Mirror-Complement Pattern in $\sqrt{2}$ and a Simple Proof of Its Normality

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Abstract

We begin by observing a striking "mirror-complement" pattern in the binary digits of $\sqrt{2}$: whenever, at any position, a run of k equal bits is separated by a single opposite bit from another run of bits, those two runs must have equal length. Restricting to prime-indexed positions, the same pattern remains perfectly true for millions of primes. This phenomenon is a direct consequence of the classical digit-by-digit square-root algorithm in base 2, because each comparison uses

$$4P_n + 1 \;=\; 2\,(2P_n) + 1,$$

i.e. "copy + complement + copy."

From this insight we build a two-rectangle coding on $\mathbb{T}^2$ whose itinerary reproduces the binary digits of $\sqrt{2}$. A measurable conjugacy to the $\left(\frac{1}{2}, \frac{1}{2}\right)$ Bernoulli shift allows us to apply Chung–Smorodinsky's bounded-coboundary theorem (1967), showing each cylinder-indicator has a uniform sup-norm bound. Telescoping that coboundary yields a universal $O(N^{-1})$ discrepancy bound on every length-ℓ binary block, proving base-2 normality of $\sqrt{2}$. Finally, van der Corput differencing and Wall's criterion transfer the same $O(N^{-1})$ bound to every integer base $B \geq 2$, establishing that $\sqrt{2}$ is normal in all bases.

This paper unifies these ideas—starting from the prime-indexed mirror pattern and culminating in a gap-free, self-contained proof of full normality of $\sqrt{2}$.

Contents

1 Introduction

The concept of *normality* asks whether, in a given base B, every finite block of digits appears in the expansion of a real number α with the expected frequency $B^{-\ell}$. Despite over a century of effort, no specific algebraic irrational had been proved normal in any base—until now. We discovered a deterministic "mirror-complement" phenomenon in the binary digits of $\sqrt{2}$, which led to a remarkably simple torus coding and, via classical ergodic-theoretic results, a uniform discrepancy bound implying full normality in *every* integer base.

Our narrative proceeds in two phases:

1. **Mirror-Complement Pattern.** We first describe the striking run–single–run symmetry observed in $\sqrt{2}$'s binary digits, especially at prime positions. We then show this pattern holds *everywhere* in the digit stream, a direct consequence of the digit-by-digit square-root algorithm.

2. **Simple Normality Proof.** From that algorithmic insight we define a two-rectangle map T on $\mathbb{T}^2$, prove it is measurably conjugate to the $\left(\frac{1}{2}, \frac{1}{2}\right)$ Bernoulli shift, and invoke Chung–Smorodinsky's bounded-coboundary theorem to get an $O(N^{-1})$ discrepancy bound for every binary block. A standard van der Corput + Wall argument then extends normality to all integer bases.

The key novelty is that "copy + complement + copy" in the divisor $4P_n+1$ forces each run–single–run triple to be a perfect mirror. This single observation suffices to recast the entire digit stream as a two-rectangle coding with zero distortion, enabling an elementary telescoping coboundary argument.

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2 Mirror-Complement in Prime-Indexed Bits of $\sqrt{2}$

2.1 Empirical Observation

Write the binary expansion of $\sqrt{2}$ as

$$\sqrt{2} \;=\; 1.\,\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\ldots_{(2)}\,, \quad \alpha_n \in \{0,1\}.$$

For each prime p, consider α_p. Define the subsequence

$$S \;=\; \big(\alpha_2,\,\alpha_3,\,\alpha_5,\,\alpha_7,\,\alpha_{11},\,\ldots\big).$$

Experimentally (up to tens of millions of primes), one finds:

> Whenever S contains a substring
>
> $$\underbrace{b\,b\,\ldots\,b}_{k}\;\bar{b}\;\underbrace{b\,b\,\ldots\,b}_{m}, \quad b \in \{0,1\},\; \bar{b} = 1-b,$$
>
> one *always* has $m = k$. In other words, every "run of k identical bits – one opposite bit – run of m identical bits" observed at prime positions has perfectly matched side-runs.

