My Proofs of Conjectures on Number Theory

Abdelmajid Ben Hadj Salem

Abstract

In this booklet, I present my proofs of open conjectures on the theory of numbers. It concerns the following conjectures:

- The Riemann Hypothesis.

- Beal's conjecture.

- The conjecture *c<rad^{1.63}(abc).*

- The explicit *abc* conjecture of Alan Baker.

- Two proofs of the *abc* conjecture.

- The conjecture *c<rad^2(abc)*.

May 24, 2025

Abdelmajid BEN HADJ SALEM

MY PROOFS OF CONJECTURES ON NUMBER THEORY

$-Version\ 1.(May\ 2025)$

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To the memory of my Parents.
To my Teachers, my Professors, my Colleagues and my Friends

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- The Riemann Hypothesis.
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***Résumé* (Mes Démonstrations de Conjectures de la Théorie des Nombres, mai 2025)**

Dans ce fascicule, je présente mes démonstrations des conjectures ouvertes de la théorie des nombres. Elles concernent les conjectures suivantes:
- L'hypothèse de Riemann.
- La conjecture de Beal.
- La conjecture $c < rad^{1.63}(abc)$.
- L'explicite conjecture *abc* d'Alan Baker.
- La conjecture *abc* (deux démonstrations).
- La conjecture $c < rad^2(abc)$.

CONTENTS

CHAPTER 1

THE RIEMANN HYPOTHESIS IS TRUE: THE END OF THE MYSTERY

Abstract. — In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros) $s=\sigma+it$ of the zeta function, defined by:*

$$\zeta(s)=\sum_{n=1}^{+\infty}\frac{1}{n^s},\ \text{for}\quad \Re(s)>1$$

have real part $\sigma=\dfrac{1}{2}$. In this note, I give the proof that $\sigma=\dfrac{1}{2}$ using an equivalent statement of the Riemann Hypothesis: the Dirichlet η function.

The paper is under reviewing.

This paper is dedicated to the memory of my ***Father*** *who taught me arithmetic,*
To my wife ***Wahida****, my daughter* ***Sinda*** *and my son* ***Mohamed Mazen***

'I feel that these aren't the right techniques to solve the Riemann hypothesis itself, it's going to need some big idea from somewhere else.'

James Maynard (07/15/2024)[**1**]

1.1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [**2**] known Riemann Hypothesis:

Conjecture 1. — *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give the proof that $\sigma = \dfrac{1}{2}$.

1.1.1. The function zeta(s)

We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [**3**]:

Theorem 2. — *The function $\zeta(s)$ satisfies the following :*
1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;
4. the nontrivial zeros lie inside the region $0 \le \Re(s) \le 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \dfrac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line.

For our proof, we will use the function presented by G.H. Hardy [**4**] namely Dirichlet eta function [**3**]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [**3**].

We have also the theorem (see page 16, [**4**]):

Theorem 3. — *For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.*

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

1.1.2. A Equivalent statement to the Riemann Hypothesis

Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet eta function which is stated as follows [**3**]:

Equivalence 4. — *The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :*

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \tag{1}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.

The series (1) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([**4**], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{2}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{3}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

1.2. Preliminaries of the proof of the zeros of $\eta(s)$ are on $\Re(s) = 1/2$

Proof. — We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we denote it $s = \beta + i\gamma$, then we obtain $0 < \beta < 1$ and

$\eta(s) = 0 \Longleftrightarrow (1-2^{1-s})\zeta(s) = 0$. We verify easily the two propositions:

(4) $\boxed{s \text{ is one zero of } \eta(s) \text{ that falls in the critical strip, is also one zero of}}$ $\boxed{\zeta(s) \text{ in the critical strip}}$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1-2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

(5) $\boxed{s \text{ is one zero of } \zeta(s) \text{ that falls in the critical strip, is also one zero of}}$ $\boxed{\eta(s) \text{ in the critical strip}}$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}.e^{-itLogn}$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i\sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let $s = \beta + i\gamma$ with $0 < \beta < 1$ be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \left| \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s} \right| < \epsilon'$$

It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(\gamma Logk)}{k^\beta} = 0$$
$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(\gamma Logk)}{k^\beta} = 0$$

Using the definition of the limit of a sequence, we can write:

(6) $\forall \epsilon_1 > 0\, \exists n_r, \forall N > n_r, \mid \Re(\eta(s)_N) \mid < \epsilon_1 \Longrightarrow \Re^2(\eta(s)_N) < {\epsilon_1}^2$

(7) $\forall \epsilon_2 > 0\, \exists n_i, \forall N > n_i, \mid \Im(\eta(s)_N) \mid < \epsilon_2 \Longrightarrow \Im^2(\eta(s)_N) < {\epsilon_2}^2$

Then:

$$(8) 0 < \sum_{k=1}^{N} \frac{cos^2(\gamma Logk)}{k^{2\beta}} + 2 \sum_{k,k'=1;k\neq k'}^{N} \frac{(-1)^{k+k'} cos(\gamma Logk).cos(\gamma Logk')}{k^{\beta} k'^{\beta}} < \epsilon_1^2$$

$$(9) 0 < \sum_{k=1}^{N} \frac{sin^2(\gamma Logk)}{k^{2\beta}} + 2 \sum_{k,k'=1;k\neq k'}^{N} \frac{(-1)^{k+k'} sin(\gamma Logk).sin(\gamma Logk')}{k^{\beta} k'^{\beta}} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$(10) \qquad 0 < \sum_{k=1}^{N} \frac{1}{k^{2\beta}} + 2 \sum_{k,k'=1;k\neq k'}^{N} (-1)^{k+k'} \frac{cos(\gamma Log(k/k'))}{k^{\gamma} k'^{\beta}} < 2\epsilon^2$$

In detail, we rewrite the above equation (10) as:

$$(11) \qquad 0 < \sum_{k=1}^{N} \frac{1}{k^{2\beta}} + 2 \sum_{k=1}^{k=N-1} \frac{(-1)^k}{k^{\beta}} \left(\sum_{k'=2,k'>k}^{k'=N} (-1)^{k'} \frac{cos(\gamma Log(k/k'))}{k'^{\beta}} \right) < 2\epsilon^2$$

We denote:

$$(12) \qquad S_N(\beta,\gamma) = \sum_{k=1}^{k=N-1} \frac{(-1)^k}{k^{\beta}} \left(\sum_{k'=2,k'>k}^{k'=N} (-1)^{k'} \frac{cos(\gamma Log(k/k'))}{k'^{\beta}} \right)$$

We can write the above equation as :

$$(13) \qquad 0 < \rho_N^2 < 2\epsilon^2$$

or $\rho(s) = 0$.

1.3. Case $0 < \Re(s) < 1/2$

Suppose there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (10):

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k\neq k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2$$

or:

$$(14) \qquad -\frac{1}{2} \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < \sum_{k,k'=1;k\neq k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < \epsilon^2 - \frac{1}{2} \sum_{k=1}^{N} \frac{1}{k^{2\sigma}}$$

But $2\sigma < 1$, it follows that $lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}\longrightarrow+\infty$ and then, we obtain :

$$\sum_{k,k'=1;k\neq k'}^{+\infty}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}=-\infty \tag{15}$$

1.4. Case $\Re(s)=1/2$

We suppose that $\sigma=\dfrac{1}{2}$. Let's start by recalling Hardy's theorem (1914) ([**3**], page 24):

Theorem 5. — *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (4-5), it follows the proposition :

Proposition 6. — *There are infinitely many zeros of $\eta(s)$ on the critical line.*

Let $s_j=\frac{1}{2}+it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j)=0$. The equation (10) is written for s_j:

$$0<\sum_{k=1}^{N}\frac{1}{k}+2\sum_{k,k'=1;k\neq k'}^{N}(-1)^{k+k'}\frac{cos(t_jLog(k/k'))}{\sqrt{k}\sqrt{k'}}<2\epsilon^2$$

or:

$$-\frac{1}{2}\sum_{k=1}^{N}\frac{1}{k}<\sum_{k,k'=1;k\neq k'}^{N}(-1)^{k+k'}\frac{cos(t_jLog(k/k'))}{\sqrt{k}\sqrt{k'}}<\epsilon^2-\frac{1}{2}\sum_{k=1}^{N}\frac{1}{k} \tag{16}$$

If $N\longrightarrow+\infty$, the series $\sum_{k=1}^{N}\frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty}\frac{1}{k}\leq 2\epsilon^2-2\sum_{k,k'=1;k\neq k'}^{+\infty}(-1)^{k+k'}\frac{cos(t_jLog(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$lim_{N\longrightarrow+\infty}\sum_{k,k'=1;k\neq k'}^{N}(-1)^{k+k'}\frac{cos(t_jLog(k/k'))}{\sqrt{k}\sqrt{k'}}=-\infty \tag{17}$$

if not, we will have a contradiction with the fact that :

$$lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}(-1)^{k-1}\frac{1}{k^{s_j}}=0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j=\frac{1}{2}+it_j$$

1.5. Case $1/2 < \Re(s) < 1$

Let $s=\sigma+it$ be the zero of $\eta(s)$ in $0<\Re(s)<\frac{1}{2}$, object of the section 1.3. From the proposition (4), $\zeta(s)=0$. According to point 4 of theorem 2, the complex number $s'=1-\sigma+it=\sigma'+it'$ with $\sigma'=1-\sigma$, $t'=t$ and $\frac{1}{2}<\sigma'<1$ verifies $\zeta(s')=0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2}<\Re(s)<1$, it follows from the proposition (5) that $\eta(s')=0 \Longrightarrow \rho(s')=0$. By applying (10), we get:

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma'}}+2\sum_{k,k'=1;k\neq k'}^{N}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}}<2\epsilon^2$$

$$-\frac{1}{2}\sum_{k=1}^{N}\frac{1}{k^{2\sigma'}}<\sum_{k,k'=1;k\neq k'}^{N}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}}<\epsilon^2-\frac{1}{2}\sum_{k=1}^{N}\frac{1}{k^{2\sigma'}} \tag{18}$$

As $0<\sigma<\frac{1}{2}\Longrightarrow 2>2\sigma'=2(1-\sigma)>1$, then the series $\sum_{k=1}^{N}\dfrac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2<1/k^{2\sigma'}$ for all $k>0$, then :

$$0<\zeta(2)=\frac{\pi^2}{6}=\sum_{k=1}^{+\infty}\frac{1}{k^2}\leq\sum_{k=1}^{+\infty}\frac{1}{k^{2\sigma'}}=C(\sigma')=\zeta_1(2\sigma')=\zeta(2\sigma')$$

From the equation (18), it follows that :

$$\boxed{\sum_{k,k'=1;k\neq k'}^{+\infty}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}}=-\frac{C(\sigma')}{2}=-\frac{\zeta(2\sigma')}{2}>-\infty} \tag{19}$$

1.5.0.1. Case $t=0$

We suppose that $t=0\Longrightarrow t'=0$. We known the following proposition:

Proposition 7. — *For all $s=\sigma$ real with $0<\sigma<1$, $\eta(s)>0$ and $\zeta(s)<0$.*

We deduce the contradiction with the hypothesis $s'=\sigma'$ is a zero of $\eta(s)$ and:

(20) The equation (19) is false for the case $t'=t=0$.

1.5.0.2. Case $t' = t \neq 0$

We suppose that $t' \neq 0$. Let $s' = \sigma' + it' = 1 - \sigma + it$ a zero of $\eta(s)$, we have:

$$\sum_{k,k'=1;k\neq k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{21}$$

the left member of the equation (21) above is finite and depends of σ' and t', but the right member is a function only of σ' equal to $-\zeta(2\sigma')/2$.

We recall the following theorem (see page 140, [**4**]):

Theorem 8. —

$$lim_{T\longrightarrow+\infty}\frac{1}{T}\int_1^T \mid \zeta(\sigma" + i\tau) \mid^2 d\tau = \zeta(2\sigma") \quad (\sigma" > \frac{1}{2}) \tag{22}$$

Let t_0 so that $t_0 \geq 1$. As the integral of the left member of the above equation is convergent, the equation (108) can be written as:

$$lim_{T\longrightarrow+\infty}\frac{1}{T}\int_{t_0}^T |\zeta(\sigma" + i\tau)|^2 d\tau = \zeta(2\sigma")$$

and $\zeta(2\sigma")$ is independent of any t_0 then in particular for $t_0 = t'$. As $\sigma"$ is any $\sigma" > 1/2$, I choose $\sigma" = \sigma'$ and $t_0 = t'$, it follows that $\zeta(2\sigma')$ does not depend of t' so that $s' = \sigma' + it'$ is a root of η. Hence, the contradiction with equation (19). Then the equation (21) is false.

$$\boxed{\text{It follows that the equation (21) is false for the case } t' \neq 0.} \tag{23}$$

It follows that the equation (19) is false and $\eta(s')$ does not vanish for $\sigma' \in]1/2, 1[$.

From (20-23), we conclude that the function $\eta(s)$ has no zeros for all $s' = \sigma' + it'$ with $\sigma' \in]1/2, 1[$, it follows that the case of the section (1.3) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false too. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \frac{1}{2}$. From the equivalent statement (343), it follows that **the Riemann hypothesis is verified**. □

We therefore announce the important theorem as follows:

Theorem 9. — *The Riemann Hypothesis is true:*
All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \frac{1}{2}$.

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CHAPTER 2

A COMPLETE PROOF OF BEAL'S CONJECTURE

Abstract. — In 1997, Andrew Beal announced the following conjecture: *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If $A^m + B^n = C^l$ then $A, B,$ and C have a common factor.* We begin to construct the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ with p, q integers depending on A^m, B^n and C^l. We resolve $x^3 - px + q = 0$ and we obtain the three roots x_1, x_2, x_3 as functions of p and a parameter θ. Since $A^m, B^n, -C^l$ are the only roots of $x^3 - px + q = 0$, we discuss the conditions that x_1, x_2, x_3 are integers and have or do have not a common factor. Three numerical examples are given.

The paper is under reviewing.

To the memory of my ***Father*** *who taught me arithmetic, To my wife* ***Wahida****, my daughter* ***Sinda*** *and my son* ***Mohamed Mazen***

2.1. Introduction

In 1997, Andrew Beal [**1**] announced the following conjecture :

> ***Conjecture 10.*** — *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*
>
> (24) $$A^m + B^n = C^l$$
>
> *then $A, B,$ and C have a common factor.*

The purpose of this paper is to give a complete proof of Beal's conjecture. Our idea is to construct a polynomial $P(x)$ of order three having as roots A^m, B^n and $-C^l$ with the condition (24). We obtain $P(x) = x^3 - px + q$ where p, q are depending of A^m, B^n and C^l. Then we express $A^m, B^n, -C^l$ the roots of $P(x) = 0$ in function of p and a parameter θ that depends of the A, B, C. The calculations give that $A^{2m} = \dfrac{4p}{3}cos^2\dfrac{\theta}{3}$. As A^{2m} is an integer, it follows that $cos^2\dfrac{\theta}{3}$ must be written as $\dfrac{a}{b}$

where a, b are two positive coprime integers. Beside the trivial cases, there are two main hypothesis to study:
- the first hypothesis is: $3 \mid a \quad and \quad b \mid 4p$,
- the second hypothesis is: $3 \mid p \quad and \quad b \mid 4p$.

We discuss the conditions of divisibility of p, a, b so that the expression of A^{2m} is an integer. Depending of each individual case, we obtain that A, B, C have or do have not a common factor. Our proof of the conjecture contains many cases to study. there are many cases where we use elementary number theory and some cases need more research to obtain finally the solution.

The paper is organized as follows. In section 1, it is an introduction of the paper. The trivial case, where $A^m = B^n$, is studied in section 2. The preliminaries needed for the proof are given in section 3 where we consider the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$. The section 4 is the preamble of the proof of the main theorem. Section 5 treats the cases of the first hypothesis $3 \mid a$ and $b \mid 4p$. We study the cases of the second hypothesis $3 \mid p$ and $b \mid 4p$ in section 6. Finally, we present three numerical examples and the conclusion in section 7.

In 1997, Andrew Beal [**1**] announced the following conjecture :

Conjecture 11. — *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

(25) $$A^m + B^n = C^l$$

then $A, B,$ and C have a common factor.

2.2. Trivial Case

We consider the trivial case when $A^m = B^n$. The equation (25) becomes:

(26) $$2A^m = C^l$$

then $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^q.C_1$ with $q \geq 1$, $2 \nmid C_1$ and $2A^m = 2^{ql}C_1^l \Longrightarrow A^m = 2^{ql-1}C_1^l$. As $l > 2$, $q \geq 1$, then $2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^r A_1$ with $r \geq 1$ and $2 \nmid A_1$. The equation (26),becomes:

(27) $$2 \times 2^{rm} A_1^m = 2^{ql} C_1^l$$

As $2 \nmid A_1$ and $2 \nmid C_1$, we obtain the first condition :

(28) there exists two positive integers r, q with $r.q \geq 1$ so that $\boxed{ql = mr + 1}$

Then from (27):

(29) $$A_1^m = C_1^l$$

2.2.1. Case 1 $A_1 = 1 \Longrightarrow C_1 = 1$

Using the condition (28) above, we obtain $2.(2^r)^m = (2^q)^l$ and the Beal conjecture is verified.

2.2.2. Case 2 $A_1 > 1 \Longrightarrow C_1 > 1$

From the fundamental theorem of the arithmetic, we can write:

$$A_1 = a_1^{\alpha_1} \dots a_I^{\alpha_I}, \quad a_1 < a_2 < \dots < a_I \Longrightarrow A_1^m = a_1^{m\alpha_1} \dots a_I^{m\alpha_I} \tag{30}$$

$$C_1 = c_1^{\beta_1} \dots c_J^{\beta_J}, \quad c_1 < c_2 < \dots < c_J \Longrightarrow C_1^l = c_1^{l\beta_1} \dots c_J^{l\beta_J} \tag{31}$$

where a_i (respectively c_j) are distinct positive prime numbers and α_i (respectively β_j) are integers > 0.

From (29) and using the uniqueness of the factorization of A_1^m and C_1^l, we obtain necessary:

$$\begin{cases} I = J \\ a_i = c_i, \quad i = 1, 2, \dots, I \\ m\alpha_i = l\beta_i \end{cases} \tag{32}$$

As one $a_i \mid A^m \Longrightarrow a_i \mid B^m \Longrightarrow a_i \mid B$ and in this case, the Beal conjecture is verified.

We suppose in the following that $A^m > B^n$.

2.3. Preliminaries

Let $m, n, l \in \mathbb{N}^* \ > 2$ and $A, B, C \in \mathbb{N}^*$ such:

$$A^m + B^n = C^l \tag{33}$$

We call:

$$\begin{aligned} P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l) \\ +x[A^m B^n - C^l(A^m + B^n)] + C^l A^m B^n \end{aligned} \tag{34}$$

Using the equation (33), $P(x)$ can be written as:

$$\boxed{P(x) = x^3 + x[A^m B^n - (A^m + B^n)^2] + A^m B^n (A^m + B^n)} \tag{35}$$

We introduce the notations:

$$p = (A^m + B^n)^2 - A^m B^n = A^{2m} + A^m B^n + B^{2n}$$
$$q = A^m B^n (A^m + B^n)$$

As $A^m \neq B^n$, we have $p > (A^m - B^n)^2 > 0$. Equation (35) becomes:

$$P(x) = x^3 - px + q$$

Using the equation (34), $P(x)=0$ has three different real roots : A^m, B^n and $-C^l$.

Now, let us resolve the equation:

(36) $$P(x)=x^3-px+q=0$$

To resolve (36) let:

$$x=u+v$$

Then $P(x)=0$ gives:

(37) $$P(x)=P(u+v)=(u+v)^3-p(u+v)+q=0 \Longrightarrow u^3+v^3+(u+v)(3uv-p)+q=0$$

To determine u and v, we obtain the conditions:

$$u^3+v^3=-q$$
$$uv=p/3>0$$

Then u^3 and v^3 are solutions of the second order equation:

(38) $$X^2+qX+p^3/27=0$$

Its discriminant Δ is written as :

$$\Delta=q^2-4p^3/27=\frac{27q^2-4p^3}{27}=\frac{\bar{\Delta}}{27}$$

Let:

$$\begin{aligned}\bar{\Delta}=27q^2-4p^3&=27(A^mB^n(A^m+B^n))^2-4[(A^m+B^n)^2-A^mB^n]^3\\ &=27A^{2m}B^{2n}(A^m+B^n)^2-4[(A^m+B^n)^2-A^mB^n]^3\end{aligned} \quad (39)$$

Denoting :

$$\alpha=A^mB^n>0$$
$$\beta=(A^m+B^n)^2$$

we can write (39) as:

(40) $$\bar{\Delta}=27\alpha^2\beta-4(\beta-\alpha)^3$$

As $\alpha\neq 0$, we can also rewrite (40) as :

$$\bar{\Delta}=\alpha^3\left(27\frac{\beta}{\alpha}-4\left(\frac{\beta}{\alpha}-1\right)^3\right)$$

We call t the parameter :

$$t=\frac{\beta}{\alpha}$$

$\bar{\Delta}$ becomes :

$$\bar{\Delta}=\alpha^3(27t-4(t-1)^3)$$

Let us calling :

$$y=y(t)=27t-4(t-1)^3$$

Since $\alpha > 0$, the sign of $\bar{\Delta}$ is also the sign of $y(t)$. Let us study the sign of y. We obtain $y'(t)$:

$$y'(t) = y' = 3(1+2t)(5-2t)$$

$y' = 0 \Longrightarrow t_1 = -1/2$ and $t_2 = 5/2$, then the table of variations of y is given below:

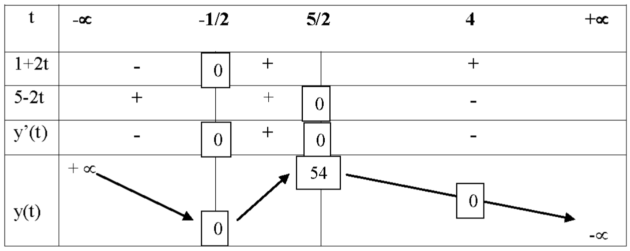

FIGURE 1. The table of variations

The table of the variations of the function y shows that $y < 0$ for $t > 4$. In our case, we are interested for $t > 0$. For $t = 4$ we obtain $y(4) = 0$ and for $t \in]0, 4[\Longrightarrow y > 0$. As we have $t = \frac{\beta}{\alpha} > 4$ as $A^m \neq B^n$:

$$(A^m - B^n)^2 > 0 \Longrightarrow \beta = (A^m + B^n)^2 > 4\alpha = 4A^mB^n$$

Then $y < 0 \Longrightarrow \bar{\Delta} < 0 \Longrightarrow \Delta < 0$. Then, the equation (38) does not have real solutions u^3 and v^3. Let us find the solutions u and v with $x = u + v$ is a positive or a negative real and $u.v = p/3$.

2.3.1. Expressions of the roots

Proof. — The solutions of (38) are:

$$X_1 = \frac{-q + i\sqrt{-\Delta}}{2}$$
$$X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2}$$

We may resolve:

$$u^3 = \frac{-q + i\sqrt{-\Delta}}{2}$$
$$v^3 = \frac{-q - i\sqrt{-\Delta}}{2}$$

Writing X_1 in the form:

$$X_1 = \rho e^{i\theta}$$

with:

$$\rho = \frac{\sqrt{q^2-\Delta}}{2} = \frac{p\sqrt{p}}{3\sqrt{3}}$$
$$\text{and } sin\theta = \frac{\sqrt{-\Delta}}{2\rho} > 0$$
$$cos\theta = -\frac{q}{2\rho} < 0$$

Then $\theta\,[2\pi] \in]+\frac{\pi}{2}, +\pi[$, let:

(41) $$\boxed{\frac{\pi}{2} < \theta < +\pi \Rightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < cos\frac{\theta}{3} < \frac{\sqrt{3}}{2}}$$

and:

(42) $$\boxed{\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4}}$$

hence the expression of X_2:

(43) $$X_2 = \rho e^{-i\theta}$$

Let:

(44) $$u = re^{i\psi}$$

(45) $$\text{and } j = \frac{-1+i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}}$$

(46) $$j^2 = e^{i\frac{4\pi}{3}} = -\frac{1+i\sqrt{3}}{2} = \bar{j}$$

j is a complex cubic root of the unity $\Longleftrightarrow j^3 = 1$. Then, the solutions u and v are:

(47) $$u_1 = re^{i\psi_1} = \sqrt[3]{\rho}e^{i\frac{\theta}{3}}$$

(48) $$u_2 = re^{i\psi_2} = \sqrt[3]{\rho}je^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+2\pi}{3}}$$

(49) $$u_3 = re^{i\psi_3} = \sqrt[3]{\rho}j^2e^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{+i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+4\pi}{3}}$$

and similarly:

(50) $$v_1 = re^{-i\psi_1} = \sqrt[3]{\rho}e^{-i\frac{\theta}{3}}$$

(51) $$v_2 = re^{-i\psi_2} = \sqrt[3]{\rho}j^2e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi-\theta}{3}}$$

(52) $$v_3 = re^{-i\psi_3} = \sqrt[3]{\rho}je^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{2\pi-\theta}{3}}$$

We may now choose u_k and v_h so that $u_k + v_h$ will be real. In this case, we have necessary :

(53) $$v_1 = \overline{u_1}$$

(54) $$v_2 = \overline{u_2}$$

(55) $$v_3 = \overline{u_3}$$

We obtain as real solutions of the equation (37):

$$x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\frac{\theta}{3} > 0 \tag{56}$$

$$x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\tfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right) < 0 \tag{57}$$

$$x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\tfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right) > 0 \tag{58}$$

We compare the expressions of x_1 and x_3, we obtain:

$$2\sqrt[3]{p}cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt[3]{p}\left(-cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right)$$

$$3cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt{3}sin\tfrac{\theta}{3} \tag{59}$$

As $\frac{\theta}{3} \in]+\frac{\pi}{6}, +\frac{\pi}{3}[$, then $sin\frac{\theta}{3}$ and $cos\frac{\theta}{3}$ are > 0. Taking the square of the two members of the last equation, we get:

$$\frac{1}{4} < cos^2\frac{\theta}{3} \tag{60}$$

which is true since $\frac{\theta}{3} \in]+\frac{\pi}{6}, +\frac{\pi}{3}[$ then $x_1 > x_3$. As A^m, B^n and $-C^l$ are the only real solutions of (36), we consider, as A^m is supposed great than B^n, the expressions:

$$\boxed{\begin{cases} A^m = x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\dfrac{\theta}{3} \\ B^n = x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\dfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \\ -C^l = x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\dfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \end{cases}} \tag{61}$$

□

2.4. Preamble of the Proof of the Main Theorem

Theorem 12. — *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{62}$$

then $A, B,$ and C have a common factor.

Proof. — $A^m = 2\sqrt[3]{\rho}cos\frac{\theta}{3}$ is an integer $\Rightarrow A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3}$ is also an integer. But :

$$\sqrt[3]{\rho^2} = \frac{p}{3} \tag{63}$$

Then:

$$A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3} = p.\frac{4}{3}.cos^2\frac{\theta}{3} \tag{64}$$

As A^{2m} is an integer and p is an integer, then $cos^2\frac{\theta}{3}$ must be written under the form:

$$\boxed{cos^2\frac{\theta}{3} = \frac{1}{b} \quad or \quad cos^2\frac{\theta}{3} = \frac{a}{b}} \tag{65}$$

with $b \in \mathbb{N}^*$; for the last condition $a \in \mathbb{N}^*$ and a, b coprime.

Notations: In the following of the paper, the scalars $a, b, ..., z$, $\alpha, \beta, ...$, $A, B, C, ...$ and $\Delta, \Phi, ...$ represent positive integers except the parameters θ, ρ, or others cited in the text, are reals.

2.4.1. Case $cos^2\frac{\theta}{3} = \frac{1}{b}$

We obtain:

$$A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p}{3.b} \tag{66}$$

As $\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4} \Rightarrow \frac{1}{4} < \frac{1}{b} < \frac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3.$

2.4.1.1. $b = 1$

$b = 1 \Rightarrow 4 < 3$ which is impossible.

2.4.1.2. $b = 2$

$b = 2 \Rightarrow A^{2m} = p.\frac{4}{3}.\frac{1}{2} = \frac{2.p}{3} \Rightarrow 3 \mid p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, we obtain:

$$A^{2m} = (A^m)^2 = \frac{2p}{3} = 2.p' \Longrightarrow 2 \mid p' \Longrightarrow p' = 2^{\alpha}p_1^2$$
$$with \quad 2 \nmid p_1, \quad \alpha + 1 = 2\beta$$
$$A^m = 2^{\beta}p_1 \tag{67}$$
$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = p' = 2^{\alpha}p_1^2 \tag{68}$$

From the equation (67), it follows that $2 \mid A^m \Longrightarrow A = 2^iA_1$, $i \geq 1$ and $2 \nmid A_1$. Then, we have $\beta = i.m = im$. The equation (68) implies that $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

2.4.1.2.1. Case $2 \mid B^n$: — - If $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$ with $2 \nmid B_1$. The expression of $B^n C^l$ becomes:

$$B_1^n C^l = 2^{2im-1-jn} p_1^2$$

- If $2im - 1 - jn \geq 1$, $2 \mid C^l \Longrightarrow 2 \mid C$ according to $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (34) is verified.
- If $2im - 1 - jn \leq 0 \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

2.4.1.2.2. Case $2 \mid C^l$: — If $2 \mid C^l$: with the same method used above, we obtain the identical results.

2.4.1.3. $b = 3$

$b = 3 \Rightarrow A^{2m} = p.\dfrac{4}{3}.\dfrac{1}{3} = \dfrac{4p}{9} \Rightarrow 9 \mid p \Rightarrow p = 9p'$ with $p' \neq 1$, as $9 \ll p$ then $A^{2m} = 4p'$. If p' is prime, it is impossible. We suppose that p' is not a prime, as $m \geq 3$, it follows that $2 \mid p'$, then $2 \mid A^m$. But $B^n C^l = 5p'$ and $2 \mid (B^n C^l)$. Using the same method for the case $b = 2$, we obtain the identical results.

2.4.2. Case $a > 1,\ cos^2\dfrac{\theta}{3} = \dfrac{a}{b}$

We have:

$$cos^2\frac{\theta}{3} = \frac{a}{b}\,; \quad A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p.a}{3.b} \tag{69}$$

where a, b verify one of the two conditions:

$$\boxed{\{3 \mid a \quad and \quad b \mid 4p\}} \text{ or } \boxed{\{3 \mid p \quad and \quad b \mid 4p\}} \tag{70}$$

and using the equation (42), we obtain a third condition:

$$\boxed{b < 4a < 3b} \tag{71}$$

For these conditions, $A^{2m} = 4\sqrt[3]{\rho^2} cos^2\frac{\theta}{3} = 4\dfrac{p}{3}.cos^2\dfrac{\theta}{3}$ is an integer.

Let us study the conditions given by the equation (70) in the following two sections.

2.5. Hypothesis : $\{3 \mid a \quad and \quad b \mid 4p\}$

We obtain :

$$3 \mid a \Longrightarrow \exists a' \in \mathbb{N}^* \ / \ a = 3a' \tag{72}$$

2.5.1. Case $b = 2$ and $3 \mid a$

A^{2m} is written as:

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = \frac{4p}{3}.\frac{a}{2} = \frac{2.p.a}{3} \tag{73}$$

Using the equation (72), A^{2m} becomes :

$$A^{2m} = \frac{2.p.3a'}{3} = 2.p.a' \tag{74}$$

but $cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2} > 1$ which is impossible, then $b \neq 2$.

2.5.2. Case $b = 4$ and $3 \mid a$

A^{2m} is written :

$$A^{2m} = \frac{4.p}{3}cos^2\frac{\theta}{3} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.p}{3}.\frac{a}{4} = \frac{p.a}{3} = \frac{p.3a'}{3} = p.a' \tag{75}$$

$$\text{and} \quad cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3.a'}{4} < \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} \Longrightarrow a' < 1 \tag{76}$$

which is impossible. Then the case $b = 4$ is impossible.

2.5.3. Case $b = p$ and $3 \mid a$

We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p} \tag{77}$$

and:

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{3a'}{p} = 4a' = (A^m)^2 \tag{78}$$

$$\exists a" \ / \ a' = a"^2 \tag{79}$$

$$and \quad B^nC^l = p - A^{2m} = b - 4a' = b - 4a"^2 \tag{80}$$

The calculation of A^mB^n gives :

$$A^mB^n = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a'$$

$$or \quad A^mB^n + 2a' = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{81}$$

The left member of (81) is an integer and p also, then $2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3}$ is written under the form :

$$2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{82}$$

where k_1, k_2 are two coprime integers and $k_2 \mid p \Longrightarrow p = b = k_2.k_3$, $k_3 \in \mathbb{N}^*$.

2.5.3.1. We suppose that $k_3 \neq 1$

We obtain :

$$A^m(A^m + 2B^n) = k_1.k_3 \tag{83}$$

Let μ be a prime integer with $\mu \mid k_3$, then $\mu \mid b$ and $\mu \mid A^m(A^m + 2B^n) \Longrightarrow \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** A-1-1- If $\mu \mid A^m \Longrightarrow \mu \mid A$ and $\mu \mid A^{2m}$, but $A^{2m} = 4a' \Longrightarrow \mu \mid 4a' \Longrightarrow (\mu = 2$, but $2 \mid a')$ or $(\mu \mid a')$. Then $\mu \mid a$ it follows the contradiction with a, b coprime.

** A-1-2- If $\mu \mid (A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. We write $\mu \mid (A^m + 2B^n)$ as:

$$A^m + 2B^n = \mu.t' \tag{84}$$

It follows :

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{85}$$

As $p = b = k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid b \Longrightarrow \exists \mu'$ and $b = \mu\mu'$, so we can write:

$$\mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m) \tag{86}$$

From the last equation, we obtain $\mu \mid B^n(B^n - A^m) \Longrightarrow \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** A-1-2-1- If $\mu \mid B^n$ which is in contradiction with $\mu \nmid B^n$.

** A-1-2-2- If $\mu \mid (B^n - A^m)$ and using that $\mu \mid (A^m + 2B^n)$, we arrive to :

$$\mu \mid 3B^n \begin{cases} \mu \mid B^n \\ or \\ \mu = 3 \end{cases} \tag{87}$$

** A-1-2-2-1- If $\mu \mid B^n \Longrightarrow \mu \mid B$, it is the contradiction with $\mu \nmid B$ cited above.

** A-1-2-2-2- If $\mu = 3$, then $3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

2.5.3.2. We assume now $k_3 = 1$

Then :

(88) $$A^{2m} + 2A^mB^n = k_1$$

(89) $$b = k_2$$

(90) $$\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{b}$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3}sin^2\frac{2\theta}{3} = \frac{k_1^2}{b^2}$$
$$\frac{16}{3}sin^2\frac{\theta}{3}cos^2\frac{\theta}{3} = \frac{k_1^2}{b^2}$$
$$\frac{16}{3}sin^2\frac{\theta}{3}.\frac{3a'}{b} = \frac{k_1^2}{b^2}$$

Finally:

(91) $$4^2a'(p-a) = k_1^2$$

but $a' = a"^2$, then $p-a$ is a square. Let:

(92) $$\lambda^2 = p - a = b - a = b - 3a"^2 \Longrightarrow \lambda^2 + 3a"^2 = b$$

The equation (91) becomes:

(93) $$4^2a"^2\lambda^2 = k_1^2 \Longrightarrow k_1 = 4a"\lambda$$

taking the positive root, but $k_1 = A^m(A^m + 2B^n) = 2a"(A^m + 2B^n)$, then :

(94) $$A^m + 2B^n = 2\lambda \Longrightarrow \lambda = a" + B^n$$

** A-2-1- As $A^m = 2a" \Longrightarrow 2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^iA_1$, with $i \geq 1$ and $2 \nmid A_1$, then $A^m = 2a" = 2^{im}A_1^m \Longrightarrow a" = 2^{im-1}A_1^m$, but $im \geq 3 \Longrightarrow 4 \mid a"$. As $\lambda = a" + B^n$, taking its square, we obtain $\lambda^2 = a"^2 + 2a".B^n + B^{2n} \Longrightarrow \lambda^2 \equiv B^{2n}(\text{mod } 4) \Longrightarrow \lambda^2 \equiv B^{2n} \equiv 0(\text{mod } 4)$ or $\lambda^2 \equiv B^{2n} \equiv 1(\text{mod } 4)$.

