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# ON VALIDITY IN NON STANDARD ANALYSIS OF RIEMANN- DINI THEOREM

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## INTRODUCTION

In this article we will use some fundamental concept of non standard analysis founded by mathematical logician Abraham Robinson in the sixties of twenty century subsequently it was simplified by Jerome Keisler, these concepts have already been proposed in a previous article of mine [1] we demonstrate that with the use of non-standard analysis and the definition of non-standard rearrangement introduced in [1] we overcome some paradoxes that often arise with numerical series .

## COMMENT

This article is an extension and an application of arguments exposed and defined in another article of mine [1] however for completeness some definitions are repeated also in this article

## DEFINITION AND PRELIMINARIES

We give here some concept and definition that we will use for the continuations of this article. Fundamentally Keisler approach is based on the following two principles [2][3].

### THE EXTENSION PRINCIPLE

- a) The real numbers form a subset of the hyperreal numbers, and the order relation  $x < y$  for the real numbers is a subset of the order relation for the hyperreal numbers
- b) There is a hyperreal number that is greater than zero but less than every positive real number
- c) For every real function  $f$  of one or more variables we are given a corresponding hyperreal function  $f^*$  of the same number of variables ,  $f^*$  is called the natural extension of  $f$  (in this article  $f^*$  is called the extension of  $f$  at non standard model of analysis) . Furthermore with each relation  $X$  on  $R$  there is corresponding relation  $X^*$  on  $R^*$  called the natural extension of  $X$  [7]

### THE TRANSFER PRINCIPLE

Every real statement that hold for one or more particular real function holds for the hyperreal natural extensions of these functions, the transfer principle is equivalent to Leibniz' principle , which is the property that for each real bounded sentence  $\phi \in L$ , is true if and only if  $\phi^*$  is true .  $L$  is the language of the first order predicate [3] we still give the following definitions always of Keisler [2]:

## DEFINITIONS

A hyperreal number  $b$  is said to be:  
positive infinitesimal if  $b$  is positive but less than every positive real number,  
negative infinitesimal if  $b$  is negative but greater than every negative real number.  
A hyperreal number  $b$  is said to be:

finite if  $b$  is between two real numbers,  
 positive infinite if  $b$  is greater than every real number,  
 negative infinite if  $b$  is less than every real number.

### ABSTRACT

By a simple extension and application of rearrangement definition of a simply convergent series, at non standard model of analysis called "non standard rearrangement" already introduced by [1] we overcome some paradoxes that often arise with numerical series to this end we give three significant examples of "standard" and "non standard rearrangement" of the harmonic series with alternate signs.

Instead notable result is that with the definition of " non standard rearrangement " introduced in [1] the commutative property of addition continues to hold even for simply convergent series (such as harmonic series with alternate) contrary to what is stated by Riemann-Dini theorem or Riemann rearrangement theorem,

### TEXT

Stated harmonic series with alternate signs :

$$S = \sum_1^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2 \quad (1)$$

extending it at "non standard model" we have

$$S^* = \sum_1^{\omega} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2 \quad (1)^*$$

with  $\omega$  = infinite number positive , \* = asterisk indicates the extension of  $\zeta(s)$  at non standard model of analysis (see definitions and preliminaries) . We call "non standard rearranged" series of  $(1)^*$  a series with both the conditions following ,the first condition is: "the rearranged series of  $(1)^*$  must consist in a permutation of all standard terms (i.e with form  $\frac{1}{N}$ ) contained in  $(1)^*$  " the second condition is : "the rearranged series must consist in a permutation of all non standard terms ( i.e with form  $\frac{1}{N^*}$ ) contained in  $(1)^*$  and no another terms "( $N$  is the Natural numbers set ,  $N^*$  is the infinite hypernatural numbers set i.e the extension of  $N$  at non standard model of analysis). Since real numbers are a subset of hyperreals ( see extension principle) the non standard rearrangement could be defined by a single condition ,but to Better highlight the difference between two rearrangements two were written. We give an example of standard rearrangement (using only (1) condition) and correspondent "non standard rearrangement" (using (1) and (2) conditions)

$$S = \sum_1^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{1}{2n-1} - \sum_1^{\infty} \frac{1}{2n} \quad (1).$$

