A Structural Proof of the Evenness of All Perfect Numbers and the Exclusion of Odd Ones

YoungHwan Yun

Zero Theoretical Physics Lab, Seoul, Republic of Korea.

yhyun690@naver.com
younghwan.yun@snu.ac.kr

Abstract

We introduce a structural decomposition framework for perfect numbers, based on additive and multiplicative symmetries among their proper divisors. Under this model, we derive a system of proportional identities that all known even perfect numbers satisfy. By analyzing the integer conditions required for recursive consistency in the divisor sequences, we prove that such a structure necessarily implies the presence of the even divisor 2, thereby excluding the possibility of an odd perfect number conforming to this model. Although primarily developed for perfect numbers, the additive portion of the framework may also encompass semiperfect numbers, which share similar but relaxed divisor-sum properties. Our results support the long-standing conjecture that no odd perfect numbers exist and suggest a broader structure for divisor-based classification of integers.

1. Historical Overview of Perfect Numbers

The study of *perfect numbers* has captivated mathematicians for over two millennia. A perfect number is a positive integer equal to the sum of its proper divisors, excluding itself.

1.1 Ancient Foundations: Euclid and Nicomachus The notion of perfect numbers dates back to ancient Greece. In *Elements*, Book IX, Euclid provided a construction for even perfect numbers[1]:

$$n = 2^{p-1}(2^p - 1),$$

where $2^p - 1$ must be a prime, now known as a Mersenne prime. Later, Nicomachus of Gerasa (1st century CE) discussed perfect numbers such as 6, 28, 496, and 8128, embedding them in numerological contexts[2].

1.2 Renaissance and Enlightenment Era: Mersenne and Euler In the 17th century, Marin Mersenne compiled a list of potential primes of the form $2^p - 1$, known as Mersenne primes. Leonhard Euler, in the 18th century, proved that all even perfect numbers must be of Euclid's form, showing that[3][4]:

$$\text{If } n \text{ is even and perfect, then } n = 2^{p-1}(2^p - 1), \text{ where } 2^p - 1 \text{ is prime.}$$

1.3 Modern Number Theory: Search for Odd Perfect Numbers Despite extensive effort, no odd perfect number has been found. Several important results include:

- Touchard (1953): An odd perfect number must be of the form $12k + 1$ or $36k + 9$[5].
- Nielsen (2007): Any odd perfect number must have at least 75 prime factors[6].
- Ochem and Rao (2012): Raised this lower bound to 101 distinct prime factors[7].

1.4 Computational Era: GIMPS and Large Perfect Numbers Modern searches are powered by distributed computing through the GIMPS (Great Internet Mersenne Prime Search) project. As of 2024, 51 even perfect numbers have been discovered, each associated with a known Mersenne prime. No odd perfect number has yet been identified[8].

2. Structural Model and Integer Constraints of Perfect Numbers

Definition 2.1. *A natural number $N \in \mathbb{N}$ is called* perfect *if the sum of its proper divisors equals the number itself [9]:*

$$\sigma(N) - N = N \quad \textit{or equivalently,} \quad \sigma(N) = 2N,$$

where $\sigma(N)$ denotes the sum-of-divisors function.

Lemma 2.1 (Additive Decomposition)**.** *Let N be a perfect number. Then its proper divisors can be partitioned into two strictly increasing sequences:*

$$1 < a_1 < a_2 < \cdots < a_k < b_1 < b_2 < \cdots < b_k < N,$$

such that the sum of all these divisors equals N:

$$1 + a_1 + a_2 + \cdots + a_k + b_1 + b_2 + \cdots + b_k = N. \tag{2.1}$$

Theorem 2.1 (Multiplicative Symmetry)**.** *Under the decomposition above, the following identity must also hold:*

$$a_k b_1 = a_{k-1} b_2 = \cdots = a_1 b_k = N. \tag{2.2}$$

Figure 1 shows the equation (2.2)