** A-2-1-1- We suppose that $\lambda^2 \equiv B^{2n} \equiv 0(\text{mod } 4) \Longrightarrow 4 \mid \lambda^2 \Longrightarrow 2 \mid (b-a)$. But $2 \mid a$ because $a = 3a' = 3a"^2 = 3 \times 2^{2(im-1)}A_1^{2m}$ and $im \geq 3$. Then $2 \mid b$, it follows the contradiction with a, b coprime.

** A-2-1-2- We suppose now that $\lambda^2 \equiv B^{2n} \equiv 1(\text{mod} 4)$. As $A^m = 2^{im-1}A_1^m$ and $im - 1 \geq 2$, then $A^m \equiv 0(\text{mod} 4)$. As $B^{2n} \equiv 1(\text{mod} 4)$, then B^n verifies $B^n \equiv 1(\text{mod } 4)$ or $B^n \equiv 3(\text{mod } 4)$ which gives for the two cases $B^nC^l \equiv 1(\text{mod } 4)$.

We have also $p = b = A^{2m} + A^mB^n + B^{2n} = 4a' + B^n.C^l = 4a"^2 + B^nC^l \Longrightarrow B^nC^l = \lambda^2 - a"^2 = B^n.C^l$, then $\lambda, a" \in \mathbb{N}^*$ are solutions of the Diophantine equation :

(95) $$x^2 - y^2 = N$$

with $N = B^n C^l > 0$. Let $Q(N)$ be the number of the solutions of (95) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the equation (95) (see theorem 27.3 in [**2**]):
- If $N \equiv 2(\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3(\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0(\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.

$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

In our case, we have $N = B^n.C^l \equiv 1(\bmod 4)$, then $Q(N) = [\tau(N)/2]$. As $\lambda, a"$ is a couple of solutions of the Diophantine equation (95), then $\exists\ d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2\lambda \tag{96}$$

$$d - d' = 2a" \tag{97}$$

** A-2-1-2-1- As $C^l > B^n$, we take $d = C^l$ and $d' = B^n$. It follows:

$$C^l + B^n = 2\lambda = A^m + 2B^n \tag{98}$$

$$C^l - B^n = 2a" = A^m \tag{99}$$

Then the case $d = C^l$ and $d' = B^n$ gives *a priory* no contradictions.

** A-2-1-2-2- Now, we consider the case $d = B^n C^1$ and $d' = 1$. We rewrite the equations (96-97):

$$B^n C^l + 1 = 2\lambda \tag{100}$$

$$B^n C^l - 1 = 2a" \tag{101}$$

We obtain $1 = \lambda - a"$, but from (94), we have $\lambda = a" + B^n$, it follows $B^n = 1$ and $C^l - A^m = 1$, we know [**4**] that the only positive solution of the last equation is $C = 3, A = 2, m = 3$ and $l = 2 < 3$, then the contradiction.

** A-2-1-2-3- Now, we consider the case $d = c_1^{lr-1} C_1^l$ where c_1 is a prime integer with $c_1 \nmid C_1$ and $C = c_1^r C_1$, $r \geq 1$. It follows that $d' = c_1.B^n$. We rewrite the equations (96-97):

$$c_1^{lr-1} C_1^l + c_1.B^n = 2\lambda \tag{102}$$

$$c_1^{lr-1} C_1^l - c_1.B^n = 2a" \tag{103}$$

As $l \geq 3$, from the last two equations above, it follows that $c_1 \mid (2\lambda)$ and $c_1 \mid (2a")$. Then $c_1 = 2$, or $c_1 \mid \lambda$ and $c_1 \mid a"$.

** A-2-1-2-3-1- We suppose $c_1 = 2$. As $2 \mid A^m$ and $2 \mid C^l$ because $l \geq 3$, it follows $2 \mid B^n$, then $2 \mid (p = b)$. Then the contradiction with a, b coprime.

** A-2-1-2-3-2- We suppose $c_1 \neq 2$ and $c_1 \mid a"$ and $c_1 \mid \lambda$. $c_1 \mid a" \Longrightarrow c_1 \mid a$ and $c_1 \mid (A^m = 2a")$. $B^n = C^l - A^m \Longrightarrow c_1 \mid B^n$. It follows that $c_1 \mid (p = b)$. Then the contradiction with a, b coprime.

The other cases of the expressions of d and d' with d, d' not coprime so that $N = B^n C^l = d.d'$ give also contradictions.

** A-2-1-2-4- Now, let $C = c_1^r C_1$ with c_1 a prime, $r \geq 1$ and $c_1 \nmid C_1$, we consider the case $d = C_1^l$ and $d' = c_1^{rl} B^n$ so that $d > d'$. We rewrite the equations (96-97):

$$C_1^l + c_1^{rl} B^n = 2\lambda \tag{104}$$

$$C_1^l - c_1^{rl} B^n = 2a" \tag{105}$$

We obtain $c_1^{rl} B^n = \lambda - a" = B^n \Longrightarrow c_1^{rl} = 1$, then the contradiction.

** A-2-1-2-5- Now, let $C = c_1^r C_1$ with c_1 a prime, $r \geq 1$ and $c_1 \nmid C_1$, we consider the case $d = C_1^l B^n$ and $d' = c_1^{rl}$ so that $d > d'$. We rewrite the equations (96-97):

$$C_1^l B^l + c_1^{rl} = 2\lambda \tag{106}$$

$$C_1^l B^l - c_1^{rl} = 2a" \tag{107}$$

We obtain $c_1^{rl} = \lambda - a" = B^n \Longrightarrow c_1 \mid B^n$, then $c_1 \mid A^m = 2a"$. If $c_1 = 2$, the contradiction with $B^n C^l \equiv 1 (\mathrm{mod}\, 4)$. Then $c_1 \mid a" \Longrightarrow c_1 \mid a \Longrightarrow c_1 \mid (p = b)$, it follows a, b are not coprime, then the contradiction.

Cases like $d' < C^l$ a divisor of C^l or $d' < B^l$ a divisor of B^n with $d' < d$ and $d.d' = N = B^n C^l$ give contradictions.

** A-2-1-2-6- Now, we consider the case $d = b_1.C^l$ where b_1 is a prime integer with $b_1 \nmid B_1$ and $B = b_1^r B_1$, $r \geq 1$. It follows that $d' = b_1^{nr-1} B_1^n$. We rewrite the equations (96-97):

$$b_1 C^l + b_1^{nr-1} B_1^n = 2\lambda \tag{108}$$

$$b_1 C^l - b_1^{nr-1} B_1^n = 2a" \tag{109}$$

As $n \geq 3$, from the last two equations above, it follows that $b_1 \mid 2\lambda$ and $b_1 \mid (2a")$. Then $b_1 = 2$, or $b_1 \mid \lambda$ and $b_1 \mid a"$.

** A-2-1-2-6-1- We suppose $b_1 = 2 \Longrightarrow 2 \mid B^n$. As $2 \mid (A^m = 2a" \Longrightarrow 2 \mid a" \Longrightarrow 2 \mid a$, but $2 \mid B^n$ and $2 \mid A^m$ then $2 \mid (p = b)$. It follows the contradiction with a, b coprime.

** A-2-1-2-6-2- We suppose $b_1 \neq 2$, then $b_1 \mid \lambda$ and $b_1 \mid a" \Longrightarrow b_1 \mid A^m$ and $b_1 \mid a" \Longrightarrow b_1 \mid a$, but $b_1 \mid B^n$ and $b_1 \mid A^m$ then $b_1 \mid (p = b)$. It follows the contradiction with a, b coprime.

The other cases of the expressions of d and d' with d, d' not coprime and $d > d'$ so that $N = C^l B^m = d.d'$ give also contradictions.

Finally, from the cases studied in the above paragraph A-2-1-2, we have found one suitable factorization of N that gives a priory no contradictions, it is the case $N = B^n.C^l = d.d'$ with $d = C^l, d' = B^n$ but $1 \ll \tau(N)$, it follows the contradiction with $Q(N) = [\tau(N)/2] \leq 1$. We conclude that the case A-2-1-2 is to reject.

Hence, the case $k_3 = 1$ is impossible.

Let us verify the condition (71) given by $b < 4a < 3b$. In our case, the condition becomes :

$$p < 3A^{2m} < 3p \quad with \quad p = A^{2m} + B^{2n} + A^m B^n \tag{110}$$

and $3A^{2m} < 3p \Longrightarrow A^{2m} < p$ that is verified. If :

$$p < 3A^{2m} \Longrightarrow 2A^{2m} - A^m B^n - B^{2n} \overbrace{>}^{?} 0$$

Studying the sign of the polynomial $Q(Y) = 2Y^2 - B^n Y - B^{2n}$ and taking $Y = A^m > B^n$, the condition $2A^{2m} - A^m B^n - B^{2n} > 0$ is verified, then the condition $b < 4a < 3b$ is true.

In the following of the paper, we verify easily that the condition $b < 4a < 3b$ implies to verify that $A^m > B^n$ which is true.

2.5.4. Case $b \mid p \Rightarrow \ p = b.p', p' > 1$, $b \neq 2$, $b \neq 4$ and $3 \mid a$

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.b.p'.3.a'}{3.b} = 4.p'a' \tag{111}$$

We calculate $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{112}$$

but $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$, we obtain:

$$B^n C^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = p'(b - 4a') \tag{113}$$

As $p = b.p'$, and $p' > 1$, so we have :

$$B^n C^l = p'(b - 4a') \tag{114}$$

$$\text{and} \quad A^{2m} = 4.p'.a' \tag{115}$$

** B-1- We suppose that p' is prime, then $A^{2m} = 4a'p' = (A^m)^2 \Longrightarrow p' \mid a'$. But $B^n C^l = p'(b - 4a') \Longrightarrow p' \mid B^n$ or $p' \mid C^l$.

** B-1-1- If $p' \mid B^n \Longrightarrow p' \mid B \Longrightarrow B = p'B_1$ with $B_1 \in \mathbb{N}^*$. Hence : $p'^{n-1}B_1^nC^l = b - 4a'$. But $n > 2 \Rightarrow (n-1) > 1$ and $p' \mid a'$, then $p' \mid b \Longrightarrow a$ and b are not coprime, then the contradiction.

** B-1-2- If $p' \mid C^l \Longrightarrow p' \mid C$. The same method used above, we obtain the same results.

** B-2- We consider that p' is not a prime integer.

** B-2-1- p', a are supposed coprime: $A^{2m} = 4a'p' \Longrightarrow A^m = 2a".p_1$ with $a' = a"^2$ and $p' = p_1^2$, then $a", p_1$ are also coprime. As $A^m = 2a".p_1$ then $2 \mid a"$ or $2 \mid p_1$.

** B-2-1-1- $2 \mid a"$, then $2 \nmid p_1$. But $p' = p_1^2$.

** B-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a".p_1$.

** B-2-1-1-2- We suppose that p_1 is not prime, we can write it as $p_1 = \omega^m \Longrightarrow p' = \omega^{2m}$, then: $B^nC^l = \omega^{2m}(b - 4a')$.

** B-2-1-1-2-1- If ω is prime, it is different of 2, then $\omega \mid (B^nC^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** B-2-1-1-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^jB_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(b - 4a')$.

** B-2-1-1-2-1-1-1- If $2m - n.j = 0$, we obtain $B_1^n.C^l = b - 4a'$. As $C^l = A^m + B^n \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, and $\omega \mid (b - 4a')$. But $\omega \neq 2$ and ω is coprime with a' then coprime with a, then $\omega \nmid b$. The conjecture (34) is verified.

** B-2-1-1-2-1-1-2- If $2m - nj \geq 1$, in this case with the same method, we obtain $\omega \mid C^l \Longrightarrow \omega \mid C$ and $\omega \mid (b - 4a')$ and $\omega \nmid a$ and $\omega \nmid b$. The conjecture (34) is verified.

** B-2-1-1-2-1-1-3- If $2m - nj < 0 \Longrightarrow \omega^{n.j-2m}B_1^n.C^l = b - 4a'$. As $\omega \mid C$ using $C^l = A^m + B^n$ then $C = \omega^h.C_1 \Longrightarrow \omega^{n.j-2m+h.l}B_1^n.C_1^l = b - 4a'$. If $n.j - 2m + h.l < 0 \Longrightarrow \omega \mid B_1^nC_1^l$, it follows the contradiction that $\omega \nmid B_1$ or $\omega \nmid C_1$. Then if $n.j - 2m + h.l > 0$ and $\omega \mid (b - 4a')$ with ω, a, b coprime and the conjecture (34) is verified.

** B-2-1-1-2-1-2- We obtain the same results if $\omega \mid C^l$.

** B-2-1-1-2-2- Now, $p' = \omega^{2m}$ and ω not prime, we write $\omega = \omega_1^f.\Omega$ with ω_1 prime $\nmid \Omega$ and $f \geq 1$ an integer, and $\omega_1 \mid A$. Then $B^nC^l = \omega_1^{2f.m}\Omega^{2m}(b-4a') \Longrightarrow \omega_1 \mid (B^nC^l) \Longrightarrow \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** B-2-1-1-2-2-1- If $\omega_1 \mid B^n \Longrightarrow \omega_1 \mid B \Longrightarrow B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2mf-nj}\Omega^{2m}(b-4a')$:

** B-2-1-1-2-2-1-1- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(b-4a')$. As $C^l = A^m + B^n \Longrightarrow \omega_1 \mid C^l \Longrightarrow \omega_1 \mid C \Longrightarrow \omega_1 \mid (b-4a')$. But $\omega_1 \neq 2$ and ω_1 is coprime with a', then coprime with a, we deduce $\omega_1 \nmid b$. Then the conjecture (34) is verified.

** B-2-1-1-2-2-1-2- If $2f.m - n.j \geq 1$, we have $\omega_1 \mid C^l \Longrightarrow \omega_1 \mid C \Longrightarrow \omega_1 \mid (b-4a')$ and $\omega_1 \nmid a$ and $\omega_1 \nmid b$. The conjecture (34) is verified.

** B-2-1-1-2-2-1-3- If $2f.m - n.j < 0 \Longrightarrow \omega_1^{n.j-2m.f}B_1^n.C^l = \Omega^{2m}(b-4a')$. As $\omega_1 \mid C$ using $C^l = A^m + B^n$, then $C = \omega_1^h.C_1 \Longrightarrow \omega^{n.j-2m.f+h.l}B_1^n.C_1^l = \Omega^{2m}(b-4a')$. If $n.j - 2m.f + h.l < 0 \Longrightarrow \omega_1 \mid B_1^nC_1^l$, it follows the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (b-4a')$ with ω_1, a, b coprime and the conjecture (34) is verified.

** B-2-1-1-2-2-2- We obtain the same results if $\omega_1 \mid C^l$.

** B-2-1-2- If $2 \mid p_1$, then $2 \mid p_1 \Longrightarrow 2 \nmid a' \Longrightarrow 2 \nmid a$. But $p' = p_1^2$.

** B-2-1-2-1- If $p_1 = 2$, we obtain $A^m = 4a" \Longrightarrow 2 \mid a"$ as $m \geq 3$, then the contradiction with a, b coprime.

** B-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$, as $A^m = 2a"p_1$, p_1 is written as $p_1 = 2^{m-1}\omega^m \Longrightarrow p' = 2^{2m-2}\omega^{2m}$. It follows $B^nC^l = 2^{2m-2}\omega^{2m}(b-4a') \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** B-2-1-2-2-1- If $2 \mid B^n \Longrightarrow 2 \mid B$, as $2 \mid A$, then $2 \mid C$. From $B^nC^l = 2^{2m-2}\omega^{2m}(b-4a')$, it follows if $2 \mid (b-4a') \Longrightarrow 2 \mid b$ but as $2 \nmid a'$, there is no contradiction with a, b coprime and the conjecture (34) is verified.

** B-2-1-2-2-2- If $2 \mid C^l$, using the same method as above, we obtain the identical results.

** B-2-2- p', a' are supposed not coprime. Let ω be a prime integer so that $\omega \mid a'$ and $\omega \mid p'$.

** B-2-2-1- We suppose firstly $\omega = 3$. As $A^{2m} = 4a'p' \Longrightarrow 3 \mid A$, but $3 \mid p' \Longrightarrow 3 \mid p$, as $p = A^{2m} + B^{2n} + A^m B^n \Longrightarrow 3 \mid B^{2n} \Longrightarrow 3 \mid B$, then $3 \mid C^l \Longrightarrow 3 \mid C$. We write $A = 3^i A_1$, $B = 3^j B_1$, $C = 3^h C_1$ and 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im} A_1^{2m} + 3^{2nj} B_1^{2n} + 3^{im+jn} A_1^m B_1^n = 3^k.g$ with $k = min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid p'$ that gives $a = 3^\alpha a_1 = 3a' \Longrightarrow a' = 3^{\alpha-1} a_1$, $3 \nmid a_1$ and $p' = 3^\mu p_1$, $3 \nmid p_1$ with $A^{2m} = 4a'p' = 3^{2im} A_1^{2m} = 4 \times 3^{\alpha-1+\mu}.a_1.p_1 \Longrightarrow \alpha + \mu - 1 = 2im$. As $p = bp' = b.3^\mu p_1 = 3^\mu.b.p_1$. The exponent of the term 3 of p is k, the exponent of the term 3 of the left member of the last equation is μ. If $3 \mid b$ it is a contradiction with a, b coprime. Then, we suppose that $3 \nmid b$, and the equality of the exponents: $min(2im, 2jn, im + jn) = \mu$, recall that $\alpha + \mu - 1 = 2im$. But $B^n C^l = p'(b - 4a')$ that gives $3^{(nj+hl)} B_1^n C_1^l = 3^\mu p_1 (b - 4 \times 3^{(\alpha-1)} a_1)$. We have also $A^m + B^n = C^l$ gives $3^{im} A_1^m + 3^{jn} B_1^n = 3^{hl} C_1^l$. Let $\epsilon = min(im, jn)$, we have $\epsilon = hl = min(im, jn)$. Then, we obtain the conditions:

$$k = min(2im, 2jn, im + jn) = \mu \tag{116}$$

$$\alpha + \mu - 1 = 2im \tag{117}$$

$$\epsilon = hl = min(im, jn) \tag{118}$$

$$3^{(nj+hl)} B_1^n C_1^l = 3^\mu p_1 (b - 4 \times 3^{(\alpha-1)} a_1) \tag{119}$$

** B-2-2-1-1- $\alpha = 1 \Longrightarrow a = 3a_1 = 3a'$ and $3 \nmid a_1$, the equation (117) becomes:

$$\mu = 2im$$

and the first equation (116) is written as:

$$k = min(2im, 2jn, im + jn) = 2im$$

- If $k = 2im$, then $2im \leq 2jn \Longrightarrow im \leq jn \Longrightarrow hl = im$, and (119) gives $\mu = 2im = nj + hl = im + nj \Longrightarrow im = jn = hl$. Hence $3 \mid A, 3 \mid B$ and $3 \mid C$ and the conjecture (34) is verified.
- If $k = 2jn \Longrightarrow 2jn = 2im \Longrightarrow im = jn = hl$. Hence $3 \mid A, 3 \mid B$ and $3 \mid C$ and the conjecture (34) is verified.
- If $k = im + jn = 2im \Longrightarrow im = jn \Longrightarrow \epsilon = hl = im = jn$ case that is seen above and we deduce that $3 \mid A, 3 \mid B$ and $3 \mid C$, and the conjecture (34) is verified.

** B-2-2-1-2- $\alpha > 1 \Longrightarrow \alpha \geq 2$ and $a' = 3^{\alpha-1} a_1$.
- If $k = 2im \Longrightarrow 2im = \mu$, but $\mu = 2im + 1 - \alpha$ that is impossible.
- If $k = 2jn = \mu \Longrightarrow 2jn = 2im + 1 - \alpha$. We obtain $2jn < 2im \Longrightarrow jn < im \Longrightarrow 2jn < im + jn$, $k = 2jn$ is just the minimum of $(2im, 2jn, im + jn)$. We obtain $jn = hl < im$ and the equation (119) becomes:

$$B_1^n C_1^l = p_1 (b - 4 \times 3^{(\alpha-1)} a_1)$$

The conjecture (34) is verified.

- If $k = im + jn \leq 2im \Longrightarrow jn \leq im$ and $k = im + jn \leq 2jn \Longrightarrow im \leq jn \Longrightarrow im = jn \Longrightarrow k = im + jn = 2im = \mu$ but $\mu = 2im + 1 - \alpha$ that is impossible.
- If $k = im + jn < 2im \Longrightarrow jn < im$ and $2jn < im + jn = k$ that is a contradiction with $k = min(2im, 2jn, im + jn)$.

** B-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $p' = \omega^\mu p_1$ with $\omega \nmid p_1$. As $A^{2m} = 4a'p' = 4\omega^{\alpha+\mu}.a_1.p_1 \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$, $\omega \nmid A_1$. But $B^nC^l = p'(b - 4a') = \omega^\mu p_1(b - 4a') \Longrightarrow \omega \mid B^nC^l \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** B-2-2-2-1- $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$. As $p = bp' = \omega^\mu bp_1 = \omega^k(\omega^{2im-k}A_1^{2m} + \omega^{2jn-k}B_1^{2n} + \omega^{im+jn-k}A_1^mB_1^n)$ with $k = min(2im, 2jn, im + jn)$. Then :
- If $\mu = k$, then $\omega \nmid b$ and the conjecture (34) is verified.
- If $k > \mu$, then $\omega \mid b$, but $\omega \mid a$ we deduce the contradiction with a, b coprime.
- If $k < \mu$, it follows from :

$$\omega^\mu bp_1 = \omega^k(\omega^{2im-k}A_1^{2m} + \omega^{2jn-k}B_1^{2n} + \omega^{im+jn-k}A_1^mB_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ that is a contradiction with the hypothesis.

** B-2-2-2-2- If $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid (C^l - A^m) \Longrightarrow \omega \mid B$. Then, we obtain the same results as B-2-2-2-1- above.

2.5.5. Case $b = 2p$ and $3 \mid a$

We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{2p} = 2a' = (A^m)^2 \Longrightarrow 2 \mid a' \Longrightarrow 2 \mid a$$

Then $2 \mid a$ and $2 \mid b$ that is a contradiction with a, b coprime.

2.5.6. Case $b = 4p$ and $3 \mid a$

We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{4p} = a' = (A^m)^2 = a"^2 \quad \text{with} \quad A^m = a"$$

Let us calculate A^mB^n, we obtain:

$$A^mB^n = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{2p}{3}cos^2\frac{\theta}{3} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{a'}{2} \Longrightarrow$$
$$A^mB^n + \frac{A^{2m}}{2} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3}$$

Let:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{120}$$

The left member of (120) is an integer and p is an integer, then $\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3}$ will be written as :

$$\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two integers coprime and $k_2 \mid p \Longrightarrow p = k_2.k_3$.

** C-1- Firstly, we suppose that $k_3 \neq 1$. Then :

$$A^{2m} + 2A^mB^n = k_3.k_1$$

Let μ be a prime integer and $\mu \mid k_3$, then $\mu \mid A^m(A^m + 2B^n) \Longrightarrow \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** C-1-1- If $\mu \mid (A^m = a") \Longrightarrow \mu \mid (a"^2 = a') \Longrightarrow \mu \mid (3a' = a)$. As $\mu \mid k_3 \Longrightarrow \mu \mid p \Longrightarrow \mu \mid (4p = b)$, then the contradiction with a, b coprime.

** C-1-2- If $\mu \mid (A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then:

$$\mu \neq 2 \quad and \quad \mu \nmid B^n \tag{121}$$

$\mu \mid (A^m + 2B^n)$, we write:

$$A^m + 2B^n = \mu.t'$$

Then:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$
$$\Longrightarrow p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m)$$

As $b = 4p = 4k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid b \Longrightarrow \exists \mu'$ so that $b = \mu.\mu'$, we obtain:

$$\mu'.\mu = \mu(4\mu t'^2 - 8t'B^n) + 4B^n(B^n - A^m)$$

The last equation implies $\mu \mid 4B^n(B^n - A^m)$, but $\mu \neq 2$ then $\mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** C-1-1-1- If $\mu \mid B^n \Longrightarrow$ then the contradiction with (121).

** C-1-1-2- If $\mu \mid (B^n - A^m)$ and using $\mu \mid (A^m + 2B^n)$, we have :

$$\mu \mid 3B^n \Longrightarrow \begin{cases} \mu \mid B^n \\ or \\ \mu = 3 \end{cases}$$

** C-1-1-2-1- If $\mu \mid B^n$ then the contradiction with (121).

** C-1-1-2-2- If $\mu = 3$, then $3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

** C-2- We assume now that $k_3 = 1$, then:

$$A^{2m} + 2A^m B^n = k_1 \tag{122}$$

$$p = k_2$$

$$\frac{2\sqrt{3}}{3} sin \frac{2\theta}{3} = \frac{k_1}{p}$$

We take the square of the last equation, we obtain :

$$\frac{4}{3} sin^2 \frac{2\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} cos^2 \frac{\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} . \frac{3a'}{b} = \frac{k_1^2}{p^2}$$

Finally:

$$a'(4p - 3a') = k_1^2 \tag{123}$$

but $a' = a"^2$, then $4p - 3a'$ is a square. Let :

$$\lambda^2 = 4p - 3a' = 4p - a = b - a$$

The equation (123) becomes :

$$a"^2 \lambda^2 = k_1^2 \Longrightarrow k_1 = a"\lambda \tag{124}$$

taking the positive root. Using (122), we have:

$$k_1 = A^m(A^m + 2B^n) = a"(A^m + 2B^n)$$

Then :

$$A^m + 2B^n = \lambda$$

Now, we consider that $b - a = \lambda^2 \Longrightarrow \lambda^2 + 3a"^2 = b$, then the couple $(\lambda, a")$ is a solution of the Diophantine equation:

$$X^2 + 3Y^2 = b \tag{125}$$

with $X = \lambda$ and $Y = a"$. But using one theorem on the solutions of the equation given by (125), b is written under the form (see theorem 37.4 in [**3**]):

$$b = 2^{2s} \times 3^t . p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

where p_i are prime integers so that $p_i \equiv 1 (\bmod 6)$, the q_j are also prime integers so that $q_j \equiv 5 (\bmod 6)$. Then, as $b = 4p$:

- If $t \geq 1 \Longrightarrow 3 \mid b$, but $3 \mid a$, then the contradiction with a, b coprime.

** C-2-2-1- Hence, we suppose that p is written under the form:

$$p = p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

with $p_i \equiv 1 (\bmod 6)$ and $q_j \equiv 5 (\bmod 6)$. Finally, we obtain that :

(126) $$p \equiv 1 (\bmod 6)$$

We will verify if this condition does not give contradictions.

We will present the table of the value modulo 6 of $p = A^{2m} + A^m B^n + B^{2n}$ in function of the values of $A^m, B^n (\bmod 6)$. We obtain the table below:

TABLE 1. Table of $p \,(\bmod 6)$

A^m, B^n	0	1	2	3	4	5
0	0	**1**	4	3	4	**1**
1	**1**	3	**1**	**1**	3	**1**
2	4	**1**	0	**1**	4	3
3	3	**1**	**1**	3	**1**	**1**
4	4	3	4	**1**	0	**1**
5	**1**	**1**	3	**1**	**1**	3

** C-2-2-1-1- Case $A^m \equiv 0 (\bmod 6) \Longrightarrow 2 \mid (A^m = a") \Longrightarrow 2 \mid (a' = a"^2) \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime. All the cases of the first line of the table 1 are to reject.

** C-2-2-1-2- Case $A^m \equiv 1 (\bmod 6)$ and $B^n \equiv 0 (\bmod 6)$, then $2 \mid B^n \Longrightarrow B^n = 2B'$ and p is written as $p = (A^m + B')^2 + 3B'^2$ with $(p, 3) = 1$, if not $3 \mid p$, then $3 \mid b$, but $3 \mid a$, then the contradiction with a, b coprime. Hence, the pair $(A^m + B', B')$ verifies the equation:

(127) $$(A^m + B')^2 + 3B'^2 = p$$

that we can write it as:

(128) $$(A^m + B')^2 - B'^2 = p - 4B'^2 = A^{2m} + B^{2n} + A^m B^n - B^{2n} = C^l A^m = N$$

Then $(A^m + B', B')$ is a solution of the Diophantine equation:

(129) $$x^2 - y^2 = N$$

where $N = C^l A^m \equiv 1(\bmod 6)$. Let $Q(N)$ be the number of the solutions of (129) and $\tau(N)$ is the number of suitable factorization of N, then we recall the following result concerning the solutions of the equation (129) (see theorem 27.3 in [**2**]):

- If $N \equiv 2(\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3(\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0(\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.

As $N = C^l A^m \equiv 1(mod 6) \Longrightarrow N$ is odd, the cases $Q(N) = 0$ and $Q(N) = [\tau(N/4)/2]$ are rejected, then $N \equiv 1$ or $N \equiv 3(mod 4)$, it follows $Q(N) = [\tau(N)/2]$.

As $A^m + B', B'$ is a couple of solutions of the Diophantine equation (129), then $\exists\, d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2(A^m + B') \tag{130}$$

$$d - d' = 2B' = B^n \tag{131}$$

We will use the same method used for the paragraph above A-2-1-2-.

** C-2-2-1-2-1- As $C^l > A^m$, we take $d = C^l$ and $d' = A^m$. It follows:

$$C^l + A^m = 2(A^m + B') = 2A^m + B^n$$

$$C^l - A^m = B^n = 2B'$$

Then the case $d = C^l$ and $d' = A^m$ gives *a priory* no contradictions.

** C-2-2-1-2-2- Now, we consider the case $d = C^l A^m$ and $d' = 1$. We rewrite the equations (130-131):

$$C^l A^m + 1 = 2(A^m + B') \tag{132}$$

$$C^l A^m - 1 = 2B' \tag{133}$$

We obtain $1 = A^m$, it follows $C^l - B^n = 1$, we know [**4**] that the only positive solution of the last equation is $C = 3, B = 2, n = 3$ and $l = 2 < 3$, then the contradiction.

** C-2-2-1-2-3- Now, we consider the case $d = c_1^{lr-1} C_1^l$ where c_1 is a prime integer with $c_1 \nmid C_1$ and $C = c_1^r C_1$, $r \geq 1$. It follows that $d' = c_1.A^m$. We rewrite the equations (130-131):

$$c_1^{lr-1} C_1^l + c_1.A^m = 2(A^m + B') \tag{134}$$

$$c_1^{lr-1} C_1^l - c_1.A^m = 2B' = B^n \tag{135}$$

As $l \geq 3$, from the last two equations above, it follows that $c_1 \mid 2(A^m + B')$ and $c_1 \mid (2B')$. Then $c_1 = 2$, or $c_1 \mid (A^m + B')$ and $c_1 \mid B'$.

** C-2-2-1-2-3-1- We suppose $c_1 = 2$. As $l \geq 3$, from the equation (135) it follows that $2 \mid B^n$, then $2 \mid (A^m = a") \Longrightarrow 2 \mid (a"^2 = a') \Longrightarrow 2 \mid (a = 3a')$, but $b = 4p$ (see

2.5.6), then the contradiction with a, b coprime.

** C-2-2-1-2-3-2- We suppose $c_1 \neq 2$, then $c_1 \mid (A^m + B')$ and $c_1 \mid B'$. It follows $c_1 \mid A^m$ and $c_1 \mid (B^n = 2B') \Longrightarrow c_1 \mid p \Longrightarrow c_1 \mid b = 4p$. From $c_1 \mid (A^m = a") \Longrightarrow c_1 \mid (a"^2 = a') \Longrightarrow c_1 \mid (a = 3a')$, then the contradiction with a, b coprime.

The other cases of the expressions of d and d' with d, d' not coprime and $d > d'$ so that $N = C^l A^m = d.d'$ give also contradictions.

** C-2-2-1-2-4- Now, we consider the case $d = a_1.C^l$ where a_1 is a prime integer with $a_1 \nmid A_1$ and $A = a_1^r A_1$, $r \geq 1$. It follows that $d' = a_1^{mr-1} A_1^m$. We rewrite the equations (130-131):

$$a_1 C^l + a_1^{mr-1} A_1^m = 2(A^m + B') \tag{136}$$

$$a_1 C^l - a_1^{mr-1} A_1^m = 2B' = B^n \tag{137}$$

As $m \geq 3$, from the last two equations above, it follows that $a_1 \mid 2(A^m + B')$ and $a_1 \mid (2B')$. Then $a_1 = 2$, or $a_1 \mid (A^m + B')$ and $a_1 \mid B'$.

** C-2-2-1-2-4-1- We suppose $a_1 = 2 \Longrightarrow 2 \mid (A^m = a'') \Longrightarrow a_1 \mid (a''^2 = a') \Longrightarrow a_1 \mid (a = 3a')$. But $b = 4p$, then the contradiction with a, b coprime.

** C-2-2-1-2-4-2- We suppose $a_1 \neq 2$, then $a_1 \mid (A^m + B')$ and $a_1 \mid B'$. It follows $a_1 \mid A^m$ and $a_1 \mid (B^n = 2B') \Longrightarrow a_1 \mid p \Longrightarrow a_1 \mid b = 4p$. From $a_1 \mid (A^m = a'') \Longrightarrow a_1 \mid (a''^2 = a') \Longrightarrow a_1 \mid (a = 3a')$, then the contradiction with a, b coprime.

The other cases of the expressions of d and d' with d, d' not coprime and $d > d'$ so that $N = C^l A^m = d.d'$ give also contradictions.

** C-2-2-1-2-5- Now, let $C = c_1^r C_1$ with c_1 a prime, $r \geq 1$ and $c_1 \nmid C_1$, we consider the case $d = C_1^l$ and $d' = c_1^{rl} A^m$ so that $d > d'$. We rewrite the equations (130-131):

$$C_1^l + c_1^{rl} A^m = 2(A^m + B') \tag{138}$$

$$C_1^l - c_1^{rl} A^m = 2B' = B^n \tag{139}$$

We obtain $c_1^{rl} A^m = A^m \Longrightarrow c_1^{rl} = 1$, then the contradiction.

** C-2-2-1-2-6- Now, let $C = c_1^r C_1$ with c_1 a prime, $r \geq 1$ and $c_1 \nmid C_1$, we consider the case $d = C_1^l A^m$ and $d' = c_1^{rl}$ so that $d > d'$. We rewrite the equations (130-131):

$$C_1^l A^m + c_1^{rl} = 2(A^m + B') \tag{140}$$

$$C_1^l A^m - c_1^{rl} = 2B' = B^n \tag{141}$$

We obtain $c_1^{rl} = A^m \Longrightarrow c_1 \mid A^m$, then $c_1 \mid A^m = a" \Longrightarrow c_1 \mid (a"^2 = a') \Longrightarrow c_1 \mid (a = 3a')$. As $c_1 \mid C$ and $c_1 \mid A^m \Longrightarrow c_1 \mid B^n$, it follows $c_1 \mid (p = b)$, then the contradiction with a, b coprime.

The other cases of the expressions of d and d' with d, d' coprime and $d > d'$ so that $N = C^l A^m = d.d'$ give also contradictions.

Finally, from the cases studied in the above paragraph C-2-2-1-2, we have found one suitable factorization of N that gives *a priory* no contradictions, it is the case $N = C^l.A^m$, but $1 \ll \tau(N)$, it follows the contradiction with $Q(N) = [\tau(N)/2] \leq 1$. We conclude that the case $A^m \equiv 1(\bmod 6)$ and $B^n \equiv 0(\bmod 6)$ of the paragraph C-2-2-1-2 is to reject.

** C-2-2-1-3- Case $A^m \equiv 1(\bmod 6)$ and $B^n \equiv 2(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-4- Case $A^m \equiv 1(\bmod 6)$ and $B^n \equiv 3(\bmod 6)$, then $3 \mid B^n \Longrightarrow B^n = 3B'$. As $p = A^{2m} + A^m B^n + B^{2n} \Longrightarrow p \equiv 5(\bmod 6) \not\equiv 1(\bmod 6)$ (see (126)), then the contradiction and the case C-2-2-1-4- is to reject.