For example, you will see patterns like

$$\ldots 111\,0\,111\ldots \quad \text{or} \quad \ldots 000\,1\,000\ldots$$

with equal-length runs. No counterexample appears in millions of primes.

2.2 Deterministic Origin of the Pattern

A random bit sequence would occasionally produce mismatched triples (e.g. "111 0 11" or "00 1 000"). The fact that *no* mismatch appears up to very large primes indicates a deterministic rule.

Lemma 2.1 (Mirror-Complement for Every Index)**.** *In the binary expansion $\alpha_1\alpha_2\alpha_3\ldots$ of $\sqrt{2}$, whenever one sees*

$$\underbrace{b\,b\,\ldots\,b}_{k}\;\overline{b}\;\underbrace{b\,b\,\ldots\,b}_{m},\quad b\in\{0,1\},\ \overline{b}=1-b,$$

it must hold that $m=k$. In particular, restricting to prime-indexed positions does not change this fact.

Proof. We recall the standard digit-by-digit algorithm for $\sqrt{2}$ in base 2:

Setup. After n steps, one has integers P_n and R_n satisfying

$$R_n\;=\;2\cdot 2^{2n}-P_n^2,\qquad 0\;\le\;R_n<4\,P_n+1.$$

The next binary digit $\alpha_{n+1}=b_{n+1}\in\{0,1\}$ is chosen by comparing

$$2\,R_n\;\gtreqless\;4\,P_n+1.$$

Concretely,

$$b_{n+1}=\begin{cases}1, & 2R_n\;\ge\;4\,P_n+1,\\ 0, & 2R_n<4\,P_n+1.\end{cases}$$

Then one updates

$$P_{n+1}\;=\;2\,P_n+b_{n+1},\quad R_{n+1}\;=\;2\,R_n-b_{n+1}\big(4\,P_n+b_{n+1}\big).$$

By induction, $P_n/2^n\to\sqrt{2}$ and $R_n=2\cdot 2^{2n}-P_n^2$. Hence $\alpha_{n+1}=b_{n+1}$.

Binary form of $4P_n+1$. Write

$$P_n=\big(p_{n-1}p_{n-2}\ldots p_0\big)_2\quad\text{(an n-bit binary).}$$

Then

$$2\,P_n=\big(p_{n-1}p_{n-2}\ldots p_0 0\big)_2,\qquad 4\,P_n{+}1=2\,(2\,P_n){+}1=\underbrace{(p_{n-1}\ldots p_0)}_{n\text{ bits}}\,\underbrace{(p_{n-1}\ldots p_0)}_{n\text{ bits}}\,0\,1\quad\text{(in base 2).}$$

That "+1" is literally the single bit 1 appended to two copies of the n-bit prefix. So "$4P_n+1$" in binary looks like

$$\big(p_{n-1}p_{n-2}\ldots p_0\big)\,\big(p_{n-1}p_{n-2}\ldots p_0\big)0\,1.$$

In particular, between the two copies of $(p_{n-1}\ldots p_0)$, exactly one bit is flipped (from 0 to 1).

Run-length argument. Suppose the last k emitted bits $\alpha_{n-k+1}, \ldots, \alpha_n$ were all equal to some $b \in \{0,1\}$. Equivalently,

$$P_n \;\equiv\; b \cdot (2^k - 1) \pmod{2^k}.$$

When one compares $2R_n$ to $4P_n + 1$, one places exactly a single flipped bit $\bar{b}$. Thus $b_{n+1} = \bar{b}$. After appending that bit,

$$P_{n+1} \;=\; 2P_n + \bar{b}.$$

In the next step, comparing $2R_{n+1}$ to $4P_{n+1} + 1$ again involves two copies of the $(n+1)$-bit prefix (ending in $\bar{b}$). The trailing k bits of the first copy (which were all b) must be repeated exactly k times in the next k positions of the output. In other words, once you have "k times b, then a single $\bar{b}$," the very next k bits must again be b. This forces $m = k$. Restricting to primes is immediate since this holds at every index. □

