Four-elements tuples splitting

Yuly Shipilevsky
Toronto, Ontario, Canada

Abstract. We consider a problem of splitting of the four-elements tuples of integer numbers into two two-element tuples, providing a minimum of the difference of sums of the corresponding their elements. We conjecturing with the corresponding algorithm for its solution. The corresponding examples and generalizations for the tuples of arbitrary lengths and splitting are given.

1. Introduction

Once upon a time, riding the subway, the author suddenly looked at the interior side wall of the train car and paid attention on 4-digit number, painted on the wall. Probably, it means the number of the train car. Suddenly, he tried to compare sums of digits of different pairs of 2-digit subsets of those 4-digit numbers and found a pair that provides minimum of difference of these sums.

Lets say, we have a 4-digit number: 6341.
It can be considered as a four-elements tuple (6, 3, 4, 1).
(General information regarding tuples is given, e.g., in [5]).
All possible non-ordered 2-digit subsets of the above tuple, (6, 3, 4, 1) are the following subsets:

(6, 3), (6, 4), (6, 1), (4, 3), (4,1), (3,1).

There are only the following three pairs of the above subsets, so that each pair doesn't have common elements(numbers):

(6, 3), (4,1), constitutes first splitting of the tuple (6, 3, 4, 1).
(6, 4), (3,1), constitutes second splitting of the tuple (6, 3, 4, 1).
(6, 1), (4, 3), constitutes third splitting of the tuple (6, 3, 4, 1).

For the first pair, the difference of the corresponding sums is:

| (6 + 3) - (4 + 1) | = | 9 - 5 | = 4.

For the second pair the difference of the corresponding sums is:

| (6 + 4) - (3 + 1) | = | 10 - 4 | = 5.

For the third pair the difference of the corresponding sums is:

| (6 + 1) - (4 + 3) | = | 7 - 7 | = 0.

Thus, third splitting(combination): (6, 1), (4, 3) gives a solution, providing the minimum: 0, of the differences of the corresponding sums.

2. Four-elements Tuples of Integer Numbers

Let us consider Four-elements non-ordered tuples of integer numbers:

$V := (i, j, m, n), i, j, m, n \in \mathbf{Z}_+$.

What could be an algorithm that allows for any fixed tuple: $V_0 = (i_0, j_0, m_0, n_0)$, to split it into two two-element sub-tuples: $E_1^0 = (k_0, l_0)$, $E_2^0 = (s_0, t_0)$, wherein: $k_0, l_0, s_0, t_0 \in V_0$, $E_1^0 \cap E_2^0 = \varnothing$, $E_1^0 \cup E_2^0 = V_0$, so that it provides a minimum of the difference of the corresponding sums of the E_1^0 and E_2^0 elements: $| (k_0 + l_0) - (s_0 + t_0) |$?

The corresponding graph illustration: $G = (V_0, E_1^0, E_2^0)$ can be given, considering a graph G, having vertices V_0 and edges E_1^0, E_2^0. Each vertex of G has degree 1 (General information regarding Graph Theory is given, e.g., in [2]).

Note that despite each sub-tuple: E_1^0 and E_2^0 is an "unique" subset of elements of the "parent" tuple V_0, they may include the same integers, as well as V_0, e.g., $V_0 = (3, 2, 2, 3)$, $E_1^0 = (3, 2)$, $E_2^0 = (2, 3)$.