** C-2-2-1-5- Case $A^m \equiv 1(\bmod 6)$ and $B^n \equiv 5(\bmod 6)$, then $C^l \equiv 0(\bmod 6) \Longrightarrow 2 \mid C^l$, see C-2-2-1-2-.

** C-2-2-1-6- Case $A^m \equiv 2(\bmod 6) \Longrightarrow 2 \mid a" \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** C-2-2-1-7- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 1(\bmod 6)$, then $C^l \equiv 4(\bmod 6) \Longrightarrow 2 \mid C^l \Longrightarrow C^l = 2C'$, and C is even, see C-2-2-1-2-.

** C-2-2-1-8- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 2(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-9- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 4(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-10- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 5(\bmod 6)$, then $C^l \equiv 2(\bmod 6) \Longrightarrow 2 \mid C^l$, and C is even, see C-2-2-1-2-.

** C-2-2-1-11- Case $A^m \equiv 4(\bmod 6) \Longrightarrow 2 \mid a" \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** C-2-2-1-12- Case $A^m \equiv 5(\bmod 6)$ and $B^n \equiv 0(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-13- Case $A^m \equiv 5 (\bmod 6)$ and $B^n \equiv 1 (\bmod 6)$, then $C^l \equiv 0 (\bmod 6) \Longrightarrow 2 \mid C^l$, C is even, see C-2-2-1-2-.

** C-2-2-1-14- Case $A^m \equiv 5 (\bmod 6)$ and $B^n \equiv 3 (\bmod 6)$, then $C^l \equiv 2 (\bmod 6) \Longrightarrow 2 \mid C^l \Longrightarrow C^l = 2C'$, C is even, C-2-2-1-2-.

** C-2-2-1-15- Case $A^m \equiv 5 (\bmod 6)$ and $B^n \equiv 4 (\bmod 6)$, then B^n is even, see C-2-2-1-2-.

We have achieved the study all the cases of the table 1 giving contradictions.

Then the case $k_3 = 1$ is impossible.

2.5.7. Case $3 \mid a$ and $b = 2p',\ b \neq 2$ with $p' \mid p$

$3 \mid a \Longrightarrow a = 3a'$, $b = 2p'$ with $p = k.p'$, then:

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{6p'} = 2.k.a'$$

We calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right)$$

but $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, then using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - 2a')$$

As $p = b.p'$, and $p' > 1$, then we have:

$$B^nC^l = k(p' - 2a') \tag{142}$$

$$\text{and} \quad A^{2m} = 2k.a' \tag{143}$$

** D-1- We suppose that k is prime.

** D-1-1- If $k = 2$, then we have $p = 2p' = b \Longrightarrow 2 \mid b$, but $A^{2m} = 4a' = (A^m)^2 \Longrightarrow A^m = 2a"$ with $a' = a"^2$, then $2 \mid a" \Longrightarrow 2 \mid (a = 3a"^2)$, it follows the contradiction with a, b coprime.

** D-1-2- We suppose $k \neq 2$. From $A^{2m} = 2k.a' = (A^m)^2 \Longrightarrow k \mid a'$ and $2 \mid a' \Longrightarrow a' = 2.k.a"^2 \Longrightarrow A^m = 2.k.a"$. Then $k \mid A^m \Longrightarrow k \mid A \Longrightarrow A = k^i.A_1$ with $i \geq 1$ and $k \nmid A_1$. $k^{im}A_1^m = 2ka" \Longrightarrow 2a" = k^{im-1}A_1^m$. From $B^nC^l = k(p' - 2a') \Longrightarrow k \mid (B^nC^l) \Longrightarrow k \mid B^n$ or $k \mid C^l$.

** D-1-2-1- We suppose that $k \mid B^n \Longrightarrow k \mid B \Longrightarrow B = k^j.B_1$ with $j \geq 1$ and $k \nmid B_1$. It follows $k^{nj-1}B_1^nC^l = p' - 2a' = p' - 4ka"^2$. As $n \geq 3 \Longrightarrow nj - 1 \geq 2$, then $k \mid p'$ but $k \neq 2 \Longrightarrow k \mid (2p' = b)$, but $k \mid a' \Longrightarrow k \mid (3a' = a)$. It follows the contradiction with a, b coprime.

** D-1-2-2- If $k \mid C^l$ we obtain the identical results.

** D-2- We suppose that k is not prime. Let ω be an integer prime so that $k = \omega^s.k_1$, with $s \geq 1$, $\omega \nmid k_1$. The equations (142-143) become:

$$B^nC^l = \omega^s.k_1(p' - 2a')$$
$$\text{and} \quad A^{2m} = 2\omega^s.k_1.a'$$

** D-2-1- We suppose that $\omega = 2$, then we have the equations:

$$A^{2m} = 2^{s+1}.k_1.a' \tag{144}$$
$$B^nC^l = 2^s.k_1(p' - 2a') \tag{145}$$

** D-2-1-1- Case: $2 \mid a' \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** D-2-1-2- Case: $2 \nmid a'$. As $2 \nmid k_1$, the equation (144) gives $2 \mid A^{2m} \Longrightarrow A = 2^iA_1$, with $i \geq 1$ and $2 \nmid A_1$. It follows that $2im = s + 1$.

** D-2-1-2-1- We suppose that $2 \nmid (p' - 2a') \Longrightarrow 2 \nmid p'$. From the equation (145), we obtain that $2 \mid B^nC^l \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** D-2-1-2-1-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^jB_1$ with $2 \nmid B_1$ and $j \geq 1$, then $B_1^nC^l = 2^{s-jn}k_1(p' - 2a')$:

- If $s - jn \geq 1$, then $2 \mid C^l \Longrightarrow 2 \mid C$, and no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$, and the conjecture (34) is verified.
- If $s - jn \leq 0$, from $B_1^nC^l = 2^{s-jn}k_1(p' - 2a') \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$.

** D-2-1-2-1-2- Using the same method of the proof above, we obtain the identical results if $2 \mid C^l$.

** D-2-1-2-2- We suppose now that $2 \mid (p' - 2a') \Longrightarrow p' - 2a' = 2^\mu.\Omega$, with $\mu \geq 1$ and $2 \nmid \Omega$. We recall that $2 \nmid a'$. The equation (145) is written as:

$$B^nC^l = 2^{s+\mu}.k_1.\Omega$$

This last equation implies that $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** D-2-1-2-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^jB_1$ with $j \geq 1$ and $2 \nmid B_1$. Then $B_1^nC^l = 2^{s+\mu-jn}.k_1.\Omega$:

- If $s+\mu-jn \geq 1$, then $2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$, and the conjecture (34) is verified.
- If $s+\mu-jn \leq 0$, from $B_1^nC^l = 2^{s+\mu-jn}k_1.\Omega \Longrightarrow 2 \nmid C^l$, then contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$.

** D-2-1-2-2-2- We obtain the identical results if $2 \mid C^l$.

** D-2-2- We suppose that $\omega \neq 2$. We have then the equations:

$$A^{2m} = 2\omega^s.k_1.a' \tag{146}$$

$$B^nC^l = \omega^s.k_1.(p'-2a') \tag{147}$$

As $\omega \neq 2$, from the equation (146), we have $2 \mid (k_1.a')$. If $2 \mid a' \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** D-2-2-1- Case: $2 \nmid a'$ and $2 \mid k_1 \Longrightarrow k_1 = 2^\mu.\Omega$ with $\mu \geq 1$ and $2 \nmid \Omega$. From the equation (146), we have $2 \mid A^{2m} \Longrightarrow 2 \mid A \Longrightarrow A = 2^iA_1$ with $i \geq 1$ and $2 \nmid A_1$, then $2im = 1+\mu$. The equation (147) becomes:

$$B^nC^l = \omega^s.2^\mu.\Omega.(p'-2a') \tag{148}$$

From the equation (148), we obtain $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** D-2-2-1-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^jB_1$, with $j \in \mathbb{N}^*$ and $2 \nmid B_1$.

** D-2-2-1-1-1- We suppose that $2 \nmid (p'-2a')$, then we have $B_1^nC^l = \omega^s2^{\mu-jn}\Omega(p'-2a')$:
- If $\mu - jn \geq 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (34) is verified.
- If $\mu - jn \leq 0 \Longrightarrow 2 \nmid C^l$ then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** D-2-2-1-1-2- We suppose that $2 \mid (p'-2a') \Longrightarrow p'-2a' = 2^\alpha.P$, with $\alpha \in \mathbb{N}^*$ and $2 \nmid P$. It follows that $B_1^nC^l = \omega^s2^{\mu+\alpha-jn}\Omega.P$:
- If $\mu+\alpha-jn \geq 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (34) is verified.
- If $\mu+\alpha-jn \leq 0 \Longrightarrow 2 \nmid C^l$ then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** D-2-2-1-2- We suppose now that $2 \mid C^n \Longrightarrow 2 \mid C$. Using the same method described above, we obtain the identical results.

2.5.8. Case $3 \mid a$ and $b = 4p',\ b \neq 4$ with $p' \mid p$

$3 \mid a \Longrightarrow a = 3a'$, $b = 4p'$ with $p = k.p'$, $k \neq 1$ if not $b = 4p$ this case has been studied (see paragraph 2.5.6), then we have :

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{12p'} = k.a'$$

We calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right)$$

but $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, then using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - a')$$

As $p = b.p'$, and $p' > 1$, we have :

$$B^nC^l = k(p' - a') \tag{149}$$

$$\text{and} \quad A^{2m} = k.a' \tag{150}$$

** E-1- We suppose that k is prime. From $A^{2m} = k.a' = (A^m)^2 \Longrightarrow k \mid a'$ and $a' = k.a"^2 \Longrightarrow A^m = k.a"$. Then $k \mid A^m \Longrightarrow k \mid A \Longrightarrow A = k^i.A_1$ with $i \geq 1$ and $k \nmid A_1$. $k^{mi}A_1^m = ka" \Longrightarrow a" = k^{mi-1}A_1^m$. From $B^nC^l = k(p' - a') \Longrightarrow k \mid (B^nC^l) \Longrightarrow k \mid B^n$ or $k \mid C^l$.

** E-1-1- We suppose that $k \mid B^n \Longrightarrow k \mid B \Longrightarrow B = k^j.B_1$ with $j \geq 1$ and $k \nmid B_1$. Then $k^{n.j-1}B_1^nC^l = p' - a'$. As $n.j - 1 \geq 2 \Longrightarrow k \mid (p' - a')$. But $k \mid a' \Longrightarrow k \mid a$, then $k \mid p' \Longrightarrow k \mid (4p' = b)$ and we arrive to the contradiction that a, b are coprime.

** E-1-2- We suppose that $k \mid C^l$, using the same method with the above hypothesis $k \mid B^n$, we obtain the identical results.

** E-2- We suppose that k is not prime.

** E-2-1- We take $k = 4 \Longrightarrow p = 4p' = b$, it is the case 2.5.3 studied above.

** E-2-2- We suppose that $k \geq 6$ not prime. Let ω be a prime so that $k = \omega^s.k_1$, with $s \geq 1$, $\omega \nmid k_1$. The equations (149-150) become:

$$B^nC^l = \omega^s.k_1(p' - a') \tag{151}$$

$$\text{and} \quad A^{2m} = \omega^s.k_1.a' \tag{152}$$

** E-2-2-1- We suppose that $\omega = 2$.

** E-2-2-1-1- If $2 \mid a' \Longrightarrow 2 \mid (3a' = a)$, but $2 \mid (4p' = b)$, then the contradiction with a, b coprime.

** E-2-2-1-2- We consider that $2 \nmid a'$. From the equation (152), it follows that $2 \mid A^{2m} \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$ with $2 \nmid A_1$ and:

$$B^n C^l = 2^s k_1 (p' - a')$$

** E-2-2-1-2-1- We suppose that $2 \nmid (p' - a')$, from the above expression, we have $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** E-2-2-1-2-1-1- If $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$ with $2 \nmid B_1$. Then $B_1^n C^l = 2^{2im-jn} k_1 (p' - a')$:

- If $2im - jn \geq 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (34) is verified.
- If $2im - jn \leq 0 \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n \Longrightarrow 2 \mid C^l$.

** E-2-2-1-2-1-2- If $2 \mid C^l \Longrightarrow 2 \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-1-2-2- We suppose that $2 \mid (p' - a')$. As $2 \nmid a' \Longrightarrow 2 \nmid p'$, $2 \mid (p' - a') \Longrightarrow p' - a' = 2^\alpha . P$ with $\alpha \geq 1$ and $2 \nmid P$. The equation (151) is written as :

$$B^n C^l = 2^{s+\alpha} k_1 . P = 2^{2im+\alpha} k_1 . P \tag{153}$$

then $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** E-2-2-1-2-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$, with $2 \nmid B_1$. The equation (153) becomes $B_1^n C^l = 2^{2im+\alpha-jn} k_1 P$:

- If $2im + \alpha - jn \geq 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (34) is verified.
- If $2im + \alpha - jn \leq 0 \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n \Longrightarrow 2 \mid C^l$.

** E-2-2-1-2-2-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C$. Using the same method described above, we obtain the identical results.

** E-2-2-2- We suppose that $\omega \neq 2$. We recall the equations:

$$A^{2m} = \omega^s . k_1 . a' \tag{154}$$

$$B^n C^l = \omega^s . k_1 (p' - a') \tag{155}$$

** E-2-2-2-1- We suppose that ω, a' are coprime, then $\omega \nmid a'$. From the equation (154), we have $\omega \mid A^{2m} \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$ with $\omega \nmid A_1$ and $s = 2im$.

** E-2-2-2-1-1- We suppose that $\omega \nmid (p' - a')$. From the equation (155) above, we have $\omega \mid (B^n C^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-1-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$. Then $B_1^n C^l = 2^{2im-jn} k_1 (p' - a')$:

- If $2im - jn \geq 1 \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (34) is verified.
- If $2im - jn \leq 0 \Longrightarrow \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n \Longrightarrow \omega \mid C^l$.

** E-2-2-2-1-1-2- If $\omega \mid C^l \Longrightarrow \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-1-2- We suppose that $\omega \mid (p' - a') \Longrightarrow \omega \nmid p'$ as ω and a' are coprime. $\omega \mid (p' - a') \Longrightarrow p' - a' = \omega^\alpha . P$ with $\alpha \geq 1$ and $\omega \nmid P$. The equation (155) becomes :

$$B^n C^l = \omega^{s+\alpha} k_1 . P = \omega^{2im+\alpha} k_1 . P \tag{156}$$

then $\omega \mid (B^n C^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-1-2-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$, with $\omega \nmid B_1$. The equation (156) is written as $B_1^n C^l = 2^{2im+\alpha-jn} k_1 P$:

- If $2im + \alpha - jn \geq 1 \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (34) is verified.
- If $2im + \alpha - jn \leq 0 \Longrightarrow \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n \Longrightarrow \omega \mid C^l$.

** E-2-2-2-1-2-2- We suppose that $\omega \mid C^l \Longrightarrow \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-2- We suppose that ω, a' are not coprime, then $a' = \omega^\beta . a"$ with $\omega \nmid a"$. The equation (154) becomes:

$$A^{2m} = \omega^s k_1 a' = \omega^{s+\beta} k_1 . a"$$

We have $\omega \mid A^{2m} \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$ with $\omega \nmid A_1$ and $s + \beta = 2im$.

** E-2-2-2-2-1- We suppose that $\omega \nmid (p' - a') \Longrightarrow \omega \nmid p' \Longrightarrow \omega \nmid (b = 4p')$. From the equation (155), we obtain $\omega \mid (B^n C^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$. Then $B_1^n C^l = 2^{s-jn} k_1 (p' - a')$:

- If $s - jn \geq 1 \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (34) is verified.

- If $s - jn \leq 0 \Longrightarrow \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n \Longrightarrow \omega \mid C^l$.

** E-2-2-2-2-1-2- If $\omega \mid C^l \Longrightarrow \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-2-2- We suppose that $\omega \mid (p' - a' = p' - \omega^\beta . a") \Longrightarrow \omega \mid p' \Longrightarrow \omega \mid (4p' = b)$, but $\omega \mid a' \Longrightarrow \omega \mid a$. Then the contradiction with a, b coprime.

The study of the cases of 2.5.8 is achieved.

2.5.9. Case $3 \mid a$ and $b \mid 4p$

$a = 3a'$ and $4p = k_1 b$. As $A^{2m} = \dfrac{4p}{3} cos^2 \dfrac{\theta}{3} = \dfrac{4p}{3}\dfrac{3a'}{b} = k_1 a'$ and $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3a'}{b}\right) = \frac{k_1}{4}(b - 4a')$$

As $B^n C^l$ is an integer, we must obtain $4 \mid k_1$, or $4 \mid (b - 4a')$ or ($2 \mid k_1$ and $2 \mid (b - 4a')$).

** F-1- If $k_1 = 1 \Rightarrow b = 4p$: it is the case 2.5.6.

** F-2- If $k_1 = 4 \Rightarrow p = b$: it is the case 2.5.3.

** F-3- If $k_1 = 2$ and $2 \mid (b - 4a')$: in this case, we have $A^{2m} = 2a' \Longrightarrow 2 \mid a' \Longrightarrow 2 \mid a$. $2 \mid (b - 4a') \Longrightarrow 2 \mid b$ then the contradiction with a, b coprime.

** F-4- If $2 \mid k_1$ and $2 \mid (b - 4a')$: $2 \mid (b - 4a') \Longrightarrow b - 4a' = 2^\alpha \lambda$, α and $\lambda \in \mathbb{N}^* \geq 1$ with $2 \nmid \lambda$; $2 \mid k_1 \Longrightarrow k_1 = 2^t k_1'$ with $t \geq 1 \in \mathbb{N}^*$ with $2 \nmid k_1'$ and we have:

$$A^{2m} = 2^t k_1' a' \tag{157}$$

$$B^n C^l = 2^{t+\alpha-2} k_1' \lambda \tag{158}$$

From the equation (157), we have $2 \mid A^{2m} \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$, $i \geq 1$ and $2 \nmid A_1$.

** F-4-1- We suppose that $t = \alpha = 1$, then the equations (157-158) become :

$$A^{2m} = 2k_1' a' \tag{159}$$

$$B^n C^l = k_1' \lambda \tag{160}$$

From the equation (159) it follows that $2 \mid a' \Longrightarrow 2 \mid (a = 3a')$. But $b = 4a' + 2\lambda \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

** F-4-2- We suppose that $t + \alpha - 2 \geq 1$ and we have the expressions:

$$A^{2m} = 2^t k'_1 a' \tag{161}$$

$$B^n C^l = 2^{t+\alpha-2} k'_1 \lambda \tag{162}$$

** F-4-2-1- We suppose that $2 \mid a' \Longrightarrow 2 \mid a$, but $b = 2^\alpha \lambda + 4a' \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

** F-4-2-2- We suppose that $2 \nmid a'$. From (161), we have $2 \mid A^{2m} \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$ and $B^n C^l = 2^{t+\alpha-2} k'_1 \lambda \Longrightarrow 2 \mid B^n C^l \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** F-4-2-2-1- We suppose that $2 \mid B^n$. We have $2 \mid B \Longrightarrow B = 2^j B_1$, $j \geq 1$ and $2 \nmid B_1$. The equation (162) becomes $B_1^n C^l = 2^{t+\alpha-2-jn} k'_1 \lambda$:

- If $t + \alpha - 2 - jn > 0 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (34) is verified.
- If $t + \alpha - 2 - jn < 0 \Longrightarrow 2 \mid k'_1 \lambda$, but $2 \nmid k'_1$ and $2 \nmid \lambda$. Then this case is impossible.
- If $t + \alpha - 2 - jn = 0 \Longrightarrow B_1^n C^l = k'_1 \lambda \Longrightarrow 2 \nmid C^l$ then it is a contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** F-4-2-2-2- We suppose that $2 \mid C^l$. We use the same method described above, we obtain the identical results.

** F-5- We suppose that $4 \mid k_1$ with $k_1 > 4 \Rightarrow k_1 = 4k'_2$, we have :

$$A^{2m} = 4k'_2 a' \tag{163}$$

$$B^n C^l = k'_2 (b - 4a') \tag{164}$$

** F-5-1- We suppose that k'_2 is prime, from (163), we have $k'_2 \mid a'$. From (164), $k'_2 \mid (B^n C^l) \Longrightarrow k'_2 \mid B^n$ or $k'_2 \mid C^l$.

** F-5-1-1- We suppose that $k'_2 \mid B^n \Longrightarrow k'_2 \mid B \Longrightarrow B = k'^{\beta}_2 . B_1$ with $\beta \geq 1$ and $k'_2 \nmid B_1$. It follows that we have $k'^{n\beta-1}_2 B_1^n C^l = b - 4a' \Longrightarrow k'_2 \mid b$ then the contradiction with a, b coprime.

** F-5-1-2- We obtain identical results if we suppose that $k'_2 \mid C^l$.

** F-5-2- We suppose that k'_2 is not prime.

** F-5-2-1- We suppose that k'_2 and a' are coprime. From (163), k'_2 can be written under the form $k'_2 = q_1^{2j} . q_2^2$ and $q_1 \nmid q_2$ and q_1 prime. We have

$A^{2m} = 4q_1^{2j}.q_2^2 a' \Longrightarrow q_1 \mid A$ and $B^nC^l = q_1^{2j}.q_2^2(b-4a') \Longrightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-5-2-1-1- We suppose that $q_1 \mid B^n \Longrightarrow q_1 \mid B \Longrightarrow B = q_1^f.B_1$ with $q_1 \nmid B_1$. We obtain $B_1^nC^l = q_1^{2j-fn}q_2^2(b-4a')$:
- If $2j - f.n \geq 1 \Longrightarrow q_1 \mid C^l \Longrightarrow q_1 \mid C$ but $C^l = A^m + B^n$ gives also $q_1 \mid C$ and the conjecture (34) is verified.
- If $2j - f.n = 0$, we have $B_1^nC^l = q_2^2(b-4a')$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b-4a')$. As q_1 and a' are coprime, then $q_1 \nmid b$, and the conjecture (34) is verified.
- If $2j - f.n < 0 \Longrightarrow q_1 \mid (b-4a') \Longrightarrow q_1 \nmid b$ because a' is coprime with q_1, and $C^l = A^m + B^n$ gives $q_1 \mid C$, and the conjecture (34) is verified.

** F-5-2-1-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-5-2-2- We suppose that k'_2, a' are not coprime. Let q_1 be a prime so that $q_1 \mid k'_2$ and $q_1 \mid a'$. We write k'_2 under the form $q_1^j.q_2$ with $j \geq 1$, $q_1 \nmid q_2$. From $A^{2m} = 4k'_2a' \Longrightarrow q_1 \mid A^{2m} \Longrightarrow q_1 \mid A$. Then from $B^nC^l = q_1^jq_2(b-4a')$, it follows that $q_1 \mid (B^nC^l) \Longrightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-5-2-2-1- We suppose that $q_1 \mid B^n \Longrightarrow q_1 \mid B \Longrightarrow B = q_1^\beta.B_1$ with $\beta \geq 1$ and $q_1 \nmid B_1$. Then, we have $q_1^{n\beta}B_1^nC^l = q_1^jq_2(b-4a') \Longrightarrow B_1^nC^l = q_1^{j-n\beta}q_2(b-4a')$.
- If $j - n\beta \geq 1$, then $q_1 \mid C^l \Longrightarrow q_1 \mid C$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then the conjecture (34) is verified.
- If $j - n\beta = 0$, we obtain $B_1^nC^l = q_2(b-4a')$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b-4a') \Longrightarrow q_1 \mid b$ because $q_1 \mid a' \Longrightarrow q_1 \mid a$, then the contradiction with a, b coprime.
- If $j - n\beta < 0 \Longrightarrow q_1 \mid (b-4a') \Longrightarrow q_1 \mid b$, because $q_1 \mid a' \Longrightarrow q_1 \mid a$, then the contradiction with a, b coprime.

** F-5-2-2-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-6- If $4 \nmid (b-4a')$ and $4 \nmid k_1$ it is impossible. We suppose that $4 \mid (b-4a') \Rightarrow 4 \mid b$, and $b - 4a' = 4^t.g$, $t \geq 1$ with $4 \nmid g$, then we have :

$$A^{2m} = k_1a'$$
$$B^nC^l = k_1.4^{t-1}.g$$

** F-6-1- We suppose that k_1 is prime. From $A^{2m} = k_1a'$ we deduce easily that $k_1 \mid a'$. From $B^nC^l = k_1.4^{t-1}.g$ we obtain that $k_1 \mid (B^nC^l) \Longrightarrow k_1 \mid B^n$ or $k_1 \mid C^l$.

** F-6-1-1- We suppose that $k_1 \mid B^n \Longrightarrow k_1 \mid B \Longrightarrow B = k_1^j.B_1$ with $j > 0$ and $k_1 \nmid B_1$, then $k_1^{n.j}B_1^nC^l = k_1.4^{t-1}.g \Longrightarrow k_1^{n.j-1}B_1^nC^l = 4^{t-1}.g$. But $n \geq 3$ and $j \geq 1$, then $n.j - 1 \geq 2$. We deduce as $k_1 \neq 2$ that $k_1 \mid g \Longrightarrow k_1 \mid (b-4a')$, but

$k_1 \mid a' \Longrightarrow k_1 \mid b$, then the contradiction with a, b coprime.

** F-6-1-2- We obtain identical results if we suppose that $k_1 \mid C^l$.

** F-6-2- We suppose that k_1 is not prime $\neq 4$, ($k_1 = 4$ see case F-2, above) with $4 \nmid k_1$.

** F-6-2-1- If $k_1 = 2k'$ with k' odd > 1. Then $A^{2m} = 2k'a' \Longrightarrow 2 \mid a' \Longrightarrow 2 \mid a$, as $4 \mid b$ it follows the contradiction with a, b coprime.

** F-6-2-2- We suppose that k_1 is odd with k_1 and a' coprime. We write k_1 under the form $k_1 = q_1^j.q_2$ with $q_1 \nmid q_2$, q_1 prime and $j \geq 1$. $B^nC^l = q_1^j.q_2 4^{t-1}g \Longrightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-6-2-2-1- We suppose that $q_1 \mid B^n \Longrightarrow q_1 \mid B \Longrightarrow B = q_1^f.B_1$ with $q_1 \nmid B_1$. We obtain $B_1^nC^l = q_1^{j-f.n}q_2 4^{t-1}g$.
- If $j - f.n \geq 1 \Longrightarrow q_1 \mid C^l \Longrightarrow q_1 \mid C$, but $C^l = A^m + B^n$ gives also $q_1 \mid C$ and the conjecture (34) is verified.
- If $j - f.n = 0$, we have $B_1^nC^l = q_2 4^{t-1}g$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b - 4a')$. As q_1 and a' are coprime then $q_1 \nmid b$ and the conjecture (34) is verified.
- If $j - f.n < 0 \Longrightarrow q_1 \mid (b - 4a') \Longrightarrow q_1 \nmid b$ because q_1, a' are primes. $C^l = A^m + B^n$ gives $q_1 \mid C$ and the conjecture (34) is verified.

** F-6-2-2-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-6-2-3- We suppose that k_1 and a' are not coprime. Let q_1 be a prime so that $q_1 \mid k_1$ and $q_1 \mid a'$. We write k_1 under the form $q_1^j.q_2$ with $q_1 \nmid q_2$. From $A^{2m} = k_1a' \Longrightarrow q_1 \mid A^{2m} \Longrightarrow q_1 \mid A$. From $B^nC^l = q_1^jq_2(b - 4a')$, it follows that $q_1 \mid (B^nC^l) \Longrightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-6-2-3-1- We suppose that $q_1 \mid B^n \Longrightarrow q_1 \mid B \Longrightarrow B = q_1^{\beta}.B_1$ with $\beta \geq 1$ and $q_1 \nmid B_1$. Then we have $q_1^{n\beta}B_1^nC^l = q_1^jq_2(b - 4a') \Longrightarrow B_1^nC^l = q_1^{j-n\beta}q_2(b - 4a')$:
- If $j - n\beta \geq 1$, then $q_1 \mid C^l \Longrightarrow q_1 \mid C$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, and the conjecture (34) is verified.
- If $j - n\beta = 0$, we obtain $B_1^nC^l = q_2(b - 4a')$, but $q_1 \mid A$ and $q_1 \mid B$ then $q_1 \mid C$ and we obtain $q_1 \mid (b - 4a') \Longrightarrow q_1 \mid b$ because $q_1 \mid a' \Longrightarrow q_1 \mid a$, then the contradiction with a, b coprime.
- If $j - n\beta < 0 \Longrightarrow q_1 \mid (b - 4a') \Longrightarrow q_1 \mid b$, then the contradiction with a, b coprime.

** F-6-2-3-2- We obtain identical results as above if we suppose that $q_1 \mid C^l$.

2.6. Hypothèse: $\{3 \mid p \quad and \quad b \mid 4p\}$

2.6.1. Case $b = 2$ and $3 \mid p$

$3 \mid p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3b} = \frac{4.p'.a}{2} = 2.p'.a$$

As:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \Rightarrow 1 < 2a < 3 \Rightarrow a = 1 \Longrightarrow cos^2\frac{\theta}{3} = \frac{1}{2}$$

but this case was studied (see case 2.4.1.2).

2.6.2. Case $b = 4$ and $3 \mid p$

we have $3 \mid p \Longrightarrow p = 3p'$ with $p' \in \mathbb{N}^*$, it follows :

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3 \times 4} = p'.a$$

and:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \Rightarrow 1 < a < 3 \Rightarrow a = 2$$

as a, b are coprime, then the case $b = 4$ and $3 \mid p$ is impossible.

2.6.3. Case: $b \neq 2, b \neq 4, b \neq 3$, $b \mid p$ and $3 \mid p$

As $3 \mid p$, then $p = 3p'$ and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4 \times 3p'}{3}\frac{a}{b} = \frac{4p'a}{b}$$

We consider the case: $b \mid p' \Longrightarrow p' = bp"$ and $p" \neq 1$ (If $p" = 1$, then $p = 3b$, see paragraph 2.6.8 Case $k' = 1$). Finally, we obtain:

$$A^{2m} = \frac{4bp"a}{b} = 4ap"; \quad B^nC^l = p".(3b - 4a)$$

** G-1- We suppose that $p"$ is prime, then $A^{2m} = 4ap" = (A^m)^2 \Longrightarrow p" \mid a$. But $B^nC^l = p"(3b - 4a) \Longrightarrow p" \mid B^n$ or $p" \mid C^l$.

** G-1-1- If $p" \mid B^n \Longrightarrow p" \mid B \Longrightarrow B = p"B_1$ with $B_1 \in \mathbb{N}^*$. Then $p"^{n-1}B_1^nC^l = 3b - 4a$. As $n > 2$, then $(n-1) > 1$ and $p" \mid a$, then $p" \mid 3b \Longrightarrow p" = 3$ or $p" \mid b$.

** G-1-1-1- If $p" = 3 \Longrightarrow 3 \mid a$, with a that we write as $a = 3a'^2$, but $A^m = 6a' \Longrightarrow 3 \mid A^m \Longrightarrow 3 \mid A \Longrightarrow A = 3A_1$, then $3^{m-1}A_1^m = 2a' \Longrightarrow 3 \mid a' \Longrightarrow a' = 3a"$. As $p"^{n-1}B_1^nC^l = 3^{n-1}B_1^nC^l = 3b - 4a \Longrightarrow 3^{n-2}B_1^nC^l = b - 36a"^2$. As $n > 2 \Longrightarrow n - 2 \geq 1$, then $3 \mid b$ and the contradiction with a, b coprime.

** G-1-1-2- We suppose that $p" \mid b$, as $p" \mid a$, then the contradiction with a, b coprime.

** G-1-2- If we suppose $p" \mid C^l$, we obtain identical results (contradictions).

** G-2- We consider now that $p"$ is not prime.

** G-2-1- $p", a$ coprime: $A^{2m} = 4ap" \Longrightarrow A^m = 2a'.p_1$ with $a = a'^2$ and $p" = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$, then $2 \mid a'$ or $2 \mid p_1$.

** G-2-1-1- We suppose that $2 \mid a'$, then $2 \mid a' \Longrightarrow 2 \nmid p_1$, but $p" = p_1^2$.

** G-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a'.p_1$.

** G-2-1-1-2- We suppose that p_1 is not prime so we can write $p_1 = \omega^m \Longrightarrow p" = \omega^{2m}$. Then $B^nC^l = \omega^{2m}(3b - 4a)$.

** G-2-1-1-2-1- If ω is prime, $\omega \neq 2$, then $\omega \mid (B^nC^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** G-2-1-1-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(3b - 4a)$.

** G-2-1-1-2-1-1-1- If $2m - n.j = 0$, we obtain $B_1^n.C^l = 3b - 4a$. As $C^l = A^m + B^n \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, and $\omega \mid (3b - 4a)$. But $\omega \neq 2$ and ω, a' are coprime, then ω, a are coprime, it follows $\omega \nmid (3b)$, then $\omega \neq 3$ and $\omega \nmid b$, the conjecture (34) is verified.

** G-2-1-1-2-1-1-2- If $2m - nj \geq 1$, using the method as above, we obtain $\omega \mid C^l \Longrightarrow \omega \mid C$ and $\omega \mid (3b - 4a)$ and $\omega \nmid a$ and $\omega \neq 3$ and $\omega \nmid b$, then the conjecture (34) is verified.

** G-2-1-1-2-1-1-3- If $2m - nj < 0 \Longrightarrow \omega^{n.j-2m}B_1^n.C^l = 3b - 4a$. From $A^m + B^n = C^l \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, then $C = \omega^h.C_1$, with $\omega \nmid C_1$, we obtain $\omega^{n.j-2m+h.l}B_1^n.C_1^l = 3b - 4a$. If $n.j - 2m + h.l < 0 \Longrightarrow \omega \mid B_1^nC_1^l$ then the contradiction with $\omega \nmid B_1$ or $\omega \nmid C_1$. It follows $n.j - 2m + h.l > 0$ and $\omega \mid (3b - 4a)$ with ω, a, b coprime and the conjecture is verified.

** G-2-1-1-2-1-2- Using the same method above, we obtain identical results if $\omega \mid C^l$.

** G-2-1-1-2-2- We suppose that $p" = \omega^{2m}$ and ω is not prime. We write $\omega = \omega_1^f.\Omega$ with ω_1 prime $\nmid \Omega$, $f \geq 1$, and $\omega_1 \mid A$. Then $B^nC^l = \omega_1^{2f.m}\Omega^{2m}(3b - 4a) \Longrightarrow \omega_1 \mid$

$(B^nC^l) \Longrightarrow \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** G-2-1-1-2-2-1- If $\omega_1 \mid B^n \Longrightarrow \omega_1 \mid B \Longrightarrow B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2.m-nj}\Omega^{2m}(3b-4a)$:

** G-2-1-1-2-2-1-1- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(3b-4a)$. As $C^l = A^m + B^n \Longrightarrow \omega_1 \mid C^l \Longrightarrow \omega_1 \mid C$, and $\omega_1 \mid (3b-4a)$. But $\omega_1 \neq 2$ and ω_1, a' are coprime, then ω, a are coprime, it follows $\omega_1 \nmid (3b)$, then $\omega_1 \neq 3$ and $\omega_1 \nmid b$, and the conjecture (34) is verified.

** G-2-1-1-2-2-1-2- If $2f.m - n.j \geq 1$, we have $\omega_1 \mid C^l \Longrightarrow \omega_1 \mid C$ and $\omega_1 \mid (3b-4a)$ and $\omega_1 \nmid a$ and $\omega_1 \neq 3$ and $\omega_1 \nmid b$, it follows that the conjecture (34) is verified.

** G-2-1-1-2-2-1-3- If $2f.m-n.j < 0 \Longrightarrow \omega_1^{n.j-2m.f}B_1^n.C^l = \Omega^{2m}(3b-4a)$. As $\omega_1 \mid C$ using $C^l = A^m + B^n$, then $C = \omega_1^h.C_1 \Longrightarrow \omega^{n.j-2m.f+h.l}B_1^n.C_1^l = \Omega^{2m}(3b-4a)$. If $n.j - 2m.f + h.l < 0 \Longrightarrow \omega_1 \mid B_1^nC_1^l$, then the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (3b-4a)$ with ω_1, a, b coprime and the conjecture (34) is verified.