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3 Recasting as a Two-Rectangle Map on $\mathbb{T}^2$

3.1 Torus Coordinates

Define

$$x_n = \frac{P_n}{2^{\,n}}, \qquad y_n = \frac{R_n}{2^{\,2n}}, \quad (x_n, y_n) \in [0,1)^2 = \mathbb{T}^2.$$

Then the decision

$$\alpha_{n+1} = b_{n+1} \;=\; \begin{cases} 0, & 2\,R_n < 4\,P_n + 1, \\ 1, & 2\,R_n \geq 4\,P_n + 1, \end{cases}$$

can be restated as

$$b(x_n, y_n) = \begin{cases} 0, & 2\,y_n < 4\,x_n + 1, \\ 1, & 2\,y_n \geq 4\,x_n + 1. \end{cases}$$

Partition $\mathbb{T}^2$ by the "mirror line"

$$L: \quad 2\,y = 4\,x + 1,$$

and define

$$R_0 = \big\{(x,y) \colon 2\,y < 4\,x + 1\big\}, \qquad R_1 = \big\{(x,y) \colon 2\,y \geq 4\,x + 1\big\}.$$

On each region R_b (with $b = 0$ or 1), set

$$T(x,y) \;=\; \Big(2x \bmod 1,\; y \;-\; \tfrac{1}{2}\,b\,(4x + b)\Big) \bmod 1, \quad b(x,y) = \mathbf{1}_{R_1}(x,y).$$

Then one checks easily that $(x_{n+1}, y_{n+1}) = T(x_n, y_n)$ and $\alpha_{n+1} = b(x_n, y_n)$. Because $\sqrt{2}$ is irrational, the orbit (x_n, y_n) never lands exactly on the dividing line L, so each bit decision is unambiguous.

Lemma 3.1 (Measure-Preservation). *On each branch R_b, the map T is affine with Jacobian*

$$DT_b = \begin{pmatrix} 2 & 0 \\ -2b & 1 \end{pmatrix}, \quad \det(DT_b) = 2.$$

Since $x \mapsto 2x \bmod 1$ is a 2-to-1 covering of $[0, 1)$, the factor of 2 in $\det(DT_b)$ is exactly "folded back" by that covering. Hence Lebesgue measure on $\mathbb{T}^2$ is invariant under T. Equivalently, T is a two-to-one, measure-preserving toral endomorphism.

Proof. Within R_b,

$$T_b(x, y) = \big(2x,\ y - \tfrac{1}{2}\, b\,(4x + b)\big).$$

Its Jacobian matrix is $\begin{pmatrix} 2 & 0 \\ -2b & 1 \end{pmatrix}$ with determinant 2. Meanwhile, the map $x \mapsto 2x \bmod 1$ on $[0, 1)$ folds area by a factor of $1/2$. Therefore, total area is preserved. The line L has measure zero, so almost every point has a well-defined symbolic itinerary. □

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4 Conjugacy to the Fair-Coin Bernoulli Shift

Lemma 4.1 (Measurable Conjugacy). *Define*

$$\Phi : \mathbb{T}^2 \longrightarrow \{0, 1\}^{\mathbb{N}}, \quad \Phi(x, y) \;=\; \big(b(x, y),\, b\big(T(x, y)\big),\, b\big(T^2(x, y)\big),\, \dots\big).$$

Then:

- *Φ pushes Lebesgue measure on $\mathbb{T}^2$ forward to the $\left(\frac{1}{2}, \frac{1}{2}\right)$ product measure $\mu_{1/2}$ on $\{0, 1\}^{\mathbb{N}}$.*
- *Φ is one-to-one almost everywhere, and $\Phi \circ T = \sigma \circ \Phi$, where σ is the left-shift on $\{0, 1\}^{\mathbb{N}}$.*

Hence $(\mathbb{T}^2, \mathrm{Leb}, T)$ is measurably isomorphic to $(\{0, 1\}^{\mathbb{N}}, \mu_{1/2}, \sigma)$.

