The cubic curve know as Witch of Agnesi

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ABSTRACT

This type of cubic curve, in the United Kingdom know as the *Witch of Agnesi*, it is approached from a geometric, trigonometric and analytical point of view; using the scheme: model, algorithm, resolution, graph.

CONTENTS

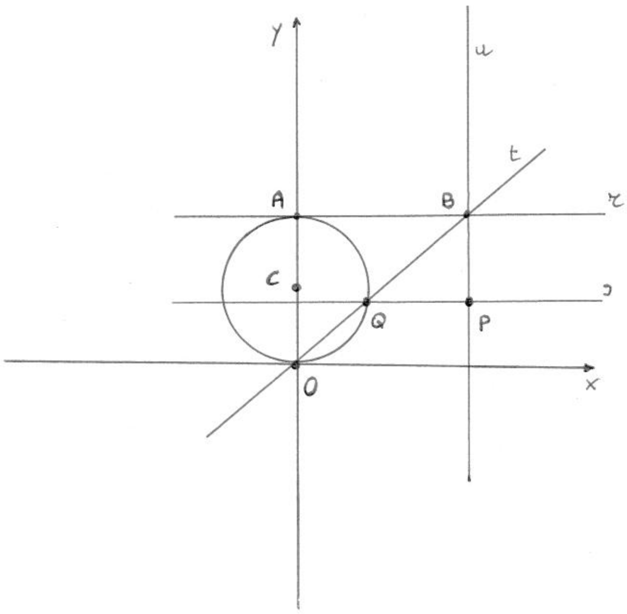
1. Introduction

This cubic was studied by Pierre de Fermat (Beaumont de Lomagne 1601, Castres 1665). He tries to square it, but in vain like the circle. It was taken up again from the Italian mathematician Guido Grandi (Cremona 1671, Pisa 1742) in his book *Quadratura circuli et hyperbolae*, published 1703, in which he defines it *versoria*, from the Latin *sinus versus*, i.e. curve with the sine towards, that is the opposite. *Versoria* it's a term that indicate the rope tied at the end of a sail for tacking a boat. Was Maria Gaetana Agnesi (Milano 1718 – 1799) to study this cubic curve (named *versiera*) in detail in her two volumes entitled *Istituzioni Analitiche ad uso della Gioventù Italiana* published in 1748. They were used as textbooks in High Schools. Still today some contents are present in High School mathematics books. The work of Agnesi was a great success and was translated into French (1775) and English (1801). In the English version the term *versiera* was understood as an abbreviation of *avversiera*, which in the Middle Age indicated one *avversaria di Dio*, or *moglie del diavolo*, that is *strega*. Still today in Anglo-Saxon countries it's called *the witch of Agnesi*.

2. Analytic geometry

Given the circumference γ with center C(0;a) and radius R, let be r the tangent line at γ in A(0;2a) parallel to the x axis. From O we draw a generic straight line which intersects γ at the point Q. The Agnesi curve (also called *versiera*) is the geometric locus of the points P of intersection of the straight line s parallel to the x axis conducted by Q and the straight line parallel to the y axis conducted by B = r ∩ t.

Model :



Algorithm :

1) We find the equation of the circumference γ
2) Bundle of lines F for O
3) Let's arrange an algebraic system between γ and F : Q = γ ∩ F
4) We find the equations of the lines r ∧ s
5) Let's calculate the coordinates of B = r ∩ t ∈ F
6) We find the equation of the straight line u
7) Let's calculate the coordinates of P = u ∩ s

Resolution :

1) $(x - 0)^2 + (y - a)^2 = a^2 \rightarrow \gamma : x^2 + y^2 - 2ay = 0$
2) F : y = mx
3) $\begin{cases} F = 0 \\ \gamma = 0 \end{cases} \rightarrow x^2 + m^2x^2 - 2amx = 0 \rightarrow (1 + m^2)x^2 + 2amx = 0 \rightarrow x = 0 \wedge x = 2am/(1 + m^2) \rightarrow$

$$Q \equiv (2am/(1 + m^2) \, ; \, 2am^2/(1 + m^2))$$

4) $r : y = 2a \wedge s : y = 2am^2/(1 + m^2)$

5) $\begin{cases} y = mx \\ y = 2a \end{cases}$ → mx = 2a → x = 2a/m → B ≡ (2a/m ; 2a)