** G-2-1-1-2-2-2- Using the same method above, we obtain identical results if $\omega_1 \mid C^l$.

** G-2-1-2- We suppose that $2 \mid p_1$: then $2 \mid p_1 \Longrightarrow 2 \nmid a' \Longrightarrow 2 \nmid a$, but $p" = p_1^2$.

** G-2-1-2-1- We suppose that $p_1 = 2$, we obtain $A^m = 4a' \Longrightarrow 2 \mid a'$, then the contradiction with a, b coprime.

** G-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$. As $A^m = 2a'p_1$, p_1 can written as $p_1 = 2^{m-1}\omega^m \Longrightarrow p" = 2^{2m-2}\omega^{2m}$. Then $B^nC^l = 2^{2m-2}\omega^{2m}(3b-4a) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** G-2-1-2-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B$. As $2 \mid A$, then $2 \mid C$. From $B^nC^l = 2^{2m-2}\omega^{2m}(3b-4a)$ it follows that if $2 \mid (3b-4a) \Longrightarrow 2 \mid b$ but as $2 \nmid a$ there is no contradiction with a, b coprime and the conjecture (34) is verified.

** G-2-1-2-2-2- We suppose that $2 \mid C^l$, using the same method above, we obtain identical results.

** G-2-2- We suppose that $p", a$ are not coprime: let ω be a prime integer so that $\omega \mid a$ and $\omega \mid p"$.

** G-2-2-1- We suppose that $\omega = 3$. As $A^{2m} = 4ap" \Longrightarrow 3 \mid A$, but $3 \mid p$. As $p = A^{2m} + B^{2n} + A^m B^n \Longrightarrow 3 \mid B^{2n} \Longrightarrow 3 \mid B$, then $3 \mid C^l \Longrightarrow 3 \mid C$. We write $A = 3^i A_1$, $B = 3^j B_1$, $C = 3^h C_1$ with 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im} A_1^{2m} + 3^{2nj} B_1^{2n} + 3^{im+jn} A_1^m B_1^n = 3^k.g$ with $k = min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid p"$ that gives $a = 3^\alpha a_1$, $3 \nmid a_1$ and $p" = 3^\mu p_1$, $3 \nmid p_1$ with $A^{2m} = 4ap" = 3^{2im} A_1^{2m} = 4 \times 3^{\alpha+\mu}.a_1.p_1 \Longrightarrow \alpha + \mu = 2im$. As $p = 3p' = 3b.p" = 3b.3^\mu p_1 = 3^{\mu+1}.b.p_1$, the exponent of the factor 3 of p is k, the exponent of the factor 3 of the left member of the last equation is $\mu + 1$ added of the exponent β of 3 of the term b, with $\beta \geq 0$, let $min(2im, 2jn, im + jn) = \mu + 1 + \beta$ and we recall that $\alpha + \mu = 2im$. But $B^n C^l = p"(3b - 4a)$, we obtain $3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_1 (b - 4 \times 3^{(\alpha-1)} a_1) = 3^{\mu+1} p_1 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1)$, $3 \nmid b_1$. We have also $A^m + B^n = C^l \Longrightarrow 3^{im} A_1^m + 3^{jn} B_1^n = 3^{hl} C_1^l$. We call $\epsilon = min(im, jn)$, we have $\epsilon = hl = min(im, jn)$. We obtain the conditions:

$$k = min(2im, 2jn, im + jn) = \mu + 1 + \beta \tag{165}$$
$$\alpha + \mu = 2im \tag{166}$$
$$\epsilon = hl = min(im, jn)$$
$$3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_1 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1)$$

** G-2-2-1-1- $\alpha = 1 \Longrightarrow a = 3a_1$ and $3 \nmid a_1$, the equation (166) becomes:

$$1 + \mu = 2im$$

and the first equation (165) is written as:

$$k = min(2im, 2jn, im + jn) = 2im + \beta$$

- If $k = 2im \Longrightarrow \beta = 0$ then $3 \nmid b$. We obtain $2im \leq 2jn \Longrightarrow im \leq jn$, and $2im \leq im + jn \Longrightarrow im \leq jn$. The third equation gives $hl = im$ and the last equation gives $nj + hl = \mu + 1 = 2im \Longrightarrow im = nj$, then $im = nj = hl$ and $B_1^n C_1^l = p_1(b - 4a_1)$. As a, b are coprime, the conjecture (34) is verified.

- If $k = 2jn$ or $k = im + jn$, we obtain $\beta = 0$, $im = jn = hl$ and $B_1^n C_1^l = p_1(b - 4a_1)$. As a, b are coprime, the conjecture (34) is verified.

** G-2-2-1-2- $\alpha > 1 \Longrightarrow \alpha \geq 2$.

- If $k = 2im \Longrightarrow 2im = \mu + 1 + \beta$, but $\mu = 2im - \alpha$ that gives $\alpha = 1 + \beta \geq 2 \Longrightarrow \beta \neq 0 \Longrightarrow 3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

- If $k = 2jn = \mu + 1 + \beta \leq 2im \Longrightarrow \mu + 1 + \beta \leq \mu + \alpha \Longrightarrow 1 + \beta \leq \alpha \Longrightarrow \beta \geq 1$. If $\beta \geq 1 \Longrightarrow 3 \mid b$ but $3 \mid a$, then the contradiction with a, b coprime.

- If $k = im + jn \Longrightarrow im + jn \leq 2im \Longrightarrow jn \leq im$, and $im + jn \leq 2jn \Longrightarrow im \leq jn$, then $im = jn$. As $k = im + jn = 2im = 1 + \mu + \beta$ and $\alpha + \mu = 2im$, we obtain $\alpha = 1 + \beta \geq 2 \Longrightarrow \beta \geq 1 \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime.

** G-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $p" = \omega^\mu p_1$ with $\omega \nmid p_1$. As $A^{2m} = 4ap" = 4\omega^{\alpha+\mu}.a_1.p_1 \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$, $\omega \nmid A_1$. But

$B^nC^l = p"(3b-4a) = \omega^\mu p_1(3b-4a) \Longrightarrow \omega \mid B^nC^l \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** G-2-2-2-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$. As $p = bp' = 3bp" = 3\omega^\mu bp_1 = \omega^k(\omega^{2im-k}A_1^{2m} + \omega^{2jn-k}B_1^{2n} + \omega^{im+jn-k}A_1^mB_1^n)$ with $k = min(2im, 2jn, im+jn)$. Then:

- If $k = \mu$, then $\omega \nmid b$ and the conjecture (34) is verified.
- If $k > \mu$, then $\omega \mid b$, but $\omega \mid a$ then the contradiction with a, b coprime.
- If $k < \mu$, it follows from:

$$3\omega^\mu bp_1 = \omega^k(\omega^{2im-k}A_1^{2m} + \omega^{2jn-k}B_1^{2n} + \omega^{im+jn-k}A_1^mB_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ then the contradiction with $\omega \nmid A_1$ or $\omega \nmid B_1$.

** G-2-2-2-2- If $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid (C^l - A^m) \Longrightarrow \omega \mid B$. Then, using the same method as for the case G-2-2-2-1-, we obtain identical results.

2.6.4. Case $b = 3$ and $3 \mid p$

As $3 \mid p \Longrightarrow p = 3p'$, We write :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4\times 3p'}{3}\frac{a}{3} = \frac{4p'a}{3}$$

As A^{2m} is an integer and a, b are coprime and $cos^2\dfrac{\theta}{3} < 1$ (see equation (41)), then we have necessary $3 \mid p' \Longrightarrow p' = 3p"$ with $p" \neq 1$, if not $p = 3p' = 3\times 3p" = 9$, but $9 \ll (p = A^{2m} + B^{2n} + A^mB^n)$, the hypothesis $p" = 1$ is impossible, then $p" > 1$, and we obtain:

$$A^{2m} = \frac{4p'a}{3} = \frac{4\times 3p"a}{3} = 4p"a\,; \quad B^nC^l = p".(9-4a)$$

As $\dfrac{1}{4} < cos^2\dfrac{\theta}{3} = \dfrac{a}{b} = \dfrac{a}{3} < \dfrac{3}{4} \Longrightarrow 3 < 4a < 9 \Longrightarrow$ as $a > 1, a = 2$ and we obtain:

(167) $$A^{2m} = 4p"a = 8p"\,; \quad B^nC^l = \frac{3p"(9-4a)}{3} = p"$$

The two last equations above imply that $p"$ is not a prime. We can write $p"$ as : $p" = \prod_{i\in I} p_i^{\alpha_i}$ where p_i are distinct primes, α_i elements of $\mathbb{N}^*$ and $i \in I$ a finite set of indexes. We can write also $p" = p_1^{\alpha_1}.q_1$ with $p_1 \nmid q_1$. From (167), we have $p_1 \mid A$ and $p_1 \mid B^nC^l \Longrightarrow p_1 \mid B^n$ or $p_1 \mid C^l$.

** H-1- We suppose that $p_1 \mid B^n \Longrightarrow B = p_1^{\beta_1}.B_1$ with $p_1 \nmid B_1$ and $\beta_1 \geq 1$. Then, we obtain $B_1^nC^l = p_1^{\alpha_1 - n\beta_1}.q_1$ with the following cases:

- If $\alpha_1 - n\beta_1 \geq 1 \Longrightarrow p_1 \mid C^l \Longrightarrow p_1 \mid C$, in accord with $p_1 \mid (C^l = A^m + B^n)$, it follows that the conjecture (34) is verified.

- If $\alpha_1 - n\beta_1 = 0 \Longrightarrow B_1^n C^l = q_1 \Longrightarrow p_1 \nmid C^l$, it is a contradiction with $p_1 \mid (A^m - B^n) \Longrightarrow p_1 \mid C^l$. Then this case is impossible.
- If $\alpha_1 - n\beta_1 < 0$, we obtain $p_1^{n\beta_1-\alpha_1} B_1^n C^l = q_1 \Longrightarrow p_1 \mid q_1$, it is a contradiction with $p_1 \nmid q_1$. Then this case is impossible.

** H-2- We suppose that $p_1 \mid C^l$, using the same method as for the case $p_1 \mid B^n$, we obtain identical results.

2.6.5. Case $3 \mid p$ and $b = p$

We have $cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{p}$ and:

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{p} = \frac{4a}{3}$$

As A^{2m} is an integer, it implies that $3 \mid a$, but $3 \mid p \Longrightarrow \ 3 \mid b$. As a and b are coprime, then the contradiction and the case $3 \mid p$ and $b = p$ is impossible.

2.6.6. Case $3 \mid p$ and $b = 4p$

$3 \mid p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, then $b = 4p = 12p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{a}{3} \Longrightarrow 3 \mid a$$

as A^{2m} is an integer. But $3 \mid p \Longrightarrow \ 3 \mid [(4p) = b]$, then the contradiction with a, b coprime and the case $b = 4p$ is impossible.

2.6.7. Case $3 \mid p$ and $b = 2p$

$3 \mid p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, then $b = 2p = 6p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{2a}{3} \Longrightarrow 3 \mid a$$

as A^{2m} is an integer. But $3 \mid p \Longrightarrow \ 3 \mid (2p) \Longrightarrow \ 3 \mid b$, then the contradiction with a, b coprime and the case $b = 2p$ is impossible.

2.6.8. Case $3 \mid p$ and $b \neq 3$ a divisor of p

We have $b = p' \neq 3$, and p is written as $p = kp'$ with $3 \mid k \Longrightarrow k = 3k'$ and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = 4ak'$$

$$B^nC^l = \frac{p}{3}.\left(3 - 4cos^2\frac{\theta}{3}\right) = k'(3p' - 4a) = k'(3b - 4a)$$

** I-1- $k' \neq 1$:

** I-1-1- We suppose that k' is prime, then $A^{2m} = 4ak' = (A^m)^2 \Longrightarrow k' \mid a$. But $B^nC^l = k'(3b-4a) \Longrightarrow k' \mid B^n$ or $k' \mid C^l$.

** I-1-1-1- If $k' \mid B^n \Longrightarrow k' \mid B \Longrightarrow B = k'B_1$ with $B_1 \in \mathbb{N}^*$. Then $k'^{n-1}B_1^nC^l = 3b-4a$. As $n > 2$, then $(n-1) > 1$ and $k' \mid a$, then $k' \mid 3b \Longrightarrow k' = 3$ or $k' \mid b$.

** I-1-1-1-1- If $k' = 3 \Longrightarrow 3 \mid a$, with a that we can write it under the form $a = 3a'^2$. But $A^m = 6a' \Longrightarrow 3 \mid A^m \Longrightarrow 3 \mid A \Longrightarrow A = 3A_1$ with $A_1 \in \mathbb{N}^*$. Then $3^{m-1}A_1^m = 2a' \Longrightarrow 3 \mid a' \Longrightarrow a' = 3a"$. But $k'^{n-1}B_1^nC^l = 3^{n-1}B_1^nC^l = 3b-4a \Longrightarrow 3^{n-2}B_1^nC^l = b - 36a"^2$. As $n \geq 3 \Longrightarrow n-2 \geq 1$, then $3 \mid b$. Hence the contradiction with a, b coprime.

** I-1-1-1-2- We suppose that $k' \mid b$, but $k' \mid a$, then the contradiction with a, b coprime.

** I-1-1-2- We suppose that $k' \mid C^l$, using the same method as for the case $k' \mid B^n$, we obtain identical results.

** I-1-2- We consider that k' is not a prime.

** I-1-2-1- We suppose that k', a coprime: $A^{2m} = 4ak' \Longrightarrow A^m = 2a'.p_1$ with $a = a'^2$ and $k' = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$ then $2 \mid a'$ or $2 \mid p_1$.

** I-1-2-1-1- We suppose that $2 \mid a'$, then $2 \mid a' \Longrightarrow 2 \nmid p_1$, but $k' = p_1^2$.

** I-1-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a'.p_1$.

** I-1-2-1-1-2- We suppose that p_1 is not prime and it can be written as $p_1 = \omega^m \Longrightarrow k' = \omega^{2m}$. Then $B^nC^l = \omega^{2m}(3b-4a)$.

** I-1-2-1-1-2-1- If ω is prime $\neq 2$, then $\omega \mid (B^nC^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** I-1-2-1-1-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^jB_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(3b-4a)$.

- If $2m - n.j = 0$, we obtain $B_1^n.C^l = 3b-4a$, as $C^l = A^m + B^n \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, and $\omega \mid (3b-4a)$. But $\omega \neq 2$ and ω, a' are coprime, then $\omega \nmid (3b) \Longrightarrow \omega \neq 3$ and $\omega \nmid b$. Hence, the conjecture (34) is verified.

- If $2m-nj \geq 1$, using the same method, we have $\omega \mid C^l \Longrightarrow \omega \mid C$ and $\omega \mid (3b-4a)$ and $\omega \nmid a$ and $\omega \neq 3$ and $\omega \nmid b$. Then the conjecture (34) is verified.

- If $2m-nj<0 \Longrightarrow \omega^{n.j-2m}B_1^n.C^l=3b-4a$. As $C^l=A^m+B^n \Longrightarrow \omega \mid C$ then $C=\omega^h.C_1 \Longrightarrow \omega^{n.j-2m+h.l}B_1^n.C_1^l=3b-4a$. If $n.j-2m+h.l<0 \Longrightarrow \omega \mid B_1^nC_1^l$, then the contradiction with $\omega \nmid B_1$ or $\omega \nmid C_1$. If $n.j-2m+h.l>0 \Longrightarrow \omega \mid (3b-4a)$ with ω, a, b coprime, it implies that the conjecture (34) is verified.

** I-1-2-1-1-2-1-2- We suppose that $\omega \mid C^l$, using the same method as for the case $\omega \mid B^n$, we obtain identical results.

** I-1-2-1-1-2-2- Now $k'=\omega^{2m}$ and ω not a prime, we write $\omega=\omega_1^f.\Omega$ with ω_1 a prime $\nmid \Omega$ and $f \geq 1$ an integer, and $\omega_1 \mid A$, then $B^nC^l=\omega_1^{2f.m}\Omega^{2m}(3b-4a) \Longrightarrow \omega_1 \mid (B^nC^l) \Longrightarrow \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** I-1-2-1-1-2-2-1- If $\omega_1 \mid B^n \Longrightarrow \omega_1 \mid B \Longrightarrow B=\omega_1^jB_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l=\omega_1^{2.fm-nj}\Omega^{2m}(3b-4a)$.

- If $2f.m-n.j=0$, we obtain $B_1^n.C^l=\Omega^{2m}(3b-4a)$. As $C^l=A^m+B^n \Longrightarrow \omega_1 \mid C^l \Longrightarrow \omega_1 \mid C$, and $\omega_1 \mid (3b-4a)$. But $\omega_1 \neq 2$ and ω_1, a' are coprime, then ω, a are coprime, then $\omega_1 \nmid (3b) \Longrightarrow \omega_1 \neq 3$ and $\omega_1 \nmid b$. Hence, the conjecture (34) is verified.

- If $2f.m-n.j \geq 1$, we have $\omega_1 \mid C^l \Longrightarrow \omega_1 \mid C$ and $\omega_1 \mid (3b-4a)$ and $\omega_1 \nmid a$ and $\omega_1 \neq 3$ and $\omega_1 \nmid b$, then the conjecture (34) is verified.

- If $2f.m-n.j<0 \Longrightarrow \omega_1^{n.j-2m.f}B_1^n.C^l=\Omega^{2m}(3b-4a)$. As $C^l=A^m+B^n \Longrightarrow \omega_1 \mid C$, then $C=\omega_1^h.C_1 \Longrightarrow \omega^{n.j-2m.f+h.l}B_1^n.C_1^l=\Omega^{2m}(3b-4a)$. If $n.j-2m.f+h.l<0 \Longrightarrow \omega_1 \mid B_1^nC_1^l$, then the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j-2m.f+h.l>0$ and $\omega_1 \mid (3b-4a)$ with ω_1, a, b coprime, then the conjecture (34) is verified.

** I-1-2-1-1-2-2-2- As in the case $\omega_1 \mid B^n$, we obtain identical results if $\omega_1 \mid C^l$.

** I-1-2-1-2- If $2 \mid p_1$: then $2 \mid p_1 \Longrightarrow 2 \nmid a' \Longrightarrow 2 \nmid a$, but $k'=p_1^2$.

** I-1-2-1-2-1- If $p_1=2$, we obtain $A^m=4a' \Longrightarrow 2 \mid a'$, then the contradiction with $2 \nmid a'$. Case to reject.

** I-1-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$. As $A^m=2a'p_1$, p_1 is written under the form $p_1=2^{m-1}\omega^m \Longrightarrow p_1^2=2^{2m-2}\omega^{2m}$. Then $B^nC^l=k'(3b-4a)=2^{2m-2}\omega^{2m}(3b-4a) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** I-1-2-1-2-2-1- If $2 \mid B^n \Longrightarrow 2 \mid B$, as $2 \mid A \Longrightarrow 2 \mid C$. From $B^nC^l=2^{2m-2}\omega^{2m}(3b-4a)$ it follows that if $2 \mid (3b-4a) \Longrightarrow 2 \mid b$ but as $2 \nmid a$, there is no contradiction with a, b coprime and the conjecture (34) is verified.

** I-1-2-1-2-2-2- We obtain identical results as above if $2 \mid C^l$.

** I-1-2-2- We suppose that k', a are not coprime: let ω be a prime integer so that $\omega \mid a$ and $\omega \mid p_1^2$.

** I-1-2-2-1- We suppose that $\omega = 3$. As $A^{2m} = 4ak' \Longrightarrow 3 \mid A$, but $3 \mid p$. As $p = A^{2m} + B^{2n} + A^m B^n \Longrightarrow 3 \mid B^{2n} \Longrightarrow 3 \mid B$, then $3 \mid C^l \Longrightarrow 3 \mid C$. We write $A = 3^i A_1$, $B = 3^j B_1$, $C = 3^h C_1$ with 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im} A_1^{2m} + 3^{2nj} B_1^{2n} + 3^{im+jn} A_1^m B_1^n = 3^s.g$ with $s = min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid k'$ that give $a = 3^\alpha a_1$, $3 \nmid a_1$ and $k' = 3^\mu p_2$, $3 \nmid p_2$ with $A^{2m} = 4ak' = 3^{2im} A_1^{2m} = 4 \times 3^{\alpha+\mu}.a_1.p_2 \Longrightarrow \alpha + \mu = 2im$. As $p = 3p' = 3b.k' = 3b.3^\mu p_2 = 3^{\mu+1}.b.p_2$. The exponent of the factor 3 of p is s, the exponent of the factor 3 of the left member of the last equation is $\mu + 1$ added of the exponent β of 3 of the factor b, with $\beta \geq 0$, let $min(2im, 2jn, im+jn) = \mu + 1 + \beta$, we recall that $\alpha + \mu = 2im$. But $B^n C^l = k'(4b - 3a)$ that gives $3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_2 (b - 4 \times 3^{(\alpha-1)} a_1) = 3^{\mu+1} p_2 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1)$, $3 \nmid b_1$. We have also $A^m + B^n = C^l$ that gives $3^{im} A_1^m + 3^{jn} B_1^n = 3^{hl} C_1^l$. We call $\epsilon = min(im, jn)$, we obtain $\epsilon = hl = min(im, jn)$. We have then the conditions:

$$s = min(2im, 2jn, im+jn) = \mu + 1 + \beta \tag{168}$$

$$\alpha + \mu = 2im \tag{169}$$

$$\epsilon = hl = min(im, jn) \tag{170}$$

$$3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_2 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1) \tag{171}$$

** I-1-2-2-1-1- $\alpha = 1 \Longrightarrow a = 3a_1$ and $3 \nmid a_1$, the equation (169) becomes:

$$1 + \mu = 2im$$

and the first equation (168) is written as :

$$s = min(2im, 2jn, im+jn) = 2im + \beta$$

- If $s = 2im \Longrightarrow \beta = 0 \Longrightarrow 3 \nmid b$. We obtain $2im \leq 2jn \Longrightarrow im \leq jn$, and $2im \leq im + jn \Longrightarrow im \leq jn$. The third equation (170) gives $hl = im$. The last equation (171) gives $nj + hl = \mu + 1 = 2im \Longrightarrow im = jn$, then $im = jn = hl$ and $B_1^n C_1^l = p_2(b - 4a_1)$. As a, b are coprime, the conjecture (34) is verified.

- If $s = 2jn$ or $s = im + jn$, we obtain $\beta = 0$, $im = jn = hl$ and $B_1^n C_1^l = p_2(b - 4a_1)$. Then as a, b are coprime, the conjecture (34) is verified.

** I-1-2-2-1-2- $\alpha > 1 \Longrightarrow \alpha \geq 2$.

- If $s = 2im \Longrightarrow 2im = \mu + 1 + \beta$, but $\mu = 2im - \alpha$ it gives $\alpha = 1 + \beta \geq 2 \Longrightarrow \beta \neq 0 \Longrightarrow 3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime and the conjecture (34) is not verified.

- If $s = 2jn = \mu+1+\beta \leq 2im \Longrightarrow \mu+1+\beta \leq \mu+\alpha \Longrightarrow 1+\beta \leq \alpha \Longrightarrow \beta = 1$. If $\beta = 1 \Longrightarrow 3 \mid b$ but $3 \mid a$, then the contradiction with a, b coprime and the conjecture (34) is not verified.

- If $s = im+jn \Longrightarrow im+jn \leq 2im \Longrightarrow jn \leq im$, and $im+jn \leq 2jn \Longrightarrow im \leq jn$, then $im = jn$. As $s = im + jn = 2im = 1 + \mu + \beta$ and $\alpha + \mu = 2im$ it gives $\alpha = 1+\beta \geq 2 \Longrightarrow \beta \geq 1 \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime and the conjecture (34) is not verified.

** I-1-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $k' = \omega^\mu p_2$ with $\omega \nmid p_2$. As $A^{2m} = 4ak' = 4\omega^{\alpha+\mu}.a_1.p_2 \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$, $\omega \nmid A_1$. But $B^nC^l = k'(3b-4a) = \omega^\mu p_2(3b-4a) \Longrightarrow \omega \mid B^nC^l \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** I-1-2-2-2-1- $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B^n = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$. As $p = bp' = 3bk' = 3\omega^\mu bp_2 = \omega^s(\omega^{2im-s}A_1^{2m} + \omega^{2jn-s}B_1^{2n} + \omega^{im+jn-s}A_1^mB_1^n)$ with $s = min(2im, 2jn, im+jn)$. Then:

- If $s = \mu$, then $\omega \nmid b$ and the conjecture (34) is verified.

- If $s > \mu$, then $\omega \mid b$, but $\omega \mid a$ then the contradiction with a, b coprime and the conjecture (34) is not verified.

- If $s < \mu$, it follows from:

$$3\omega^\mu bp_1 = \omega^s(\omega^{2im-s}A_1^{2m} + \omega^{2jn-s}B_1^{2n} + \omega^{im+jn-s}A_1^mB_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ that is the contradiction with the hypothesis and the conjecture (34) is not verified.

** I-1-2-2-2-2- If $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid (C^l - A^m) \Longrightarrow \omega \mid B$. Then we obtain identical results as the case above I-1-2-2-2-1-.

** I-2- We suppose $k' = 1$: then $k' = 1 \Longrightarrow p = 3b$, then we have $A^{2m} = 4a = (2a')^2 \Longrightarrow A^m = 2a'$, then $a = a'^2$ is even and :

$$A^mB^n = 2\sqrt[3]{\rho}cos\frac{\theta}{3}.\sqrt[3]{\rho}\left(\sqrt{3}sin\frac{\theta}{3} - cos\frac{\theta}{3}\right) = \frac{p\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a$$

and we have also:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} = 2b\sqrt{3}sin\frac{2\theta}{3} \tag{172}$$

The left member of the equation (172) is a naturel number and also b, then $2\sqrt{3}sin\frac{2\theta}{3}$ can be written under the form :

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two natural numbers coprime and $k_2 \mid b \Longrightarrow b = k_2.k_3$.

** I-2-1- $k' = 1$ and $k_3 \neq 1$: then $A^{2m} + 2A^mB^n = k_3.k_1$. Let μ be a prime integer so that $\mu \mid k_3$. If $\mu = 2 \Rightarrow 2 \mid b$, but $2 \mid a$, it is a contradiction with a, b coprime. We suppose that $\mu \neq 2$ and $\mu \mid k_3$, then $\mu \mid A^m(A^m+2B^n) \Longrightarrow \mu \mid A^m$ or $\mu \mid (A^m+2B^n)$.

** I-2-1-1- $\mu \mid A^m$: If $\mu \mid A^m \Longrightarrow \mu \mid A^{2m} \Longrightarrow \mu \mid 4a \Longrightarrow \mu \mid a$. As $\mu \mid k_3 \Longrightarrow \mu \mid b$, the contradiction with a, b coprime.

** I-2-1-2- $\mu \mid (A^m + 2B^n)$: If $\mu \mid (A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu \mid (A^m + 2B^n)$, we can write $A^m + 2B^n = \mu.t'$. It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p, we obtain:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m)$$

As $p = 3b = 3k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid p \Longrightarrow\ p = \mu.\mu'$, then we obtain:

$$\mu'.\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m)$$

and $\mu \mid B^n(B^n - A^m) \Longrightarrow \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** I-2-1-2-1- $\mu \mid B^n$: If $\mu \mid B^n \Longrightarrow \mu \mid B$, that is the contradiction with I-2-1-2- above.

** I-2-1-2-2- $\mu \mid (B^n - A^m)$: If $\mu \mid (B^n - A^m)$ and using that $\mu \mid (A^m + 2B^n)$, we obtain :

$$\mu \mid 3B^n \Longrightarrow \begin{cases} \mu \mid B^n \Longrightarrow \mu \mid B \\ or \\ \mu = 3 \end{cases}$$

** I-2-1-2-2-1- $\mu \mid B^n$: If $\mu \mid B^n \Longrightarrow \mu \mid B$, that is the contradiction with I-2-1-2- above.

** I-2-1-2-2-2- $\mu = 3$: If $\mu = 3 \Longrightarrow 3 \mid k_3 \Longrightarrow k_3 = 3k'_3$, and we have $b = k_2k_3 = 3k_2k'_3$, it follows $p = 3b = 9k_2k'_3$, then $9 \mid p$, but $p = (A^m - B^n)^2 + 3A^mB^n$ then:

$$9k_2k'_3 - 3A^mB^n = (A^m - B^n)^2$$

that we write as:

$$3(3k_2k'_3 - A^mB^n) = (A^m - B^n)^2 \tag{173}$$

then:

$$3 \mid (3k_2k'_3 - A^mB^n) \Longrightarrow\ 3 \mid A^mB^n \Longrightarrow 3 \mid A^m \ or\ 3 \mid B^n$$

** I-2-1-2-2-2-1- $3 \mid A^m$: If $3 \mid A^m \Longrightarrow 3 \mid A$ and we have also $3 \mid A^{2m}$, but $A^{2m} = 4a \Longrightarrow 3 \mid 4a \Longrightarrow 3 \mid a$. As $b = 3k_2k'_3$ then $3 \mid b$, but a, b are coprime, then the contradiction and $3 \nmid A$.

** I-2-1-2-2-2-2- $3 \mid B^m$: If $3 \mid B^n \Longrightarrow 3 \mid B$, but the equation (173) implies $3 \mid (A^m - B^n)^2 \Longrightarrow 3 \mid (A^m - B^n) \Longrightarrow 3 \mid A^m \Longrightarrow 3 \mid A$. The last case above has given that $3 \nmid A$. Then the case $3 \mid B^m$ is to reject.

Finally the hypothesis $k_3 \neq 1$ is impossible.

** I-2-2- Now, we suppose that $k_3 = 1 \Longrightarrow b = k_2$ and $p = 3b = 3k_2$, then we have:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{b} \tag{174}$$

with k_1, b coprime. We write (174) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_1}{b}$$

Taking the square of the two members and replacing $cos^2\frac{\theta}{3}$ by $\frac{a}{b}$, we obtain:

$$3 \times 4^2.a(b-a) = k_1^2 \Longrightarrow k_1^2 = 3 \times 4^2.a'^2(b-a)$$

it implies that :

$$b - a = 3\alpha^2,\ \alpha \in \mathbb{N}^* \Longrightarrow b = a'^2 + 3\alpha^2 \Longrightarrow k_1 = 12a'\alpha$$

As:

$$k_1 = 12a'\alpha = A^m(A^m + 2B^n) \Longrightarrow 3\alpha = a' + B^n$$

We consider now that $3 \mid (b-a)$ with $b = a'^2 + 3\alpha^2$. The case $\alpha = 1$ gives $a' + B^n = 3$ that is impossible. We suppose $\alpha > 1$, the pair (a', α) is a solution of the Diophantine equation:

$$X^2 + 3Y^2 = b \tag{175}$$

with $X = a'$ and $Y = \alpha$. But using a theorem on the solutions of the equation given by (175), b is written as (see theorem in [**2**]):

$$b = 2^{2s} \times 3^t.p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

where p_i are prime numbers verifying $p_i \equiv 1 (\mathrm{mod}\ 6)$, the q_j are also prime numbers so that $q_j \equiv 5 (\mathrm{mod}\ 6)$, then :

- If $s \geq 1 \Longrightarrow 2 \mid b$, as $2 \mid a$, then the contradiction with a, b coprime.
- If $t \geq 1 \Longrightarrow 3 \mid b$, but $3 \mid (b-a) \Longrightarrow 3 \mid a$, then the contradiction with a, b coprime.

** I-2-2-1- We suppose that b is written as :

$$b = p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

with $p_i \equiv 1(\mod 6)$ and $q_j \equiv 5(\mod 6)$. Finally, we obtain that $b \equiv 1(\mod 6)$. We will verify then this condition.

** I-2-2-1-1- We present the table below giving the value of $A^m + B^n = C^l$ modulo 6 in function of the value of $A^m, B^n(\mod 6)$. We obtain the table below after retiring the lines (respectively the colones) of $A^m \equiv 0(\mod 6)$ and $A^m \equiv 3(\mod 6)$ (respectively of $B^n \equiv 0(\mod 6)$ and $B^n \equiv 3(\mod 6)$), they present cases with contradictions:

Table 2. Table of $C^l(\mod 6)$

A^m, B^n	1	2	4	5
1	2	3	5	0
2	3	4	0	1
4	5	0	2	3
5	0	1	3	4

** I-2-2-1-1-1- For the case $C^l \equiv 0(\mod 6)$ and $C^l \equiv 3(\mod 6)$, we deduce that $3 \mid C^l \Longrightarrow 3 \mid C \Longrightarrow C = 3^h C_1$, with $h \geq 1$ and $3 \nmid C_1$. It follows that $p - B^n C^l = 3b - 3^{lh} C_1^l B^n = A^{2m} \Longrightarrow 3 \mid (A^{2m} = 4a) \Longrightarrow 3 \mid a \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-2- For the case $C^l \equiv 0(\mod 6)$, $C^l \equiv 2(\mod 6)$ and $C^l \equiv 4(\mod 6)$, we deduce that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h C_1$, with $h \geq 1$ and $2 \nmid C_1$. It follows that $p = 3b = A^{2m} + B^n C^l = 4a + 2^{lh} C_1^l B^n \Longrightarrow 2 \mid 3b \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-3- We consider the cases $A^m \equiv 1(\mod 6)$ and $B^n \equiv 4(\mod 6)$ (respectively $B^n \equiv 2(\mod 6)$): then $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$ with $j \geq 1$ and $2 \nmid B_1$. It follows from $3b = A^{2m} + B^n C^l = 4a + 2^{jn} B_1^n C^l$ that $2 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-4- We consider the case $A^m \equiv 5(\mod 6)$ and $B^n \equiv 2(\mod 6)$: then $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$ with $j \geq 1$ and $2 \nmid B_1$. It follows that $3b = A^{2m} + B^n C^l = 4a + 2^{jn} B_1^n C^l$, then $2 \mid b$ and we obtain the contradiction with a, b coprime.

** I-2-2-1-1-5- We consider the case $A^m \equiv 2(\mod 6)$ and $B^n \equiv 5(\mod 6)$: as $A^m \equiv 2(\mod 6) \Longrightarrow A^m \equiv 2(\mod 3)$, then A^m is not a square and also for B^n. Hence, we can write A^m and B^n as:

$$A^m = a_0.\mathcal{A}^2$$
$$B^n = b_0 \mathcal{B}^2$$

where a_0 (respectively b_0) regroups the product of the prime numbers of A^m with exponent 1 (respectively of B^n) with not necessary $(a_0, \mathcal{A}) = 1$ and $(b_0, \mathcal{B}) = 1$. We have also $p = 3b = A^{2m} + A^mB^n + B^{2n} = (A^m - B^n)^2 + 3A^mB^n \Longrightarrow 3 \mid (b - A^mB^n) \Longrightarrow A^mB^n \equiv b(\text{mod}\, 3)$ but $b = a + 3\alpha^2 \Longrightarrow b \equiv a \equiv a'^2(\text{mod}\, 3)$, then $A^mB^n \equiv a'^2(\text{mod}3)$. But $A^m \equiv 2(\text{mod}6) \Longrightarrow 2a' \equiv 2(\text{mod}6) \Longrightarrow 4a'^2 \equiv 4(\text{mod}6) \Longrightarrow a'^2 \equiv 1(\text{mod}3)$. It follows that A^mB^n is a square, let $A^mB^n = \mathcal{N}^2 = \mathcal{A}^2.\mathcal{B}^2.a_0.b_0$. We call $\mathcal{N}_1^2 = a_0.b_0$. Let p_1 be a prime number so that $p_1 \mid a_0 \Longrightarrow a_0 = p_1.a_1$ with $p_1 \nmid a_1$. $p_1 \mid \mathcal{N}_1^2 \Longrightarrow p_1 \mid \mathcal{N}_1 \Longrightarrow \mathcal{N}_1 = p_1^t\mathcal{N}_1'$ with $t \geq 1$ and $p_1 \nmid \mathcal{N}_1'$, then $p_1^{2t-1}\mathcal{N}_1'^2 = a_1.b_0$. As $2t \geq 2 \Longrightarrow 2t - 1 \geq 1 \Longrightarrow p_1 \mid a_1.b_0$ but $(p_1, a_1) = 1$, then $p_1 \mid b_0 \Longrightarrow p_1 \mid B^n \Longrightarrow p_1 \mid B$. But $p_1 \mid (A^m = 2a')$, and $p_1 \neq 2$ because $p_1 \mid B^n$ and B^n is odd, then the contradiction. Hence, $p_1 \mid a' \Longrightarrow p_1 \mid a$. If $p_1 = 3$, from $3 \mid (b - a) \Longrightarrow 3 \mid b$ then the contradiction with a, b coprime. Then $p_1 > 3$ a prime that divides A^m and B^n, then $p_1 \mid (p = 3b) \Longrightarrow p_1 \mid b$, it follows the contradiction with a, b coprime, knowing that $p = 3b \equiv 3(\text{mod}\, 6)$ and we choose the case $b \equiv 1(\text{mod}\, 6)$ of our interest.

