Proofs for Collatz Conjecture Behavior of Kaakuma Sequence

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Abstract

The objective of this study is to present rigorous proofs for Collatz conjecture and introduce some interesting behavior of the Kaakuma sequence that is a vast generalized form of Collatz sequence. We analyze the behavior of Kaakuma sequence such as scaling up, scaling down, translation, function iteration and uniform growth of inverse tree. In addition to this we investigate relationship of increasing rate, number of iterations of cycles, gap in cycles, and densities of cycles of the Kaakuma sequence and evaluate consistency of tree size density after scaling.

Our investigation culminates in the formulation of a set of conjectures encompassing lemmas and postulates, which we rigorously prove using a combination of analytical reasoning, numerical evidence, and exhaustive case analysis. These results provide compelling evidence for the veracity of the Collatz conjecture and contribute to our understanding of the underlying mathematical structure.

Keywords: Collatz Conjecture, Number Theory, Recursive Sequences, Tree Growth, Modular Arithmetic, Kaakuma Sequence, Qodaa Ratio Test, Stopping Time

1 Introduction

The Collatz Conjecture, also known as the 3n+1 Conjecture, Hailstone Problem, Kakutani's Conjecture, Ulam's Conjecture, Hasse's Algorithm, and the Syracuse Problem, is a long-standing and unsolved mathematical problem that has fascinated mathematicians for around a century. It is one of the most dangerous unsolved problems in mathematics. The conjecture is named after the German mathematician Lothar Collatz, who first proposed it in 1937.

Statement of the Conjecture

The Collatz conjecture originally states an iterative sequence of natural numbers. Take a natural number n. If n is even, make it half. If n is odd, multiply it by 3 and add 1. Continue the process repeatedly, taking the result as the next input, and continue iterating. The conjecture states that regardless of the starting value, the sequence of numbers will eventually reach the value 1. For example:

$$14 \to 7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5$$

$$\to 16 \to 8 \to 4 \to 2 \to 1$$

Historical Background and Significance

The Collatz Conjecture has captured the minds of mathematicians for almost a century. Many have attempted to prove or disprove it, employing various techniques and approaches. Despite its apparent simplicity, the conjecture has resisted all attempts at a definitive solution. The search for a solution to the Collatz conjecture continues, driven by the allure of a seemingly simple problem harboring immense complexity. It serves as a reminder that even in the vast realm of mathematics, profound mysteries still await discovery.

Even though the Collatz conjecture is simple to express and understand, it has tantalized scientists for around a century. Mathematicians have extensively tested the conjecture using computers for billions of billions of values, and it holds true for all tested cases. The Collatz Conjecture has fascinated mathematicians because of its apparent simplicity combined with its elusiveness. Many attempts have been made to prove or disprove the conjecture, involving various mathematical techniques and concepts. However, conjecture remains one of the most enduring unsolved problems in mathematics.

Heuristic Argument

A heuristic argument, sometimes stated as a probabilistic approach, attempts to show that the conjecture is true for infinitely diverging cases, not for non-trivial cycles, especially if the number of iterations is small to make a cycle. The probabilistic approach concerns how often each case will happen in mean to get lower or upper values of the starting number after a number of iterations. The ratio is $3/4$ and $n \to 3n/4$. This forms a basic study of research, working with varied examples.

Improved Results and Further Research

Almost all initial values n in which we perform our Collatz function T conclusively iterate to a value less than n. Studies indicate that 99.99% of the starting values iterate to a value less than the starting value. Allouche and Korec have improved this result by proving that for an initial value n, it iterates to a value less than $n^{0.869}$ and more improved to a value less than $n^{0.7925}$, respectively. Terras's paper "A Stopping-Time Problem on the Positive Integers" (Terras, 1976) provides initial derivation.

Allouche proves that almost all values iterate to a value less than $n^{0.869}$ and states that not just asymptotic behavior is required to determine the periodicity of the function, with periodicity referring to repeating points and intervals between them. The ideas used in Allouche's paper are based on those used by Terras in his original proof and are continued by Ivan Korec (Korec, 1994).

Tao's contribution to the Collatz Conjecture (Tao, 2019) represents a significant breakthrough. His main result, "Collatz orbits have almost bound values," states that for any function $f(n)$ such that when n tends to infinity, $f(n)$ also tends to positive infinity, the minimum term within a given Collatz orbit of n will be less than $f(n)$ for almost all values of n.

Kaakuma Sequence

Kaakuma sequence is a piecewise-defined recursive integer sequence:.

$$f(n) = \begin{cases} \frac{k_1 n + c_1}{b_1} & \text{Case } 1 \\ \frac{k_2 n + c_2}{b_2} & \text{Case } 2 \\ \frac{k_3 n + c_3}{b_3} & \text{Case } 3 \\ \vdots & \vdots \\ \frac{k_i n + c_i}{b_i} & \text{Case } i \end{cases}$$

Kaakum sequence is an iterative sequence of integers with cases based on modulo conditions, it is a vast general form of Collatz sequence.

2 Expressions of Collatz sequence

The Collatz conjecture can be represented in different ways while retaining the same meaning. Below are various notations used to describe the conjecture.

a) General Notation

$$n_{i+1} = \begin{cases} 3n_i + 1 & \text{if } n_i \text{ is odd} \\ \frac{n_i}{2} & \text{if } n_i \text{ is even} \end{cases}$$

Here, n_0 is any number that begins an orbit and eventually reaches 1 by iterating rule at n_T.

b) Function Notation

$$f(n) = \begin{cases} 3n + 1 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

In this notation, the result is used as the next value for iteration until the value reaches 1.

c) Simplified Notation

$$n = \begin{cases} 3n + 1 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

This notation is often used in coding assignments. The right side of the equation is the input, and the left side is the output. The iteration continues using the output as the next input until reaching 1.

d) Shorter Form

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

e) Modular Form

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

This notation expresses the conditions of iteration in modular form.

f) Inverse of the Collatz Conjecture

The inverse of the Collatz conjecture states that if you start from 1 as a root of a tree, and for each number, you double it in all cases and divide a number minus one by three when it is possible to get a positive integer, then all natural numbers are traced in the tree map. This implies that no natural number is left out of the inverse tree map.

$$f(n) = \begin{cases} \frac{n-1}{3} & \text{if } n \equiv 1 \pmod 3 \\ 2n & \forall n \ (n \in \mathbb{N}) \end{cases}$$

Table 1: Tabular form of Inverse Tree Map

$n \equiv 4 \pmod 6$	$f(n) = \frac{n-1}{3}$	$f(n) = 2n$
4	1	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024
16	5	10, 20, 40, 80, 160, 320, 640, 1280
10	3	6, 12, 24, 48, 96, 192, 384, 768, 1536
40	13	26, 52, 104, 208, 416, 832, 1664
52	17	34, 68, 136, 272, 544, 1088
34	11	22, 44, 88, 176, 352, 704, 1408
22	7	14, 28, 56, 112, 224, 448, 896, 1792
28	9	18, 24, 48, 96, 192, 384, 768, 1536
64	21	42, 84, 164, 328, 656, 1312
88	29	58, 116, 232, 464, 928, 1856
58	19	38, 76, 152, 304, 608, 1216
76	25	50, 100, 200, 400, 800, 1600
112	39	78, 156, 312, 624, 1248

In this tabular form of the inverse tree of the Collatz function, the nodes make new branches from values in the form $6k + 4$ from existing nodes.

3 Behavior of the Collatz Sequence

Before proceeding with the proof of the Collatz conjecture, it is essential to understand some basic behaviors of the Collatz sequence.