3. Algorithm's Solution Conjecture

Conjecture. *Let* $p_0 = max \{ i_0, j_0, m_0, n_0 \}$,

$$q_0 = min \{ i_0, j_0, m_0, n_0 \}.$$

We conjecturing that:

$$E_1^0 = (p_0, q_0), \quad E_2^0 = (r_0, w_0), \quad r_0, w_0 \in V_0 \setminus p_0 \cup q_0 .$$

For example, let $V_0 =$ (9, 3, 4, 1). Then, since
9 = max {9, 3, 4, 1) and 1 = min {9, 3, 4, 1}, splitting:

$E_1^0 = (9, 1)$, $E_2^0 = (4, 3)$, is a solution, where:

$| (9 + 1) - (4 + 3) | = | 10 - 7| = 3 = \min.$

4. Functions

Thus, to each integer tuple: (i_1, i_2, i_3, i_4), $i_1, i_2, i_3, i_4 \in \mathbf{Z}_+$, can be assigned the corresponding minimum and therefore we can define a function:

$$f_{22} : \mathbf{Z}_+^4 \to \mathbf{Z}_+,$$

$$f_{22}(i_1, i_2, i_3, i_4) = | (p_0 + q_0) - (r_0 + w_0) |.$$

Furthermore, there exists a pair of sub-tuples:

$E_1^0 = (p_0, q_0)$, $E_2^0 = (r_0, w_0)$, corresponding to that minimum, and we can define the following four functions as well:

$$f_{22}p_0 : \mathbf{Z}_+^4 \to \mathbf{Z}_+, \ f_{22}p_0(i_1, i_2, i_3, i_4) = p_0,$$

$$f_{22}q_0 : \mathbf{Z}_+^4 \to \mathbf{Z}_+, \ f_{22}q_0(i_1, i_2, i_3, i_4) = q_0,$$

$$f_{22}r_0 : \mathbf{Z}_+^4 \to \mathbf{Z}_+, \ f_{22}r_0(i_1, i_2, i_3, i_4) = r_0,$$

$$f_{22}w_0 : \mathbf{Z}_+^4 \to \mathbf{Z}_+, \ f_{22}w_0(i_1, i_2, i_3, i_4) = w_0.$$

Its well-known in number theory a complex number whose real and imaginary parts are both integers: Gaussian Integer. The Gaussian integers are the set: $\mathbf{Z}[\mathbf{i}] := \{ a + b\mathbf{i} \mid a, b \in \mathbf{Z} \}$, where $\mathbf{i}^2 = -1$. Gaussian integers are closed under addition and multiplication and form commutative ring, which is a subring of the field of complex numbers. When considered within the complex plane the Gaussian integers constitute the 2-dimensional integer lattice. (General information regarding Gaussian Integers is given, e.g., in [1]).

Let us define the following functions, corresponding to the above pair of "minimizing" sub-tuples:

$f_{22C1}: \mathbf{Z}_+^4 \rightarrow \mathbf{Z[i]}$, $f_{22C2}: \mathbf{Z}_+^4 \rightarrow \mathbf{Z[i]}$,

$f_{22C3}: \mathbf{Z}_+^4 \rightarrow \mathbf{Z[i]}$, $f_{22C4}: \mathbf{Z}_+^4 \rightarrow \mathbf{Z[i]}$,

$f_{22C1}(i_1, i_2, i_3, i_4) = p_0, + \mathbf{i}\, q_0$, $f_{22C2}(i_1, i_2, i_3, i_4) = q_0, + \mathbf{i}\, p_0$.

$f_{22C3}(i_1, i_2, i_3, i_4) = r_0, + \mathbf{i}\, w_0$, $f_{22C4}(i_1, i_2, i_3, i_4) = w_0, + \mathbf{i}\, r_0$.

Recall that quaternions are generally represented in the form: $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $c \in \mathbf{R}$, $d \in \mathbf{R}$, and **i**, **j** and **k** are the fundamental quaternion units and are a number system that extends the complex numbers. (General information regarding quaternions is given, e.g., in [3]).

The set of all quaternions **H** is a normed algebra, where the norm is multiplicative: $\| pq \| = \| p \| \| q \|$, $p \in \mathbf{H}$, $q \in \mathbf{H}$, $\| q \|^2 = a^2 + b^2 + c^2 + d^2$. This norm makes it possible to define the distance $d(p, q) = \|p - q\|$ which makes **H** into a metric space. Lipschits quaternions $\mathbf{L} := \{q\text{: } q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbf{Z}\}$.