** I-2-2-1-1-6- We consider the last case of the table above $A^m \equiv 4(\text{mod}\, 6)$ and $B^n \equiv 1(\text{mod}\, 6)$. We return to the equation (175) that b verifies :

$$b = X^2 + 3Y^2 \tag{176}$$
$$with \quad X = a'; \quad Y = \alpha$$
$$and \quad 3\alpha = a' + B^n$$

But $p = A^{2m} + A^mB^n + B^{2n} = 3b = 3(3\alpha^2 + a'^2) \Longrightarrow A^{2m} + C^lB^n = 3a'^2 + 9\alpha^2$. As $A^{2m} = (2a')^2 = 4a'^2$, we obtain:

$$9\alpha^2 - a'^2 = C^l.B^n$$

Then the pair $(3\alpha, a') \in \mathbb{N}^* \times \mathbb{N}^*$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{177}$$

where $N = C^l.B^m > 0$.

Let $Q(N)$ be the number of the solutions of (177) and $\tau(N)$ the number of ways to write the factors of N, then we announce the following result concerning the number of the solutions of (177) (see theorem 27.3 in [**2**]):
- If $N \equiv 2(\text{mod}\, 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3(\text{mod}\, 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0(\text{mod}\, 4)$, then $Q(N) = [\tau(N/4)/2]$.

As $A^m = 2a', m \geq 3 \Longrightarrow A^m \equiv 0(\text{mod}\, 4)$. Concerning B^n, for $B^n \equiv 0(\text{mod}\, 4)$ or $B^n \equiv 2(\text{mod}\, 4)$, we find that $2 \mid B^n \Longrightarrow 2 \mid \alpha \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

For the last case $B^n \equiv 3(\text{mod}4) \Longrightarrow C^l \equiv 3(\text{mod}4) \Longrightarrow N = B^nC^l \equiv 1(\text{mod } 4) \Longrightarrow Q(N) = [\tau(N)/2]$.

As $(3\alpha, a')$ is a couple of solutions of the Diophantine equation (177) and $3\alpha > a'$, then $\exists\ d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 6\alpha \tag{178}$$

$$d - d' = 2a' \tag{179}$$

We will use the same method used in the above paragraph A-2-1-2-

** I-2-2-1-1-6-1- As $C^l > B^n$, we take $d = C^l$ and $d' = B^n$. It follows:

$$C^l + B^n = 6\alpha = 2a' + 2B^n = A^m + 2B^n \tag{180}$$

$$C^l - B^n = 2a' = A^m \tag{181}$$

Then the case $d = C^l$ and $d' = B^n$ gives *a priory* no contradictions.

** I-2-2-1-1-6-2- Now, we consider the case $d = B^nC^l$ and $d' = 1$. We rewrite the equations (255-256):

$$B^nC^l + 1 = 6\alpha \tag{182}$$

$$B^nC^l - 1 = 2a' \tag{183}$$

We obtain $1 = B^n$, it follows $C^l - A^m = 1$, we know [**4**] that the only positive solution of the last equation is $C = 3, A = 2, m = 3$ and $l = 2 < 3$, then the contradiction.

** I-2-2-1-1-6-3- Now, we consider the case $d = c_1^{lr-1}C_1^l$ where c_1 is a prime integer with $c_1 \nmid C_1$ and $C = c_1^rC_1$, $r \geq 1$. It follows that $d' = c_1.B^n$. We rewrite the equations (255-256):

$$c_1^{lr-1}C_1^l + c_1.B^n = 6\alpha \tag{184}$$

$$c_1^{lr-1}C_1^l - c_1.B^n = 2a' \tag{185}$$

As $l \geq 3$, from the last two equations above, it follows that $c_1 \mid (6\alpha)$ and $c_1 \mid (2a')$. Then $c_1 = 2$, or $c_1 = 3$ and $3 \mid a'$ or $c_1 \neq 3 \mid \alpha$ and $c_1 \mid a'$.

** I-2-2-1-1-6-3-1- We suppose $c_1 = 2$. As $2 \mid (A^m = 2a') \Rightarrow 2 \mid (a = a'^2$ and $2 \mid C^l$ because $l \geq 3$, it follows $2 \mid B^n$, then $2 \mid (p = 3b)$. Then the contradiction with a, b coprime.

** I-2-2-1-1-6-3-2- We suppose $c_1 = 3 \Rightarrow c_1 \mid 2a' \Longrightarrow c_1 \mid a' \Longrightarrow c_1 \mid (a = a'^2)$. It follows that $(c_1 = 3) \mid (b = a'^2 + 3\alpha^2)$, then the contradiction with a, b coprime.

** I-2-2-1-1-6-3-3- We suppose $c_1 \neq 3$ and $c_1 \mid 3\alpha$ and $c_1 \mid a'$. It follows that $c_1 \mid a$ and $c_1 \mid (b = a'^2 + 3\alpha^2$, then the contradiction with a, b coprime.

The other cases of the expressions of d and d' not coprime so that $N = B^nC^l = d.d'$ give also contradictions.

** I-2-2-1-1-6-4- Now, let $C = c_1^r C_1$ with c_1 a prime, $r \geq 1$ and $c_1 \nmid C_1$, we consider the case $d = C_1^l$ and $d' = c_1^{rl} B^n$ so that $d > d'$. We rewrite the equations (255-256):

$$C_1^l + c_1^{rl} B^n = 6\alpha \quad (186)$$

$$C_1^l - c_1^{rl} B^n = 2a' \quad (187)$$

We obtain $c_1^{rl} B^n = B^n \Longrightarrow c_1^{rl} = 1$, then the contradiction.

** I-2-2-1-1-6-5- Now, let $C = c_1^r C_1$ with c_1 a prime, $r \geq 1$ and $c_1 \nmid C_1$, we consider the case $d = C_1^l B^n$ and $d' = c_1^{rl}$ so that $d > d'$. We rewrite the equations (255-256):

$$C_1^l B^l + c_1^{rl} = 6\alpha \quad (188)$$

$$C_1^l B^l - c_1^{rl} = 2a' \quad (189)$$

We obtain $c_1^{rl} = B^n \Longrightarrow c_1 \mid B^n$, as $c_1 \mid C$ then $c_1 \mid A^m = 2a'$. If $c_1 = 2$, the contradiction with $B^nC^l \equiv 1 (\bmod 4)$. Then $c_1 \mid a' \Longrightarrow c_1 \mid (a = a'^2) \Longrightarrow c_1 \mid (p = b)$, it follows a, b are not coprime, then the contradiction.

Cases like $d' < C^l$ a divisor of C^l or $d' < B^l$ a divisor of B^n with $d' < d$ and $d.d' = N = B^nC^l$ give contradictions.

** I-2-2-1-1-6-6- Now, we consider the case $d = b_1.C^l$ where b_1 is a prime integer with $b_1 \nmid B_1$ and $B = b_1^r B_1$, $r \geq 1$. It follows that $d' = b_1^{nr-1} B_1^n$. We rewrite the equations (255-256):

$$b_1 C^l + b_1^{nr-1} B_1^n = 6\alpha \quad (190)$$

$$b_1 C^l - b_1^{nr-1} B_1^n = 2a' \quad (191)$$

As $n \geq 3$, from the last two equations above, it follows that $b_1 \mid 6\alpha$ and $b_1 \mid (2a')$. Then $b_1 = 2$, or $b_1 \mid \alpha$ and $b_1 \mid a'$ or $b_1 = 3$ and $3 \mid a'$.

** I-2-2-1-1-6-6-1- We suppose $b_1 = 2 \Longrightarrow 2 \mid B^n$. As $2 \mid (A^m = 2a' \Longrightarrow 2 \mid a' \Longrightarrow 2 \mid a$, but $2 \mid B^n$ and $2 \mid A^m$ then $2 \mid (p = 3b)$. It follows the contradiction with a, b coprime.

** I-2-2-1-1-6-6-2- We suppose $b_1 \neq 2, 3$, then $b_1 \mid \alpha$ and $b_1 \mid a' \Longrightarrow b_1 \mid (a = a'^2)$, then $b_1 \mid (b = 3\alpha^2 + a'^2)$, it follows the contradiction with a, b coprime.

** I-2-2-1-1-6-6-3- We suppose $b_1 = 3 \Longrightarrow 3 \mid 6\alpha$, and $3 \mid (A^m = 2a') \Longrightarrow 3 \mid (a = a'^2)$, then $3 \mid (b = 3\alpha^2 + a'^2)$, it follows the contradiction with a, b coprime.

The other cases of the expressions of d and d' with d, d' not coprime and $d > d'$ so that $N = C^l B^m = d.d'$ give also contradictions.

Finally, from the cases studied in the above paragraph I-2-2-1-1-6-, we have found one suitable factorization of N that gives a priory no contradictions, it is the case $N = B^n.C^l = d.d'$ with $d = C^l, d' = B^n$ but $1 \ll \tau(N)$, it follows the contradiction with $Q(N) = [\tau(N)/2] \leq 1$. The last case $A^m \equiv 4(\bmod 6)$ and $B^n \equiv 1(\bmod 6)$ gives contradictions.

It follows that the condition $3 \mid (b - a)$ is a contradiction.

The study of the case 2.6.8 is achieved.

2.6.9. Case $3 \mid p$ and $b \mid 4p$

The following cases have been soon studied:
* $3 \mid p$, $b = 2 \Longrightarrow b \mid 4p$: case 2.6.1,
* $3 \mid p$, $b = 4 \Longrightarrow b \mid 4p$: case 2.6.2,
* $3 \mid p \Longrightarrow p = 3p'$, $b \mid p' \Longrightarrow p' = bp"$, $p" \neq 1$: case 2.6.3,
* $3 \mid p$, $b = 3 \Longrightarrow b \mid 4p$: case 2.6.4,
* $3 \mid p \Longrightarrow p = 3p'$, $b = p' \Longrightarrow b \mid 4p$: case 2.6.8.

** J-1- Particular case: $b = 12$. In fact $3 \mid p \Longrightarrow p = 3p'$ and $4p = 12p'$. Taking $b = 12$, we have $b \mid 4p$. But $b < 4a < 3b$, that gives $12 < 4a < 36 \Longrightarrow 3 < a < 9$. As $2 \mid b$ and $3 \mid b$, the possible values of a are 5 and 7.

** J-1-1- $a = 5$ and $b = 12 \Longrightarrow 4p = 12p' = bp'$. But $A^{2m} = \dfrac{4p}{3}.\dfrac{a}{b} = \dfrac{5bp'}{3b} = \dfrac{5p'}{3} \Longrightarrow 3 \mid p' \Longrightarrow p' = 3p"$ with $p" \in \mathbb{N}^*$, then $p = 9p"$, we obtain the expressions:

$$A^{2m} = 5p" \tag{192}$$

$$B^n C^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = 4p" \tag{193}$$

As $n, l \geq 3$, we deduce from the equation (193) that $2 \mid p" \Longrightarrow p" = 2^\alpha p_1$ with $\alpha \geq 1$ and $2 \nmid p_1$. Then (192) becomes: $A^{2m} = 5p" = 5 \times 2^\alpha p_1 \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$, $i \geq 1$ and $2 \nmid A_1$. We have also $B^n C^l = 2^{\alpha+2} p_1 \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-1-1-1- We suppose that $2 \mid B^n \Longrightarrow B = 2^j B_1$, $j \geq 1$ and $2 \nmid B_1$. We obtain $B_1^n C^l = 2^{\alpha+2-jn} p_1$:

- If $\alpha + 2 - jn > 0 \Longrightarrow 2 \mid C^l$, there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$ and the conjecture (34) is verified.
- If $\alpha + 2 - jn = 0 \Longrightarrow B_1^n C^l = p_1$. From $C^= 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$ that implies that $2 \mid p_1$, then the contradiction with $2 \nmid p_1$.
- If $\alpha + 2 - jn < 0 \Longrightarrow 2^{jn-\alpha-2}B_1^n C^l = p_1$, it implies that $2 \mid p_1$, then the contradiction as above.

** J-1-1-2- We suppose that $2 \mid C^l$, using the same method above, we obtain the identical results.

** J-1-2- We suppose that $a = 7$ and $b = 12 \Longrightarrow 4p = 12p' = bp'$. But $A^{2m} = \frac{4p}{3}.\frac{a}{b} = \frac{12p'}{3}.\frac{7}{12} = \frac{7p'}{3} \Longrightarrow 3 \mid p' \Longrightarrow p = 9p"$, we obtain:

$$A^{2m} = 7p"$$
$$B^n C^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = 2p"$$

The last equation implies that $2 \mid B^n C^l$. Using the same method as for the case J-1-1- above, we obtain the identical results.

We study now the general case. As $3 \mid p \Rightarrow p = 3p'$ and $b \mid 4p \Rightarrow \exists k_1 \in \mathbb{N}^*$ and $4p = 12p' = k_1 b$.

** J-2- $k_1 = 1$: If $k_1 = 1$ then $b = 12p'$, ($p' \neq 1$, if not $p = 3 \ll A^{2m} + B^{2n} + A^m B^n$). But $A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{12p'}{3}\frac{a}{b} = \frac{4p'.a}{12p'} = \frac{a}{3} \Rightarrow 3 \mid a$ because A^{2m} is a natural number, then the contradiction with a, b coprime.

** J-3- $k_1 = 3$: If $k_1 = 3$, then $b = 4p'$ and $A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{k_1.a}{3} = a = (A^m)^2 = a'^2 \Longrightarrow A^m = a'$. The term $A^m B^n$ gives $A^m B^n = \frac{p\sqrt{3}}{3}sin\frac{2\theta}{3} - \frac{a}{2}$, then:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} = 2p'\sqrt{3}sin\frac{2\theta}{3} \qquad (194)$$

The left member of (194) is an integer number and also p', then $2\sqrt{3}sin\frac{2\theta}{3}$ can be written under the form:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_2}{k_3}$$

where k_2, k_3 are two integer numbers and are coprime and $k_3 \mid p' \Longrightarrow p' = k_3.k_4$.

** J-3-1- $k_4 \neq 1$: We suppose that $k_4 \neq 1$, then:

$$A^{2m} + 2A^m B^n = k_2.k_4 \qquad (195)$$

Let μ be a prime number so that $\mu \mid k_4$, then $\mu \mid A^m(A^m+2B^n) \Longrightarrow \mu \mid A^m$ or $\mu \mid (A^m+2B^n)$.

** J-3-1-1- $\mu \mid A^m$: If $\mu \mid A^m \Longrightarrow \mu \mid A^{2m} \Longrightarrow \mu \mid a$. As $\mu \mid k_4 \Longrightarrow \mu \mid p' \Rightarrow \mu \mid (4p'=b)$. But a,b are coprime, then the contradiction.

** J-3-1-2- $\mu \mid (A^m+2B^n)$: If $\mu \mid (A^m+2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu \mid (A^m+2B^n)$, we can write $A^m+2B^n=\mu.t'$. It follows:

$$A^m+B^n=\mu t'-B^n \Longrightarrow A^{2m}+B^{2n}+2A^mB^n=\mu^2t'^2-2t'\mu B^n+B^{2n}$$

Using the expression of p, we obtain $p=t'^2\mu^2-2t'B^n\mu+B^n(B^n-A^m)$. As $p=3p'$ and $\mu \mid p' \Rightarrow \mu \mid (3p') \Rightarrow \mu \mid p$, we can write : $\exists \mu'$ and $p=\mu\mu'$, then we arrive to:

$$\mu'.\mu=\mu(\mu t'^2-2t'B^n)+B^n(B^n-A^m)$$

and $\mu \mid B^n(B^n-A^m) \Longrightarrow \mu \mid B^n$ or $\mu \mid (B^n-A^m)$.

** J-3-1-2-1- $\mu \mid B^n$: If $\mu \mid B^n \Longrightarrow \mu \mid B$, it is in contradiction with J-3-1-2-.

** J-3-1-2-2- $\mu \mid (B^n-A^m)$: If $\mu \mid (B^n-A^m)$ and using $\mu \mid (A^m+2B^n)$, we obtain :

$$\mu \mid 3B^n \Longrightarrow \begin{cases} \mu \mid B^n \\ or \\ \mu = 3 \end{cases}$$

** J-3-1-2-2-1- $\mu \mid B^n$: If $\mu \mid B^n \Longrightarrow \mu \mid B$, it is in contradiction with J-3-1-2-.

** J-3-1-2-2-2- $\mu = 3$: If $\mu = 3 \Longrightarrow 3 \mid k_4 \Longrightarrow k_4 = 3k_4'$, and we have $p'=k_3k_4=3k_3k_4'$, it follows that $p=3p'=9k_3k_4'$, then $9 \mid p$, but $p=(A^m-B^n)^2+3A^mB^n$, then we obtain:

$$9k_3k_4'-3A^mB^n=(A^m-B^n)^2$$

that we write : $3(3k_3k_4'-A^mB^n)=(A^m-B^n)^2$, then : $3 \mid (3k_3k_4'-A^mB^n) \Longrightarrow 3 \mid A^mB^n \Longrightarrow 3 \mid A^m$ or $3 \mid B^n$.

** J-3-1-2-2-2-1- $3 \mid A^m$: If $3 \mid A^m \Longrightarrow 3 \mid A^{2m} \Rightarrow 3 \mid a$, but $3 \mid p' \Rightarrow 3 \mid (4p') \Rightarrow 3 \mid b$, then the contradiction with a,b coprime and $3 \nmid A$.

** J-3-1-2-2-2-2- $3 \mid B^n$: If $3 \mid B^n$ but $A^m=\mu t'-2B^n=3t'-2B^n \Longrightarrow 3 \mid A^m$, it is in contradiction with $3 \nmid A$.

Then the hypothesis $k_4 \neq 1$ is impossible.

** J-3-2- $k_4 = 1$: We suppose now that $k_4 = 1 \Longrightarrow p' = k_3 k_4 = k_3$. Then we have:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_2}{p'} \tag{196}$$

with k_2, p' coprime, we write (196) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_2}{p'}$$

Taking the square of the two members and replacing $cos^2\frac{\theta}{3}$ by $\frac{a}{b}$ and $b = 4p'$, we obtain:

$$3.a(b-a) = k_2^2$$

As $A^{2m} = a = a'^2$, it implies that :

$$3 \mid (b-a), \quad and \quad b - a = b - a'^2 = 3\alpha^2$$

As $k_2 = A^m(A^m + 2B^n)$ following the equation (195) and that $3 \mid k_2 \Longrightarrow 3 \mid A^m(A^m + 2B^n) \Longrightarrow 3 \mid A^m$ or $3 \mid (A^m + 2B^n)$.

** J-3-2-1- $3 \mid A^m$: If $3 \mid A^m \Longrightarrow 3 \mid A^{2m} \Longrightarrow 3 \mid a$, but $3 \mid (b-a) \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime.

** J-3-2-2- $3 \mid (A^m + 2B^n) \Longrightarrow 3 \nmid A^m$ and $3 \nmid B^n$. As $k_2^2 = 9a\alpha^2 = 9a'^2\alpha^2 \Longrightarrow k_2 = 3a'\alpha = A^m(A^m + 2B^n)$, then :

$$3\alpha = A^m + 2B^n \tag{197}$$

As b can be written under the form $b = a'^2 + 3\alpha^2$, then the pair (a', α) is a solution of the Diophantine equation:

$$x^2 + 3y^2 = b \tag{198}$$

As $b = 4p'$, then :

** J-3-2-2-1- If x, y are even, then $2 \mid a' \Longrightarrow 2 \mid a$, it is a contradiction with a, b coprime.

** J-3-2-2-2- If x, y are odd, then a', α are odd, it implies $A^m = a' \equiv 1(\bmod 4)$ or $A^m \equiv 3(\bmod 4)$. If u, v verify (198), then $b = u^2 + 3v^2$, with $u \neq a'$ and $v \neq \alpha$, then u, v do not verify (197): $3v \neq u + 2B^n$, if not, $u = 3v - 2B^n \Longrightarrow b = (3v - 2B^n)^2 + 3v^2 = a'^2 + 3\alpha^2$, the resolution of the obtained equation of second degree in v gives the positive root $v_1 = \alpha$, then $u = 3v - 2B^n = 3\alpha - 2B^n = a'$, then the uniqueness of the representation of b by the equation (198).

** J-3-2-2-2-1- We suppose that $A^m \equiv 1 (\bmod 4)$ and $B^n \equiv 0 (\bmod 4)$, then B^n is even and $B^n = 2B'$. The expression of p becomes:

$$p = a'^2 + 2a'B' + 4B'^2 = (a' + B')^2 + 3B'^2 = 3p' \Longrightarrow 3 \mid (a' + B') \Longrightarrow a' + B' = 3B"$$
$$p' = B'^2 + 3B"^2 \Longrightarrow b = 4p' = (2B')^2 + 3(2B")^2 = a'^2 + 3\alpha^2$$

as b has an unique representation, it follows $2B' = B^n = a' = A^m$, then the contradiction with $A^m > B^n$.

** J-3-2-2-2-2- We suppose that $A^m \equiv 1 (\bmod 4)$ and $B^n \equiv 1 (\bmod 4)$, then C^l is even and $C^l = 2C'$. The expression of p becomes:

$$p = C^{2l} - C^l B^n + B^{2n} = 4C'^2 - 2C'B^n + B^{2n} = (C' - B^n)^2 + 3C'^2 = 3p'$$
$$\Longrightarrow 3 \mid (C' - B^n) \Longrightarrow C' - B^n = 3C"$$
$$p' = C'^2 + 3C"^2 \Longrightarrow b = 4p' = (2C')^2 + 3(2C")^2 = a'^2 + 3\alpha^2$$

as b has an unique representation, it follows $2C' = C^l = a' = A^m$, then the contradiction.

** J-3-2-2-2-3- We suppose that $A^m \equiv 1 (\bmod 4)$ and $B^n \equiv 2 (\bmod 4)$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-4- We suppose that $A^m \equiv 1 (\bmod 4)$ and $B^n \equiv 3 (\bmod 4)$, then C^l is even, see J-3-2-2-2-2-.

** J-3-2-2-2-5- We suppose that $A^m \equiv 3 (\bmod 4)$ and $B^n \equiv 0 (\bmod 4)$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-6- We suppose that $A^m \equiv 3 (\bmod 4)$ and $B^n \equiv 1 (\bmod 4)$, then C^l is even, see J-3-2-2-2-2-.

** J-3-2-2-2-7- We suppose that $A^m \equiv 3 (\bmod 4)$ and $B^n \equiv 2 (\bmod 4)$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-8- We suppose that $A^m \equiv 3 (\bmod 4)$ and $B^n \equiv 3 (\bmod 4)$, then C^l is even, see J-3-2-2-2-2-.

We have achieved the study of the case J-3-2-2- . It gives contradictions.

** J-4- We suppose that $k_1 \neq 3$ and $3 \mid k_1 \Longrightarrow k_1 = 3k'_1$ with $k'_1 \neq 1$, then $4p = 12p' = k_1 b = 3k'_1 b \Rightarrow 4p' = k'_1 b$. A^{2m} can be written as $A^{2m} = \dfrac{4p}{3} cos^2 \dfrac{\theta}{3} = \dfrac{3k'_1 b}{3} \dfrac{a}{b} = k'_1 a$ and $B^n C^l = \dfrac{p}{3}\left(3 - 4cos^2 \dfrac{\theta}{3}\right) = \dfrac{k'_1}{4}(3b - 4a)$. As $B^n C^l$ is an integer number, we must have $4 \mid (3b-4a)$ or $4 \mid k_1'$ or $[2 \mid k_1'$ and $2 \mid (3b-4a)]$.

** J-4-1- We suppose that $4 \mid (3b-4a)$.

** J-4-1-1- We suppose that $3b-4a = 4 \Longrightarrow 4 \mid b \Longrightarrow 2 \mid b$. Then, we have:

$$A^{2m} = k_1' a$$
$$B^n C^l = k_1'$$

** J-4-1-1-1- If k_1' is prime, from $B^n C^l = k_1'$, it is impossible.

** J-4-1-1-2- We suppose that $k_1' > 1$ is not prime. Let ω be a prime number so that $\omega \mid k_1'$.

** J-4-1-1-2-1- We suppose that $k_1' = \omega^s$, with $s \geq 6$. Then we have :

$$A^{2m} = \omega^s.a \tag{199}$$
$$B^n C^l = \omega^s \tag{200}$$

** J-4-1-1-2-1-1- We suppose that $\omega = 2$. If a, k_1' are not coprime , then $2 \mid a$, as $2 \mid b$, it is the contradiction with a, b coprime.

** J-4-1-1-2-1-2- We suppose $\omega = 2$ and a, k_1' are coprime, then $2 \nmid a$. From (200), we deduce that $B = C = 2$ and $n + l = s$, and $A^{2m} = 2^s.a$, but $A^m = 2^l - 2^n \Longrightarrow A^{2m} = (2^l - 2^n)^2 = 2^{2l} + 2^{2n} - 2(2^{l+n}) = 2^{2l} + 2^{2n} - 2 \times 2^s = 2^s.a \Longrightarrow 2^{2l} + 2^{2n} = 2^s(a+2)$. If $l = n$, we obtain $a = 0$ then the contradiction. If $l \neq n$, as $A^m = 2^l - 2^n > 0 \Longrightarrow n < l \Longrightarrow 2n < s$, then $2^{2n}(1 + 2^{2l-2n} - 2^{s+1-2n}) = 2^n 2^l.a$. We call $l = n + n_1 \Longrightarrow 1 + 2^{2l-2n} - 2^{s+1-2n} = 2^{n_1}.a$, but the left member is odd and the right member is even, then the contradiction. Then the case $\omega = 2$ is impossible.

** J-4-1-1-2-1-3- We suppose that $k_1' = \omega^s$ with $\omega \neq 2$:

** J-4-1-1-2-1-3-1- Suppose that a, k_1' are not coprime, then $\omega \mid a \Longrightarrow a = \omega^t.a_1$ and $t \nmid a_1$. Then, we have:

$$A^{2m} = \omega^{s+t}.a_1 \tag{201}$$
$$B^n C^l = \omega^s \tag{202}$$

From (202), we deduce that $B^n = \omega^n$, $C^n = \omega^l$, $s = n + l$ and $A^m = \omega^l - \omega^n > 0 \Longrightarrow l > n$. We have also $A^{2m} = \omega^{s+t}.a_1 = (\omega^l - \omega^n)^2 = \omega^{2l} + \omega^{2n} - 2 \times \omega^s$. As $\omega \neq 2 \Longrightarrow \omega$ is odd, then $A^{2m} = \omega^{s+t}.a_1 = (\omega^l - \omega^n)^2$ is even, then $2 \mid a_1 \Longrightarrow 2 \mid a$, it is in contradiction with a, b coprime, then this case is impossible.

** J-4-1-1-2-1-3-2- Suppose that a, k'_1 are coprime, with :

$$A^{2m} = \omega^s.a \tag{203}$$

$$B^n C^l = \omega^s \tag{204}$$

From (204), we deduce that $B^n = \omega^n$, $C^l = \omega^l$ and $s = n + l$. As $\omega \neq 2 \Longrightarrow \omega$ is odd and $A^{2m} = \omega^s.a = (\omega^l - \omega^n)^2$ is even, then $2 \mid a$. It follows the contradiction with a, b coprime and this case is impossible.

** J-4-1-1-2-2- We suppose that $k'_1 = \omega^s.k_2$, with $s \geq 6$, $\omega \nmid k_2$. We have :

$$A^{2m} = \omega^s.k_2.a$$
$$B^n C^l = \omega^s.k_2$$

** J-4-1-1-2-2-1- If k_2 is prime, from the last equation above, $\omega = k_2$, it is in contradiction with $\omega \nmid k_2$. Then this case is impossible.

** J-4-1-1-2-2-2- We suppose that $k'_1 = \omega^s.k_2$, with $s \geq 6$, $\omega \nmid k_2$ and k_2 not a prime. Then, we have:

$$A^{2m} = \omega^s.k_2.a$$
$$B^n C^l = \omega^s.k_2 \tag{205}$$

** J-4-1-1-2-2-2-1- We suppose that ω, a are coprime, then $\omega \nmid a$. As $A^{2m} = \omega^s.k_2.a \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i.A_1$ with $i \geq 1$ and $\omega \nmid A_1$, then $s = 2i.m$. From (205), we have $\omega \mid (B^n C^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-1-2-2-2-1-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j.B_1$ with $j \geq 1$ and $\omega \nmid B_1$. then :

$$B_1^n C^l = \omega^{2im-jn} k_2$$

- If $2im - jn > 0$, $\omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (34) is verified.
- If $2im - jn = 0 \Longrightarrow B_1^n C^l = k_2$, as $\omega \nmid k_2 \Longrightarrow \omega \nmid C^l$, then the contradiction with $\omega \mid (C^l = A^m + B^n)$.
- If $2im - jn < 0 \Longrightarrow \omega^{jn-2im} B_1^n C^l = k_2 \Longrightarrow \omega \mid k_2$, then the contradiction with $\omega \nmid k_2$.

** J-4-1-1-2-2-2-1-2- We suppose that $\omega \mid C^l$. Using the same method used above, we obtain identical results.

** J-4-1-1-2-2-2-2- We suppose that a, ω are not coprime, then $\omega \mid a \Longrightarrow a = \omega^t.a_1$ and $\omega \nmid a_1$. So we have :

$$A^{2m} = \omega^{s+t}.k_2.a_1 \tag{206}$$

$$B^n C^l = \omega^s.k_2 \tag{207}$$

As $A^{2m} = \omega^{s+t}.k_2.a_1 \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$ with $i \geq 1$ and $\omega \nmid A_1$, then $s+t = 2im$. From (207), we have $\omega \mid (B^n C^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-1-2-2-2-2-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $j \geq 1$ and $\omega \nmid B_1$. then:

$$B_1^n C^l = \omega^{2im-t-jn} k_2$$

- If $2im - t - jn > 0$, $\omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (34) is verified.

- If $2im - t - jn = 0 \Longrightarrow B_1^n C^l = k_2$, As $\omega \nmid k_2 \Longrightarrow \omega \nmid C^l$, then the contradiction with $\omega \mid (C^l = A^m + B^n)$.

- If $2im - t - jn < 0 \Longrightarrow \omega^{jn+t-2im} B_1^n C^l = k_2 \Longrightarrow \omega \mid k_2$, then the contradiction with $\omega \nmid k_2$.

** J-4-1-1-2-2-2-2-2- We suppose that $\omega \mid C^l$. Using the same method used above, we obtain identical results.

** J-4-1-2- $3b - 4a \neq 4$ and $4 \mid (3b - 4a) \Longrightarrow 3b - 4a = 4^s \Omega$ with $s \geq 1$ and $4 \nmid \Omega$. We obtain:

$$A^{2m} = k_1' a \tag{208}$$

$$B^n C^l = 4^{s-1} k_1' \Omega \tag{209}$$

** J-4-1-2-1- We suppose that $k_1' = 2$. From (208), we deduce that $2 \mid a$. As $4 \mid (3b - 4a) \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime and this case is impossible.

** J-4-1-2-2- We suppose that $k_1' = 3$. From (208) we deduce that $3^3 \mid A^{2m}$. From (209), it follows that $3^3 \mid B^n$ or $3^3 \mid C^l$. In the last two cases, we obtain $3^3 \mid p$. But $4p = 3k_1' b = 9b \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime. Then this case is impossible.

** J-4-1-2-3- We suppose that k_1' is prime ≥ 5:

** J-4-1-2-3-1- Suppose that k_1' and a are coprime. The equation (208) gives $(A^m)^2 = k_1'.a$, that is impossible with $k_1' \nmid a$. Then this case is impossible.

** J-4-1-2-3-2- Suppose that k_1' and a are not coprime. Let $k_1' \mid a \Longrightarrow a = k_1'^{\alpha} a_1$ with $\alpha \geq 1$ and $k_1' \nmid a_1$. The equation (208) is written as :

$$A^{2m} = k_1' a = k_1'^{\alpha+1} a_1$$

The last equation gives $k_1' \mid A^{2m} \Longrightarrow k_1' \mid A \Longrightarrow A = k_1'^{i}.A_1$, with $k_1' \nmid A_1$. If $2i.m \neq (\alpha + 1)$, it is impossible. We suppose that $2i.m = \alpha + 1$, then $k_1' \mid A^m$. We return to the equation (209). If k_1' and Ω are coprime, it is impossible. We suppose that k'_1 and Ω are not coprime, then $k'_1 \mid \Omega$ and the exponent of k'_1 in Ω is so the equation (209) is satisfying. We deduce easily that $k'_1 \mid B^n$. Then $k'^2_1 \mid (p = A^{2m} + B^{2n} + A^mB^n)$, but $4p = 3k'_1b \Longrightarrow k'_1 \mid b$, then the contradiction with a, b coprime.

** J-4-1-2-4- We suppose that $k'_1 \geq 4$ is not a prime.

** J-4-1-2-4-1- We suppose that $k'_1 = 4$, we obtain then $A^{2m} = 4a$ and $B^nC^l = 3b - 4a = 3p' - 4a$. This case was studied in the paragraph 2.6.8, case ** I-2-.

** J-4-1-2-4-2- We suppose that $k'_1 > 4$ is not a prime.

** J-4-1-2-4-2-1- We suppose that a, k'_1 are coprime. From the expression $A^{2m} = k'_1.a$, we deduce that $a = a_1^2$ and $k'_1 = k"^2_1$. It gives :

$$A^m = a_1.k"_1$$
$$B^nC^l = 4^{s-1}k"^2_1.\Omega$$

Let ω be a prime so that $\omega \mid k"_1$ and $k"_1 = \omega^t.k"_2$ with $\omega \nmid k"_2$. The last two equations become :

$$A^m = a_1.\omega^t.k"_2 \quad (210)$$
$$B^nC^l = 4^{s-1}\omega^{2t}.k"^2_2.\Omega \quad (211)$$

From (210), $\omega \mid A^m \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i.A_1$ with $\omega \nmid A_1$ and $im = t$. From (211), we obtain $\omega \mid B^nC^l \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-2-4-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j.B_1$ with $\omega \nmid B_1$. From (210), we have $B_1^nC^l = \omega^{2t-j.n}4^{s-1}.k"^2_2.\Omega$.

** J-4-1-2-4-2-1-1-1- If $\omega = 2$ and $2 \nmid \Omega$, we have $B_1^nC^l = 2^{2t+2s-j.n-2}k"^2_2.\Omega$:

- If $2t + 2s - jn - 2 \leq 0$ then $2 \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$.
- If $2t + 2s - jn - 2 \geq 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$ and the conjecture (34) is verified.

** J-4-1-2-4-2-1-1-2- If $\omega = 2$ and if $2 \mid \Omega \Longrightarrow \Omega = 2.\Omega_1$ because $4 \nmid \Omega$, we have $B_1^nC^l = 2^{2t+2s+1-j.n-2}k"^2_2\Omega_1$:

- If $2t + 2s - jn - 3 \leq 0$ then $2 \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$.
- If $2t + 2s - jn - 3 \geq 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$ and the conjecture (34) is verified.

** J-4-1-2-4-2-1-1-3- If $\omega \neq 2$, we have $B_1^nC^l = \omega^{2t-j.n}4^{s-1}.k"^2_2.\Omega$:

-If $2t - jn \leq 0 \Longrightarrow \omega \nmid C^l$ it is in contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$.

-If $2t - jn \geq 1 \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$ and the conjecture (34) is verified.

** J-4-1-2-4-2-1-2- If $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h.C_1$, with $\omega \nmid C_1$. Using the same method as in the case J-4-1-2-4-2-1-1 above, we obtain identical results.