3.1 Transformation

3.1.1 Translation

Translation is a transformation that shifts each value in the orbit by a fixed distance forward or backward. For example:

$$\text{Original sequence: } 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1$$

$$\text{Shifted by two forward: } 9, 24, 13, 36, 19, 54, 28, 15, 42, 22, 12, 5, 18, 10, 6, 4, 3$$

$$\text{Shifted by three backward: } 4, 19, 8, 31, 14, 49, 23, 10, 37, 17, 7, 2, 13, 5, 1$$

The function $f(n)$ and its translated version $g(n)$ can be expressed as:

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

$$g(n) = f(n) + 2 = \begin{cases} \frac{3n-3}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n+2}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

Similarly,

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

$$g(n) = f(n) - 3 = \begin{cases} \frac{3n+7}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n+2}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

During translation of a sequence, only the constant terms are changed. The transformation can be expressed as:

$$f(c) = c - l(k - d)$$

If a conditional equation is $\frac{kn+c}{d}$ and is translated by length l, then the translated equation becomes:

$$\frac{kn + c - l(k - d)}{d}$$

This formula is applied in all cases and is used with its sign or direction.

*Lemma 1

The next term of n after shifting by translating length l is:

$$\frac{kn+c}{d}+l=\frac{kn+c+dl}{d}$$

Using the direct formula:

$$\frac{k(n+l)+c-l(k-d)}{d}=\frac{kn+c+dl}{d}$$

Proof carried out by induction.

For a short form of the Collatz sequence translated forward by 1:

$$f(n)=\begin{cases}\frac{3n+1}{2} & \text{if } n\equiv 1 \pmod 2\\ \frac{n}{2} & \text{if } n\equiv 0 \pmod 2\end{cases}$$

$$g(n)=f(n)+1=\begin{cases}\frac{3n}{2} & \text{if } n\equiv 0 \pmod 2\\ \frac{n+1}{2} & \text{if } n\equiv 1 \pmod 2\end{cases}$$

This form and its inverse is used for its simplicity in this study.

3.1.2 Reflection on the Y-Axis

A reflection of the Collatz orbit on the y-axis involves multiplying constant terms by -1 and starting the sequence with the reflected value:

$$-1\times\frac{kn+c}{d}\longleftrightarrow\frac{kn-c}{d}$$

For the functions:

$$f(n)=\begin{cases}\frac{3n}{2} & \text{if } n\equiv 0 \pmod 2\\ \frac{n+1}{2} & \text{if } n\equiv 1 \pmod 2\end{cases}$$

$$g(n)=-f(n)=f(-n)=\begin{cases}\frac{3n}{2} & \text{if } n\equiv 0 \pmod 2\\ \frac{n-1}{2} & \text{if } n\equiv 1 \pmod 2\end{cases}$$

Example sequence for negative integers:

$$-8,-12,-18,-27,-14,-21,-11,-6,-9,-5,-3,-2$$

This converges to the $-2,-3$ cycle.

3.1.3 Scaling Up Mapping

Scaling involves multiplying the sequence by a fixed value s. This is done by multiplying the constant terms by the scaling up factor (natural number):

$$s \times \frac{kn+c}{d} \longleftrightarrow \frac{kn+sc}{d}$$

When the Collatz orbit is scaled up by s, e.g., multiplying by 5:

$$8, 12, 18, 27, 14, 21, 11, 6, 9, 5, 3, 2$$

multiplied by 5 yields:

$$40, 60, 90, 135, 70, 105, 55, 30, 45, 25, 15, 10$$

For the function:

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

and

$$g(n) = 5 \times f(n) = f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+5}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

The scaled map of the Collatz sequence by a number different from a power of 3 has two or more cycles:

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+3^i}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

The trajectory converges to 2×3^i or $(2 \times 3^i, 3^{i+1})$ cycle for all positive integers. For instance:

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+27}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

This converges to 54 or $(54, 81)$ cycle.

3.1.4 Scaling Down mapping

Scaling down is inverse of scaling up mapping, it is a transformation that scales down a sequence by scaling down factor. all divisible numbers by scaling factor in a sequence divided by scaling factor and the rest removed. When it is required we can use translation before scaling down.

When $f(n) \equiv 0 \pmod 3$

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

and

$$g(n) = \frac{f(n)}{3} \quad \text{if} \quad f(n) \equiv 0 \pmod 3 = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{3n+1}{4} & \text{if } n \equiv 1 \pmod 4 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

It converges to 1 for all natural numbers, if Collatz conjecture is true.
8, 12, 18, 27, 14, 21, 11, 6, 9, 5, 3, 2 maps to 4, 6, 9, 7, 2, 3, 1
The Equation for inverse tree of scaled down by scaling down factor 3.

$$f(n) = \begin{cases} \dfrac{2n}{3} & \text{if } n \equiv 0 \pmod 3 \\ \dfrac{4n-1}{3} & \text{if } n \equiv 1 \pmod 4 \\ 4n-1 & \forall n \ (n \in \mathbb{N}) \end{cases}$$

When we try to demonstrate scaling down graphically.