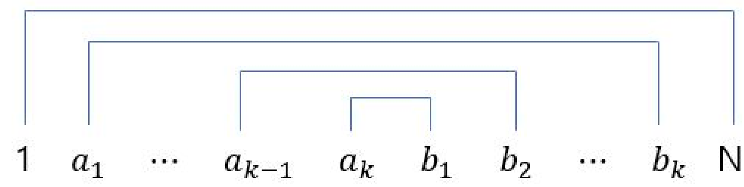

Figure 1: Divisors for N

Lemma 2.2 (Recursive Proportional Relations)**.** *To ensure compatibility between equations (2.1) and (2.2), we require that:*

$$1 + a_1 + \cdots + a_k + b_1 + \cdots + b_{k-1} = \alpha b_k \tag{2.3}$$
$$1 + a_1 + \cdots + a_k + b_1 + \cdots + b_{k-2} = \beta b_{k-1} \tag{2.4}$$
$$1 + a_1 + \cdots + a_k + b_1 + \cdots + b_{k-3} = \gamma b_{k-2} \tag{2.5}$$
$$\vdots$$

where $\alpha, \beta, \gamma, \ldots \in \mathbb{N}$ *are proportional constants.*

Proof. For equation (2.3), if we take the left-hand side of equation (2.1) and exclude the term b_k, then making the remaining sum a multiple of b_k, as in (2.3), ensures that the identity $a_1 \cdot b_k = N$ in equation (2.2) holds. That is,

$$\alpha \cdot b_k + b_k = N \Rightarrow (\alpha + 1)b_k = N, \tag{2.6}$$

which confirms that N is a multiple of b_k.

Similarly, for equation (2.4), if we exclude b_{k-1} from the left-hand side of (2.3), and make the remaining terms a multiple of b_{k-1}, the identity $a_2 \cdot b_{k-1} = N$ in (2.2) can be satisfied. That is,

$$\beta \cdot b_{k-1} + b_{k-1} = \alpha b_k \Rightarrow (\beta + 1)b_{k-1} = \alpha b_k = \alpha \frac{N}{\alpha + 1}. \tag{2.7}$$

For equation (2.5), excluding b_{k-2} from the left-hand side of (2.4), and expressing the remaining terms as a multiple of b_{k-2}, leads to the identity $a_3 \cdot b_{k-2} = N$. Thus,

$$\gamma \cdot b_{k-2} + b_{k-2} = \beta b_{k-1} \Rightarrow (\gamma + 1)b_{k-2} = \beta b_{k-1} = \alpha\beta \frac{N}{\alpha + 1} \frac{1}{\beta + 1}. \tag{2.8}$$

This logic can be continued recursively, producing a consistent system of proportional relations that ensure all identities in equation (2.2) are satisfied. $\square$

Remark 2.1. These identities ensure that removing each b_j from the additive decomposition yields an integer multiple of b_j, thereby aligning with the multiplicative identity in (2.2).

Theorem 2.2 (Integer Consistency and Evenness)**.** *If N satisfies the structural system in (2.6)–(2.8), then the elements a_i are defined recursively as:*

$$\begin{aligned} a_1 &= \alpha + 1 \\ a_2 &= \frac{(\beta+1)(\alpha+1)}{\alpha} \\ a_3 &= \frac{(\gamma+1)(\beta+1)(\alpha+1)}{\alpha\beta} \\ &\vdots \end{aligned}$$

These expressions are integers only when $\alpha = \beta = \gamma = \cdots = 1$. Therefore, $a_1 = 2$, implying:

$$2 \mid N.$$

Hence, any such perfect number must be even.

Proof. By comparing equation (2.6) with equation (2.2), $a_1 b_k = N$, we obtain:

$$a_1 = \alpha + 1$$

By comparing equation (2.7) with equation (2.2), $a_2 b_{k-1} = N$, we obtain:

$$a_2 = \frac{(\beta+1)(\alpha+1)}{\alpha}$$

By comparing equation (2.8) with equation (2.2), $a_3 b_{k-2} = N$, we obtain:

$$a_3 = \frac{(\gamma+1)(\beta+1)(\alpha+1)}{\alpha\beta}$$

Since a_2 must be an integer, the expression $\frac{\alpha+1}{\alpha}$ must also be an integer. This condition holds if and only if $\alpha = 1$.

Similarly, for a_3 to be an integer, $\frac{\beta+1}{\beta}$ must be an integer, which implies $\beta = 1$.

