THE EXPLICIT abc CONJECTURE OF ALAN BAKER IMPLIES c SMALLER THAN R^2 TRUE

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Abstract. In this paper, we assume that the explicit abc conjecture of Alan Baker (2004) is true, and prove that $c < rad^2(abc)$ is true; it is one of the keys to resolve the mystery of the abc conjecture.

1. Introduction and Notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers, and $\alpha_i \geq 1$ positive integers. The radical of a the integer $\prod_i a_i$ is denoted as $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{1.2}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [5]. It describes the distribution of the prime factors of the two integers along with their sum. The abc conjecture is defined as follows:

Conjecture 1.1. *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers are relatively prime with $c = a + b$, then:*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{1.3}$$

where K is a constant that depends only on ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [4]. The best example given by Reyssat [4] is as follows:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{1.4}$$

It was proposed that $c < rad^2(abc)$ [2]. In 2004, Baker [5], [1] proposed an explicit version of the abc conjecture, namely,

Conjecture 1.2. *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} \tag{1.5}$$

with $R = rad(abc)$ and $\omega = \omega(abc)$ the number of distinct prime factors of abc.

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In the following, we assume that the conjecture of Alan Barker is true, I will give a very elementary proof of the conjecture $c < rad^2(abc)$ that constitutes one key to resolve the open abc conjecture.

2. The Proof of The $c < R^2$ Conjecture

Proof. Let one triplet (a, b, c) of positive integers be relatively prime with $c = a+b$ and:

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!}$$

is true. We know the following inequality:

$$\forall x \geq Log2, max_{k\geq 0}\frac{x^k}{k!} \leq \frac{5}{6}e^x$$

We take $x = LogR, R \geq 2$, then :

$$\forall \omega > 0, \frac{(LogR)^\omega}{\omega!} \leq \frac{5}{6}e^{LogR} = \frac{5}{6}R$$

It follows:

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} \leq \frac{6}{5}R.\frac{5}{6}R \Longrightarrow c < R^2 \tag{2.1}$$

The proof of the conjecture $c < R^2$ is finished.

Q.E.D

□

We give below some numerical examples.

3. Examples

3.1. **Example 1. of Eric Reyssat.** We give an example of Eric Reyssat [5], which is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6\,436\,343 \tag{3.1}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19\,683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6\,436\,343 \Rightarrow rad(c) = 23$. Then $R = rad(abc) = 2\times3\times109\times23 = 15\,042 \Longrightarrow R^2 = 226\,261\,764$.
$\omega = 4 \Longrightarrow A = \dfrac{(LogR)^4}{4!} = 356.64$, $R^2 > \dfrac{6}{5}R\dfrac{(LogR)^\omega}{\omega!} = 6\,437\,590.238 > (c = 6\,436\,343)$.

3.2. **Example 2. of Nitaj.** See [3]:

$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79$

$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311$

$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953$

$R = rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110$

$\Longrightarrow R^2 = 831\,072\,936\,124\,776\,471\,158\,132\,100 > (c = 613\,474\,845\,886\,230\,468\,750)$

$\omega = 10 \Longrightarrow A = \dfrac{(LogR)^{10}}{10!} = 225\,312\,992.556 \Longrightarrow$

$$R^2 > \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} = 7\,794\,478\,289\,809\,729\,132\,015,590 > (c = 613\,474\,845\,886\,230\,468\,750),$$

3.3. **Example 3.** The example is of Ralf Bonse, see [4] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983$$

$$a = 2543^4.182587.2802983.85813163$$

$$b = 2^{15}.3^{77}.11.173$$

$$c = 5^{56}.245983$$

$$R = rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$

$$R = 1.5683959920004546031461002610848e + 33 \Longrightarrow$$

$$R^2 = 2.4598659877230900595045886864952e + 66$$

$$\omega = 10 \Longrightarrow A = \frac{(LogR)^{10}}{10!} = 1\,875\,772\,681\,108.203 \Longrightarrow$$

$$R^2 > \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} = 3.5303452259448631166310839830891e + 45 >$$

$$c = 3.4136998783296235160378273576498e + 44,$$

4. Conclusion

Assuming that the explicit *abc* conjecture is true, we provide an elementary proof that the $c < R^2$ conjecture holds. The important theorem is as follows:

Theorem 4.1. *Assuming the explicit abc conjecture of Baker true, then the $c < R^2$ conjecture is true.*

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