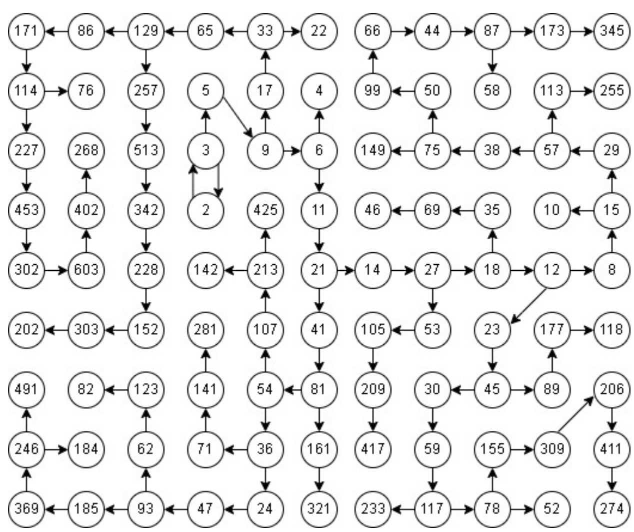

Figure 1: Original Inverse Tree

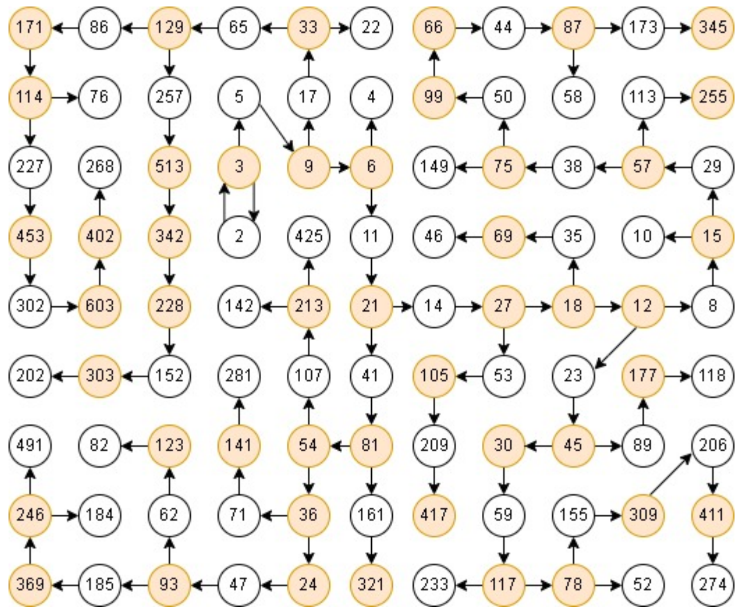

Figure 2: when We select only three factors

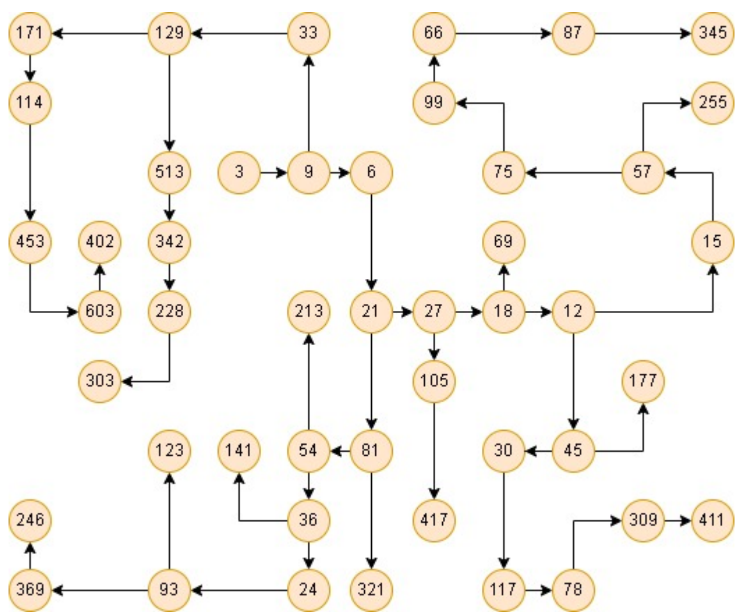

Figure 3: when we remove non-three factors

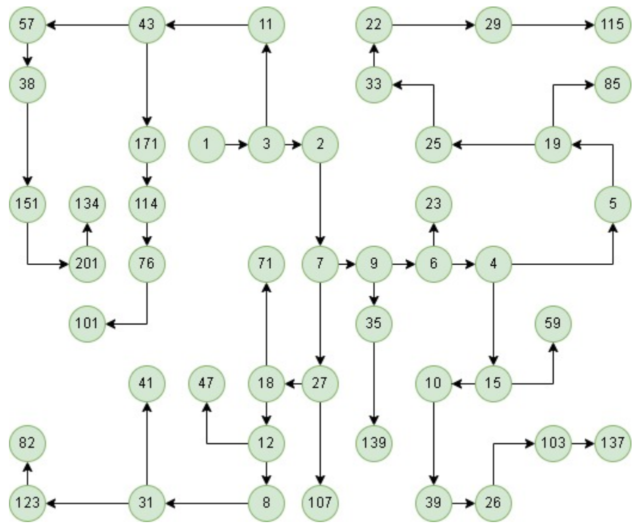

Figure 4: when we divide the rest by three

When $f(n) \equiv 0 \pmod 5$

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

and

$$g(n) = \frac{f(n)}{5} \quad \text{if} \quad f(n) \equiv 0 \pmod 5 = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n & \text{if } n \equiv 3 \pmod 4 \\ \frac{3n+1}{4} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+3}{16} & \text{if } n \equiv 13 \pmod{16} \\ \frac{9n+7}{4} & \text{if } n \equiv 5 \pmod{32} \\ \frac{9n+11}{8} & \text{if } n \equiv 53 \pmod{64} \\ \frac{9n+67}{64} & \text{if } n \equiv 21 \pmod{64} \end{cases}$$

It converges to 1 for all natural numbers if collatz conjecture is true.
28, 42, 63, 32, 48, 72, 108, 162, 243, 122, 183, 92, 138, 207, 104, 156, 234, 351, 176, 264, 396, 594, 891, 446, 669, 335, 168, 252, 378, 567, 284, 426, 639, 320, 480, 720, 1080, 1620, 2430, 3645, 1823, 912, 1368, 2052, 3078, 4617, 2309, 1155, 578, 867, 434, 651, 326, 489, 245, 123, 62, 93, 47 maps to:

11, 33, 25, 19, 57, 43, 129, 97, 73, 55, 165, 373, 421, 949, 1069, 67, 201, 151, 453, 1021, 64, 96, 144, 216, 324, 486, 729, 547, 1641, 1231, 3693, 231, 693, 781, 49, 37

When $f(n) \equiv 0 \pmod 9$

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

and

$$g(n) = \frac{f(n)}{9} \quad \text{if} \quad f(n) \equiv 0 \pmod 9 = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n & \text{if } n \equiv 3 \pmod 8 \\ \frac{3n+1}{8} & \text{if } n \equiv 5 \pmod 8 \\ \frac{9n+1}{8} & \text{if } n \equiv 7 \pmod 8 \\ \frac{3n+1}{4} & \text{if } n \equiv 1 \pmod{32} \\ \frac{3n+5}{32} & \text{if } n \equiv 9 \pmod{32} \\ \frac{9n+7}{32} & \text{if } n \equiv 17 \pmod{32} \\ \frac{n+7}{64} & \text{if } n \equiv 57 \pmod{64} \\ \frac{9n+31}{128} & \text{if } n \equiv 25 \pmod{128} \\ \frac{3n+5}{16} & \text{if } n \equiv 89 \pmod{128} \end{cases}$$

It converges to 1 for all natural numbers, if Collatz conjecture is true.

28, 42, 63, 32, 48, 72, 108, 162, 243, 122, 183, 92, 138, 207, 104, 156, 234, 351, 176, 264, 396, 594, 891, 446, 669, 335, 168, 252, 378, 567, 284, 426, 639, 320, 480, 720, 1080, 1620, 2430, 3645, 1823, 912, 1368, 2052, 3078, 4617, 2309, 1155, 578, 867, 434, 651, 326, 489, 245, 123, 62, 93, 47
maps to:
7, 8, 12, 18, 27, 81, 23, 26, 39, 44, 66, 99, 297, 28, 42, 63, 71, 80, 120, 180, 270, 405, 152, 228, 342, 513, 385, 289, 217, 41, 4

3.1.5 Function Iteration.

$$f(n) = \begin{cases} \frac{9n}{4} & \text{if } n \equiv 0 \pmod 4 \\ \frac{3n+2}{4} & \text{if } n \equiv 2 \pmod 4 \\ \frac{3n+3}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 4 \end{cases}$$
$$\text{Eg: } 8, 18, 14, 11, 9, 3, 3$$

$$f(n) = \begin{cases} \frac{27n}{8} & \text{if } n \equiv 0 \pmod 8 \\ \frac{9n+4}{8} & \text{if } n \equiv 4 \pmod 8 \\ \frac{9n+6}{8} & \text{if } n \equiv 2 \pmod 8 \\ \frac{3n+6}{8} & \text{if } n \equiv 6 \pmod 8 \\ \frac{9n+9}{8} & \text{if } n \equiv 7 \pmod 8 \\ \frac{3n+7}{8} & \text{if } n \equiv 3 \pmod 8 \\ \frac{3n+9}{8} & \text{if } n \equiv 5 \pmod 8 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \end{cases}$$
$$\text{Eg: } 8, 27, 11, 5, 3, 2$$

$$f(n) = \frac{3n + 3 \times 2^{i-1} - 3}{2^i} \text{ if } n = 2^i k + 2^{i-1} + 1$$

where i ranges from 1 to ∞ as we divide the last case into two cases infinitely.

3.2 Proportional Distribution of Powers of 3 or 2

When mapping the inverse tree of the Collatz trajectory, There are two occurrences of 3^{i-1} factors situated between two instances of 3^i factors on onward tree) .

- There are only two $3^i k$ numbers between two $3^{i+1} k$ numbers.
- The maximum number of $3^{i+j} k$ numbers between two $3^i k$ numbers is only one, for i and j greater than 1.
- All $3k$ numbers are separated by only one $3k + 2$ number.