By exchanging the terms of series (1) so as to have a sequence of two positive terms followed by a negative term we have:

$$(1 + \frac{1}{3}) - \frac{1}{2} + (\frac{1}{5} + \frac{1}{7}) - \frac{1}{4} + (\frac{1}{9} + \frac{1}{11}) - \frac{1}{6} + \dots \quad (2)$$

The (2) is an example of standard rearrangement of the (1). We demonstrate that (2) convergent to  $\frac{3}{2}S = \frac{3}{2}Ln2$ , in fact dividing (1) by 2 we have

$$\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \dots \quad (3)$$

adding term by term the (2) and the (3)

$$S + \frac{1}{2}S = \frac{3}{2}S = \frac{3}{2}Ln2 = 1 + 0 - \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + 0 - \frac{1}{4} - \frac{1}{4} + \frac{1}{5} + 0 - \frac{1}{6} + \frac{1}{6} + \frac{1}{7} + 0 - \frac{1}{8} - \frac{1}{8} + \dots \quad (4). \text{ or}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = S$$

$$\text{That is } \frac{3}{2}S = S \quad (4)$$

We can arrive at the same result by applying Ohm's rearrangement theorem with it is shown that in (1) taking  $p$  positive and  $q$  negative terms we have the sum :  $S = \ln(2) + (1/2)\ln(p/q)$ , in previous example was  $p=2$  and  $q = 1$ , instead with our definition of "non standard rearrangement" it is always  $p^* = q^*$  both infinite numbers [2][3] as seen in [1]). The result (4) is in according with Riemann theorem of rearrangement, instead in non standard analysis using the two definitions of the non standard rearrangement of series (1\*) we have:

$$S^* = \sum_1^{\omega} \frac{(-1)^{n+1}}{n} = \sum_1^{\frac{\omega}{2}} \frac{1}{2n-1} - \sum_1^{\frac{\omega}{2}} \frac{1}{2n} \quad (1^*)$$

Dividing by 2 the (1 \*) we have:

$$\frac{1}{2}S^* = \sum_1^{\frac{\omega}{2}} \frac{1}{4n-2} - \sum_1^{\frac{\omega}{2}} \frac{1}{4n} \quad (2^*)$$

( if we associate as in ( 2) for each pair of positive numbers p a single negative number q we should write ( 2\*) as:

$$\sum_1^{\frac{\omega}{2}} \frac{1}{2n-1} - \sum_1^{\frac{\omega}{4}} \frac{1}{2n}$$

it would be the standard rearrangement extended to the non-standard model regardless of the definition of non-standard rearrangement ,in fact since  $p^* = \omega/2$  while  $q^* = \omega/4$  therefore  $p^* \neq q^*$  and the ( 2\*) could not be a non-standard rearrangement of ( 2) since it should always be  $p^* = q^*$  ). Adding  $S^*$  with  $\frac{1}{2}S^*$  we have:

$$S^* + \frac{1}{2}S^* = \sum_1^{\frac{\omega}{2}} \frac{1}{2n-1} - \sum_1^{\frac{\omega}{2}} \frac{1}{2n} + \sum_1^{\frac{\omega}{2}} \frac{1}{4n-2} - \sum_1^{\frac{\omega}{2}} \frac{1}{4n} \quad (3^*)$$

as it is  $\sum_1^{\frac{\omega}{2}} \frac{1}{4n} = \frac{1}{2} \sum_1^{\frac{\omega}{2}} \frac{1}{2n}$  and

$$\sum_1^{\frac{\omega}{2}} \frac{1}{4n-2} = \frac{1}{2} \sum_1^{\frac{\omega}{2}} \frac{1}{2n-1} \quad \text{we have:}$$

$$S^* + \frac{1}{2}S^* = (1 + \frac{1}{2}) \sum_1^{\frac{\omega}{2}} \frac{1}{2n-1} - \sum_1^{\frac{\omega}{2}} \frac{1}{2n} = (1 + \frac{1}{2}) \sum_1^{\frac{\omega}{2}} \frac{1}{2n} = (1 + \frac{1}{2}) [\sum_1^{\frac{\omega}{2}} (\frac{1}{2n-1} - \frac{1}{2n})] \quad (4^*)$$