6) u : x = 2a/m

7) $\begin{cases} x = \frac{2a}{m} \\ y = \frac{2am^2}{1+m^2} \end{cases}$ Put m = k → $\begin{cases} x = \frac{2a}{k} \\ y = \frac{2ak^2}{1+k^2} \end{cases}$

Parametric equations of the curve.
By eliminating the parameter k we have :

$k = 2a/x \rightarrow y = 8a^3 / (x^2 + 4a^2)$

Hence the *Witch of Agnesi* has the explicit equation : $y = \frac{8a^3}{x^2 + 4a^2}$

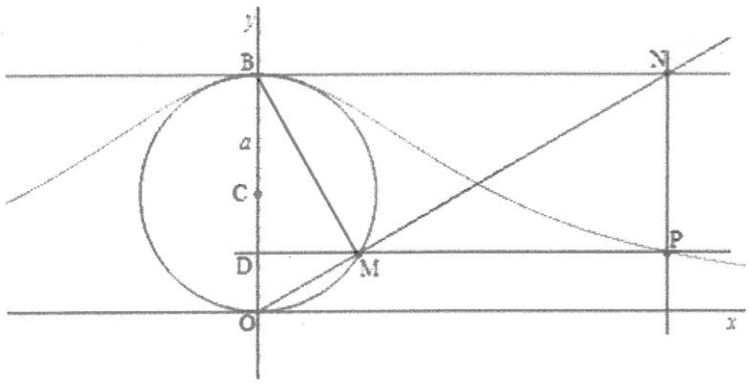
And in canonical form :

$$\mathbf{x^2y + 4a^2y - 8a^3 = 0}$$

Rational curve of the third order, cubic.

3. Elementary geometry

Model :



With reference to the figure above it is easy to establish that right-angled triangles ODM and OBM are similar. So : OD : DM = OB : BN. Being BN = DP → **OD : DM = OB : DP**.
We put P(x;y) : x = DP e y = OD.
The triangle OBM is a right angle because it's inscribed in a semicircle, so, by Euclid's second theorem we have : $DM^2 = OD \cdot BD = y(2a - y) \rightarrow DM = \sqrt{y(2a-y)}$.
Substituting into the initial proportion we obtain :

$y : \sqrt{y(2a-y)} = 2a : x \rightarrow y/\sqrt{y(2a-y)} = 2a/x \rightarrow y^2/y(2a-y) = 4a^2/x^2 \rightarrow$
$y/2a - y = 4a^2/x^2 \rightarrow$ $\mathbf{x^2y + 4a^2y - 8a^3 = 0}$ equation of the cubic curve *Witch of Agnesi*.

If the center is in C(0;-a) and radius R = a, the equation it's obtain by an axial symmetry with y = 0 (x axis) : x = x ∧ y = -y : $\mathbf{x^2y + 4a^2x + 8a^3 = 0}$

If the center is in C(a;0)) and radius R = a, the equation it's obtain by the substitutions x = y ∧ y = x : $\mathbf{x^2y + 4a^2x + 8a^3 = 0}$

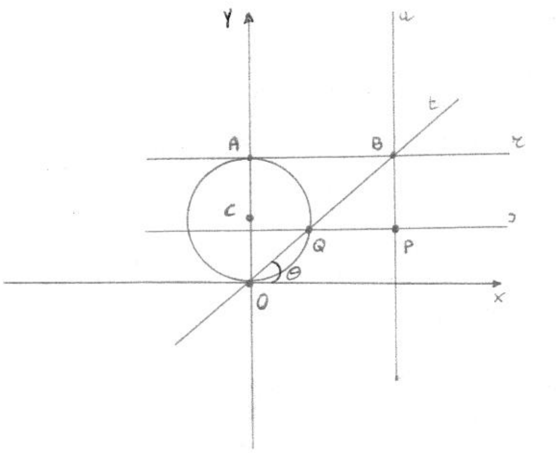
From this by an axial symmetry with x = 0 (y axis) we obtain the equation of Witch of Agnesi with center in C(-a;0) and radius R = a : $\mathbf{xy^2 + 4a^2x - 8a^3 = 0}$

Note

The second Euclid's theorem in right-angled triangle affirm that the height is medium proportional between the two projections of the two cathetus on the hypotenuse.