Example:
27, 53, 105, 209, 417, 833, 1665, 3329, 6657, 13313, 26625, 53249, 106497, 212993, 425985, 851969, 1703937, 3407873, 6815745

Lemma 2 For $3k$, $6k-1$, and $12k-3$, all pairs of $3k$ numbers are separated by one $3k+2$ number. From this, when we formulate sequences of $3k$ numbers:

$$f(n) = 4n - 3$$

$9k$, $36k-3$, $144k-15$, $576k-63$, all pairs of $9k$ numbers are separated by two $3k$ numbers. From this, when we formulate sequences of $9k$ numbers:

$$f(n) = 64n - 63$$

By following the same principle, $3^i k$ can be formulated by $2^{2j}n - 3^i l$. If we start the sequence with $3^{i+1}k$, the sequence is:

$$\begin{aligned}
&3^{i+1}k, \\
&2^{2j}3^{i+1}k - 3^i l, \\
&2^{4j}3^{i+1}k - 2^{2j}3^i l - 3^i l, \\
&2^{6j}3^{i+1}k - 2^{4j}3^i l - 2^{2j}3^i l - 3^i l
\end{aligned}$$

where $j_1 = 1$ and $j_{i+1} = 3j_i$.

The fourth term is a factor of 3^{i+1} because j is even and $2^{4j} + 2^{2j} + 1$ is a factor of 3:

$$\begin{aligned}
2^{4j}n &\equiv 1 \pmod 3, \\
2^{2j}n &\equiv 1 \pmod 3, \\
1 &\equiv 1 \pmod 3
\end{aligned}$$

Adding them:

$$2^{4j} + 2^{2j} + 1 \equiv 0 \pmod 3$$

Thus:

$$2^{6j}3^{i+1}k - 2^{4j}3^i l - 2^{2j}3^i l - 3^i l = 3^i\left(3 \times 2^{6j} - (2^{4j}l + 2^{2j}l + 1)\right) = 3^{i+1}m$$

Therefore, the size of a tree or branches of the inverse tree of the Collatz function has a proportional growth rate based on the initial condition. This behavior of the Collatz sequence maintains the proportionality of the size of branches and prevents the occurrence of an unbalanced growth rate of a tree or branches. This property is one of crucial properties to decide the collatz conjecture. This property of equivalent distribution of numbers and their powers works for any number and any Kaakuma sequence.

3.3 Constants

3.3.1 Nearly Constant Expansion Rate of Inverse Tree Map

The average growth rate of the Collatz inverse tree map is $\frac{1}{3}$.

$$f(n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \pmod 3 \\ 2n-1 & \forall n \ (n \in \mathbb{N}) \end{cases}$$

Let us start from 2 as the root of the tree and ignore recycling because the 2 and 3 cycling cases duplicate data. The main root of the tree is 2, {2}, {3}, {5}, {9}, {6, 17}, {4, 11, 33}, {7, 21, 22, 65}, ...

Expansion Rate Analysis

The expansion rate, on average, is $\frac{1}{3}$. For lists with more than 30 elements $\frac{1}{3}$ of the numbers are $3k$, $\frac{1}{3}$ are $3k+1$, and $\frac{1}{3}$ are $3k+2$. Among these, $3k$ creates double nodes $6k-1$ and $2k$. That is why the expansion rate is $\frac{1}{3}$.

H, LC, TS is for Height, Leaf Count and Tree Size respectively

H	LC	TS	Leafs
1	1	1	2
2	1	2	3
3	1	3	5
4	1	4	9
5	2	6	6, 17
6	3	9	4, 11, 33
7	4	13	7, 21, 22, 65
8	5	18	13, 14, 41, 43, 129
9	6	24	25, 27, 81, 85, 86, 257
10	8	32	49, 18, 53, 54, 161, 169, 171, 513
11	12	44	97, 12, 35, 105, 36, 107, 321, 337, 114, 341, 342, 1025
12	18	62	193, 8, 23, 69, 70, 209, 24, 71, 213, 214, 641, 673, 76, 227, 681, 228, 683, 2049
13	24	86	385, 15, 45, 46, 137, 139, 417, 16, 47, 141, 142, 425, 427, 1281, 1345, 151, 453, 454, 1361, 152, 455, 1365, 1366, 4097
14	31	117	769, 10, 29, 30, 89, 91, 273, 277, 278, 833, 31, 93, 94, 281, 283, 849, 853, 854, 2561, 2689, 301, 302, 905, 907, 2721, 303, 909, 910, 2729, 2731, 8193
15	39	156	1537, 19, 57, 20, 59, 177, 181, 182, 545, 553, 555, 1665, 61, 62, 185, 187, 561, 565, 566, 1697, 1705, 1707, 5121, 5377, 601, 603, 1809, 1813, 1814, 5441, 202, 605, 606, 1817, 1819, 5457, 5461, 5462, 16385
16	50	206	3073, 37, 38, 113, 39, 117, 118, 353, 361, 363, 1089, 1105, 370, 1109, 1110, 3329, 121, 123, 369, 373, 374, 1121, 1129, 1131, 3393, 3409, 1138, 3413, 3414, 10241, 10753, 1201, 402, 1205, 1206, 3617, 3625, 3627, 10881, 403, 1209, 404, 1211, 3633, 3637, 3638, 10913, 10921, 10923, 32769

Table 2: Tree growth data

The table above shows the leaf count in each step with new branches approaching a size $\frac{1}{3}$ of the previous leaf count.

H	L Count	T Size	Rate	H	L Count	T Size	Rate
1	1	1		30	2829	11301	33.317
2	1	2	0	31	3765	15066	33.085
3	1	3	0	32	5014	20080	33.1739
4	1	4	0	33	6682	26762	33.266
5	2	6	100	34	8902	35664	33.223
6	3	9	50	35	11878	47542	33.430
7	4	13	33.333	36	15844	63386	33.389
8	5	18	25	37	21122	84508	33.312
9	6	24	20	38	28150	112658	33.273
10	8	32	33.333	39	37536	150194	33.342
11	12	44	50	40	50067	200261	33.383
12	18	62	50	41	66763	267024	33.347
13	24	86	33.333	42	89009	356033	33.320
14	31	117	29.166	43	118631	474664	33.279
15	39	156	25.806	44	158171	632835	33.330
16	50	206	28.205	45	210939	843774	33.361
17	68	274	36	46	281334	1125108	33.372
18	91	365	33.823	47	375129	1500237	33.339
19	120	485	31.868	48	500106	2000343	33.315
20	159	644	32.5	49	666725	2667068	33.316
21	211	855	32.704	50	888947	3556015	33.330
22	282	1137	33.649	51	1185305	4741320	33.338
23	381	1518	35.106	52	1580518	6321838	33.342
24	505	2023	32.545	53	2107346	8429184	33.332
25	665	2688	31.683	54	2809845	11239029	33.335
26	885	3573	33.082	55	3746399	14985428	33.331
27	1187	4760	34.124	56	4995078	19980506	33.330
28	1590	6350	33.951	57	6660211	26640717	33.335
29	2122	8472	33.459				

Table 3: Growth Rate of Leaf Count with Heights

The table above shows the leaf count and tree size at each height with their corresponding rate of expansion. The average expansion rate remains close to $\frac{1}{3}$ as the tree grows.

3.3.2 Average Stopping Time

$$f(n) = \begin{cases} 3n+1 & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

Average stopping time of this sequence is 3.49269.

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{3n+1}{4} & \text{if } n \equiv 1 \pmod 4 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Average stopping time of this sequence is 3.037.