Since in the square bracket we have the harmonic series with alternating signs calculated up to  $\omega/2$  it converges to  $\ln 2$  for the Cauchy convergence test therefore have  $S^* + \frac{1}{2}S^* = \frac{3}{2}S^*$ , by the Cauchy convergence test we see that the harmonic series with alternating signs always converges to the same quantity (standard) if calculated for any infinite number [1][2]. Now we generalize and formalize the two conditions which together constitute the new concept of "non standard rearrangement" of a series (1\*) for the first condition that characterizes the classical standard rearrangement of a series we

have: give the series  $\sum_k a_k$  with real or complex terms and one bijective

function  $\pi : N \rightarrow N$  it's called rearranged series of  $\sum_k a_k$  according  $\pi$  the series

$\sum_k a_{\pi(k)}$ . The second condition is the following: given the series  $\sum_k a_k^*$  with non standard and complex terms (infinitesimal complex and infinitesimal real numbers) and one bijective non standard function (see definition and preliminaries)  $\pi^* : N^* \rightarrow N^*$  ( $N^*$  are also called infinite hypernatural numbers [6])

It is called rearranged series according to  $\pi^*$  the series:  $\sum_k a_{\pi^*(k)}^*$ . The

biunivocity of  $\pi$  and  $\pi^*$  ensure in particular that the number of terms of the rearranged series have the same number of terms (positives and negatives) as the originals series respectively in  $N$  and in  $N^*$ . It is shown in standard analysis (Ohm's rearrangement theorem) that in (1) taking  $p$  positive and  $q$  negative terms we have the sum :  $S = \ln(2) + (1/2)\ln(p/q)$ . instead with our definition of "non standard rearrangement" it is always  $p^* = q^*$  both infinite numbers . The validity of the commutative property in (1)\* should not be surprising as this property is obviously valid for real numbers, according to the transfer principle it is also valid for hyperreal numbers in fact we have:

$$(\forall x \forall y) \in \mathbb{R} (x+y=y+x)$$

it is true in  $\mathbb{R}$ , extending it at non standard model we have :  $(\forall x^* \forall y^*) \in \mathbb{R}^*$   $(x^*+y^*=y^*+x^*)$  ( $\mathbb{R}^*$ = set of hyperreal numbers)( Following common usage we omit the asterisk for the sum between hyperreal numbers). Why is Riemann -Dini theorem valid with " standard rearrangement" despite the commutativity of addition? The answer is simple in fact formalizing this theorem (by the first

order predicate logic) we have: let  $\sum_k a_k$  a simply convergent series

$$(\forall x \in \mathbb{R} \cup \{-\infty, +\infty\}) \exists \pi : N \leftrightarrow N:$$

$$\lim_{N \rightarrow \infty} \sum_{k=1}^N a_{\pi(k)} = x$$

Extending at non standard model we have:

$$(\forall x^* \in \mathbb{R}^* \exists \pi^* : N^* \leftrightarrow N^*) :$$

$$\sum_{k=1}^{\omega} a_{\pi^*(k)} = x^*$$

It is precisely the limit with  $N \rightarrow \infty$   $\sum_{k=1}^N a_{\pi(k)}$  or in non standard model

$\sum_{k=1}^{\omega} a_{\pi^*(k)}$  that introduces (or omits) an infinite quantity of infinitesimals whose sum is different from zero ( and from being infinitesimal) not existing in the original series (1)\* these quantities are excluded only with the " non standard rearrangement" as seen before. We give another example of standard rearrangement in which apparently as many positive terms as negative terms are counted ,still obtaining a different result from (1) [or ln2] which however with non standard rearrangement the result of this rearrangement respects the commutative property of addition for simply convergent series. Be given

$$S = (1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots) - (\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots) =$$

$$(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots) + (\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots) - 2(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots) =$$

$$(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots) - (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots) = 0$$

Or

$$S = \sum_{n=1}^{\infty} \frac{1}{2n-1} - \sum_{n=1}^{\infty} \frac{1}{2n} = \sum_{n=1}^{\infty} \frac{1}{2n-1} + \sum_{n=1}^{\infty} \frac{1}{2n} - 2 \sum_{n=1}^{\infty} \frac{1}{2n} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n} = 0$$

Therefore for this standard rearrangement  $S = 0$ .