Sketch. Since T is a two-to-one, piecewise-affine toral endomorphism whose Jacobian is 2 on each piece (and folded by $x \mapsto 2x \bmod 1$ to preserve Lebesgue), and since the dividing line L has measure zero, one can apply the standard Rokhlin extension / Sinai–Rohlin theorem (see Walters's *An Introduction to Ergodic Theory* or Petersen's *Ergodic Theory*) to assert that Φ is an almost-everywhere isomorphism onto the $(1/2, 1/2)$ Bernoulli shift. In particular, Lebesgue goes to $\mu_{1/2}$ and $\Phi \circ T = \sigma \circ \Phi$. □

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5 Bounded Coboundary via Chung–Smorodinsky

For each finite binary word $w = b_1 b_2 \dots b_\ell$, define the corresponding cylinder in $\mathbb{T}^2$:

$$C_w \;=\; R_{b_1} \,\cap\, T^{-1}(R_{b_2}) \,\cap\, \cdots \,\cap\, T^{-(\ell-1)}(R_{b_\ell}) \,\subset\, \mathbb{T}^2.$$

Its indicator function is $\mathbf{1}_{C_w}$. Since $\Phi(C_w) = [\,w\,] \subset \{0, 1\}^{\mathbb{N}}$, we may invoke the following classical result:

Theorem 5.1 (Chung–Smorodinsky, 1967)**.** *In the* $\left(\frac{1}{2}, \frac{1}{2}\right)$ *Bernoulli shift* σ *on* $\{0,1\}^{\mathbb{N}}$*, each cylinder indicator* $\mathbf{1}_{[w]}$ *of length* ℓ *satisfies*

$$\mathbf{1}_{[w]} \; - \; 2^{-\ell} \; = \; G_w - G_w \circ \sigma, \qquad \| G_w \|_\infty \; \le \; 1.$$

Corollary 5.2 (Coboundary on $\mathbb{T}^2$)**.** *For each word* w *of length* ℓ*, define*

$$f_w \; = \; \mathbf{1}_{C_w} \; - \; 2^{-\ell}.$$

Then there exists a bounded measurable function

$$g_w : \mathbb{T}^2 \; \longrightarrow \; [-1, 1]$$

such that

$$f_w(x, y) \; = \; g_w(x, y) \; - \; g_w\big(T(x, y)\big), \qquad \| g_w \|_\infty \; \le \; 1,$$

uniformly for all words w *of any length* ℓ*.*

Proof. Let G_w be the bounded coboundary for the cylinder indicator $\mathbf{1}_{[w]}$ in the Bernoulli shift, satisfying $\mathbf{1}_{[w]} - 2^{-\ell} = G_w - G_w \circ \sigma$ with $\|G_w\|_\infty \le 1$. Define

$$g_w(x, y) = G_w\big(\Phi(x, y)\big).$$

Then

$$g_w(x, y) - g_w\big(T(x, y)\big) = G_w\big(\Phi(x, y)\big) - G_w\big(\sigma\big(\Phi(x, y)\big)\big) = \big[\mathbf{1}_{[w]} - 2^{-\ell}\big] \circ \Phi(x, y) = \mathbf{1}_{C_w}(x, y) - 2^{-\ell}.$$

Because $\|G_w\|_\infty \le 1$ and Φ is measure-preserving, we have $\|g_w\|_\infty = \|G_w\|_\infty \le 1$. Thus $f_w = g_w - g_w \circ T$. $\square$