3.3.3 Ratio of Stopping Time

The ratio of stopping time to $\log_2(n)$ is bounded. Specifically, this ratio is less than 10 for large starting numbers (more than 8 digits). For such large numbers, the ratio is bounded and typically less than 6. For a starting number 2^p and stopping time t, the ratio ranges from 3.67 to 5.15.

p	187	188	189	190	191	192	193	194	195	196	197
t	693	690	753	753	753	749	994	994	994	994	747
t/p	3.71	3.67	3.98	3.96	3.94	3.90	5.15	5.12	5.10	5.07	3.79

Table 4: Ratio of stopping time t to $\log_2(n)$

3.4 Stopping Time Iteration Groups

When we group the numbers by iteration, some numbers have the same number of stopping times and are grouped by $2^t k + c$. If the iterations of c's stopping time is t and $2^t > c$, then all the numbers noted $2^t k + c$ have stopping time t.

t	4	7	5	7	5	59	56	8	54	7	54	51	8	45	8
c	4	8	12	16	24	28	32	40	48	60	64	72	80	92	96

Table 5: Stopping Time Iteration Groups

For corresponding values t to c, $2^t k + c$ has the same stopping time of t. For example:

- The stopping time t of $2^5 k + 12$ is 5,
- The stopping time t of $2^7 k + 16$ is 7,
- The stopping time t of $2^5 k + 24$ is 5,
- The stopping time t of $2^5 9k + 28$ is 59.

Riho Terras (1976) showed that almost all initial values (more than 99.99%) eventually become a value less than n. This is 100 times the sum of the reciprocals of stopping times grouped: $100 \times \sum 1/2^t$.

3.5 Connection of $2n$ in Collatz Iteration

$2n$ is connected with n or $4n$. When we iterate the Collatz function in the translated format using $\frac{3n}{2}$ and $\frac{n+1}{2}$, $2n$ is connected with n or $4n$.

If there is a new cycle, $2n$ must be connected with $4n$ because a new cycle's starting value cannot be connected with a smaller one. This property makes the Collatz structure very special and provides a visible framework for a potential proof of the Collatz conjecture.

Lemma 3

Let $2^{i+1} \cdot o$ be the first number of a non-trivial cycle (where o denotes an odd integer). Then $2^{i+2} \cdot o$ is also in the new cycle.

We define:

$$\begin{aligned} 2^i \cdot o &\to 3^i \cdot o = m \\ 2^{i+1} \cdot o &\to 3^{i+1} \cdot o = 3m \\ 2^{i+2} \cdot o &\to 3^{i+2} \cdot o = 9m \end{aligned}$$

Now we can compare m, $3m$, and $9m$, all odd numbers since o is odd.

Case 1: Let $m = 4k + 3$

$$\begin{aligned} m = 4k + 3 &\to 2k + 2 \to 3k + 3 \\ 3m = 12k + 9 &\to 6k + 5 \to 3k + 3 \end{aligned}$$

In this case, $3m$ is connected with a smaller number, so it cannot be the start of a new cycle.

Case 2: Let $m = 4k + 1$

$$m = 4k + 1 \rightarrow 2k + 1 \rightarrow n + 1$$
$$3m = 12k + 3 \rightarrow 6k + 2 \rightarrow 9k + 3$$
$$9m = 36k + 9 \rightarrow 18k + 5 \rightarrow 9k + 3$$

Now $3m$ is connected with $9m$, meaning if there is a non-trivial cycle that starts with n, then $2n$ is also a node in it.

4 Proofs

4.1 Contradiction in Tree Size Density

Tree size is defined as the number of nodes connected to a cycle in the inverse of the Collatz sequence. **Tree size density** refers to the number of nodes relative to the whole set of natural numbers. The tree size density of a cycle is the ratio of nodes connected to it to the whole set of natural numbers. If a Kaakum sequence has a single cycle, its tree size is the whole set of natural numbers, and its density is $1 or 100\%$.

The **root** of a tree is the smallest natural number in any cycle.

Factors contributing for Tree size Density in a Sequences

- **The gap between roots:** As the gap between roots of trees in a sequence increases, the density of the previous cycle increases.
- **Number of iterations of a loop:** As the number of iterations in a loop increases, the corresponding tree becomes denser.
- **Increasing rate of increasing portion:** increasing rate of the increasing part of a sequence provides an opportunity to occurrence of a large gap of a Kaakuma sequence.
- **Initially obtaining more nodes:** If the tree of a cycle initially produces more nodes, it tends to be denser overall.

4.1.1 Proof 1: Contradiction of Tree Size Density Before and After Scaling Down

This proof is conducted via **proof by contradiction**.

Let **non-trivial cycle exists** we can investigate its tree size density in different ways and evaluate consistency of the tree size density.

The **Tree size densities** before scaling down Assume 2^{80} is the first non-Collatz number. We consider the minimal portion of the tree leaves of cycle 1 that can be traced before 2^{80}.

Growing the tree until the largest leaf in the Collatz inverse tree exceeds 2^{80}, we find that this occurs at height 81. At this point, the leaf count exceeds 6.6×10^9. The largest leaf at height 81 is $2^{80} + 1$. This is a conservative estimate for the leaf count compared to a non-trivial cycle, since $2^{80} - 3.3 \times 10^{10}$ nodes are traced back makes the tree denser, while 2^{80} never traces a smaller number.

This is to mean that if $3^{50} < 2^{80}$ is a leaf at height 81 that was not previously traced, it would generate many more leaves less than 2^{80} at different heights. Hence, the density of the trivial cycle of Collatz inverse tree map is significantly more than 6.6×10^9 times than the density of a hypothetical non-trivial cycle. we can also use projection of tree growth in 3.3.1 before height 80 and it is 117,869,914,517 continuing from height 57. from this tree size density of the trivial cycle is $1 - 10^{-11}$ and tree size density of non-trivial cycle is 10^{-11}

The **Tree size densities** After scaling down We observe a contradiction when we scale down, as seen in Property 3.1.4. Scaling down removes the gap that allows the trivial cycle to dominate before reaching the non-trivial one. We also examine the initially connected nodes by the non-trivial cycle.

$$f(n) = \begin{cases} \dfrac{2n}{3} & \text{if } n \equiv 0 \pmod 3 \\ 2n - 1 & \forall n \in \mathbb{N} \end{cases}$$

Assume $2p$ is the first non-Collatz number, and define a scaled-down function:

$$g(n) = \frac{f(n)}{p} \quad \text{if } f(n) \equiv 0 \pmod p$$

Then, the first non-trivial cycle maps to 2. The values $2, 3, 4, 6$, and 9 are known nodes in the non-trivial inverse tree from the beginning, making it denser than the trivial cycle tree map with only the known root 1.

If $2p$ is the root of the non-trivial cycle, then $4p$ is also a node in that cycle. That is why $2, 3, 4, 6$, and 9 are nodes in the non-trivial cycle after scaling down. If we want, we may displace initially connected nodes by changing the scaling down factor. For example, scaling down by $2p$ results in $1, 2, 3$ as the known initial nodes of the non-trivial cycle. Here, the density of trivial cycle is less than 0.5 and the density of non-trivial cycle is greater than 0.5 after scaling down.

In addition, a non-trivial cycle has a high number of iterations in the first loop, making the corresponding cycle denser.

Conclusion There exists a significant contradiction in the density relationship of the inverse tree sizes for the trivial and non-trivial cycles of the Collatz sequence, before and after scaling down. This shows that a non-trivial cycle in the Collatz sequence never exist, supporting the truth of the Collatz Conjecture.

Further insight can be gained by constructing related sequences for illustration and clarification. To assess their relevance, we utilize various Kaakuma sequences closely related to the Collatz sequence.

Semi-Cycled Sequences of Collatz Sequences

To obtain meaningful results, the starting number of a cycle should be the smallest number of the semi-cycle in the forward sequence. We use the *stopping point* to identify sub-cycles. When using the inverse tree, the starting number corresponds to a local minimum or a saddle point.