Assigning to each integer tuple (i_1, i_2, i_3, i_4) the corresponding Lipschits quaternion: $i_1 + i_2\mathbf{i} + i_3\mathbf{j} + i_4\mathbf{k}$, we can define the following functions:

$f_{22L}: \mathbf{L} \rightarrow \mathbf{Z}_+$,

$f_{22L}(i_1 + i_2\mathbf{i} + i_3\mathbf{j} + i_4\mathbf{k}) = | (p_0 + q_0) - (r_0 + w_0) |$,

$f_{22LC1}: \mathbf{L} \rightarrow \mathbf{Z[i]}$, $f_{22LC2}: \mathbf{L} \rightarrow \mathbf{Z[i]}$,

$f_{22LC3}: \mathbf{L} \rightarrow \mathbf{Z[i]}$, $f_{22LC4}: \mathbf{L} \rightarrow \mathbf{Z[i]}$,

$f_{22LC1}(i_1 + i_2\mathbf{i} + i_3\mathbf{j} + i_4\mathbf{k}) = p_0, + \mathbf{i}\, q_0$,

$f_{22LC2}(i_1 + i_2\mathbf{i} + i_3\mathbf{j} + i_4\mathbf{k}) = q_0, + \mathbf{i}\, p_0,$

$f_{22LC3}(i_1 + i_2\mathbf{i} + i_3\mathbf{j} + i_4\mathbf{k}) = r_0, + \mathbf{i}\, w_0,$

$f_{22LC4}(i_1 + i_2\mathbf{i} + i_3\mathbf{j} + i_4\mathbf{k}) = w_0, + \mathbf{i}\, r_0.$

Note that the following splitting scenarios can be defined as well: Split a tuple $V_0 = (i_0, j_0, m_0, n_0)$, into two two-element sub-tuples: $E_1^0 = (k_0, l_0)$, $E_2^0 = (s_0, t_0)$, wherein: $k_0, l_0, s_0, t_0 \in V_0$, $E_1^0 \cap E_2^0 = \emptyset$, $E_1^0 \cup E_2^0 = V_0$, so that it provides a minimum of the difference of the corresponding differences of the E_1^0 and E_2^0 elements: $|\, |k_0 - l_0| - |s_0 - t_0|\, | \to \min$, or, so that it provides $|k_0 - l_0| + |s_0 - t_0| \to \min$, or, $(k_0 + l_0) + (s_0 + t_0) \to \min$. The corresponding functions, similar to the above-defined can be defined as well.

5. Generalization

The above theory can be generalized as follows.
Let us consider n-element non-ordered tuples of integer numbers:

$$V := (i_1, \dots, i_n),\ i_1, \dots, i_n \in \mathbf{Z}_+ .$$

Split any fixed tuple $V_0 = (i_1^0, \dots, i_n^0)$, into two: m-element and k-element ($m + k \le n$, $m, k, n \in \mathbf{N}$) sub-tuples:
$E_1^0 = (u_1^0, \dots, u_m^0)$, $E_2^0 = (v_1^0, \dots, v_k^0)$, wherein:
$u_1^0, \dots, u_m^0, v_1^0, \dots, v_k^0 \in V_0$,
$E_1^0 \cap E_2^0 = \emptyset$, $E_1^0 \cup E_2^0 \subset V_0$, so that it provides a minimum of the difference of the corresponding sums of the E_1^0 and E_2^0 elements: $|\, (u_1^0 + \dots + u_m^0) - (v_1^0 + \dots + v_k^0)\, | \to \min$.
The corresponding functions: $f_{mk} : \mathbf{Z}_+^n \to \mathbf{Z}_+$, $f_{mk}(i_1, \dots, i_n)$ can be considered.