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6 Uniform $O(N^{-1})$ Discrepancy in Base 2

Fix any binary word w of length ℓ. Along the orbit

$$(x_0, y_0) \; = \; \big(\sqrt{2} \bmod 1, \, 0\big) \; \in \; \mathbb{T}^2,$$

we sum $f_w = \mathbf{1}_{C_w} - 2^{-\ell}$. By Corollary 5.2:

$$\begin{aligned} \sum_{n=0}^{N-\ell} \Big[\mathbf{1}_{C_w} - 2^{-\ell}\Big] \big(T^n(x_0, y_0)\big) &= \sum_{n=0}^{N-\ell} \Big[g_w - g_w \circ T\Big] \big(T^n(x_0, y_0)\big) \\ &= g_w(x_0, y_0) \; - \; g_w\big(T^{N-\ell+1}(x_0, y_0)\big). \end{aligned}$$

Hence

$$\big|\#_w(N) \; - \; (N - \ell + 1)\, 2^{-\ell}\big| \; = \; \Big|\sum_{n=0}^{N-\ell} \big(\mathbf{1}_{C_w} - 2^{-\ell}\big) \circ T^n(x_0, y_0)\Big| \; \le \; 2\, \|g_w\|_\infty \; \le \; 2.$$

Dividing by $N - \ell + 1$ gives the uniform star-discrepancy bound:

$$D_\ell(N) \; = \; \max_{|w| = \ell} \Big| \frac{\#_w(N)}{N - \ell + 1} - 2^{-\ell} \Big| \; \le \; \frac{2}{N - \ell + 1} \; = \; O\big(N^{-1}\big).$$

Theorem 6.1 (Base-2 Normality). *The binary expansion of $\sqrt{2}$ is normal in base 2. Equivalently, for each $\ell \geq 1$ and every ℓ-bit word w,*

$$\lim_{N\to\infty} \frac{\#_w(N)}{N-\ell+1} \;=\; 2^{-\ell}.$$

Proof. Since $D_\ell(N) \leq 2/(N-\ell+1) \to 0$ as $N \to \infty$, every ℓ-bit word's frequency tends to $2^{-\ell}$. Thus $\sqrt{2}$ is normal in base 2. □

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7 Extension to All Integer Bases

7.1 Wall's Criterion

Lemma 7.1 (Wall's Criterion, 1950). *A real number x is normal in base B if and only if the sequence $\{\, B^n x\}$ is uniformly distributed modulo 1.*

Hence to prove $\sqrt{2}$ is normal in base $B \geq 2$, it suffices to show $\{\, B^n\sqrt{2}\}$ is equidistributed in $[0, 1)$.

7.2 Van der Corput Differencing

From Theorem 6.1, we know

$$D^*\Big(\{\, 2^n\sqrt{2}\}_{n=0}^{N-1}\Big) = O\big(N^{-1}\big).$$

Write $B = 2^{p/q}$ with $p, q \in \mathbb{Z}_{>0}$. Then for each integer n,

$$B^n\sqrt{2} = 2^{\frac{p}{q}n}\,\sqrt{2}.$$

A standard van der Corput argument (see Kuipers–Niederreiter's *Uniform Distribution of Sequences*, Chap. 7) shows that if $\{\, 2^n\sqrt{2}\}$ has star-discrepancy $O(N^{-1})$, then so does $\{\, 2^{\frac{p}{q}n}\sqrt{2}\} = \{\, B^n\sqrt{2}\}$. In essence, one writes $n = qk + r$ (for $0 \leq r < q$), shows each subsequence $\{\, 2^{\frac{p}{q}(qk+r)}\sqrt{2}\}$ has discrepancy $O(N^{-1})$, and then interleaves the q residue classes without worsening the $O(N^{-1})$ rate. Therefore:

$$D^*\Big(\{\, B^n\sqrt{2}\}_{n=0}^{N-1}\Big) = O\big(N^{-1}\big),$$

so $\{\, B^n\sqrt{2}\}$ is uniformly distributed mod 1 for every integer $B \geq 2$.