Examples

- **Example 1:** Sub-cycle $7 \to 11 \to 17 \to 26 \to 13$ versus $1 \to 2$
  Full cycle comparison: (7–13, 4 iterations) vs (1–2, 1 iteration).
  Densities: 52.47% (non-trivial), 47.53% (trivial)

- **Example 2:** Sub-cycle $27 \to 41 \to 62 \to \ldots \to 23$ vs $1 \to 2$
  (27–23, 59 iterations) vs (1–2, 1 iteration).
  Densities: 46.115% (non-trivial), 53.885% (trivial)

- **Example 3:** Sub-cycle $191 \to 287 \to 431 \to \ldots \to 154$ vs $1 \to 2$ (191–154, 13 iterations) vs (1–2, 1 iteration).
  Densities: 5.3325% (non-trivial), 94.6675% (trivial)

These examples indicate that non-trivial cycles in the Collatz sequence tend to have comparable densities due to their large number of iterations.

The $3n - 1$ Sequence

We analyze the inverse of the $3n/2$ sequence defined as:

$$f(n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \pmod 3 \\ 2n & \forall n \in \mathbb{N} \end{cases}$$

This sequence has three roots: 0, 4, and 16, with 1, 3, and 11 iterations, respectively.

Size and Density of Trees

Scaled down by	Size1	Size2	Size3	Density1	Density2	Density3
1	2615505	2597930	2786564	32.69381	32.47413	34.83205
3	2613554	2600664	2785781	32.66943	32.50830	34.82226
9	2611674	2601054	2787271	32.64593	32.51318	34.84089
27	2609873	2602065	2788061	32.62341	32.52581	34.85076
81	2609796	2602512	2787691	32.62245	32.53140	34.84614
243	2609861	2603354	2786784	32.62326	32.54193	34.83480
225	2610226	2603335	2786438	32.62783	32.54169	34.83048

Table 6: Comparison of Tree Sizes and Densities at Different Scaling Levels

This demonstrates that the density of non-trivial cycles is approximately equal to the density of trivial cycles after appropriate scaling down.

The $3n + p$ Sequence

$$f(n) = \begin{cases} \frac{3n+p}{2} & n \equiv 1 \pmod 2 \\ \frac{n}{2} & n \equiv 0 \pmod 2 \end{cases}$$

To examine the effect of large gaps, consider the sequence:

$$f(n) = \begin{cases} \frac{3n+1394753}{2} & n \equiv 1 \pmod 2 \\ \frac{n}{2} & n \equiv 0 \pmod 2 \end{cases}$$

This sequence has only two cycles: 1 and 1394753. Their tree size densities are:

$$\text{Cycle 1: } \frac{1394752}{1394753}, \quad \text{Cycle 2: } \frac{1}{1394753}$$

After scaling down by $\frac{p-1}{2} = 697376$ or $\frac{p+1}{2} = 697377$, the gap between cycles reduces from 1394752 to 1. This shows that iteration count of the first cycle has a major impact on maintaining consistent tree size density after scaling down.

The $3n+p$ sequence helps understand how the non-trivial cycle of $3n+1$ behaves. However, as p increases p and 1 are only cycles, the iteration length of the first cycle may grow excessively.

More Sequences

Let us compare sequences with relatively large gaps but only two cycles.

Equations

$$\text{Eq1: } f(n) = \begin{cases} \dfrac{22n-54}{3} & \text{if } n \equiv 0 \pmod 3 \\ \lceil \frac{n}{3} \rceil & \text{otherwise} \end{cases}$$

$$\text{Eq2: } f(n) = \begin{cases} \dfrac{41n}{4} & \text{if } n \equiv 0 \pmod 4 \\ \dfrac{5n+2}{4} & \text{if } n \equiv 2 \pmod 4 \\ \lceil \frac{n}{4} \rceil & \text{otherwise} \end{cases}$$

$$\text{Eq3: } f(n) = \begin{cases} \dfrac{6n}{5} & \text{if } n \equiv 0 \pmod 5 \\ \dfrac{197n+2}{5} & \text{if } n \equiv 4 \pmod 5 \\ \lceil \frac{n}{5} \rceil & \text{otherwise} \end{cases}$$

$$\text{Eq4: } f(n) = \begin{cases} \dfrac{243n - 644}{4} & \text{if } n \equiv 0 \pmod 4 \\ \lceil \frac{n}{4} \rceil & \text{otherwise} \end{cases}$$

$$\text{Eq5: } f(n) = \begin{cases} \dfrac{207n - 668}{4} & \text{if } n \equiv 0 \pmod 4 \\ \lceil \frac{n}{4} \rceil & \text{otherwise} \end{cases}$$

$$f(n) = \begin{cases} \dfrac{3n + 1000209607}{2} & \text{if } n \equiv 1 \pmod 2 \\ \dfrac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases} \qquad \text{(Eq6)}$$

New Cycles and Densities

Equation	New Cycle	Density	First Few Nodes After Scaling Down
Eq1	2151	0.0725%	2, 347, 1661, 3068, 4981, 5773, 8674, 14943, 15203, 15205, 17012, 17319, 20879, 22046, 22496, 22506, 22519, 22902, 25646
Eq2	46040	0.0035%	2, 1718, 2046, 26639, 27440, 94164, 123666, 181557, 201781, 238202, 255311, 281260, 282024, 297171, 318439, 328915
Eq3	87194	0.000009%	2, 394, 156249, 166317, 239061, 526696, 572774, 599757, 807307, 831583, 868781, 940487, 1434351, 1438936, 1655078
Eq4	107612	0.000025%	2, 145471, 146994, 161451, 188779, 209516, 247514, 258076, 349152, 379525, 535323, 563743, 563973, 581883, 673291
Eq5	114288	0.003%	2, 50118, 95807, 129686, 144989, 163801, 245307, 278639, 335573, 375173, 435339, 473948, 476444, 497122, 524268
Eq6	1000209607	10^{-9}%	$1000209607k - 1000209605$ k is natural number

Table 7: Comparison of Tree Sizes Densities and Distribution of Nodes of Second Cycle After Scaling Down

Discussion

From this experiment, we observe that as the geometric mean increases and the High Variance of coefficients that valid only natural numbers, there is a potential for increasing the gap. Increasing in gap corresponds to an increase in the density of the first cycle.

However, when we compare geometric mean, variances of coefficients, gap, and density, the relationships become inaccurate. A more reliable metric for comparison is the *average stopping time, gap and density.*

Even if we assume a non-trivial Collatz cycle exists beyond 2^{68}, its density would be less than $10^{-17}\%$, and its second node after scaling down would be greater than 10^{16}. In contrast, for the non-trivial cycle associated with the classic Collatz sequence, the scaled down first few nodes are $2, 3, 4, 6, 9$, consistent with behaviors noted in Sections 3.1.4 and 3.6.

This means that if the unscaled sequence contains $2p, 3p, 4p, 6p, 9p$, after dividing by p, we can get $2, 3, 4, 6, 9$. This supports the validity of the Collatz conjecture.

Although it is highly unlikely to find a sequence exhibiting the exact behavior of the Collatz conjecture with both a connection between n and $2n$, and a small average stopping time after few iterations, the most promising approach is to search for sequences with:

- Similar average stopping time,
- Nearly the same base/modulus cases.

Combinations after Function Iteration

If we iterate the Collatz function from Section 3.2 three times, the result is a 16-branch equation, with coefficients 3^i for $i = 0$ to 4, and branch counts based on binomial coefficients:

$$1, 4, 6, 4, 1$$

Let us define:

$$f(n) = \left\lceil \frac{3^i n}{16} \right\rceil \quad \text{for } n \equiv r \pmod{16}$$

There are more than 50 million combinations when considering all module configurations for $\mod 16$, but after checking more than 200,000 of them, the maximum observed gap was 25. This suggests that if there were non-trivial cycle of $3n + 1$ its it is less than 100.

