23}\} \;=\; -\delta g_{31}$$
$$\delta g_{21} \;=\; \frac{1}{2}\,.\,\{\Upsilon\,.\,(g_{22} \;-\; g_{11})\} \;=\; -\delta g_{12}$$

Example 3.1. *Variations of an initial Minkowski's geometry*

Working with a metric with signature (+ - - -), one gets:

$$\forall\,\Upsilon$$

$$\delta\eta_{00} \;=\; \delta\eta_{11} \;=\; \delta\eta_{22} \;=\; \delta\eta_{33} \;=\; 0$$
$$\delta\eta_{01} \;=\; 0 \;=\; -\delta\eta_{10} \;=\; q\,.\,E_x$$
$$\delta\eta_{02} \;=\; 0 \;=\; -\delta\eta_{20} \;=\; q\,.\,E_y$$
$$\delta\eta_{03} \;=\; \chi \;=\; -\delta\eta_{30} \;=\; q\,.\,E_z$$
$$\delta\eta_{32} \;=\; 0 \;=\; -\delta\eta_{23} \;=\; q\,.\,H_x$$
$$\delta\eta_{13} \;=\; 0 \;=\; -\delta\eta_{31} \;=\; q\,.\,H_y$$
$$\delta\eta_{21} \;=\; 0 \;=\; -\delta\eta_{12} \;=\; q\,.\,H_z$$

And:

$$\delta[\eta] \;=\; q\,.\,\begin{bmatrix} 0 & & 0 & E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -E_z & 0 & 0 & 0 \end{bmatrix} \;=\; \begin{bmatrix} 0 & 0 & 0 & \chi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\chi & 0 & 0 & 0 \end{bmatrix} \;=\; \begin{bmatrix} 0 & 0 & 0 & \pm\Gamma^0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mp\Gamma^0 & 0 & 0 & 0 \end{bmatrix}$$

If these calculations correspond to some reality, then:

$$[\eta] \to [\eta] + \delta[\eta] = [G] = \begin{bmatrix} 1 & 0 & 0 & \pm\Gamma^0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mp\Gamma^0 & 0 & 0 & -1 \end{bmatrix}$$

As usual in mathematical physics, one must ask: "What do these equations really mean?"

Considering the definition of Γ^0, one might say - in a first try- that if a particle with a mass m enters into an empty region of the universe with a Minkowski's geometry, then (i) that particle automatically deforms this initial geometry; (ii) if Γ^0 describes this deformation, the theory of spinors allows to think that it mimics the z-component of an electromagnetic field. Does it mean that the instruments would measure an electromagnetic field? Not sure. Are the geometrical deformations induced by the presence of the particle correctly and entirely described through the quantity Γ^0? No guaranty.

4 Conclusion

Once more time, the mathematics opens theoretical doors which don't necessarily correspond to some reality. But it would be a shame to not break the habits that confine our minds in sterile territories. Who tries nothing cannot succeed. This work does not pretend to be exhaustive. It leaves many topics unexplored: "What does the matrix [B] really represent? Do the chameleons fields exist in the nature? Do they correspond to a certain type of elementary particles? Why do some ${}_{\Gamma(2)}\Phi(\mathbf{u})$ matrices coincide with Dirac's matrices?"...

4.1 My contributions

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4.2 International works

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