** J-4-1-2-4-2-2- We suppose that a, k'_1 are not coprime. Let ω be a prime so that $\omega \mid a$ and $\omega \mid k'_1$. We write:

$$a = \omega^\alpha.a_1$$
$$k'_1 = \omega^\mu.k"_1$$

with $a_1, k"_1$ coprime. The expression of A^{2m} becomes $A^{2m} = \omega^{\alpha+\mu}.a_1.k"_1$. The term B^nC^l becomes:

$$B^nC^l = 4^{s-1}.\omega^\mu.k"_1.\Omega \tag{212}$$

** J-4-1-2-4-2-2-1- If $\omega = 2 \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime, this case is impossible.

** J-4-1-2-4-2-2-2- If $\omega \geq 3$, we have $\omega \mid a$. If $\omega \mid b$ then the contradiction with a, b coprime. We suppose that $\omega \nmid b$. From the expression of A^{2m}, we obtain $\omega \mid A^{2m} \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i.A_1$ with $\omega \nmid A_1$, $i \geq 1$ and $2i.m = \alpha + \mu$. From (212), we deduce that $\omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-2-4-2-2-2-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$ and $j \geq 1$. Then, $B_1^nC^l = 4^{s-1}\omega^{\mu-jn}.k"_1.\Omega$:

* $\omega \nmid \Omega$:
- If $\mu - jn \geq 1$, we have $\omega \mid C^l \Longrightarrow \omega \mid C$, there is no contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$ and the conjecture (34) is verified.
- If $\mu - jn \leq 0$, then $\omega \nmid C^l$ and it is a contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$. Then this case is impossible.

* $\omega \mid \Omega$: we write $\Omega = \omega^\beta.\Omega_1$ with $\beta \geq 1$ and $\omega \nmid \Omega_1$. As $3b - 4a = 4^s.\Omega = 4^s.\omega^\beta.\Omega_1 \Longrightarrow 3b = 4a + 4^s.\omega^\beta.\Omega_1 = 4\omega^\alpha.a_1 + 4^s.\omega^\beta.\Omega_1 \Longrightarrow 3b = 4\omega(\omega^{\alpha-1}.a_1 + 4^{s-1}.\omega^{\beta-1}.\Omega_1)$. If $\omega = 3$ and $\beta = 1$, we obtain $b = 4(3^{\alpha-1}a_1 + 4^{s-1}\Omega_1)$ and $B_1^nC^l = 4^{s-1}3^{\mu+1-jn}.k"_1\Omega_1$.
- If $\mu - jn + 1 \geq 1$, then $3 \mid C^l$ and the conjecture (34) is verified.
- If $\mu - jn + 1 \leq 0$, then $3 \nmid C^l$ and it is the contradiction with $C^l = 3^{im}A_1^m + 3^{jn}B_1^n$.

Now, if $\beta \geq 2$ and $\alpha = im \geq 3$, we obtain $3b = 4\omega^2(\omega^{\alpha-2}a_1 + 4^{s-1}\omega^{\beta-2}\Omega_1)$. If $\omega = 3$ or not, then $\omega \mid b$, but $\omega \mid a$, then the contradiction with a, b coprime.

** J-4-1-2-4-2-2-2-2- We suppose that $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$ and $h \geq 1$. Then, $B^n C_1^l = 4^{s-1}\omega^{\mu-hl}.k"_1.\Omega$. Using the same method as above, we obtain identical results.

** J-4-2- We suppose that $4 \mid k'_1$.

** J-4-2-1- $k'_1 = 4 \Longrightarrow 4p = 3k'_1 b = 12b \Longrightarrow p = 3b = 3p'$, this case has been studied (see case I-2- paragraph 2.6.8).

** J-4-2-2- $k'_1 > 4$ with $4 \mid k'_1 \Longrightarrow k'_1 = 4^s k"_1$ and $s \geq 1$, $4 \nmid k"_1$. Then, we obtain:

$$A^{2m} = 4^s k"_1 a = 2^{2s} k"_1 a$$
$$B^n C^l = 4^{s-1} k"_1 (3b - 4a) = 2^{2s-2} k"_1 (3b - 4a)$$

** J-4-2-2-1- We suppose that $s = 1$ and $k'_1 = 4k"_1$ with $k"_1 > 1$, so $p = 3p'$ and $p' = k"_1 b$, this is the case 2.6.3 already studied.

** J-4-2-2-2- We suppose that $s > 1$, then $k'_1 = 4^s k"_1 \Longrightarrow 4p = 3 \times 4^s k"_1 b$ and we obtain:

$$A^{2m} = 4^s k"_1 a \tag{213}$$
$$B^n C^l = 4^{s-1} k"_1 (3b - 4a) \tag{214}$$

** J-4-2-2-2-1- We suppose that $2 \nmid (k"_1.a) \Longrightarrow 2 \nmid k"_1$ and $2 \nmid a$. As $(A^m)^2 = (2^s)^2.(k"_1.a)$, we call $d^2 = k"_1.a$, then $A^m = 2^s.d \Longrightarrow 2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$ with $2 \nmid A_1$ and $i \geq 1$, then: $2^{im} A_1^m = 2^s.d \Longrightarrow s = im$. From the equation (214), we have $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-1-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j.B_1$, with $j \geq 1$ and $2 \nmid B_1$. The equation (214) becomes:

$$B_1^n C^l = 2^{2s-jn-2} k"_1 (3b - 4a) = 2^{2im-jn-2} k"_1 (3b - 4a)$$

* We suppose that $2 \nmid (3b - 4a)$:
- If $2im - jn - 2 \geq 1$, then $2 \mid C^l$, there is no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (34) is verified.
- If $2im - jn - 2 \leq 0$, then $2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

* We suppose that $2^\mu \mid (3b - 4a)$, $\mu \geq 1$:
- If $2im + \mu - jn - 2 \geq 1$, then $2 \mid C^l$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (34) is verified.
- If $2im + \mu - jn - 2 \leq 0$, then $2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** J-4-2-2-2-1-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2- We suppose that $2 \mid (k"_1.a)$:

** J-4-2-2-2-2-1- We suppose that $k"_1$ and a are coprime:

** J-4-2-2-2-2-1-1- We suppose that $2 \nmid a$ and $2 \mid k"_1 \Longrightarrow k"_1 = 2^{2\mu}.k"^2_2$ and $a = a_1^2$, then the equations (213-214) become:

$$A^{2m} = 4^s.2^{2\mu}k"^2_2a_1^2 \Longrightarrow A^m = 2^{s+\mu}.k"_2.a_1 \tag{215}$$

$$B^nC^l = 4^{s-1}2^{2\mu}k"^2_2(3b-4a) = 2^{2s+2\mu-2}k"^2_2(3b-4a) \tag{216}$$

The equation (215) gives $2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i.A_1$ with $2 \nmid A_1$, $i \geq 1$ and $im = s+\mu$. From the equation (216), we have $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-1-1-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geq 1$, then $B_1^nC^l = 2^{2s+2\mu-jn-2}k"^2_2(3b-4a)$:

* We suppose that $2 \nmid (3b-4a)$:
- If $2im+2\mu-jn-2 \geq 1 \Rightarrow 2 \mid C^l$, then there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (34) is verified.
- If $2im+2\mu-jn-2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha \mid (3b-4a)$, $\alpha \geq 1$ so that a, b remain coprime:
- If $2im+2\mu+\alpha-jn-2 \geq 1 \Rightarrow 2 \mid C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (34) is verified.
- If $2im+2\mu+\alpha-jn-2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** J-4-2-2-2-2-1-1-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2-1-2- We suppose that $2 \nmid k"_1$ and $2 \mid a \Longrightarrow a = 2^{2\mu}.a_1^2$ and $k"_1 = k"^2_2$, then the equations (213-214) become:

$$A^{2m} = 4^s.2^{2\mu}a_1^2k"^2_2 \Longrightarrow A^m = 2^{s+\mu}.a_1.k"_2. \tag{217}$$

$$B^nC^l = 4^{s-1}k"^2_2(3b-4a) = 2^{2s-2}k"^2_2(3b-4a) \tag{218}$$

The equation (217) gives $2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i.A_1$ with $2 \nmid A_1$, $i \geq 1$ and $im = s+\mu$. From the equation (218), we have $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-1-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geq 1$. Then we obtain $B_1^n C^l = 2^{2s-jn-2} k"_2^2 (3b-4a)$:

* We suppose that $2 \nmid (3b-4a) \Longrightarrow 2 \nmid b$:
- If $2im - jn - 2 \geq 1 \Rightarrow 2 \mid C^l$, then no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (34) is verified.
- If $2im - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

* We suppose that $2^\alpha \mid (3b-4a)$, $\alpha \geq 1$, in this case a, b are not coprime, then the contradiction.

** J-4-2-2-2-2-1-2-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2-2- We suppose that $k"_1$ and a are not coprime $2 \mid a$ and $2 \mid k"_1$. Let $a = 2^t.a_1$ and $k"_1 = 2^\mu k"_2$ and $2 \nmid a_1$ and $2 \nmid k"_2$. From (213), we have $\mu + t = 2\lambda$ and $a_1.k"_2 = \omega^2$. The equations (213-214) become:

$$A^{2m} = 4^s k"_1 a = 2^{2s}.2^\mu k"_2.2^t.a_1 = 2^{2s+2\lambda}.\omega^2 \Longrightarrow A^m = 2^{s+\lambda}.\omega \tag{219}$$

$$B^n C^l = 4^{s-1} 2^\mu k"_2 (3b-4a) = 2^{2s+\mu-2} k"_2 (3b-4a) \tag{220}$$

From (219) we have $2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1, i \geq 1$ and $2 \nmid A_1$. From(220), $2s + \mu - 2 \geq 1$, we deduce that $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geq 1$. Then we obtain $B_1^n C^l = 2^{2s+\mu-jn-2} k"^2 (3b-4a)$:

* We suppose that $2 \nmid (3b-4a)$:

- If $2s + \mu - jn - 2 \geq 1 \Rightarrow 2 \mid C^l$, then no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (34) is verified.
- If $2s + \mu - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

* We suppose that $2^\alpha \mid (3b-4a)$, for one value $\alpha \geq 1$. As $2 \mid a$, then $2^\alpha \mid (3b-4a) \Longrightarrow 2 \mid (3b-4a) \Longrightarrow 2 \mid (3b) \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

** J-4-2-2-2-2-2-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-3- $2 \mid k'_1$ and $2 \mid (3b-4a)$: then we obtain $2 \mid k'_1 \Longrightarrow k'_1 = 2^t.k"_1$ with $t \geq 1$ and $2 \nmid k"_1$, $2 \mid (3b-4a) \Longrightarrow 3b - 4a = 2^\mu.d$ with $\mu \geq 1$ and $2 \nmid d$. We have also

$2 \mid b$. If $2 \mid a$, it is a contradition with a, b coprime.

We suppose, in the following, that $2 \nmid a$. The equations (213-214) become:

$$A^{2m} = 2^t.k"_1.a = (A^m)^2 \tag{221}$$

$$B^nC^l = 2^{t-1}k"_1.2^{\mu-1}d = 2^{t+\mu-2}k"_1.d \tag{222}$$

From (221), we deduce that the exponent t is even, let $t = 2\lambda$. Then we call $\omega^2 = k"_1.a$, it gives $A^m = 2^\lambda.\omega \Longrightarrow 2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i.A_1$ with $i \geq 1$ and $2 \nmid A_1$. From (222), we have $2\lambda+\mu-2 \geq 1$, then $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$:

** J-4-3-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^jB_1$, with $j \geq 1$ and $2 \nmid B_1$. Then we obtain $B_1^nC^l = 2^{2\lambda+\mu-jn-2}.k"_1.d$.

- If $2\lambda + \mu - jn - 2 \geq 1 \Rightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (34) is verified.
- If $2s+t+\mu-jn-2 \leq 0 \Rightarrow 2 \nmid C$, then the contradiction with $C^l = 2^{im}A_1^m+2^{jn}B_1^n$.

** J-4-3-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C$. With the same method used above, we obtain identical results.

□

The Main Theorem is proved.

2.7. Examples and Conclusion

2.7.1. Numerical Examples

2.7.1.1. Example 1:

We consider the example : $6^3 + 3^3 = 3^5$ with $A^m = 6^3$, $B^n = 3^3$ and $C^l = 3^5$. With the notations used in the paper, we obtain:

$$p = 3^6 \times 73, \quad q = 8 \times 3^{11}, \quad \bar{\Delta} = 4 \times 3^{18}(3^7 \times 4^2 - 73^3) < 0$$

$$\rho = \frac{3^8 \times 73\sqrt{73}}{\sqrt{3}}, \quad cos\theta = -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \tag{223}$$

As $A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} \Longrightarrow cos^2\frac{\theta}{3} = \frac{3A^{2m}}{4p} = \frac{3 \times 2^4}{73} = \frac{a}{b} \Longrightarrow a = 3 \times 2^4,\ b = 73$; then we obtain:

$$cos\frac{\theta}{3} = \frac{4\sqrt{3}}{\sqrt{73}}, \quad p = 3^6.b \tag{224}$$

We verify easily the equation (223) to calculate $cos\theta$ using (224). For this example, we can use the two conditions from (70) as $3 \mid a$,$b \mid 4p$ and $3 \mid p$. The cases 2.5.4 and 2.6.3 are respectively used. For the case 2.5.4, it is the case B-2-2-1- that was used and the conjecture (34) is verified. Concerning the case 2.6.3, it is the case G-2-2-1- that was used and the conjecture (34) is verified.

2.7.1.2. Example 2:

The second example is: $7^4+7^3 = 14^3$. We take $A^m = 7^4$, $B^n = 7^3$ and $C^l = 14^3$. We obtain $p = 57\times 7^6 = 3\times 19\times 7^6$, $q = 8\times 7^{10}$, $\overline{\Delta} = 27q^2-4p^3 = 27\times 4\times 7^{18}(16\times 49-19^3) = -27\times 4\times 7^{18}\times 6075 < 0$, $\rho = 19\times 7^9\times \sqrt{19}$, $cos\theta = -\dfrac{4\times 7}{19\sqrt{19}}$. As $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} \Longrightarrow cos^2\dfrac{\theta}{3} = \dfrac{3A^{2m}}{4p} = \dfrac{7^2}{4\times 19} = \dfrac{a}{b} \Longrightarrow a = 7^2,\ b = 4\times 19$, then $cos\dfrac{\theta}{3} = \dfrac{7}{2\sqrt{19}}$ and we have the two principal conditions $3 \mid p$ and $b \mid (4p)$. The calculation of $cos\theta$ from the expression of $cos\dfrac{\theta}{3}$ is confirmed by the value below:

$$cos\theta = cos3(\theta/3) = 4cos^3\frac{\theta}{3} - 3cos\frac{\theta}{3} = 4\left(\frac{7}{2\sqrt{19}}\right)^3 - 3\frac{7}{2\sqrt{19}} = -\frac{4\times 7}{19\sqrt{19}}$$

Then, we obtain $3 \mid p \Rightarrow p = 3p'$, $b \mid (4p)$ with $b \neq 2, 4$ then $12p' = k_1 b = 3\times 7^6 b$. It concerns the paragraph 2.6.9 of the second hypothesis. As $k_1 = 3\times 7^6 = 3k_1'$ with $k_1' = 7^6 \neq 1$. It is the case J-4-1-2-4-2-2- with the condition $4 \mid (3b-4a)$. So we verify :

$$3b - 4a = 3\times 4\times 19 - 4\times 7^2 = 32 \Longrightarrow 4 \mid (3b-4a)$$

with $A^{2m} = 7^8 = 7^6\times 7^2 = k_1'.a$ and k_1' not a prime, with a and k_1' not coprime with $\omega = 7 \nmid \Omega(=2)$. We find that the conjecture (34) is verified with a common factor equal to 7 (prime and divisor of $k_1' = 7^6$).

2.7.1.3. Example 3:

The third example is: $19^4 + 38^3 = 57^3$ with $A^m = 19^4$, $B^n = 38^3$ and $C^l = 57^3$. We obtain $p = 19^6\times 577$, $q = 8\times 27\times 19^{10}$, $\overline{\Delta} = 27q^2 - 4p^3 = 4\times 19^{18}(27^3\times 16\times 19^2 - 577^3) < 0$, $\rho = \dfrac{19^9\times 577\sqrt{577}}{3\sqrt{3}}$, $cos\theta = -\dfrac{4\times 3^4\times 19\sqrt{3}}{577\sqrt{577}}$. As $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} \Longrightarrow cos^2\dfrac{\theta}{3} = \dfrac{3A^{2m}}{4p} = \dfrac{3\times 19^2}{4\times 577} = \dfrac{a}{b} \Longrightarrow a = 3\times 19^2,\ b = 4\times 577$, then $cos\dfrac{\theta}{3} = \dfrac{19\sqrt{3}}{2\sqrt{577}}$ and we have the first hypothesis $3 \mid a$ and $b \mid (4p)$. Here again, the calculation of $cos\theta$ from the expression of $cos\dfrac{\theta}{3}$ is confirmed by the value below:

$$cos\theta = cos3(\theta/3) = 4cos^3\frac{\theta}{3} - 3cos\frac{\theta}{3} = 4\left(\frac{19\sqrt{3}}{2\sqrt{577}}\right)^3 - 3\frac{19\sqrt{3}}{2\sqrt{577}} = -\frac{4\times 3^4\times 19\sqrt{3}}{577\sqrt{577}}$$

Then, we obtain $3 \mid a \Rightarrow a = 3a' = 3\times 19^2$, $b \mid (4p)$ with $b \neq 2, 4$ and $b = 4p'$ with $p = kp'$ soit $p' = 577$ and $k = 19^6$. This concerns the paragraph 2.5.8 of the first hypothesis. It is the case E-2-2-2-2-1- with $\omega = 19$, a', ω not coprime and $\omega = 19 \nmid (p' - a') = (577 - 19^2)$ with $s - jn = 6 - 1\times 3 = 3 \geq 1$, and the conjecture (34) is verified.

2.7.2. Conclusion

The method used to give the proof of the conjecture of Beal has discussed many possibles cases, using elementary number theory and the results of some theorems about Diophantine equations. We have confirmed the method by three numerical examples. In conclusion, we can announce the theorem:

Theorem 13. — *Let $A, B, C, m, n,$ and l be positive natural numbers with $m, n, l > 2$. If :*

$$A^m + B^n = C^l \tag{225}$$

then $A, B,$ and C have a common factor.

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CHAPTER 3

A COMPLETE PROOF OF THE CONJECTURE $c < rad^{1.63}(abc)$

Abstract. — In this paper, we consider the *abc* conjecture, we will give the proof that the conjecture $c < rad^{1.63}(abc)$ is true. It constitutes the key to resolving the *abc* conjecture.

The paper is under reviewing.

To the memory of my ***Father*** *who taught me arithmetic,*
To my wife ***Wahida****, my daughter* ***Sinda*** *and my son* ***Mohamed Mazen***
To Prof. ***A. Nitaj*** *for his work on the abc conjecture*

3.1. Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{226}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{227}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 14. — *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{228}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**2**]. It concerned the best example given by E. Reyssat [**2**]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{229}$$

A conjecture was proposed that $c < rad^2(abc)$ [**3**]. In 2012, A. Nitaj [**4**] proposed the following conjecture:

Conjecture 15. — *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < rad^{1.63}(abc) \tag{230}$$

$$abc < rad^{4.42}(abc) \tag{231}$$

In this paper, we will give the proof of the conjecture given by (230) that constitutes the key to obtain the proof of the *abc* conjecture using classical methods with the help of some theorems from the field of the number theory.

3.2. The Proof of the conjecture $c < rad^{1.63}(abc)$

Let a, b, c be positive integers, relatively prime, with $c = a + b$, $b < a$ and $R = rad(abc)$, $c = \prod_{j'=1}^{j'=J'} c_{j'}^{\beta_{j'}}$, $\beta_{j'} \geq 1$, $c_{j'} \geq 2$ prime integers.

In the following, we will give the proof of the conjecture $c < rad^{1.63}(abc)$.

Proof. — :

3.2.1. Trivial cases:

- We suppose that $c < rad(abc)$, then we obtain:

$$c < rad(abc) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

and the condition (230) is satisfied.

- We suppose that $c = rad(abc)$, then a, b, c are not coprime, case to reject.

In the following, we suppose that $c > rad(abc)$ and a, b and c are not all prime numbers.

- We suppose $\mu_a \leq rad^{0.63}(a)$. We obtain :

$$c = a + b < 2a \leq 2rad^{1.63}(a) < rad^{1.63}(abc) \Longrightarrow c < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

Then (230) is satisfied.

- We suppose $\mu_c \leq rad^{0.63}(c)$. We obtain :

$$c = \mu_c rad(c) \leq rad^{1.63}(c) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

and the condition (230) is satisfied.

3.2.2. We suppose $\mu_c > rad^{0.63}(c)$ and $\mu_a > rad^{0.63}(a)$

3.2.2.1. Case : $rad^{0.63}(c) < \mu_c \leq rad^{1.63}(c)$ and $rad^{0.63}(a) < \mu_a \leq rad^{1.63}(a)$

We can write:

$$\left.\begin{array}{l} \mu_c \leq rad^{1.63}(c) \Longrightarrow c \leq rad^{2.63}(c) \\ \\ \mu_a \leq rad^{1.63}(a) \Longrightarrow a \leq rad^{2.63}(a) \end{array}\right\} \Longrightarrow ac \leq rad^{2.63}(ac) \Longrightarrow a^2 < ac \leq rad^{2.63}(ac)$$

$$\Longrightarrow a < rad^{1.315}(ac) \Longrightarrow c < 2a < 2rad^{1.315}(ac) < rad^{1.63}(abc)$$

$$\Longrightarrow \boxed{c = a + b < R^{1.63}}$$

3.2.2.2. Case : $rad^{1.63}(c) < \mu_c$ or $rad^{1.63}(a) < \mu_a$

I - We suppose that $rad^{1.63}(c) < \mu_c$ and $rad^{1.63}(a) < \mu_a \leq rad^2(a)$:

I-1- Case $rad(a) < rad(c)$:
In this case $a = \mu_a.rad(a) \leq rad^3(a) \leq rad^{1.63}(a)rad^{1.37}(a) < rad^{1.63}(a).rad^{1.37}(c)$ $\Longrightarrow c < 2a < 2rad^{1.63}(a).rad^{1.37}(c) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

I-2- Case $rad(c) < rad(a) < rad^{\frac{1.63}{1.37}}(c)$: As $a \leq rad^{1.63}(a).rad^{1.37}(a) < rad^{1.63}(a).rad^{1.63}(c) \Longrightarrow c < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

I-3- Case $rad^{\frac{1.63}{1.37}}(c) < rad(a)$:

I-3-1- We suppose $rad^{1.63}(c) < \mu_c \leq rad^{2.26}(c)$, we obtain:

$$c \leq rad^{3.26}(c) \Longrightarrow c \leq rad^{1.63}(c).rad^{1.63}(c) \Longrightarrow$$

$$c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(c).rad^{1.63}(a).rad^{1.63}(b) = R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$$

I-3-2- We suppose $\mu_c > rad^{2.26}(c) \Longrightarrow c > rad^{3.26}(c)$.

I-3-2-1- We consider the case $\mu_a = rad^2(a) \Longrightarrow a = rad^3(a)$ and $c = a + 1$. Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = c \tag{232}$$

I-3-2-1-1- We suppose that $c = rad^n(c)$ with $n \geq 4$, we obtain the equation:

$$rad^n(c) - rad^3(a) = 1 \tag{233}$$

But the solutions of the equation (233) are [**5**] :($rad(c) = 3, n = 2, rad(a) = +2$), it follows the contradiction with $n \geq 4$ and the case $c = rad^n(c), n \geq 4$ is to reject.

I-3-2-1-2- In the following, we will study the cases $\mu_c = A.rad^n(c)$ with $rad(c) \nmid A, n \geq 0$. The above equation (232) can be written as :

$$(X+1)(X^2 - X + 1) = c \tag{234}$$

Let δ one divisor of c so that :

$$X + 1 = \delta \tag{235}$$

$$X^2 - X + 1 = \frac{c}{\delta} = m = \delta^2 - 3X \tag{236}$$

We recall that $rad(a) > rad^{\frac{1.63}{1.37}}(c)$.

I-3-2-1-2-1- We suppose $\delta = l.rad(c)$. We have $\delta = l.rad(c) < c = \mu_c.rad(c) \Longrightarrow l < \mu_c$. As $\dfrac{c}{\delta} = \dfrac{\mu_c rad(c)}{lrad(c)} = \dfrac{\mu_c}{l} = m = \delta^2 - 3X \Longrightarrow \mu_c = l.m = l(\delta^2 - 3X)$. From $m = \delta^2 - 3X$) and $X = rad(a)$, we obtain:

$$m = l^2 rad^2(c) - 3rad(a) \Longrightarrow 3rad(a) = l^2 rad^2(c) - m$$

A- Case $3|m \Longrightarrow m = 3m', m' > 1$: As $\mu_c = ml = 3m'l \Longrightarrow 3|rad(c)$ and $(rad(c), m')$ not coprime. We obtain:

$$rad(a) = l^2 rad(c).\frac{rad(c)}{3} - m'$$

It follows that a, c are not coprime, then the contradiction.

B - Case $m = 3 \Longrightarrow \mu_c = 3l \Longrightarrow c = 3lrad(c) = 3\delta = \delta(\delta^2 - 3X) \Longrightarrow \delta^2 = 3(1+X) = 3\delta \Longrightarrow \delta = lrad(c) = 3 \Longrightarrow c = 3\delta = 9 = a + 1 \Longrightarrow a = 8 \Longrightarrow c = 9 < (2 \times 3)^{1.63} \approx 18.55$, it is a trivial case and the conjecture is true.

I-3-2-1-2-2- We suppose $\delta = l.rad^2(c), l \geq 2$. If $n = 0$ then $\mu_c = A$ and from the equation above (236):

$$m = \frac{c}{\delta} = \frac{\mu_c.rad(c)}{lrad^2(c)} = \frac{A.rad(c)}{lrad^2(c)} = \frac{A}{lrad(c)} \Rightarrow rad(c)|A$$

It follows the contradiction with the hypothesis above $rad(c) \nmid A$.

I-3-2-1-2-3- We suppose $\delta = lrad^2(c), l \geq 2$ and in the following $n > 0$. As $m = \dfrac{c}{\delta} = \dfrac{\mu_c.rad(c)}{lrad^2(c)} = \dfrac{\mu_c}{lrad(c)}$, if $lrad(c) \nmid \mu_c$ then the case is to reject. We suppose $lrad(c)|\mu_c \Longrightarrow \mu_c = m.lrad(c)$, with $m, rad(c)$ not coprime, then

$\dfrac{c}{\delta} = m = \delta^2 - 3rad(a)$.

C - Case $m = 1 = c/\delta \Longrightarrow \delta^2 - 3rad(a) = 1 \Longrightarrow (\delta-1)(\delta+1) = 3rad(a) = rad(a)(\delta+1) \Longrightarrow \delta = 2 = l.rad^2(c)$, then the contradiction.

D - Case $m = 3$, we obtain $3(1+rad(a)) = \delta^2 = 3\delta \Longrightarrow \delta = 3 = lrad^2(c)$. Then the contradiction.

E - Case $m \neq 1, 3$, we obtain: $3rad(a) = l^2rad^4(c) - m \Longrightarrow rad(a)$ and $rad(c)$ are not coprime. Then the contradiction.

I-3-2-1-2-4- We suppose $\delta = l.rad^n(c), l \geq 2$ with $n \geq 3$. $c = \mu_c.rad(c) = lrad^n(c)(\delta^2 - 3rad(a))$ and $m = \delta^2 - 3rad(a) = \delta^2 - 3X$.

F - As seen above (paragraphs C,D), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G - Case $m \neq 1, 3$. Let q be a prime that divides m (q can be equal to m), it follows $q|(\mu_c = l.m) \Longrightarrow q = c_{j'_0} \Longrightarrow c_{j'_0}|\delta^2 \Longrightarrow c_{j'_0}|3rad(a)$. Then $rad(a)$ and $rad(c)$ are not coprime. It follows the contradiction.

I-3-2-1-2-5- We suppose $\delta = \prod_{j\in J_1} c_j^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with:

$$\beta_{j_0} \geq 2, \quad rad(c) \nmid \delta \tag{237}$$

We can write:

$$\delta = \mu_\delta.rad(\delta), \quad rad(c) = r.rad(\delta), \quad r > 1, \quad (r, \mu_\delta) = 1 \tag{238}$$

Then, we obtain:

$$c = \mu_c.rad(c) = \mu_c.r.rad(\delta) = \delta(\delta^2 - 3X) = \mu_\delta.rad(\delta)(\delta^2 - 3X) \Longrightarrow$$
$$r.\mu_c = \mu_\delta(\delta^2 - 3X) \tag{239}$$

- We suppose $\mu_c = \mu_\delta \Longrightarrow r = \delta^2 - 3X = (\mu_c.rad(\delta))^2 - 3X$. As $\delta < \delta^2 - 3X \Longrightarrow r > \delta \Longrightarrow rad(c) > r > (\mu_c.rad(\delta) = A.rad^n(c)rad(\delta)) \Longrightarrow 1 > A.rad^{n-1}(\delta)$, then the contradiction.

- We suppose $\mu_c < \mu_\delta$. As $rad(a) = \delta - 1 = \mu_\delta rad(\delta) - 1$, we obtain:

$$rad(a) > \mu_c.rad(\delta) - 1 > 0 \Longrightarrow rad(ac) > c.rad(\delta) - rad(c) > 0$$

As $c = 1 + a$ and we consider the cases $c > rad(ac)$, then:

$$c > rad(ac) > c.rad(\delta) - rad(c) > 0 \Longrightarrow c > c.rad(\delta) - rad(c) > 0 \Longrightarrow$$
$$1 > rad(\delta) - \frac{rad(c)}{c} > 0, \quad rad(\delta) \geq 2 \Longrightarrow \text{The contradiction} \tag{240}$$

- We suppose $\mu_c > \mu_\delta$. In this case, from the equation (239) and as $(r, \mu_\delta) = 1$, it follows we can write:

$$\mu_c = \mu_1.\mu_2, \quad \mu_1, \mu_2 > 1,$$
$$c = \mu_c rad(c) = \mu_1.\mu_2.rad(\delta).r = \delta.(\delta^2 - 3X),$$
$$\text{We do a choice so that} \quad \mu_2 = \mu_\delta, \quad r.\mu_1 = \delta^2 - 3X \Longrightarrow \delta = \mu_2.rad(\delta).$$

** *1*- We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists\, c_{j_0}$ so that $c_{j_0}|\mu_1$ and $c_{j_0}|\mu_2$. But $\mu_\delta = \mu_2 \Rightarrow c_{j_0}^2|\delta$. From $3X = \delta^2 - r\mu_1 \Longrightarrow c_{j_0}|3X \Longrightarrow c_{j_0}|X$ or $c_{j_0} = 3$.

- If $c_{j_0}|(X = rad(a))$, it follows the contradiction with $(c, a) = 1$.

- If $c_{j_0} = 3$. We have $r\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \Longrightarrow \delta^2 - 3\delta + 3 - r.\mu_1 = 0$. As $3|\mu_1 \Longrightarrow \mu_1 = 3^k \mu_1', 3 \nmid \mu_1', k \geq 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - 3^{k-1} r\mu_1') = 0 \tag{241}$$

** *1-1*- We consider the case $k > 1 \Longrightarrow 3 \nmid (1 - 3^{k-1} r\mu_1')$. Let us recall the Eisenstein criterion [**6**]:

> ***Theorem 16.*** — ***(Eisenstein Criterion)*** *Let $f = a_0 + \cdots + a_n X^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists p$ a prime number so that $p \nmid a_n$, $p|a_i$, $(0 \leq i \leq n-1)$, and $p^2 \nmid a_0$, then f is irreducible in $\mathbb{Q}$.*

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

$$R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1} r\mu_1') \tag{242}$$

then:

- $3 \nmid 1$, - $3|(-3)$,- $3|3(1 - 3^{k-1} r\mu_1')$, and - $3^2 \nmid 3(1 - 3^{k-1} r\mu_1')$.

It follows that the polynomial $R(Z)$ is irreducible in $\mathbb{Q}$, then, the contradiction with $R(\delta) = 0$.

** *1-2*- We consider the case $k = 1$, then $\mu_1 = 3\mu_1'$ and $(\mu_1', 3) = 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - r\mu_1') = 0 \tag{243}$$

** *1-2-1*- We consider that $3 \nmid (1 - r.\mu_1')$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 - 3Z + 3(1 - r\mu_1')$$

and we find a contradiction with $R'(\delta) = 0$.

** *1-2-2*- We consider that:

$$3|(1 - r.\mu_1') \Longrightarrow r\mu_1' - 1 = 3^i.h,\ i \geq 1,\ 3 \nmid h, h \in \mathbb{N}^* \tag{244}$$

δ is an integer root of the polynomial $R'(Z)$:

$$R'(Z) = Z^2 - 3Z + 3(1 - r\mu_1') = 0 \tag{245}$$

The discriminant of $R'(Z)$ is:

$$\Delta = 3^2 + 3^{i+1} \times 4.h$$

As the root δ is an integer, it follows that $\Delta = t^2 > 0$ with t a positive integer. We obtain:

$$\Delta = 3^2(1 + 3^{i-1} \times 4h) = t^2 \tag{246}$$

$$\Longrightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^* \tag{247}$$

We can write the equation (243) as :

$$\delta(\delta - 3) = 3^{i+1}.h \Longrightarrow 3^3\mu'_1\frac{rad(\delta)}{3}.\,(\mu'_1 rad(\delta) - 1) = 3^{i+1}.h \Longrightarrow \tag{248}$$

$$\mu'_1\frac{rad(\delta)}{3}.\,(\mu'_1 rad(\delta) - 1) = h \tag{249}$$

We obtain $i = 2$ and $q^2 = 1 + 12h = 1 + 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1)$. Then, q satisfies :

$$q^2 - 1 = 12h = 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1) \Longrightarrow \tag{250}$$

$$\tfrac{(q-1)}{2}.\tfrac{(q+1)}{2} = 3h = (\mu'_1 rad(\delta) - 1).\mu'_1 rad(\delta) \Rightarrow \tag{251}$$

$$q - 1 = 2\mu'_1 rad(\delta) - 2 \tag{252}$$

$$q + 1 = 2\mu'_1 rad(\delta) \tag{253}$$

It follows that $(q = x, 1 = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{254}$$

with $N = 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1) = 12h > 0$. Let $Q(N)$ be the number of the solutions of (254) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the Diophantine equation (254) (see theorem 27.3 in [**7**]):

- If $N \equiv 2 (\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 (\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0 (\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.

$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

As $N = 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1) \Longrightarrow N \equiv 0 (\bmod 4) \Longrightarrow Q(N) = [\tau(N/4)/2]$. As $(q, 1)$ is a couple of solutions of the Diophantine equation (254), then $\exists\ d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2q \tag{255}$$

$$d - d' = 2.1 = 2 \tag{256}$$

** *1-2-2-1* As $N > 1$, we take $d = N$ and $d' = 1$. It follows:

$$\begin{cases} N + 1 = 2q \\ N - 1 = 2 \end{cases} \Longrightarrow N = 3 \Longrightarrow \text{then the contradiction with } N \equiv 0 (\bmod 4).$$

** *1-2-2-2* Now, we consider the case $d = 2\mu_1' rad(\delta)(\mu_1' rad(\delta) - 1)$ and $d' = 2$. It follows:

$$\left\{\begin{array}{l} 2\mu_1' rad(\delta)(\mu_1' rad(\delta) - 1) + 2 = 2q \\ 2\mu_1' rad(\delta)(\mu_1' rad(\delta) - 1) - 2 = 2 \end{array}\right. \Rightarrow 2\mu_1' rad(\delta)(\mu_1' rad(\delta) - 1) = q + 1$$

As $q + 1 = 2\mu_1' rad(\delta)$, we obtain $\mu_1' rad(\delta) = 2$, then the contradiction with $3|\delta$.