In general, for any Kaakuma sequence having two cycles with a big gap like 10^{20}:

- It is connected with the number of iterations of cycles and the increasing rate of increasing portion.
- If the first cycle has few iterations as in the Collatz sequence, the increasing rate must be large, and greater than 10^{10} Eqs $(1 - 5)$.
- If the increasing rate is small as in the $3n + p$ sequence, the number of iterations of the first cycle is more than 10^{14} $Eq6$ first cycle has 7901480 iterations.
- If the gap is very large, the nodes of non-trivial cycle are very dispersed even after scaled down because it scarcity. Eg Eqs $(1 - 6)$

4.1.2 proof 2 Contradiction of Tree Size Density Before Scaling Up and After Scaling Up

Scaling up is the reverse operation of scaling down, and we can compare the density of tree size with trivial cycle 1 and non-trivial cycle 2 somewhere above 20 digit numbers. When we scale up by a scaling factor usually the product of coefficient powers to avoid occurrences of new cycles, the gap between trivial cycle and non-trivial cycle increases significantly. This result unbalanced tree size density before and after scaling up.

How much can we increase the gap by scaling up, and what will happens to the density of the non-trivial cycle part after a significant increment of the gap between cycles?

Let $2^{80} - 1$ be a non-trivial root, and let $3^{1,000,000}$ be the scaling up factor. If we aim to maximize the gap between two cycles, we cannot use just any scaling up factor.

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod{2}, \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

Its non-trivial root is $2^{80} - 1$ and the smallest ratio of tree size density is $1 : 2^{80}$. To maximize the scaling factor gap between cycles, we use the inverse function after applying a scaling factor 3^x:

$$f(n) = \begin{cases} \frac{3n+3^x}{2} & \text{if } n \equiv 1 \pmod 2, \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2. \end{cases} \quad \text{(scaled up)}$$

$$f^{-1}(n) = \begin{cases} \frac{2n-3^x}{3} & \text{if } n \equiv 0 \pmod 3, \\ 2n & \text{for all } n. \end{cases} \quad \text{(inverse)}$$

Now the starting number is transformed to $3^x(2^{80}-1)$. To get the maximum value of x, we analyze the gap function:

$$f(x) = 2^{80+x} - 3^x$$
$$f'(x) = 2^{81+x}\ln 2 - 3^x \ln 3$$

Setting $f'(x) = 0$, we find:

$$2^{81+x}\ln 2 = 3^x \ln 3 \quad \Rightarrow \quad x \approx 136.3$$

The nearest integer is $x = 136$, so we can maximize the gap from 2^{80} to $2^{216} - 3^{136} \approx 2^{214}$. The density diminishes by 2^{134}. If the density of the non-trivial cycle were d before, it becomes less than $d/2^{134}$ after scaling up that is impossible and inconsistent.

Now consider sequences with two or more roots and arbitrary large scaling factors. Do they diminish their tree size densities relative to each other, or maintain consistency of their proportionality?

There are methods to maintain tree size density consistency before and after scaling up:

We can use the sequence below to make the change in densities extreme in the first proof.

$$f(n) = \begin{cases} \frac{3n+3^{216}}{2} & \text{if } n \equiv 1 \pmod 2, \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

1. Unmovable Root

If the smallest number in the scaled-up sequence remains the same, then the proportionality of tree size density remains consistent.

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2, \\ \frac{n+43}{2} & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

This sequence has two roots: 1 and 43. Their tree size density ratio is $1:43$. Scaling by 3^x does not affect their proportionality, since all factors of 43 remain connected to root 43 and the rest to root 1.

2. Immersion of Counterbalancing Values

If a sequence can be scaled by any factor (with negative constants), proportionality is preserved by interposing values. More values are immersed before root 2 to keep tree size density consistent.

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2, \\ \frac{n-1}{2} & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

This sequence has roots 0, 4, and 16. When scaled by 3^x, these become connected via:

$$\frac{2^x \cdot \{0, 4, 16\}}{3^x}$$

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2, \\ \frac{n-81}{2} & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

3. Stretching Back

If the scaling factor is too large, the second cycle stretches back sufficiently to the left to maintain proportionality.

Example 1:

$$f(n) = \begin{cases} \frac{11n+181}{2} & \text{if } n \equiv 3 \pmod 4, \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 4, \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Roots: 1, 35267, 63051.

Scaled up:

$$f(n) = \begin{cases} \frac{11n+181\cdot 11^x}{2} & \text{if } n \equiv 3 \pmod 4, \\ \frac{n+3\cdot 11^x}{4} & \text{if } n \equiv 1 \pmod 4, \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Inverse:

$$f^{-1}(n) = \begin{cases} \frac{2n - 181 \cdot 11^x}{11}, \\ 4n - 3 \cdot 11^x, \\ 2n. \end{cases}$$

Let $n_0 = 35267 \cdot 11^x$. Then,

$$n_m = 2^x \cdot 35267 - 181 \cdot 2^x + 181$$

Apply this to:

$$f(x) = 4n_m - 3 \cdot 11^x = 35086 \cdot 2^{x+2} - 3 \cdot 11^x$$

Maximum at $x = 4$, so the gap becomes 140525 (from 35267). **NB** small change comes by scaling up counterbalanced by the number of iterations in the first cycle. when we scale up by proper amount of scaling factor, number of iterations of first cycle increases and balances the change.

4.2 proof 3 The Vanishing Ratio of Binomial Sum of increasing portion in Function Iteration

When we iterate the Collatz function successively, a binomial number system pattern emerges. This pattern involves the number of powers of three as coefficients of a variable combined with the total number of cases equal to the base. When we successively iterate cases infinitely, the sum of binomials where the coefficients satisfy $\frac{3^i}{2^n} > 1$ approaches zero:

$$\sum_{\frac{3^i}{2^n} > 1} \binom{n}{i} = 0.$$

From Section 3.1.5 when we extend function iterations infinitely:- We can start from zero to show that successive iterate cases of the sequence generate a binomial form with 3^i coefficients in the numerator.

$$f(n) = 0, \quad f(n) = n, \quad \text{and so on.}$$

No.	Number of Cases	Number of 3^i (i from 0 to n)
1	0	0
2	1	1
3	2	$1, 1$
4	3	$1, 2, 1$
5	4	$1, 3, 3, 1$
6	5	$1, 4, 6, 4, 1$
7	6	$1, 5, 10, 10, 5, 1$

Lemma 4

Let 3^{i-1} and 3^i have b_{i-1} and b_i amounts for $f(n)$ in the next division of cases. The b_i amount of 3^i and b_{i-1} amount of $3 \cdot 3^{i-1} = 3^i$ results in $b_i + b_{i-1}$ amounts of 3^i generated.

This is how Pascal's triangle develops for binomials:

$$b_i = b_i + b_{i-1}.$$

Central Limit Theorem (CLT) Insight

For large n, the binomial distribution $X \sim \text{Bin}(n, 0.5)$ approximates a normal distribution:

$$X \sim N(\mu, \sigma^2), \quad \mu = \frac{n}{2}, \quad \sigma^2 = \frac{n}{4}.$$

Thus, the probability $P(X > nt)$ can be approximated using:

$$Z = \frac{X - \mu}{\sigma},$$

where Z is a standard normal random variable. The condition $X > nt$ translates to:

$$P(X > nt) = P\left(Z > \frac{nt - \mu}{\sigma}\right).$$

Substituting $\mu = \frac{n}{2}$ and $\sigma = \sqrt{\frac{n}{4}} = \frac{\sqrt{n}}{2}$, we get:

$$P(X > nt) = P\left(Z > \frac{nt - \frac{n}{2}}{\frac{\sqrt{n}}{2}}\right) = P\left(Z > \sqrt{n} \cdot (2t - 1)\right).$$

For $t = \log_2(3) \approx 1.585$, we have $2t - 1 > 1$. As $n \to \infty$, the term $\sqrt{n} \cdot (2t - 1) \to \infty$, and since the tail probability of a standard normal distribution decays exponentially:

$$P(Z > \sqrt{n} \cdot (2t - 1)) \to 0.$$

Chernoff Bound for Rigorous Proof

The Chernoff bound provides an upper bound for $P(X > nt)$:

$$P(X > nt) \leq \exp\left(-n \cdot D(t \,||\, 0.5)\right),$$

where $D(t \,||\, 0.5)$ is the relative entropy:

$$D(t \,||\, 0.5) = t \log\left(\frac{t}{0.5}\right) + (1 - t) \log\left(\frac{1 - t}{0.5}\right).$$

For $t > 0.5$, $D(t \,||\, 0.5) > 0$. Thus:

$$P(X > nt) \leq \exp\left(-n \cdot \text{constant}\right),$$

which decays exponentially as $n \to \infty$. The sum of binomial coefficients

$$\sum_{i > n \log_2(3)} \binom{n}{i}$$

normalized by 2^n approaches 0 as $n \to \infty$ because the probability

$$P\left(X > n \log_2(3)\right)$$

decays exponentially.