4. Trigonometry

With reference to the model :



We indicate with θ the angle formed by x axis and the line t in the bundle F : y = mx : k = m = tanθ. Substituting into parametric equation :

$$\begin{cases} x = \frac{2a}{k} \\ y = \frac{2ak^2}{1+k^2} \end{cases} \rightarrow \begin{cases} \frac{2a\cos\theta}{\sin\theta} \\ 2a\sin^2\theta \end{cases}$$

Polar equations of the *versiera*, which, as is easy to verify, satisfy the equation $\mathbf{x^2y + 4a^2y - 8a^3 = 0}$

5. Mathematical Analysis

We study the *Witch of Agnesi* or *versiera* $y = f(x) = \frac{8a^3}{x^2+4a^2}$ by mathematical analysis

1) Domain D ≡ $(+\infty\ ;\ -\infty)$
2) Intersections x axis : $\begin{cases} f(x) = 0 \\ y = 0 \end{cases} \rightarrow \nexists$

Intersections y axis : $\begin{cases} f(x) = 0 \\ x = 0 \end{cases}$ → y = 2a

3) y = f(x) > 0 $\forall x \in R$

4) Axial symmetry y axis : x = -x ∧ y = y → $y = f(x) = \frac{8a^3}{x^2 + 4a^2}$

The cubic is symmetric to the y axis.

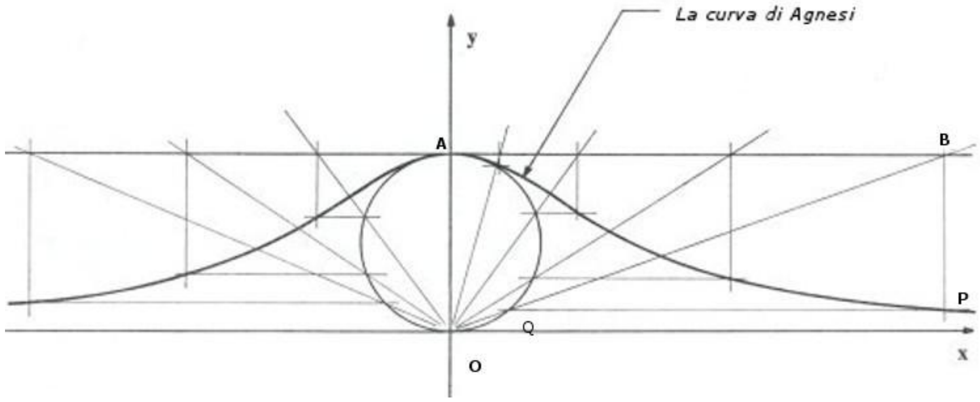
5) The cubic has only the horizontal asymptote of equation y = 0 (x axis), in fact:

$$y = \lim_{x \to \infty} \frac{8a^3}{x^2 + 4a^2} = 0$$

6) $y = f(x) = \frac{8a^3}{x^2 + 4a^2} \rightarrow y' = f'(x) = \frac{-16a^3x}{(x^2+4a^2)^2} = 0$ → x = 0 → y = 2a →
Maximum point M≡(0;2a)

7) $y'' = f''(x) = \frac{[16a^3(x^2+4a^2) - 64a^3x^2]}{(x^2+4a^2)^3}$ = 0 → $64a^5 - 48a^3x^2$ = 0 → $6x^2 - 8a^2$ = 0 →
x = $\pm\sqrt{\frac{4a^2}{3}} = \pm\frac{2a\sqrt{3}}{3}$ → y = $\frac{3a}{2}$ → Inflection points $F_1 \equiv \left(\frac{2a\sqrt{3}}{3}; \frac{3a}{2}\right)$, $F_2 \equiv \left(-\frac{2a\sqrt{3}}{3}; \frac{3a}{2}\right)$

8) Graphic :



If a = 0 the cubic curve degenerates into y = 0 (x axis)

If a = 1/2 the cubic curve has equation : $y = \frac{1}{x^2 + 1}$ = arctanx.