Applying the non standard rearrangement we have.  $S^* = \sum_1^{\frac{\omega}{2}} \frac{1}{2n-1} - \sum_1^{\frac{\omega}{2}} \frac{1}{2n} =$

$$\sum_1^{\frac{\omega}{2}} \frac{1}{2n-1} + \sum_1^{\frac{\omega}{2}} \frac{1}{2n} - 2 \sum_1^{\frac{\omega}{2}} \frac{1}{2n} = \sum_1^{\omega} \frac{1}{n} - 2 \sum_1^{\frac{\omega}{2}} \frac{1}{2n}$$

(In this step according to the standard

rearrangement this sum is zero since  $S = \sum_{n=1}^{\infty} \frac{1}{n} - 2 \sum_{n=1}^{\infty} \frac{1}{2n} = 0$ )

$$\text{Since } 2 \sum_1^{\frac{\omega}{2}} \frac{1}{2n} = \sum_1^{\frac{\omega}{2}} \frac{1}{n} \text{ we obtain } S^* = \sum_1^{\omega} \frac{1}{n} - \sum_1^{\frac{\omega}{2}} \frac{1}{n} = \sum_{\frac{\omega}{2}+1}^{\omega} \frac{1}{n}$$

Using (E.M.F) we have

$$\sum_{\frac{\omega}{2}+1}^{\omega} \frac{1}{n} \sim \left\| \frac{n^{-1}}{1} \right\|_{\frac{\omega}{2}}^{\omega} = \ln 2$$

Therefore  $S^* = \ln 2$  in accordance with commutative property, in this case  $p^*=q^*=\omega/2$ , therefore it is a non standard rearrangement of (1\*). Let us now look at a well-known result of Ramanujan in the light of non-standard rearrangement. As is known, Ramanujan arrived at the following result

$$\sum_1^{\infty} n = 1+2+3+4+\dots = -\frac{1}{12} \quad (5)$$

This result is clearly absurd, let's see how it should be interpreted according to the non-standard rearrangement introduced in [1] it obviously in non-standard analysis becomes

$$\sum_1^{\omega} n = 1+2+3+4+\dots = -\frac{1}{12} \quad (5^*)$$

Applying the formula (III), obtained with a non standard rearrangement of (1\*) see[1] we have with  $s=-1$  and  $R = \omega - 1$ ,

$$\zeta^*(-1) = \sum_1^{\omega-1} n - \frac{1}{2}\omega^2 + \frac{1}{2}\omega - \omega - \frac{1}{12}\omega^0 \quad (5^*) \text{ (The number } -\frac{1}{12} \text{ is the}$$

only real number in second member of (5\*)).

In (5\*)  $\omega^0 = e^{0\ln\omega} = e^0 = 1$ . Therefore we have ( adding and subtracting the positive infinite number  $\omega$  to the second member)  $\zeta^*(-1) =$

$$\sum_1^{\omega} n - \frac{1}{2}\omega^2 - \frac{1}{2}\omega - \frac{1}{12} \quad \text{That is ( since } \zeta^*(-1) = -\frac{1}{12} \text{)}$$

$$\sum_1^{\omega} n = \frac{1}{2}\omega^2 + \frac{1}{2}\omega = \frac{1}{2}\omega(\omega+1) \quad (6^*)$$

Where  $\zeta^*(-1)$  is the extension at non standard model of analysis of zeta function of Riemann obtained by non standard rearrangement calculated in point  $-1$  as seen in [1]. Furthermore, the following simple formula is well known.

$$\sum_1^m n = \frac{1}{2} m(m+1) \quad (\forall m, \forall n) \in N$$

it is true for all natural numbers  $m$  and  $n$ , therefore by transfert principle (see preliminaries and definitions) is true too the following formula extended at

non standard model of analysis  $\sum_1^{m^*} n = \frac{1}{2} m^*(m^*+1) \quad (\forall m^*, \forall n^*) \in N^*$  (That

is the  $(6^*)$  with  $m^* = \omega$ ). Where  $N^*$  it is the ensemble of infinite hypernatural numbers

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