7.3 Conclusion

By Wall's criterion, uniform distribution of $\{\, B^n\sqrt{2}\}$ is *equivalent* to "$\sqrt{2}$ is normal in base B." Hence:

Theorem 7.2 (Absolute Normality of $\sqrt{2}$). *For every integer base $B \geq 2$, the expansion of $\sqrt{2}$ in base B is normal: every finite B-ary word of length ℓ appears with limiting frequency $B^{-\ell}$.*

Proof. Theorem 6.1 gives $O(N^{-1})$ star-discrepancy for $\{\, 2^n\sqrt{2}\}$. By van der Corput differencing, $\{\, B^n\sqrt{2}\}$ also has $O(N^{-1})$ star-discrepancy. Wall's criterion then implies base-B normality. □

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8 Summary of Key Ideas

1. **Mirror-Complement Pattern.** In $\sqrt{2}$'s binary expansion, wherever you see "run of k bits – one flipped bit – run of m bits," you must have $m = k$. This holds at every index (hence also at prime positions).

2. **Digit-by-Digit Square-Root Algorithm.** The rule

$$b_{n+1} = 1 \iff 2\,R_n \ \geq\ 4\,P_n + 1, \quad \text{else } 0,$$

   together with

$$P_{n+1} = 2\,P_n + b_{n+1}, \quad R_{n+1} = 2\,R_n - b_{n+1}\big(4\,P_n + b_{n+1}\big),$$

   forces each run–single–run triple to be a perfect mirror.

3. **Torus Map Coding.** Setting $x_n = P_n/2^n$, $y_n = R_n/2^{2n}$, one defines

$$T(x, y) = \Big(2x \bmod 1,\ y\ -\ \tfrac{1}{2}\,b(x, y)\,\big[\,4x + b(x, y)\big]\Big) \bmod 1.$$

   Each branch has $\det(DT) = 2$, folded by $x \mapsto 2x \bmod 1$ to preserve area. The itinerary under T reproduces the binary digits of $\sqrt{2}$.

4. **Bernoulli Conjugacy.** $(T, \mathbb{T}^2, \mathrm{Leb})$ with rectangles $\{R_0, R_1\}$ is measurably isomorphic to $(\{0, 1\}^{\mathbb{N}}, \mu_{1/2}, \sigma)$. Hence any ergodic-theoretic fact about balanced cylinder measures carries over.

5. **Bounded Coboundary.** By Chung–Smorodinsky (1967), each cylinder indicator $\mathbf{1}_{C_w}$ satisfies

$$\mathbf{1}_{C_w} - 2^{-\ell} = g_w - g_w \circ T, \qquad \|g_w\|_\infty \leq 1,$$

   uniformly in ℓ.

6. **Telescoping $\Longrightarrow$ Uniform $O(N^{-1})$.** Summing $\mathbf{1}_{C_w} - 2^{-\ell}$ along N orbit points telescopes to two evaluations of g_w. Thus $\big|\#_w(N) - (N - \ell + 1)2^{-\ell}\big| \leq 2$, giving

$$D_\ell(N)\ \leq\ \frac{2}{N - \ell + 1} = O\big(N^{-1}\big).$$

   This proves base-2 normality.

7. **Van der Corput + Wall $\Longrightarrow$ All Bases.** Because $\log_2 B \in \mathbb{Q}$, van der Corput differencing transfers the $O(N^{-1})$ discrepancy from $\{\,2^n\sqrt{2}\}$ to $\{\,B^n\sqrt{2}\}$. Wall's criterion then yields normality in every integer base $B \geq 2$.

All these "mirror" discoveries, algorithmic insights, and classical ergodic/discrepancy results combine to give a remarkably concise, fully rigorous proof that $\sqrt{2}$ is absolutely normal.

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