If a = 1 the cubic curve has equation : $y = \frac{8}{x^2+4}$: M(0;2), $F_{1;2}$≡ (±2√3/3;3/2)

If e a = 2 the cubic curve has equation : $y = \frac{64}{x^2+16}$: M(0;4), $F_{1;2}$≡ (±4√3/3;3)

… …

Area

Bearing in mind that the *Witch of Agnesi* is a cubic symmetrical to the y axis (even function), the area enclosed by the curve and the abscissa axis is:

$area = 2\int_0^{+\infty} \frac{8a^3}{4a^2+x^2}\, dx = 8a^2 \int_0^{+\infty} \frac{\frac{1}{2a}}{1+\left(\frac{x}{2a}\right)^2}\, dx = 8a^2 \left[arctg\,\frac{x}{2a}\right] [0; +\infty)$

Being : $\frac{x}{2a} = \frac{\pi}{2} \rightarrow x = a\pi$ hence : *area* (*versiera*) = $8a^3\pi$

The area enclosed by the *Witch of Agnesi* and the abscissa axis is 8a·*area* γ (area of circle $\gamma = a^2\pi$).
If a = 0 the curve degenerates into the straight line y = 0 (x axis) : *area* = 0
If a = +∞ the curve degenerates into the infinity line parallel to the x axis : *area* = + ∞

Note

The surface obtained by rotating the curve *versiera* around x axis takes the shape of a horizontal spindle, while the rotation around the y axis has the shape of a vertical spindle.
Estimate the volume of these spindles presents considerable difficulties in integral calculations.
In the end we get to the result : Volume V = $8a^4\pi^2$.

Osculating circle

We verify that the osculating circle of the *Witch of Agnesi* coincides with the circle γ of radius a at its point of absolute maximum M(0;2a).

The radius of the osculating circle is : R = $\frac{[1+y'(0)^2]^{\frac{3}{2}}}{y''(0)}$

Being $y'(0) = \left[\frac{-16a^3x}{(x^2+4a^2)^2}\right]_{x=0} = 0$

and $y''(0) = \left[\frac{[16a^3(x^2+4a^2)-64a^3x^2]}{(x^2+4a^2)^3}\right]_{x=0} = \frac{64a^5}{64a^6} = \frac{1}{a}$

It follow : R = a → Q.E.T.

Notation

The *Witch of Agnesi* has application in Physics in the description of resonance phenomena, and in the context of studies on atomic and molecular spectra.

6. Historical notes

Maria Gaetana Agnesi (Milan 1718 – 1799) was a mathematician, philosopher and benefactress. She is recognized as one of he greatest minds of all time. The third of twenty-one children, from an early age she showed a strong propensity for learning foreign languages (it is said that at the age of twenty she spoke six) and extraordinary degrees in mathematics and philosophy. Her father Pietro Agnesi, a rich silk merchant, had her educated under the guidance of illustrious tutors. The Agnesi house became one of the fashionable salons of Milan, frequented by eminent intellectuals of Italy and Europe. She got into the habit of ex pressing her opinions and beliefs. These dissertations of her were published in 1738 in the book *Propositiones Philosophicae* : a collection of 191 theses ranging from logic to botany, from cosmology to physics, from ontology to neuropathy which at the time was understood as science of the spirit.

In 1748, at the age of thirty, M. G. Agnesi published in two volumes *Istituzioni Analitiche ad uso della Gioventù Italiana*, in which he gave a rigorous arrangement of the principles of algebra, analytical geometry and infinitesimal differential and integral calculus.

In clear and concise language, she described her curve called *versiera*. The two volumes were dedicated to Empress Maria Theresa of Austria. After the publication, Pope Benedict XIV appointed her professor of Mathematics and Philosophy at the University *Alma Mater Studiorum* of Bologna. She never taught there because in those times teaching was prohibited to women.
Upon the dead of her father (1752), M. G. Agnesi abandoned her studies to dedicate herself to charity, mainly caring for the mentally ill. In 1783 she was appointed director of the Pio Albergo Trivulzio in Milan, and she dedicated herself to this activity until her death.

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