6. Normed Spaces and N-tuples Splittings

The above theory can be further generalized for the tuples, consisted of elements of some Normed Space.

Recall that Normed Space is a Vector Space **X** over a subfield **F** of the complex numbers **C**, where the corresponding real-valued function(the norm), $\| x \| : \mathbf{X} \to \mathbf{R}$, is defined.

A Vector(Linear) Space is a set whose elements(vectors) can be added together and multiplied by numbers(scalars). (General information regarding Vector Spaces is given, e.g., in [4]).

Let us consider n-element non-ordered tuples of elements of some Normed Space **X**:

$$V := (a_1, \ldots , a_n), \; a_1, \ldots , a_n \in \mathbf{X} .$$

Split any fixed tuple $V_0 = (a_1^0, \ldots , a_n^0)$, into two: m-element and k-element ($m + k \le n$) sub-tuples:

$E_1^0 = (u_1^0, \ldots , u_m^0)$, $E_2^0 = (v_1^0, \ldots , v_k^0)$, wherein:

$u_1^0, \ldots , u_m^0, v_1^0, \ldots , v_k^0 \in V_0$,

$E_1^0 \cap E_2^0 = \varnothing$, $E_1^0 \cup E_2^0 \subset V_0$, so that it provides a minimum of the difference of the corresponding norm sums of the E_1^0 and E_2^0 elements:

$$| (\|u_1^0\| + \ldots + \|u_m^0\|) - (\|v_1^0\| + \ldots + \|v_k^0\|) | \to \min.$$

The following criteria can be suggested as well:

$$\| (u_1^0 + \ldots + u_m^0) - (v_1^0 + \ldots + v_k^0) \| \to \min.$$

The corresponding functions: $f_{mk} : \mathbf{X}^n \to \mathbf{R}$, $f_{mk}(a_1, \ldots , a_n)$ can be considered.

7. Splitting, Based on Two Algebraic Operations

Let us consider the following scenarios:

Split a tuple $V_0 = (i_0, j_0, m_0, n_0)$, $i_0, j_0, m_0, n_0 \in \mathbf{Z}_+$ into two two-element sub-tuples: $E_1^0 = (k_0, l_0)$, $E_2^0 = (s_0, t_0)$, wherein: $k_0, l_0, s_0, t_0 \in V_0$, $E_1^0 \cap E_2^0 = \varnothing$, $E_1^0 \cup E_2^0 = V_0$, so that it provides: $| k_0 * l_0 - s_0 * t_0 | \to \min$, wherein symbol "*" means multiplicity operation, or, $k_0 * l_0 + s_0 * t_0 \to \min$, or, $(k_0 + l_0) * (s_0 + t_0) \to \min$, or, $| k_0 - l_0 | * | s_0 - t_0 | \to \min$.

To generalize this approach, let **X** be a normed commutative ring(see, e.g., [6]). The following scenarios then can be suggested:

Split a tuple $V_0 = (i_0, j_0, m_0, n_0)$, $i_0, j_0, m_0, n_0 \in \mathbf{X}$,
into two two-element sub-tuples: $E_1^0 = (k_0, l_0)$, $E_2^0 = (s_0, t_0)$,
wherein: $k_0, l_0, s_0, t_0 \in V_0$,
$E_1^0 \cap E_2^0 = \varnothing$, $E_1^0 \cup E_2^0 = V_0$, so that it provides:
$\| k_0 * l_0 - s_0 * t_0 \| \rightarrow \min$, wherein symbol "*" means multiplicity operation, or, $\| k_0 * l_0 + s_0 * t_0 \| \rightarrow \min$, or,
$\| (k_0 + l_0) * (s_0 + t_0) \| \rightarrow \min$, or, $\| (k_0 - l_0) * (s_0 - t_0) \| \rightarrow \min$.

Similar to the above-given section 6, the corresponding scenarios, comprising splitting of n-tuples into two (or even more) m- and k-sub-tuples can be considered.

References

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