** *1-2-2-3* Now, we consider the case $d = \mu_1' rad(\delta)(\mu_1' rad(\delta) - 1)$ and $d' = 4$. It follows:

$$\left\{\begin{array}{l} \mu_1' rad(\delta)(\mu_1' rad(\delta) - 1) + 4 = 2q \\ \mu_1' rad(\delta)(\mu_1' rad(\delta) - 1) - 4 = 2 \Rightarrow \mu_1' rad(\delta)(\mu_1' rad(\delta) - 1) = 6 \end{array}\right.$$

As $\mu_1' rad(\delta) > 0 \Longrightarrow \mu_1' rad(\delta) = 3 \Longrightarrow \mu_1' = 1, \quad rad(\delta) = 3$ and $q = 5$. From $q^2 = 1 + 12h$, we obtain $h = 2$. Using the relation (244) $r\mu_1' - 1 = 3^i h$ as $\mu_1' = 1, i = 2, h = 2$, it gives $r - 1 = 9h = 18$. As δ is the positive root of the equation (243):

$$Z^2 - 3Z + 3(1 - r) = 0 \Longrightarrow \delta = 9 = 3^2$$

But $\delta = 1 + X = 1 + rad(a) \Longrightarrow rad(a) = 8 = 2^3$, then the contradiction.

** *1-2-2-4* Now, let c_{j_0} be a prime integer so that $c_{j_0}|rad\delta$, we consider the case $d = \mu_1' \dfrac{rad(\delta)}{c_{j_0}}(\mu_1' rad(\delta) - 1)$ and $d' = 4c_{j_0}$. It follows:

$$\left\{\begin{array}{l} \mu_1' \dfrac{rad(\delta)}{c_{j_0}}(\mu_1' rad(\delta) - 1) + 4c_{j_0} = 2q \\ \mu_1' \dfrac{rad(\delta)}{c_{j_0}}(\mu_1' rad(\delta) - 1) - 4c_{j_0} = 2 \end{array}\right. \Longrightarrow \mu_1' \frac{rad(\delta)}{c_{j_0}}(\mu_1' rad(\delta) - 1) = 2(1 + 2c_{j_0}) \Longrightarrow$$

Then the contradiction as the left member is greater than the right member $2(1 + 2c_{j_0})$.

** *1-2-2-5* Now, we consider the case $d = 4\mu_1' rad(\delta)$ and $d' = (\mu_1' rad(\delta) - 1)$. It follows:

$$\left\{\begin{array}{l} 4\mu_1' rad(\delta) + (\mu_1' rad(\delta) - 1) = 2q \\ 4\mu_1' rad(\delta) - (\mu_1' rad(\delta) - 1) = 2 \end{array}\right. \Longrightarrow 3\mu_1' rad(\delta) = 1 \Longrightarrow \text{Then the contradiction.}$$

** *1-2-2-6* Now, we consider the case $d = 2\mu_1' rad(\delta)$ and $d' = 2(\mu_1' rad(\delta) - 1)$. It follows:

$$\left\{\begin{array}{l} 2\mu_1' rad(\delta) + 2(\mu_1' rad(\delta) - 1) = 2q \Longrightarrow 2\mu_1' rad(\delta) - 1 = q \\ 2\mu_1' rad(\delta) - 2(\mu_1' rad(\delta) - 1) = 2 \Longrightarrow 2 = 2 \end{array}\right.$$

It follows that this case presents no contradictions a priori.

** *1-2-2-7* $\mu'_1 rad(\delta)$ and $\mu'_1 rad(\delta)-1$ are coprime, let $\mu'_1 rad(\delta)-1 = \prod_{j=1}^{j=J}\lambda_j^{\gamma_j}$, we consider the case $d = 2\lambda_{j'}\mu'_1 rad(\delta)$ and $d' = 2\dfrac{\mu'_1 rad(\delta)-1}{\lambda_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'}\mu'_1 rad(\delta) + 2\dfrac{\mu'_1 rad(\delta)-1}{\lambda_{j'}} = 2q \\ \\ 2\lambda_{j'}\mu'_1 rad(\delta) - 2\dfrac{\mu'_1 rad(\delta)-1}{\lambda_{j'}} = 2 \end{cases}$$

** *1-2-2-7-1* We suppose that $\gamma_{j'} = 1$. We consider the case $d = 2\lambda_{j'}\mu'_1 rad(\delta)$ and $d' = 2\dfrac{\mu'_1 rad(\delta)-1}{\lambda_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'}\mu'_1 rad(\delta) + 2\dfrac{\mu'_1 rad(\delta)-1}{\lambda_{j'}} = 2q \\ \\ 2\lambda_{j'}\mu'_1 rad(\delta) - 2\dfrac{\mu'_1 rad(\delta)-1}{\lambda_{j'}} = 2 \end{cases} \Longrightarrow 4\lambda_{j'}\mu'_1 rad(\delta) = 2(q+1) \Longrightarrow 2\lambda_{j'}\mu'_1 rad(\delta) = q+1$$

But from the equation (253), $q+1 = 2\mu'_1 rad(\delta)$, then $\lambda_{j'} = 1$, it follows the contradiction.

** *1-2-2-7-2* We suppose that $\gamma_{j'} \geq 2$. We consider the case $d = 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_1 rad(\delta)$ and $d' = 2\dfrac{\mu'_1 rad(\delta)-1}{\lambda_{j'}^{r'_{j'}}}$. It follows:

$$\begin{cases} 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_1 rad(\delta) + 2\dfrac{\mu'_1 rad(\delta)-1}{\lambda_{j'}^{r'_{j'}}} = 2q \\ \\ 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_1 rad(\delta) - 2\dfrac{\mu'_1 rad(\delta)-1}{\lambda_{j'}^{r'_{j'}}} = 2 \end{cases} \Longrightarrow 4\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_1 rad(\delta) = 2(q+1)$$

$$\Longrightarrow 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_1 rad(\delta) = q+1$$

As above, it follows the contradiction. It is trivial that the other cases for more factors $\prod_{j"}\lambda_{j"}^{\gamma_{j"}-r"_{j"}}$ give also contradictions.

** *1-2-2-8* Now, we consider the case $d = 4(\mu'_1 rad(\delta)-1)$ and $d' = \mu'_1 rad(\delta)$, we have $d > d'$. It follows:

$$\begin{cases} 4(\mu'_1 rad(\delta)-1) + \mu'_1 rad(\delta) = 2q \Rightarrow 5\mu'_1 rad(\delta) = 2(q+2) \\ 4(\mu'_1 rad(\delta)-1) - \mu'_1 rad(\delta) = 2 \Rightarrow \mu'_1 rad(\delta) = 2 \end{cases} \Rightarrow \begin{cases} \text{Then the contradiction as} \\ 3|\delta. \end{cases}$$

** *1-2-2-9* Now, we consider the case $d = 4u(\mu'_1 rad(\delta) - 1)$ and $d' = \dfrac{\mu'_1 rad(\delta)}{u}$, where $u > 1$ is an integer divisor of $\mu'_1 rad(\delta)$. We have $d > d'$ and:

$$\begin{cases} 4u(\mu'_1 rad(\delta) - 1) + \dfrac{\mu'_1 rad(\delta)}{u} = 2q \\ \\ 4u(\mu'_1 rad(\delta) - 1) - \dfrac{\mu'_1 rad(\delta)}{u} = 2 \end{cases} \Longrightarrow 2u(\mu'_1 rad(\delta) - 1) = \mu'_1 rad(\delta)$$

Then the contradiction as $\mu'_1 rad(\delta)$ and $(\mu'_1 rad(\delta) - 1)$ are coprime.

In conclusion, we have found only one case (** *1-2-2-6* above) where there is no contradictions *a priori*. As $\tau(N)$ is large and also $[\tau(N/4)/2]$, it follows the contradiction with $Q(N) \leq 1$ and the hypothesis $(\mu_1, \mu_2) \neq 1$ is false.

** *2-* We suppose that $(\mu_1, \mu_2) = 1$.

From the equation $r\mu_1 = \delta^2 - 3X$ and the condition $rad(a) = X > rad^{1.63/1.37}(c) \iff \delta - 1 = X > rad^{1.19}(c)$, we obtain the following inequality:

$$\delta - 1 > (r.rad(\delta))^{1.19} \Longrightarrow -3(\delta - 1) < -3r.rad(\delta).(r.rad(\delta))^{0.19} \Longrightarrow$$
$$r\mu_1 = \delta^2 - 3(\delta - 1) < (r.rad(\delta))^2 - 3r.rad(\delta).(r.rad(\delta))^{0.19} \Longrightarrow$$
$$\mu_1 < r.rad^2(\delta) - 3.rad(\delta).(r.rad(\delta))^{0.19} \Longrightarrow$$
$$\mu_1 < r.rad^2(\delta)\left(1 - \frac{3}{(r.rad(\delta))^{0.81}}\right) \tag{257}$$

As $a = rad^3(a) < c$, we can write:

$$rad^3(a) < \mu_1\mu_2 rad(c) < \mu_2.rad(\delta).rad^2(c)\left(1 - \frac{3}{(r.rad(\delta))^{0.81}}\right)$$

but $(r, rad(\delta)) = 1$, $r.rad(\delta) \geq 6 \Longrightarrow (r.rad(\delta))^{0.81} \geq (6^{0.81} \approx 4.26)$ and $\delta = \mu_2.rad(\delta)$, it follows:

$$rad^3(a) < \mu_1\mu_2 rad(c) < \mu_2.rad(\delta).rad^2(c) \Longrightarrow rad^3(a) < \delta.rad^2(c) < 1.6rad(a).rad^2(c)$$

As $rad(a) > (rad^{1.62/1.37}(c) = rad^{1.19}(c)) \Longrightarrow rad^{1.19}(c) < rad(a) < 1.27rad(c)$, then we obtain:

$$rad^{1.19}(c) < 1.27rad(c) \Longrightarrow rad(c) < 3.5 \Longrightarrow rad(c) \leq 3, \text{ but } rad(c) = r.rad(\delta) \geq 6$$

Then the contradiction.

It follows that the case $\mu_c > rad^{2.26}(c) \Rightarrow c > rad^{3.26}(c)$ and $a = rad^3(a)$ is impossible.

I-3-2-2- We consider the case $\mu_a = rad^2(a) \Longrightarrow a = rad^3(a)$ and $c = a + b$. Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = \bar{c} \tag{258}$$

with $\bar{c} = c - b + 1 = a + 1 \Longrightarrow (\bar{c}, a) = 1$. We obtain the same result as of the case **I-3-2-1-** studied above considering $rad(a) > rad^{\frac{1.63}{1.37}}(\bar{c})$.

I-3-2-3- We suppose $\mu_c > rad^{2.26}(c) \Rightarrow c > rad^{3.26}(c)$ and c large and $\mu_a < rad^2(a)$, we consider $c = a + b, b \geq 1$. Then $c = rad^3(c) + h, h > rad^3(c)$, h a positive integer and we can write $a + l = rad^3(a)$, $l > 0$. Then we obtain :

$$rad^3(c) + h = rad^3(a) - l + b \Longrightarrow rad^3(a) - rad^3(c) = h + l - b > 0 \tag{259}$$

as $rad(a) > rad^{\frac{1.63}{1.37}}(c)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h + l - b = m > 0 \tag{260}$$

Let $X = rad(a) - rad(c)$, then X is an integer root of the polynomial $H(X)$ defined as:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \tag{261}$$

To resolve the above equation, we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by:

$$G(t) = t^2 - mt - rad^3(ac) = 0 \tag{262}$$

The discriminant of $G(t)$ is $\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0$. As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the expression of the discriminant Δ, it follows that the couple $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{263}$$

with $N = 4rad^3(ac) = 4rad^3(a).rad^3(c) > 0$. Here, we will use the same method that is given in the above sub-paragraph ** *1-2-2-* of the paragraph **I-3-2-1-2-5-**. We have the two terms $rad^3(a)$ and $rad^3(c)$ coprime. As (α, m) is a couple of solutions of the Diophantine equation (263) and $\alpha > m$, then $\exists\ d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2\alpha \tag{264}$$

$$d - d' = 2m \tag{265}$$

I-3-2-3-1- Let us consider the case $d = 2rad^3(a)$, $d' = 2rad^3(c)$. It follows:

$$\begin{cases} 2rad^3(a) + 2rad^3(c) = 2\alpha \Longrightarrow \alpha = rad^3(a) + rad^3(c) \\ 2rad^3(a) - 2rad^3(c) = 2m \Longrightarrow m = rad^3(a) - rad^3(c) \end{cases}$$

It follows that this case presents *a priori* no contradictions.

I-3-2-3-2- Now, we consider for example, the case $d = 4rad^3(a)$ and $d' = rad^3(c) \Longrightarrow d > d'$. We rewrite the equations (264-265):

$$4rad^3(a) + rad^3(c) = 2(rad^3(a) + rad^3(c)) \Rightarrow 2rad^3(a) = rad^3(c))$$
$$4rad^3(a) - rad^3(c) = 2(rad^3(a) - rad^3(c)) \Longrightarrow 2rad^3(a) = -rad^3(c))$$

Then the contradiction.

I-3-2-3-3- We consider the case $d = 4rad^3(c)rad^3(a)$ and $d' = 1 \Longrightarrow d > d'$. We rewrite the equations (264-265):

$$4rad^3(c)rad^3(a) + 1 = 2(rad^3(c) + rad^3(a)) \Longrightarrow$$
$$2(2rad^3(c)rad^3(a) - rad^3(c) - rad^3(a)) = -1 \Rightarrow \text{ a contradiction}$$
$$4rad^3(c)rad^3(a) - 1 = 2(rad^3(c) - rad^3(a))$$

Then the contradiction.

I-3-2-3-4- Let c_1 be the first factor of $rad(c)$. We consider the case $d = 4c_1rad^3(a)$ and $d' = \dfrac{rad^3(c)}{c_1} \Longrightarrow d > d'$. We rewrite the equation (264):

$$4c_1rad^3(a) + \frac{rad^3(c)}{c_1} = 2(rad^3(a) + rad^3(c)) \Rightarrow$$
$$2rad^3(a)(2c_1 - 1) = \frac{rad^3(c)}{c_1}(2c_1 - 1) \Rightarrow 2rad^3(a) = rad^2(c).\frac{rad(c)}{c_1}$$

$c_1 = 2$ or not, there is a contradiction with a, c coprime.

The other cases of the expressions of d and d' not coprime so that $N = d.d'$ give also contradictions.

Let $Q(N)$ be the number of the solutions of (263), as $N \equiv 0(\text{mod} 4)$, then $Q(N) = [\tau(N/4)/2]$. From the study of the cases above, we obtain that $Q(N) \leq 1$ is $\ll [(\tau(N)/4)/2]$. It follows the contradiction.

Then the cases $\mu_a \leq rad^2(a)$ and $c > rad^{3.26}(c)$ are impossible.

II- We suppose that $rad^{1.63}(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^{1.63}(a)$:

II-1- Case $rad(c) < rad(a)$: As $c \leq rad^3(c) = rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(ac) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

II-2- Case $rad(a) < rad(c) < rad^{\frac{1.63}{1.37}}(a)$:

As $c \leq rad^3(c) \leq rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow c < rad^{1.63}(c).rad^{1.63}(a) < rad^{1.63}(abc) \Longrightarrow$ $\boxed{c < R^{1.63}}$.

II-3- Case $rad^{\frac{1.63}{1.37}}(a) < rad(c)$:

II-3-1- We suppose $rad^{1.63}(a) < \mu_a \leq rad^{2.26}(a) \Longrightarrow a \leq rad^{1.63}(a).rad^{1.63}(a)$ $\Longrightarrow a < rad^{1.63}(a).rad^{1.37}(c) \Longrightarrow c = a + b < 2a < 2rad^{1.63}(a).rad^{1.63}(c) <$ $rad^{1.63}(abc) \Longrightarrow c < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

II-3-2- We suppose $\mu_a > rad^{2.26}(a) \Longrightarrow a > rad^{3.26}(a)$ and $\mu_c \leq rad^2(c)$. Using the same method as it was explicated in the paragraphs **I-3-2-** (permuting a, c see in Appendix **II'-3-2-**), we arrive at a contradiction. It follows that the cases $\mu_c \leq rad^2(c)$ and $\mu_a > rad^{2.26}(a)$ are impossible.

3.2.2.3. Case $\mu_a > rad^{1.63}(a)$ and $\mu_c > rad^{1.63}(c)$:

Taking into account the cases studied above, it remains to see the following two cases:

- $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$,
- $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$.

III- We suppose $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a) \Longrightarrow c > rad^3(c)$ and $a > rad^{2.63}(a)$. We can write $c = rad^3(c) + h$ and $a = rad^3(a) + l$ with h a positive integer and $l \in \mathbb{Z}$.

III-1- We suppose $rad(c) < rad(a)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h - l - b = m > 0 \tag{266}$$

Let $X = rad(a) - rad(c)$, from the above equation, X is a real root of the polynomial:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \tag{267}$$

As above, to resolve (267), we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by :

$$G(t) = t^2 - mt - rad^3(ac) = 0 \tag{268}$$

The discriminant of $G(t)$ is:

$$\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0 \tag{269}$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the equation (269), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{270}$$

with $N = 4rad^3(ac) > 0$. Let $Q(N)$ be the number of the solutions of (270) and $\tau(N)$ is the number of suitable factorization of N, and using the same method as in the paragraph **I-3-2-3-** above, we obtain a contradiction.

III-2- We suppose $rad(a) < rad(c)$. We obtain the equation:

$$rad^3(c) - rad^3(a) = b + l - h = m > 0 \tag{271}$$

Let X be the variable $X = rad(c) - rad(a)$, we use the similar calculations as in the paragraph above **I-3-2-3-** permuting c, a, we find a contradiction.

It follows that the case $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$ is impossible.

IV - We suppose $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$, we obtain $a > rad^3(a)$ and $c > rad^{2.63}(c)$. We can write $a = rad^3(a) + h$ and $c = rad^3(c) + l$ with h a positive integer and $l \in \mathbb{Z}$.

The calculations are similar to those in the cases of the paragraph **III**. We obtain a contradiction.

It follows that the case $\mu_c > rad^{1.63}(c)$ and $\mu_a > rad^2(a)$ is impossible.

All possible cases are discussed. □

We can state the following important theorem:

> ***Theorem 17.*** — *Let a, b, c positive integers relatively prime with $c = a + b$, then $c < rad^{1.63}(abc)$.*

From the theorem above, we can announce also:

> ***Corollary 18.*** — *Let a, b, c positive integers relatively prime with $c = a + b$, then the conjecture $c < rad^2(abc)$ is true.*

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Appendix

II'-3-2- We suppose $\mu_a > rad^{2.26}(a) \Longrightarrow a > rad^{3.26}(a)$.

II'-3-2-1- We consider the case $\mu_c = rad^2(c) \Longrightarrow c = rad^3(c)$ and $c = a+1$. Then, we obtain that $Y = rad(c)$ is a solution in positive integers of the equation:

$$Y^3 - 1 = a \tag{272}$$

II'-3-2-1-1- We suppose that $a = rad^n(a)$ with $n \geq 4$, we obtain the equation:

$$rad^3(c) - rad^n(a) = 1 \tag{273}$$

But the solutions of the Catalan equation [**5**] $x^p - y^q = 1$ where the unknowns x, y, p and q take integer values, all ≥ 2, has only one solution $(x, y, p, q) = (3, 2, 2, 3)$, but the solution of the equation (273) are $(rad(c) = 3, rad(a) = 2, 3 \neq 2, n \geq 4)$, it follows the contradiction with $n \geq 4$ and the case $a = rad^n(a), n \geq 4$ is to reject.

II'-3-2-1-2- In the following, we will study the cases $\mu_a = A.rad^n(a)$ with $rad(a) \nmid A, n \geq 0$. The above equation (272) can be written as :

$$(Y-1)(Y^2+Y+1) = a \tag{274}$$

Let δ one divisor of a so that :

$$Y - 1 = \delta \tag{275}$$

$$Y^2 + Y + 1 = \frac{a}{\delta} = m = \delta^2 + 3Y \tag{276}$$

We recall that $rad(c) > rad^{\frac{1.63}{1.37}}(a)$.

II'-3-2-1-2-1- We suppose $\delta = l.rad(a)$. We have $\delta = l.rad(a) < a = \mu_a.rad(a) \Longrightarrow l < \mu_a$. As δ is a divisor of a, then l is a divisor of μ_a, $\frac{a}{\delta} = \frac{\mu_a rad(a)}{l.rad(a)} = \frac{\mu_a}{l} = m = \delta^2 + 3Y$, then $\mu_a = l.m$. From $\mu_a = l(\delta^2 + 3Y)$, we obtain:

$$m = l^2 rad^2(a) + 3rad(c) \Longrightarrow 3rad(c) = m - l^2 rad^2(a)$$

A'- Case $3|m \Longrightarrow m = 3m', m' > 1$: As $\mu_a = ml = 3m'l \Longrightarrow 3|rad(a)$ and $(rad(a), m')$ not coprime. We obtain:

$$rad(c) = m' - l^2 rad(a).\frac{rad(a)}{3}$$

It follows that a, c are not coprime, then the contradiction.

B' - Case $m = 3 \Longrightarrow \mu_a = 3l \Longrightarrow a = 3lrad(a) = 3\delta = \delta(\delta^2 + 3Y) \Longrightarrow \delta^2 = 3(1 - Y) = -3\delta < 0$, then the contradiction.

II'-3-2-1-2-2- We suppose $\delta = l.rad^2(a), l \geq 2$. If $n = 0$ then $\mu_a = A$ and from the equation above (276):

$$m = \frac{a}{\delta} = \frac{\mu_a.rad(a)}{lrad^2(a)} = \frac{A.rad(a)}{lrad^2(a)} = \frac{A}{lrad(a)} \Rightarrow rad(a)|A$$

It follows the contradiction with the hypothesis above $rad(a) \nmid A$.

II'-3-2-1-2-3- We suppose $\delta = lrad^2(a), l \geq 2$ and in the following $n > 0$. As $m = \dfrac{a}{\delta} = \dfrac{\mu_a.rad(a)}{lrad^2(a)} = \dfrac{\mu_a}{lrad(a)}$, if $lrad(a) \nmid \mu_a$ then the case is to reject. We suppose $lrad(a)|\mu_a \Longrightarrow \mu_a = m.lrad(a)$, with $m, rad(a)$ not coprime, then $\dfrac{a}{\delta} = m = \delta^2 + 3rad(c)$.

C' - Case $m = 1 = a/\delta \Longrightarrow \delta^2 + 3rad(c) = 1$, then the contradiction.

D' - Case $m = 3$, we obtain $3(1 - rad(c)) = \delta^2 \Longrightarrow \delta^2 < 0$. Then the contradiction.

E' - Case $m \neq 1, 3$, we obtain: $3rad(c) = m - l^2rad^4(a) \Longrightarrow rad(a)$ and $rad(c)$ are not coprime. Then the contradiction.

II'-3-2-1-2-4- We suppose $\delta = l.rad^n(a), l \geq 2$ with $n \geq 3$. From $a = \mu_a.rad(a) = lrad^n(a)(\delta^2 + 3rad(c))$, we denote $m = \delta^2 + 3rad(c) = \delta^2 + 3Y$.

F' - As seen above (paragraphs C',D'), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G' - Case $m \neq 1, 3$. Let q be a prime that divides m (q can be equal to m), it follows $q|\mu_a \Longrightarrow q = a_{j'_0} \Longrightarrow a_{j'_0}|\delta^2 \Longrightarrow a_{j'_0}|3rad(c)$. Then $rad(a)$ and $rad(c)$ are not coprime. It follows the contradiction.

II'-3-2-1-2-5- We suppose $\delta = \prod_{j \in J_1} a_j^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with:

$$\beta_{j_0} \geq 2, \quad rad(a) \nmid \delta \tag{277}$$

We can write:

$$\delta = \mu_\delta.rad(\delta), \quad rad(a) = r.rad(\delta), \quad r > 1, \quad (r, rad(\delta)) = 1 \Rightarrow (r, \mu_\delta) = 1 \tag{278}$$

Then, we obtain:

$$a = \mu_a.rad(a) = \mu_a.r.rad(\delta) = \delta(\delta^2 + 3Y) = \mu_\delta.rad(\delta)(\delta^2 + 3Y) \Longrightarrow$$
$$r.\mu_a = \mu_\delta(\delta^2 + 3Y) \tag{279}$$

- We suppose $\mu_a = \mu_\delta \Longrightarrow r = \delta^2 + 3Y = (\mu_a.rad(\delta))^2 + 3Y$. As $\delta < \delta^2 + 3Y \Longrightarrow r > \delta \Longrightarrow rad(a) > r > (\mu_a.rad(\delta) = A.rad^n(a)rad(\delta)) \Longrightarrow 1 > A.rad^{n-1}(\delta)$, then the contradiction.

- We suppose $\mu_a < \mu_\delta$. As $rad(c) = \mu_\delta rad(\delta) + 1$, we obtain:

$$rad(c) > \mu_a.rad(\delta) + 1 > 0 \Longrightarrow rad(ac) > a.rad(\delta) + rad(a) > 0$$

As $c = 1 + a$ and we consider the cases $c > rad(ac)$, then:

$$c > rad(ac) > a.rad(\delta) + rad(a) > 0 \Longrightarrow a + 1 \geq a.rad(\delta) + rad(a) > 0 \Longrightarrow$$
$$a \geq a.rad(\delta) + rad(\delta) \Longrightarrow 1 \geq rad(\delta) + \frac{rad(a)}{a} > 0, \quad rad(\delta) \geq 2 \Longrightarrow \text{The contradiction}$$

- We suppose $\mu_a > \mu_\delta$. In this case, from the equation (239) and as $(r, \mu_\delta) = 1$, it follows we can write:

$$\mu_a = \mu_1.\mu_2, \quad \mu_1, \mu_2 > 1 \tag{280}$$
$$a = \mu_a rad(a) = \mu_1.\mu_2.r.rad(\delta) = \delta.(\delta^2 + 3Y) \tag{281}$$
$$\text{so that} \quad r.\mu_1 = \delta^2 + 3Y, \quad \mu_2 = \mu_\delta \Longrightarrow \delta = \mu_2.rad(\delta) \tag{282}$$

** *1-* We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists\, a_{j_0}$ so that $a_{j_0}|\mu_1$ and $a_{j_0}|\mu_2$. But $\mu_\delta = \mu_2 \Rightarrow a_{j_0}^2|\delta$. From $3Y = r\mu_1 - \delta^2 \Longrightarrow a_{j_0}|3Y \Longrightarrow a_{j_0}|Y$ or $a_{j_0} = 3$.

- If $a_{j_0}|(Y = rad(c))$, it follows the contradiction with $(c, a) = 1$.
- If $a_{j_0} = 3$. We have $r\mu_1 = \delta^2 + 3Y = \delta^2 + 3(\delta + 1) \Longrightarrow \delta^2 + 3\delta + 3 - r.\mu_1 = 0$.

As $3|\mu_1 \Longrightarrow \mu_1 = 3^k\mu_1', 3 \nmid \mu_1', k \geq 1$, we obtain:

$$\delta^2 + 3\delta + 3(1 - 3^{k-1}r\mu_1') = 0 \tag{283}$$

** *1-1-* We consider the case $k > 1 \Longrightarrow 3 \nmid (1 - 3^{k-1}r\mu_1')$. Let us recall the Eisenstein criterion [**6**]:

> ***Theorem 19.*** — ***(Eisenstein Criterion)*** *Let $f = a_0 + \cdots + a_nX^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists p$ a prime number so that $p \nmid a_n$, $p|a_i$, $(0 \leq i \leq n-1)$, and $p^2 \nmid a_0$, then f is irreducible in $\mathbb{Q}$.*

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

$$R(Z) = Z^2 + 3Z + 3(1 - 3^{k-1}r\mu_1') \tag{284}$$

then:
- $3 \nmid 1$, - $3|(+3)$,- $3|3(1 - 3^{k-1}r\mu_1')$, and - $3^2 \nmid 3(1 - 3^{k-1}r\mu_1')$.

It follows that the polynomial $R(Z)$ is irreducible in $\mathbb{Q}$, then, the contradiction with $R(\delta) = 0$.

** *1-2-* We consider the case $k = 1$, then $\mu_1 = 3\mu_1'$ and $(\mu_1', 3) = 1$, we obtain:

$$\delta^2 + 3\delta + 3(1 - r\mu_1') = 0 \tag{285}$$

** *1-2-1-* We consider that $3 \nmid (1 - r.\mu_1')$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 + 3Z + 3(1 - r\mu_1')$$

and we find a contradiction with $R'(\delta) = 0$.

** *1-2-2-* We consider that:

$$3|(1-r.\mu_1') \Longrightarrow r\mu_1' - 1 = 3^i.h,\ i \geq 1,\ 3 \nmid h, h \in \mathbb{N}^* \tag{286}$$

δ is an integer root of the polynomial $R'(Z)$:

$$R'(Z) = Z^2 + 3Z + 3(1 - r\mu_1') = 0 \tag{287}$$

The discriminant of $R'(Z)$ is:

$$\Delta = 3^2 + 3^{i+1} \times 4.h$$

As the root δ is an integer, it follows that $\Delta = t^2 > 0$ with t a positive integer. We obtain:

$$\Delta = 3^2(1 + 3^{i-1} \times 4h) = t^2 \tag{288}$$

$$\Longrightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^* \tag{289}$$

As $\mu_\delta = \mu_2$ and $3|\mu_2 \Longrightarrow \mu_2 = 3\mu_2'$, then we can write the equation (285) as :

$$\delta(\delta+3) = 3^{i+1}.h \Longrightarrow 3^3\mu_2' \frac{rad(\delta)}{3}.\left(\mu_2' rad(\delta) + 1\right) = 3^{i+1}.h \Longrightarrow \tag{290}$$

$$\mu_2' \frac{rad(\delta)}{3}.\left(\mu_2' rad(\delta) + 1\right) = h \tag{291}$$

We obtain $i = 2$ and $q^2 = 1 + 12h = 1 + 4\mu_2' rad(\delta)(\mu_2' rad(\delta) + 1)$. Then, q satisfies :

$$q^2 - 1 = 12h = 4\mu_2' rad(\delta)(\mu_2' rad(\delta) + 1) \Longrightarrow \tag{292}$$

$$\tfrac{(q-1)}{2}.\tfrac{(q+1)}{2} = 3h = \mu_2' rad(\delta)(\mu_2' rad(\delta) + 1). \Rightarrow \tag{293}$$

$$q + 1 = 2\mu_2' rad(\delta) + 2 \tag{294}$$

$$q - 1 = 2\mu_2' rad(\delta) \tag{295}$$

It follows that $(q = x, 1 = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{296}$$

with $N = 4\mu_2' rad(\delta)(\mu_2' rad(\delta) + 1) = 12h > 0$. Let $Q(N)$ be the number of the solutions of (296) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the Diophantine equation (296) (see theorem 27.3 in [**7**]):

- If $N \equiv 2 (\mathrm{mod}\, 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 (\mathrm{mod}\, 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0 (\mathrm{mod}\, 4)$, then $Q(N) = [\tau(N/4)/2]$.

$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

As $N = 4\mu_2' rad(\delta)(\mu_2' rad(\delta) + 1) \Longrightarrow N \equiv 0 (\mathrm{mod}\, 4) \Longrightarrow Q(N) = [\tau(N/4)/2]$. As $(q, 1)$ is a couple of solutions of the Diophantine equation (296), then $\exists\ d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2q \tag{297}$$

$$d - d' = 2.1 = 2 \tag{298}$$

** *1-2-2-1* As $N>1$, we take $d=N$ and $d'=1$. It follows:

$$\begin{cases} N+1=2q \\ N-1=2 \end{cases} \Longrightarrow N=3 \Longrightarrow \text{then the contradiction with } N\equiv 0 (\operatorname{mod} 4).$$

** *1-2-2-2* Now, we consider the case $d=2\mu_2' rad(\delta)(\mu_2' rad(\delta)+1)$ and $d'=2$. It follows:

$$\begin{cases} 2\mu_2' rad(\delta)(\mu_2' rad(\delta)+1)+2=2q \\ 2\mu_2' rad(\delta)(\mu_2' rad(\delta)+1)-2=2 \end{cases} \Rightarrow \mu_2' rad(\delta)(\mu_2' rad(\delta)+1)=q-1$$

As $q-1=2\mu_2' rad(\delta)$, we obtain $\mu_2' rad(\delta)=1$, then the contradiction.

** *1-2-2-3* Now, we consider the case $d=\mu_2' rad(\delta)(\mu_2' rad(\delta)+1)$ and $d'=4$. It follows:

$$\begin{cases} \mu_2' rad(\delta)(\mu_2' rad(\delta)+1)+4=2q \\ \mu_2' rad(\delta)(\mu_2' rad(\delta)+1)-4=2 \Rightarrow \mu_2' rad(\delta)(\mu_2' rad(\delta)+1)=6 \end{cases}$$

As $\mu_2' rad(\delta)\geq 2 \Longrightarrow \mu_2' rad(\delta)=2 \Longrightarrow \mu_2'=1 \Rightarrow \mu_2=3=\mu_\delta$ and $rad(\delta)=2$ but $3\nmid 2$, then the contradiction.

** *1-2-2-4* Now, let a_{j_0} be a prime integer so that $a_{j_0}|rad\delta$, we consider the case $d=\mu_2'\dfrac{rad(\delta)}{a_{j_0}}(\mu_2' rad(\delta)+1)$ and $d'=4a_{j_0}$. It follows:

$$\begin{cases} \mu_2'\dfrac{rad(\delta)}{a_{j_0}}(\mu_2' rad(\delta)+1)+4a_{j_0}=2q \\ \mu_2'\dfrac{rad(\delta)}{a_{j_0}}(\mu_2' rad(\delta)+1)-4a_{j_0}=2 \end{cases} \Longrightarrow \mu_2'\frac{rad(\delta)}{a_{j_0}}(\mu_2' rad(\delta)+1)=2(1+2a_{j_0}) \Longrightarrow$$

Then the contradiction as the left member is greater than the right member $2(1+2a_{j_0})$.

** *1-2-2-5* Now, we consider the case $d=4\mu_2' rad(\delta)$ and $d'=(\mu_2' rad(\delta)+1)$. It follows:

$$\begin{cases} 4\mu_2' rad(\delta)+(\mu_2' rad(\delta)+1)=2q \\ 4\mu_2' rad(\delta)-(\mu_2' rad(\delta)+1)=2 \end{cases} \Longrightarrow 3\mu_2' rad(\delta)=3 \Longrightarrow \text{Then the contradiction.}$$

** *1-2-2-6* Now, we consider the case $d=2(\mu_2' rad(\delta)+1)$ and $d=2\mu_2' rad(\delta)$. It follows:

$$\begin{cases} 2(\mu_2' rad(\delta)+1)+2\mu_2' rad(\delta)=2q \Longrightarrow 2\mu_2' rad(\delta)+1=q \\ 2(\mu_2' rad(\delta)+1)-2\mu_2' rad(\delta)=2 \Longrightarrow 2=2 \end{cases}$$

It follows that this case presents no contradictions *a prior*.

** *1-2-2-7* $\mu'_2 rad(\delta)$ and $\mu'_2 rad(\delta)+1$ are coprime, let $\mu'_2 rad(\delta)+1 = \prod_{j=1}^{j=J} \lambda_j^{\gamma_j}$, we consider the case $d = 2\lambda_{j'}\mu'_2 rad(\delta)$ and $d' = 2\dfrac{\mu'_2 rad(\delta)+1}{\lambda_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'}\mu'_2 rad(\delta) + 2\dfrac{\mu'_2 rad(\delta)+1}{\lambda_{j'}} = 2q \\ \\ 2\lambda_{j'}\mu'_2 rad(\delta) - 2\dfrac{\mu'_2 rad(\delta)+1}{\lambda_{j'}} = 2 \end{cases}$$

** *1-2-2-7-1* We suppose that $\gamma_{j'} = 1$. We consider the case $d = 2\lambda_{j'}\mu'_2 rad(\delta)$ and $d' = 2\dfrac{\mu'_2 rad(\delta)+1}{\lambda_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'}\mu'_1 rad(\delta) + 2\dfrac{\mu'_1 rad(\delta)-1}{\lambda_{j'}} = 2q \\ \\ 2\lambda_{j'}\mu'_1 rad(\delta) - 2\dfrac{\mu'_1 rad(\delta)-1}{\lambda_{j'}} = 2 \end{cases} \Longrightarrow 4\lambda_{j'}\mu'_1 rad(\delta) = 2(q+1) \Longrightarrow 2\lambda_{j'}\mu'_1 rad(\delta) = q+1$$

But from the equation (253), $q+1 = 2\mu'_1 rad(\delta)$, then $\lambda_{j'} = 1$, it follows the contradiction.