The same reasoning applies to the remaining terms as well. □

Remark 2.2. In order to further clarify the conclusion that $\alpha = \beta = \gamma = \cdots = 1$, we now provide an illustrative example with additional explanation. Consider the following structural system:

$$1 + a_1 + a_2 + b_1 + b_2 = N \tag{1}$$

$$a_2 \cdot b_1 = a_1 \cdot b_2 = N \tag{2}$$

$$1 + a_1 + a_2 + b_1 = \alpha \cdot b_2 \tag{3}$$

$$1 + a_1 + a_2 = \beta \cdot b_1 \tag{4}$$

Under this system, we derive:

$$a_1 = 1 + \alpha \tag{5}$$

$$a_2 = \frac{(1+\alpha)(1+\beta)}{\alpha} \tag{6}$$

Now substituting (5) and (6) into equation (4), we obtain:

$$1 + 1 + \alpha + \frac{(1+\alpha)(1+\beta)}{\alpha} = \beta \cdot b_1 \tag{7}$$

Or equivalently,

$$b_1 = \frac{2+\alpha}{\beta} + \frac{(1+\alpha)(1+\beta)}{\alpha \cdot \beta} \tag{8}$$

From equation (8), observe that the term

$$\frac{(1+\alpha)(1+\beta)}{\alpha \cdot \beta}$$

is not an integer unless $\alpha = \beta = 1$. Otherwise, b_1 fails to be an integer.

Since b_1 must be an integer to satisfy equations (1) and (3), and it is directly involved in the computation of b_2 and N, any non-integer value of b_1 propagates inconsistency throughout the structure.

Therefore, the only values of α and β that allow all variables a_1, a_2, b_1, b_2, and N to remain integers are:

$$\boxed{\alpha = \beta = 1}$$

This confirms that the integer structure of the model is strictly preserved only under this unique assignment.

3. Nonexistence of Odd Perfect Numbers Under This Model

Theorem 3.1 (Exclusion of Odd Perfect Numbers)**.** *If a perfect number N satisfies the structural decomposition defined by equations (2.1)–(2.5), then N must be even. Thus, no odd perfect number can satisfy this structure.*

Proof. From Theorem 2.2 , the only valid integer solutions occur when all proportional constants are 1. This leads to $a_1 = 2$. Since $a_1 \mid N$, and $a_1 = 2$, it follows that $2 \mid N$. Therefore, N must be even.

Assuming the structure is a necessary property of all perfect numbers, the existence of an odd perfect number would contradict this derived evenness, leading to a contradiction. □

Remark 3.1. This result provides a structural explanation for why no odd perfect numbers have been found, and if the decomposition model holds universally, it establishes the nonexistence of odd perfect numbers.

Remark 3.2. The decomposition model presented here, while developed for perfect numbers, may also be adaptable to semiperfect numbers, as it is based on additive divisor structures. However, the strict multiplicative symmetry conditions used to eliminate odd perfect numbers may not hold for such generalized cases.

4. Conclusion

In this work, we proposed a structural decomposition model for perfect numbers, built on the additive arrangement and multiplicative symmetry of their proper divisors. The model defines two subsets of divisors, $\{a_i\}$ and $\{b_j\}$, linked through both a complete additive sum and recursive multiplicative identities. Specifically, identities such as $a_k \cdot b_1 = a_{k-1} \cdot b_2 = \cdots = a_1 \cdot b_k = N$ enforce a symmetry that strongly constrains the form of N.

Through symbolic derivation and analysis, we demonstrated that the resulting recursive formulae yield integer values for the a_i only when all proportionality constants $\alpha, \beta, \gamma, \ldots$ are equal to 1. This leads to $a_1 = 2$, and thus $2 \mid N$, proving that any number satisfying the full structure must be even. Consequently, no odd number can satisfy the perfect number structure as defined herein, supporting the long-standing conjecture that odd perfect numbers do not exist.

Remark 3.3. Although this model was originally formulated for perfect numbers, its additive structure—particularly the identity

$$1 + a_1 + a_2 + \cdots + a_k + b_1 + b_2 + \cdots + b_k = N$$

—may naturally apply to semiperfect numbers as well. These are numbers for which a subset of proper divisors sums exactly to N, without requiring full multiplicative symmetry[10]. Future work could explore whether relaxed forms of our recursive conditions can be adapted to characterize semiperfect or abundant numbers in a similar framework.

This structural approach provides not only insight into the parity of perfect numbers but also opens a pathway toward a more unified classification of integers based on divisor configurations.

References

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