** *1-2-2-7-2* We suppose that $\gamma_{j'} \geq 2$. We consider the case $d = 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_2 rad(\delta)$ and $d' = 2\dfrac{\mu'_2 rad(\delta)+1}{\lambda_{j'}^{r'_{j'}}}$. It follows:

$$\begin{cases} 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_2 rad(\delta) + 2\dfrac{\mu'_2 rad(\delta)+1}{\lambda_{j'}^{r'_{j'}}} = 2q \\ \\ 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_2 rad(\delta) - 2\dfrac{\mu'_2 rad(\delta)+1}{\lambda_{j'}^{r'_{j'}}} = 2 \end{cases} \Longrightarrow 4\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_2 rad(\delta) = 2(q+1)$$

$$\Longrightarrow 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_2 rad(\delta) = q+1$$

As above, it follows the contradiction. It is trivial that the other cases for more factors $\prod_{j"} \lambda_{j"}^{\gamma_{j"}-r"_{j"}}$ give also contradictions.

** *1-2-2-8* Now, we consider the case $d = 4(\mu'_2 rad(\delta)+1)$ and $d' = \mu'_2 rad(\delta)$, we have $d > d'$. It follows:

$$\begin{cases} 4(\mu'_2 rad(\delta)+1) + \mu'_2 rad(\delta) = 2q \Rightarrow 5\mu'_2 rad(\delta) = 2(q+2) \\ 4(\mu'_2 rad(\delta)+1) - \mu'_2 rad(\delta) = 2 \Rightarrow \mu'_2 rad(\delta) = 2 \end{cases} \Rightarrow \begin{cases} \text{Then the contradiction as} \\ 3|\delta. \end{cases}$$

** *1-2-2-9* Now, we consider the case $d = 4u(\mu'_2 rad(\delta)+1)$ and $d' = \dfrac{\mu'_2 rad(\delta)}{u}$, where $u > 1$ is an integer divisor of $\mu'_2 rad(\delta)$. We have $d > d'$ and:

$$\begin{cases} 4u(\mu'_2 rad(\delta)+1) + \dfrac{\mu'_2 rad(\delta)}{u} = 2q \\ \\ 4u(\mu'_2 rad(\delta)+1) - \dfrac{\mu'_2 rad(\delta)}{u} = 2 \end{cases} \Longrightarrow 2u(\mu'_2 rad(\delta)+1) = \mu'_2 rad(\delta)+1 \Rightarrow 2u = 1$$

Then the contradiction.

In conclusion, we have found only one case (** *1-2-2-6* above) where there is no contradictions *a prior*. As $\tau(N)$ is large and also $[\tau(N/4)/2]$, it follows the contradiction with $Q(N) \leq 1$ and the hypothesis $(\mu_1, \mu_2) \neq 1$ is false.

** *2-* We suppose that $(\mu_1, \mu_2) = 1$.

We recall that $rad(c) = Y > rad^{1.63/1.37}(a)$, $\delta+1 = Y$, $rad(a) = r.rad(\delta)$, $(r, rad(\delta)) = 1$, $\delta = \mu_2 rad(\delta)$ and $r\mu_1 = \delta^2 + 3X$, it follows:

(299) $$U(\delta) = \delta^2 + 3\delta + 3 - r\mu_1 = 0$$

** *2-1-* We suppose $3|(3 - r\mu_1)$ and $3^2 \nmid (3 - r\mu_1)$, then we use the Eisenstein criterion [**6**] to the polynomial $U(\delta)$ given by the equation (299), and the contradiction.

** *2-2-* We suppose $3|(3 - r\mu_1)$ and $3^2|(3 - r\mu_1)$. From $3|(3 - r\mu_1) \Longrightarrow 3|r\mu_1 \Longrightarrow 3|r \text{ or } 3|\mu_1$.

- If $3|r \Longrightarrow (3, rad\delta) = 1 \Longrightarrow 3 \nmid \delta$. Then the contradiction with $3|\delta^2$ by the equation (299).
- If $3|\mu_1 \Longrightarrow 3 \nmid \mu_2 \Longrightarrow 3 \nmid \delta$, it follows the contradiction with $3|\delta^2$ by the equation (299).

** *2-3-* We suppose $3 \nmid (3 - r\mu_1) \Longrightarrow 3 \nmid r\mu_1 \Longrightarrow 3 \nmid r$ and $3 \nmid \mu_1$. From the equation (299), $U(\delta) = 0 \Longrightarrow r\mu_1 \equiv \delta^2 (\text{mod} 3)$, as δ^2 is a square then $\delta^2 \equiv 1 (\text{mod} 3) \Longrightarrow r\mu_1 \equiv 1 (\text{mod} 3)$, but this result is not all verified. Then the contradiction.

It follows that the case $\mu_a > rad^{2.26}(a) \Rightarrow a > rad^{3.26}(a)$ and $c = rad^3(c)$ is impossible.

II'-3-2-2- We consider the case $\mu_c = rad^2(c) \Longrightarrow c = rad^3(c)$ and $c = a + b$. Then, we obtain that $Y = rad(c)$ is a solution in positive integers of the equation:

(300) $$Y^3 + 1 = \bar{c}$$

with $\bar{c} = a + b + 1 = c + 1 \Longrightarrow (\bar{c}, c) = 1$. We obtain the same result as of the case **I-3-2-1-** studied above considering $rad(\bar{c}) > rad^{\frac{1.63}{1.37}}(c)$.

II'-3-2-3- We suppose $\mu_a > rad^{2.26}(a) \Rightarrow a > rad^{3.26}(a)$ and c large and $\mu_c < rad^2(c)$, we consider $c = a + b, b \geq 1$. Then $a = rad^3(a) + h, h > 0$, h a positive integer and we can write $c + l = rad^3(c)$, $l > 0$. As $rad(c) > rad^{\frac{1.63}{1.37}}(a) \Longrightarrow rad(c) > rad(a) \Longrightarrow h + l + b = m > 0$, it follows:

(301) $rad^3(c) - l = rad^3(a) + h + b > 0 \Longrightarrow rad^3(c) - rad^3(a) = h + l + b = m > 0$

We obtain the same result (a contradiction) as of the case **I-3-2-3-** studied above considering $rad(c) > rad^{\frac{1.63}{1.37}}(a)$. Then, this case is to reject.

Then the cases $\mu_c \leq rad^2(c)$ and $a > rad^{3.26}(a)$ are impossible.

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CHAPTER 4

THE EXPLICIT abc CONJECTURE OF ALAN BARKER IS TRUE

Abstract. — In this paper, assuming that the conjecture $c < rad^{1.63}(abc)$ is true, we give the proof that the explicit abc conjecture of Alan Baker (2004) is true. Some numerical examples are given.

The paper is under reviewing.

To the memory of my ***Father*** *who taught me arithmetic*
To my wife ***Wahida****, my daughter* ***Sinda*** *and my son* ***Mohamed Mazen***
To Prof. ***A. Nitaj*** *for his work on the abc conjecture*

4.1. Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{302}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{303}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 20. — ***(abc Conjecture)****: For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{304}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**2**]. It concerned the best example given by E. Reyssat [**2**]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{305}$$

A conjecture was proposed that $c < rad^2(abc)$ [**3**]. In 2004, Alan Baker [**1**], [**4**] proposed the explicit version of the *abc* conjecture namely:

Conjecture 21. — *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < \frac{6}{5} R \frac{(LogR)^\omega}{\omega!} \tag{306}$$

with $R = rad(abc)$ and $\omega = \omega(abc)$ the number of distinct prime factors of abc.

In the following, we assume that the conjecture $c < rad^{1.63}(abc)$ is true, I give an elementary proof of Alan Baker's conjecture cited above. For our proof, we proceed by contradiction of the abc conjecture. We give also some numerical examples.

4.2. The Proof of the explicit *abc* conjecture

Proof. — : We proceed by contradiction. It exists at least one triplet (a, b, c) of positive integers relatively prime with $c = a + b$ and :

(307)
$$c \geq \frac{6}{5} R \frac{(LogR)^\omega}{\omega!} \Longrightarrow Logc \geq Log1.2 + 1.63LogR - 0.63LogR + Log\left[\frac{(LogR)^\omega}{\omega!}\right]$$

we assume that the conjecture $c < rad^{1.63}(abc)$ true, we can write :

$$0 > -Log\frac{R^{1.63}}{c} \geq Log1.2 - 0.63LogR + Log\left[\frac{(LogR)^\omega}{\omega!}\right] \tag{308}$$

We write $LogR$ as:

$$LogR = LogR^{0.63}\left(1 + \frac{0.37}{0.63}\right)$$

The equation (308) becomes:

$$0 > -Log\frac{R^{1.63}}{c} \geq Log1.2 - 0.63LogR + \omega Log\left(1 + \frac{0.37}{0.63}\right) + Log\left[\frac{(Log(R^{0.63}))^\omega}{\omega!}\right] > Log1.2$$
$$-0.63LogR + Log(1 + 0.5873\omega) + Log\left[\frac{(Log(R^{0.63}))^\omega}{\omega!}\right] \Longrightarrow$$

(309)
$$0 > -Log\frac{R^{1.63}}{c} > -0.63LogR + Log(1.2 + 0.70476\omega) + Log\left[\frac{(Log(R^{0.63}))^\omega}{\omega!}\right]$$

Let $A = \dfrac{(Log(R^{0.63}))^{\omega}}{\omega!}$, we obtain:

$$R^{0.63} = e^{LogR^{0.63}} = 1 + Log(R^{0.63}) + \frac{(Log(R^{0.63}))^2}{2!} + \cdots + A + \sum_{k=\omega+1}^{+\infty} \frac{(Log(R^{0.63}))^k}{k!} \Longrightarrow$$

$$A = R^{0.63} - 1 - \sum_{k=1,\neq\omega}^{+\infty} \frac{(Log(R^{0.63}))^k}{k!} \Longrightarrow$$

$$A = R^{0.63}\left(1 - \frac{1}{R^{0.63}}\left[1 + \sum_{k=1,\neq\omega}^{+\infty} \frac{(Log(R^{0.63}))^k}{k!}\right]\right) = R^{0.63}(1-B) > 0,\ 0 < B < 1$$

The equation (309) becomes:

$$0 > -Log\frac{R^{1.63}}{c} > Log(0.70476\omega + 1.2 - 0.70476B\omega - 1.2B) \tag{310}$$

Let us consider the smallest case $9 = 8 + 1 \Longrightarrow w = 2, R = 2 \times 3 = 6 < 9 = c$. The conjecture is verified $c = 9 < 11.56$, we obtain $B = 0.54 \ll 2 = w$ with $R = 6$ and $0.70476\omega + 1.2 - 0.70476B\omega - 1.2B = 1.2 > 1$. If R is large, then ω can be large and B will be small, then $B \ll \omega$, it follows that the term :

$$0.70476\omega + 1.2 - 0.70476B\omega - 1.2B > 1 \Longrightarrow 0 > -Log\frac{R^{1.63}}{c} > 0$$

Then it is the contradiction and we obtain:

$$c < \frac{6}{5}R\frac{(LogR)^{\omega}}{\omega!}$$

The proof of the explicit *abc* conjecture of Alan Baker is finished.

Q.E.D

□

We give below some numerical examples.

4.3. Examples

4.3.1. Example 1. of Eric Reyssat

We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{311}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$.
$\omega = 4 \Longrightarrow \dfrac{6}{5}R\dfrac{(LogR)^{\omega}}{\omega!} = 6\,437\,590.238 > 6\,436\,343,\ B = 0.86 < w = 4.$

4.3.2. Example 2. of Nitaj

See [**5**]:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110$$

$\omega = 10 \Longrightarrow \frac{6}{5}R\frac{(LogR)^{\omega}}{\omega!} = 7\,794\,478\,289\,809\,729\,132\,015,590 > 613\,474\,845\,886\,230\,468\,750,\ B = 0.9927 \ll (w = 10)$.

4.3.3. Example 3.

The example is of Ralf Bonse, see [**2**] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983$$
$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983$$
$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$rad(abc) = 1.5683959920004546031461002610848e + 33$$

$$\omega = 10 \Longrightarrow \frac{6}{5}R\frac{(LogR)^{\omega}}{\omega!} = 4.6712291777572705786110845974696e + 358 >$$
$$c = 3.4136998783296235160378273576498e + 44,\ B \approx 1 \ll (w = 10).$$

4.4. Conclusion

Assuming $c < R^{1.63}$ is true, we have given an elementary proof of the explicit *abc* conjecture. We can announce the important theorem:

> ***Theorem 22.*** — *Assuming $c < R^{1.63}$ is true, the explicit abc conjecture of Alan Baker is true:*
> *Let a, b, c positive integers relatively prime with $c = a + b$, then:*
>
> $$c < \frac{6}{5}R\frac{(LogR)^{\omega}}{\omega!} \quad (312)$$
>
> *where ω is the number of distinct prime factors of abc.*

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CHAPTER 5

A NEW APPROACH FOR THE PROOF OF THE *abc* CONJECTURE

Abstract. — In this paper, we assume that the explicit *abc* conjecture of Alan Baker and the conjecture $c < R^{1.63}$ are true, we give a proof of the *abc* conjecture and we propose the constant $K(\epsilon)$. Some numerical examples are given.

The paper is under reviewing.

To the Memory of my ***Mother***

5.1. Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{313}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{314}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 23. — ***(abc Conjecture)****: For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{315}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**2**]. It concerned the best example given by E. Reyssat [**2**]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{316}$$

A conjecture was proposed that $c < rad^2(abc)$ [**3**]. In 2012, A. Nitaj [**4**] proposed the following conjecture:

Conjecture 24. — *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < rad^{1.63}(abc) \tag{317}$$

$$abc < rad^{4.42}(abc) \tag{318}$$

In the following, we assume that the conjecture $c < rad^{1.63}(abc)$ is true. In 2004, Alan Baker [**1**], [**5**] proposed the explicit version of the *abc* conjecture namely:

Conjecture 25. — *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} \tag{319}$$

with $R = rad(abc)$ and ω denote the number of distinct prime factors of abc.

A proof of the conjecture (25) written by the author is under review [**6**]. In the following, we assume also that the above conjecture is true, I will give an elementary proof of the *abc* conjecture by verifying the below inequality:

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} < ... < K(\epsilon)R^{1+\epsilon} \tag{320}$$

with an adequate choice of the constant $K(\epsilon)$. Let we denote $\alpha = \dfrac{6}{5}R\dfrac{(LogR)^\omega}{\omega!}$, we have remarked from some numerical examples (see below) that $c \ll \alpha - c$ when $\omega = 10$ and R not very large. With our choice, c will be very very small comparing to $K(\epsilon)R^{1+\epsilon}$.

5.2. The Proof of the *abc* conjecture

Proof. — : Let $A = \dfrac{(Log(R^\epsilon))^\omega}{\omega!}$, and $\epsilon \in]0, 0.63[$, we obtain:

$$R^\epsilon = e^{LogR^\epsilon} = 1 + Log(R^\epsilon) + \frac{(Log(R^\epsilon))^2}{2!} + \cdots + A + \sum_{k=\omega+1}^{+\infty} \frac{(Log(R^\epsilon))^k}{k!} \Longrightarrow$$

$$A = R^\epsilon - 1 - \sum_{k=1,\neq\omega}^{+\infty} \frac{(Log(R^\epsilon))^k}{k!} \Longrightarrow$$

$$A = R^\epsilon \left(1 - \frac{1}{R^\epsilon}\left[1 + \sum_{k=1,\neq\omega}^{+\infty} \frac{(Log(R^\epsilon))^k}{k!}\right]\right) = R^\epsilon(1-B) > 0,\ 0 < B < 1 \Longrightarrow$$

$$A = \frac{(Log(R^\epsilon))^\omega}{\omega!} = R^\epsilon(1-B) > 0 \tag{321}$$

We begin from the Baker's formula below :

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} = \frac{6}{5}R.\frac{1}{\epsilon^\omega}\frac{(\epsilon LogR)^\omega}{\omega!} = \frac{6}{5}\frac{R}{\epsilon^\omega}\frac{(Log(R^\epsilon))^\omega}{\omega!}$$

Using the term $\dfrac{(Log(R^\epsilon))^\omega}{\omega!}$ from (321), the equation above becomes :

(322)

$$c < \frac{6}{5}\frac{R}{\epsilon^\omega}R^\epsilon(1-B) \overbrace{<}^{?} 1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}} R^{1+\epsilon} \Longrightarrow \text{our choice of the constant } K(\epsilon) = 1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}}$$

We recall the following proposition [**4**]:

> ***Proposition 26.*** — *Let $\epsilon \longrightarrow K(\epsilon)$ the application verifying the abc conjecture, then:*
>
> $$lim_{\epsilon\to 0}K(\epsilon) = +\infty \tag{323}$$

The chosen constant $K(\epsilon)$ verifies the proposition above. Now, is the following inequality true? :

$$\frac{6}{5}\frac{1}{\epsilon^\omega}(1-B) \overbrace{<}^{?} 1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}} \tag{324}$$

Supposing that :

$$\frac{6}{5}\frac{1}{\epsilon^\omega}(1-B) > \frac{6}{5}e^{e^{\left(\frac{1}{\epsilon^4}\right)}}. \Longrightarrow 1 > (1-B) > \epsilon^\omega.e^{e^{\left(\frac{1}{\epsilon^4}\right)}}$$

As $\omega \geq 4 \Longrightarrow \omega = 4\omega' + r,\ 0 \leq r < 4, \omega' \geq 1$, we write $\epsilon^\omega.e^{e^{(1/\epsilon)^4}}$ as:

$$\epsilon^\omega.e^{e^{(1/\epsilon)^4}} = \frac{e^{e^{(1/\epsilon)^4}}}{(1/(\epsilon^4))^{\omega'}}.\epsilon^r = \frac{e^{e^X}}{X^{\omega'}}.\epsilon^r$$

where $X = \frac{1}{\epsilon^4}$ and $1 \ll X$. Or we know that $X^{\omega'} \ll e^X \Longrightarrow X^{\omega'} \ll e^{e^X}$.

- If $\epsilon \in [0.1, 0.63[$, we obtain $\epsilon^r > 0.001$ and $e^X > 8.8e + 4342$, it follows that $\epsilon^{\omega}.e^{e^{\left(\frac{1}{\epsilon^4}\right)}} > 1$ and we obtain a contradiction and the inequality (324) is true.

- Now we consider $0 < \epsilon < 0.1$, when $\epsilon \longrightarrow 0^+, K(\epsilon) \longrightarrow +\infty$ and the inequality (324) becomes $+\infty \leq +\infty$ and the abc conjecture is true.

- For ϵ very small $\in]0, 0.10[$, e^{e^X} becomes very large, then $8.8e + 4342 \ll e^{e^X}$ and $1 \ll \frac{e^{e^X}}{X^{\omega'}}.\epsilon^r$, it follows a contradiction, then the equation (324) is true.

Finally, the choice of the constant $K(\epsilon) = 1.2e^{e^{\left(\frac{1}{\epsilon}\right)^4}}$ is acceptable for $\epsilon \in]0, 0.63[$. As we assume that the conjecture $c < R^{1+0.63}$ is true, we adopt $K(\epsilon) = 1.2$ for $\epsilon \geq 0.63$, and the *abc* conjecture is true for all $\epsilon > 0$.

The proof of the *abc* conjecture is finished.

Q.E.D

□

We give below some numerical examples.

5.3. Examples

5.3.1. Example 1. of Eric Reyssat

We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{325}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$.
$\omega = 4 \Longrightarrow \alpha = \dfrac{6}{5}R\dfrac{(LogR)^{\omega}}{\omega!} = 6\,437\,590.238 > 6\,436\,343 = c$, $B = 0.86 < w = 4$; $\alpha - c = 1\,247.238$.

$\epsilon = 0.5 \Longrightarrow \epsilon^{\omega}.e^{e^{\left(\frac{1}{\epsilon}\right)^4}} = 9.446e + 109 > 1 \Longrightarrow (1 - B) < 1$.

$\epsilon = 0.01 \Longrightarrow \epsilon^{\omega} = \epsilon^4 = 10^{-8} \ll e^{\left(\frac{1}{\epsilon}\right)^4}$ then $(1 - B) < 1$.

5.3.2. Example 2. of Nitaj

See [**4**]:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110$$
$$\omega = 10 \Longrightarrow \alpha = \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} = 7\,794\,478\,289\,809\,729\,132\,015.590 >$$
$$613\,474\,845\,886\,230\,468\,750 = c,\; B = 0.9927 \ll (w = 10);\; \alpha - c =$$
$$7\,181\,003\,443\,923\,198\,663\,265.590 \approx 11.71c$$

$\epsilon = 0.5 \Longrightarrow \epsilon^\omega = \epsilon^{10} = 0.009765625 \ll e^{1/(\epsilon^4)} \Longrightarrow (1 - B) < 1.$
$\epsilon = 0.001 \Longrightarrow \epsilon^\omega = \epsilon^{10} = 10^{-30}$, $1/(\epsilon^4) = 10^{12} \Longrightarrow \epsilon^{10}.e^{10^{12}} > 1 \Longrightarrow (1 - B) < 1.$

5.4. Conclusion

Assuming $c < R^{1.63}$ is true, and the explicit *abc* conjecture of Alan Baker true, we can announce the important theorem:

Theorem 27. — *Assuming $c < R^{1.63}$ is true and the explicit abc conjecture of Alan Baker true, then the abc conjecture is true:*
For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{326}$$

where K is a constant depending only of ϵ. For $\epsilon \in]0, 0.63[$, $K(\epsilon) = 1.2e^{e^{\left(\frac{1}{\epsilon}\right)^4}}$ and $K(\epsilon) = 1.2$ if $\epsilon \geq 0.63$.

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CHAPTER 6

A NOVEL PROOF OF THE abc CONJECTURE: IT IS EASY AS ABC!

Abstract. — In this paper, we consider the abc conjecture. Assuming that the conjecture $c < rad^{1.63}(abc)$ is true, we give the proof that the abc conjecture is true.
The paper is under reviewing.

This paper is dedicated to the memory of my ***Father*** *who taught me arithmetic,*
To my wife ***Wahida****, my daughter* ***Sinda*** *and my son* ***Mohamed Mazen***

6.1. Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{327}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{328}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 28. — *(****abc Conjecture****): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{329}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**2**]. It concerned the best example given by E. Reyssat [**2**]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{330}$$

A conjecture was proposed that $c < rad^2(abc)$ [**3**]. In 2012, A. Nitaj [**4**] proposed the following conjecture:

> ***Conjecture 29.*** — *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*
>
> $$c < rad^{1.63}(abc) \tag{331}$$
>
> $$abc < rad^{4.42}(abc) \tag{332}$$

In the following, we assume that the conjecture giving by the equation (347) is true that constitutes the key to obtain the proof of the *abc* conjecture and we consider the cases $c > R$ because the *abc* conjecture is verified if $c < R$. For our proof, we proceed by contradiction of the abc conjecture, for $\epsilon \in]0., 0.63[$.

6.2. The Proof of the *abc* conjecture

Proof. — :

6.2.1. Trivial Case $\epsilon \geq (0.63 = \epsilon_0)$.

In this case, we choose $K(\epsilon) = e$ and let a, b, c be positive integers, relatively prime, with $c = a + b$, $1 \leq b < a, R = rad(abc)$, then $c < R^{1+\epsilon_0} \leq K(\epsilon).R^{1+\epsilon} \Longrightarrow c < K(\epsilon).R^{1+\epsilon}$ and the *abc* conjecture is true.

6.2.2. Case: $0 < \epsilon < (0.63 = \epsilon_0)$.

We recall the following proposition [**4**]:

> ***Proposition 30.*** — *Let $\epsilon \longrightarrow K(\epsilon)$ the application verifying the abc conjecture, then:*
>
> $$lim_{\epsilon \to 0} K(\epsilon) = +\infty \tag{333}$$

We suppose that the abc conjecture is false, then it exists $\epsilon' \in]0, \epsilon_0[$ and for all parameter $K' = K'(\epsilon') > 0$ it exists at least one triplet (a', b', c') so a', b', c' be positive integers relatively prime with $c' = a' + b'$ and c' verifies :

$$c' > K'(\epsilon').R'^{1+\epsilon'} \tag{334}$$

From the proposition cited above, it follows that $lim_{\epsilon \longrightarrow 0} K'(\epsilon) < +\infty$, we can suppose that $K'(\epsilon)$ is an increasing parameter for $\epsilon \in]0, \epsilon_0[$.

As the parameter K' is arbitrary, we choose $K'(\epsilon) = e^{\epsilon^2}$, it is an increasing parameter. Let :

$$Y_{c'}(\epsilon) = \epsilon^2 + (1+\epsilon) LogR' - Logc', \epsilon \in]0, \epsilon_0[\tag{335}$$

About the function $Y_{c'}$, we have:

$$lim_{\epsilon \longrightarrow \epsilon_0} Y_{c'}(\epsilon) = \epsilon_0^2 + Log(R'^{1+\epsilon_0}/c') = \lambda > 0, \quad as\ c < R^{1+\epsilon_0}$$
$$lim_{\epsilon \longrightarrow 0} Y_{c'}(\epsilon) = -Log(c'/R') < 0, \quad as\ R < c$$

The function $Y_{c'}(\epsilon)$ represents a parabola and it is an increasing function for $\epsilon \in]0, \epsilon_0[$, then the equation $Y_{c'}(\epsilon) = 0$ has one root that we denote ϵ'_1, it follows the equation :

$$e^{\epsilon_1'^2} R'^{\epsilon'_1} = \frac{c'}{R'} \tag{336}$$

Discussion about the equation (336) above:

We recall the following definition:

Definition 31. — *The number ξ is called algebraic number if there is at least one polynomial:*

$$l(x) = l_0 + l_1 x + \cdots + l_m x^m, \quad l_m \neq 0 \tag{337}$$

with integral coefficients such that $l(\xi) = 0$, and it is called transcendental if no such polynomial exists.

We consider the equation (336) :

$$c' = K'(\epsilon'_1) R'^{1+\epsilon'_1} \Longrightarrow \frac{c'}{R'} = \frac{\mu'_{c'}}{rad(a'b')} = e^{\epsilon'^2_1} R'^{\epsilon'_1} \tag{338}$$

i) - We suppose that $\epsilon'_1 = \beta_1$ is an algebraic number then $\beta_0 = \epsilon'^2_1$ and $\alpha_1 = R'$ are also algebraic numbers. We obtain:

$$\frac{c'}{R'} = \frac{\mu'_{c'}}{rad(a'b')} = e^{\epsilon'^2_1} R'^{\epsilon'_1} = e^{\beta_0}.\alpha_1^{\beta_1} \tag{339}$$

From the theorem (see theorem 3, page 196 in [**5**]):

Theorem 32. — *$e^{\beta_0} \alpha_1^{\beta_1} \dots \alpha_n^{\beta_n}$ is transcendental for any nonzero algebraic numbers $\alpha_1, \dots, \alpha_n, \beta_0, \dots, \beta_n$.*

we deduce that the right member $e^{\beta_0}.\alpha_1^{\beta_1}$ of (339) is transcendental, but the term $\dfrac{\mu'_{c'}}{rad(a'b')}$ is an algebraic number, then the contradiction and the hypothesis that the *abc* conjecture is false on $\epsilon \in]0, \epsilon_0[$. It follows that the *abc* conjecture is true on $\epsilon \in]0, \epsilon_0[$, then for all $\epsilon > 0$.

ii) - We suppose that ϵ'_1 is transcendental, then ${\epsilon'}_1^2$ is transcendental. If not, ${\epsilon'}_1^2$ is an algebraic number, it verifies:

$$l(x) = l_{2m}{\epsilon'}_1^{2m} + 0 \times {\epsilon'}_1^{2m-1} + l_{2(m-1)}{\epsilon'}_1^{2(m-1)} + \cdots + l_2{\epsilon'}_1^2 + 0 \times {\epsilon'}_1 + l_0 = 0$$

From the definition (337) and the equation above, e'_1 is also an algebraic number, then the contradiction.

As $R' > 0$ is an algebraic number, we know that $LogR'$ is transcendental. We rewrite the equation (336) as:

$$\frac{c'}{R'} = e^{{\epsilon'}_1^2} R'^{\epsilon'_1} = e^{{\epsilon'}_1^2 + \epsilon'_1 LogR'} \tag{340}$$

By the theorem of Hermite (page 45, [**5**]) e is transcendental. Let $z = {\epsilon'}_1^2 + \epsilon'_1 LogR' > 0$:

- As $z \neq 0$, if z is an algebraic number it follows that e^z is transcendental by the theorem of Lindemann (page 51, [**5**]), it follows the contradiction with c'/R' an algebraic number. Then the hypothesis that the *abc* conjecture is false on $\epsilon \in]0, \epsilon_0[$ is not true. It follows that the *abc* conjecture is true on $\epsilon \in]0, \epsilon_0[$, then for all $\epsilon > 0$.

- Now we suppose that $z \neq 0$ is transcendental. We write e^z as:

$$e^z = \sum_{n=1}^{+\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots + \frac{z^N}{N!} + r(z)$$

$$\text{and } r(z) \ll \frac{z^N}{N!} \quad \text{for } \; N \text{ very large}$$

Then :

$$R'z^N + R'Nz^{(N-1)} + \cdots + R'N!z + N!(R' - c') = 0$$

It follows that z is an algebraic number $\Longrightarrow$ the contradiction avec z transcendental. Then the hypothesis that the *abc* conjecture is false on $\epsilon \in]0, \epsilon_0[$ is not true. It follows that the *abc* conjecture is true on $\epsilon \in]0, \epsilon_0[$, then for all $\epsilon > 0$.

The proof of the *abc* conjecture is finished. Assuming $c < R^{1+\epsilon_0}$ is true, we obtain that $\forall \epsilon > 0$, $\exists K(\epsilon) > 0$, if $c = a + b$ with a, b, c positive integers relatively coprime, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{341}$$

and the constant $K(\epsilon)$ depends only of ϵ.

Q.E.D

Ouf, end of the mystery!

□

6.3. Conclusion

Assuming $c < R^{1+\epsilon_0}$ is true, we have given an elementary proof of the *abc* conjecture. We can announce the important theorem:

Theorem 33. — *Assuming $c < R^{1+\epsilon_0}$ is true, the abc conjecture is true: For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then:*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{342}$$

where K is a constant depending of ϵ.

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CHAPTER 7

THE EXPLICIT abc A. BAKER'S CONJECTURE $\Longrightarrow c < R^2$ TRUE

Abstract. — In this paper, we assume that the explicit *abc* conjecture of Alan Baker (2004) is true, we give the proof that $c < rad^2(abc)$ is true, it is one of the keys to resolve the mystery of the *abc* conjecture. Some numerical examples are given.

The paper is under reviewing.

To the memory of my Father who taught me arithmetic
To my wife ***Wahida***, *my daughter* ***Sinda*** *and my son* ***Mohamed Mazen***
To Prof. ***A. Nitaj*** *for his work on the abc conjecture*

7.1. Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{343}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{344}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 34. — *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{345}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**2**]. It concerned the best example given by E. Reyssat [**2**]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{346}$$

A conjecture was proposed that $c < rad^2(abc)$ [**3**]. In 2004, Alan Baker [**1**], [**4**] proposed the explicit version of the abc conjecture namely:

Conjecture 35. — *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} \tag{347}$$

with $R = rad(abc)$ and $\omega = \omega(abc)$ the number of distinct prime factors of abc.

In the following, we assume that the conjecture of Alan Barker is true, I will give an elementary proof of the conjecture $c < rad^2(abc)$ that constitutes one key to resolve the open abc conjecture. For our proof, we proceed by contradiction of the abc conjecture. We give also some numerical examples.

7.2. The Proof of the $c < R^2$ Conjecture

Proof. — : Let one triplet (a, b, c) of positive integers relatively prime with $c = a+b$ and :

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!}$$

Let $A = \dfrac{(LogR)^\omega}{\omega!}$, $rad(a) = \prod_{i=1,I} a_i, rad(b) = \prod_{j=1,J} b_j$, and $c = \prod_{l=1,L} c_l$, then $\omega = I + J + L$. we obtain:

$$\omega \ll (LogR = \sum_{i=1,I} Loga_i + \sum_{j=1,J} Logb_j + \sum_{l=1,L} Logc_l)$$

We can write R as:

$$R = e^{LogR} = 1 + LogR + \frac{(LogR)^2}{2!} + \cdots + A + \sum_{k=\omega+1}^{+\infty} \frac{(LogR)^k}{k!} \tag{348}$$

As $\dfrac{(LogR)^n}{n!} < \dfrac{(LogR)^{n+1}}{(n+1)!}$ for $n < LogR$ and $\omega \ll LogR$, it follows :

$$A \leq 1 + LogR + \frac{(LogR)^2}{2!} + \cdots + \frac{(LogR)^{\omega-1}}{(\omega-1)!} + \sum_{k=\omega+1}^{n, n<LogR} \frac{(LogR)^k}{k!}$$

I propose that $A \leq \dfrac{5}{6}R$, then:

$$c < \frac{6}{5}R\frac{(LogR)^{\omega}}{\omega!} = \frac{6}{5}RA \leq \frac{6}{5}R.\frac{5}{6}R \Longrightarrow c < R^2 \tag{349}$$

If not, $A > \dfrac{5}{6}R$, we write $R = 1+LogR+A+r$, $r>0$, then $A > 5+5LogR+5r$, but for large R, we have $\omega \ll LogR, A \ll LogR \ll 5logR$, we obtain a contradiction. It follows $c < R^2$.

The proof of $c < R^2$ conjecture is finished.

Q.E.D

□

We give below some numerical examples.

7.3. Examples

7.3.1. Example 1. of Eric Reyssat

We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6\,436\,343 \tag{350}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19\,683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6\,436\,343 \Rightarrow rad(c) = 23$. Then $R = rad(abc) = 2 \times 3 \times 109 \times 23 = 15\,042 \Longrightarrow R^2 = 226\,261\,764$.
$\omega = 4 \Longrightarrow A = \dfrac{(LogR)^4}{4!} = 356.64$, $R^2 > \dfrac{6}{5}R\dfrac{(LogR)^{\omega}}{\omega!} = 6\,437\,590.238 > (c = 6\,436\,343)$. $\dfrac{A}{R} \approx 0.06 \ll \dfrac{5}{6} = 0.83$.

7.3.2. Example 2. of Nitaj

See [**5**]:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953$$
$$R = rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110$$
$$\Longrightarrow R^2 = 831\,072\,936\,124\,776\,471\,158\,132\,100 > (c = 613\,474\,845\,886\,230\,468\,750)$$
$$\omega = 10 \Longrightarrow A = \frac{(LogR)^{10}}{10!} = 225\,312\,992.556 \Longrightarrow$$
$$R^2 > \frac{6}{5}R\frac{(LogR)^{\omega}}{\omega!} = 7\,794\,478\,289\,809\,729\,132\,015,590 > (c = 613\,474\,845\,886\,230\,468\,750),$$
$$\frac{A}{R} = 7.815e-6 \ll \frac{5}{6} = 0.83$$

7.3.3. Example 3.

The example is of Ralf Bonse, see [**2**] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983$$
$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983$$
$$R = rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$R = 1.5683959920004546031461002610848e+33 \Longrightarrow$$
$$R^2 = 2.4598659877230900595045886864952e+66$$
$$\omega = 10 \Longrightarrow A = \frac{(LogR)^{10}}{10!} = 1\,875\,772\,681\,108.203 \Longrightarrow$$
$$R^2 > \frac{6}{5}R\frac{(LogR)^{\omega}}{\omega!} = 3.5303452259448631166310839830891e+45 >$$
$$c = 3.4136998783296235160378273576498e+44,\ \frac{A}{R} = 1.196e-21 \ll \frac{5}{6} = 0.83$$

7.4. Conclusion

Assuming that the explicit *abc* conjecture is true, we have given an elementary proof that the $c < R^2$ conjecture holds. We can announce the important theorem:

Theorem 36. — *Assuming the explicit abc conjecture of Alan Baker is true, then the $c < R^2$ conjecture is true.*

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