NB:- when we successively divide after scaling down it is mor rapid.

4.3 Proof 4 Qodaa Ratio Test

The Qodaa Ratio Test is a method of analyzing the product of coefficients of cases with their occurrences as power of a Kaakuma sequence. Kaakuma sequence is a sequence of integers that fluctuating up and down based on conditions and it is equated with two or more well defined conditions. The Kaakuma sequence is a broad generalization of the Collatz sequence. The Qodaa Ratio Test helps in determining the exact limit coefficients where diverging occurs by examining the ratio of products of numerators to denominators with their occurrences.

$$f(n) = \begin{cases} 3n+1 & \text{if } n \equiv 1 \pmod 2 \\ \frac{3n-2}{4} & \text{if } n \equiv 2 \pmod 4 \\ \frac{n}{4} & \text{if } n \equiv 0 \pmod 4 \end{cases}$$

Case1 2k+1 it is half of natural numbers, it generates only one-fourth of natural numbers of case2 and one-fourth of natural numbers of case3 with ratio case2:case3=1/4:1/4=1:1.
Case2 4k+2 it is one-fourth of natural numbers, it generates half of natural numbers of case1, one-fourth of natural numbers of case2 and one-fourth of natural numbers of case3 with ratio case1:case2:case3=1/2:14:1/4=2:1:1 based on their fractions of natural numbers and Case3 4k it is one-fourth of natural numbers, it generates in the same with case2 When we calculate them by in-out rule they may have different occurrences amount of cases relatively. The occurrences amount of each case is used as the power of cases in product of coefficients.
Before starting we need to realize some points on Qodaa ratio as much as Qodaa Ratio Test is efficient and simple to apply.

- If cases do not have proportional chances of generating other cases, then the tree size of branches on the inverse tree map of the Kaakuma sequence is not applicable and nearly constant growth of leaves is not valid. Proportional cases generation validates tree size balance and vice versa.

- If cases do not have proportional chances of generating cases, then the generating amount must be negligible to avoid overload of tree size.

- The occurrences of cases, number of iterations, and occurrence of values are not random, even if they cannot be precisely determined. It is possible to infer them from behaviors discussed in Sections 3.3 (Successive Case Division) and 3.6 (Stopping Time Iteration Group).

- Even if occurrences are probabilistic, values like 3/4 must be interpreted and defined carefully, particularly as probabilistic value approach zero.
- if we force to vary natural law of generating of cases proportionally it is impossible to set rule when altered by successive partition or selective mapping .

$$f(n) = \begin{cases} \frac{k_1 n + c_1}{b_1} & \text{Case } 1 \\ \frac{k_2 n + c_2}{b_2} & \text{Case } 2 \\ \frac{k_3 n + c_3}{b_3} & \text{Case } 3 \\ \vdots & \vdots \\ \frac{k_i n + c_i}{b_i} & \text{Case } i \end{cases}$$

Qodaa Ratio Test states that if

$$R = \prod_i \left(\frac{k_i}{b_i} \right)^{p_i}.$$

If $R < 1$, the sequence is expected to be convergent or bounded. If $R > 1$, divergence is likely. When applying the in-out rule, these cases may have different occurrences. The occurrences of each case are used as the power of cases in the product of coefficients. Kaakuma sequences have many categories. Among them, we can check simple, complex, and complicated Kaakuma sequences only for positive integers.

4.3.1 Simple Kaakuma Sequence

In a simple Kaakuma sequence, each case generates all cases, and we can simply take the ratio of the cases' fractions of natural numbers to determine the occurrences of each case relatively. This will be consistent with the rule of in and out.

Example 1: Base Two

$$f(n) = \begin{cases} \frac{kn + c}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

Case 1: $n \equiv 1 \pmod 2$ is half of the natural numbers, and Case 2: $n \equiv 0 \pmod 2$ is also half of the natural numbers.

The ratio of Case 1 to Case 2 is $1/2 : 1/2 = 1 : 1$. From the Qodaa ratio test rule:

$$\left(\frac{k}{2}\right)^1 \times \left(\frac{1}{2}\right)^1 < 1 \implies \frac{k}{4} < 1 \implies k < 4$$

The sequence $f(n)$ with $k = 3$:

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio 3/4.

Example 2: Base Three

$$f(n) = \begin{cases} \frac{kn+c}{3} & \text{if } n \equiv 2 \pmod 3 \\ \frac{n+2}{3} & \text{if } n \equiv 1 \pmod 3 \\ \frac{n}{3} & \text{if } n \equiv 0 \pmod 3 \end{cases}$$

The ratio is 1/3 :1/3 : 1/3 = 1 : 1 : 1. by using Qodaa ratio rule:

$$\left(\frac{k}{3}\right)^1 \times \left(\frac{1}{3}\right)^1 \times \left(\frac{1}{3}\right)^1 < 1 \implies k/27 < 1 \implies k < 27$$

With $k = 26$:

$$f(n) = \begin{cases} \frac{26n-25}{3} & \text{if } n \equiv 2 \pmod 3 \\ \frac{n+2}{3} & \text{if } n \equiv 1 \pmod 3 \\ \frac{n}{3} & \text{if } n \equiv 0 \pmod 3 \end{cases}$$

converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of 26/27.

Example 3: Base Four

$$f(n) = \begin{cases} \frac{255n-261}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+2}{4} & \text{if } n \equiv 2 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 4 \\ \frac{n}{4} & \text{if } n \equiv 0 \pmod 4 \end{cases}$$

converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of $255/256 = 0.996$.

Compare this with the original Collatz sequence after the first successive division:

$$f(n) = \begin{cases} \frac{9n}{4} & \text{if } n \equiv 0 \pmod 4 \\ \frac{3n+2}{4} & \text{if } n \equiv 2 \pmod 4 \\ \frac{3n+3}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 4 \end{cases}$$

converges to 2 for all $n \in \mathbb{N}$ with a Qodaa ratio of $81/256 = 0.3045$.

Example 4: Base Eight

$$f(n) = \begin{cases} \frac{16777215n-116440489}{8} & \text{if } n \equiv 7 \pmod 8 \\ \frac{n+2}{8} & \text{if } n \equiv 6 \pmod 8 \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod 8 \\ \frac{n+4}{8} & \text{if } n \equiv 4 \pmod 8 \\ \frac{n+5}{8} & \text{if } n \equiv 3 \pmod 8 \\ \frac{n+6}{8} & \text{if } n \equiv 2 \pmod 8 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n}{8} & \text{if } n \equiv 0 \pmod 8 \end{cases}$$

converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of $16777215/16777216 = 0.99999994$.

Compare this with the original Collatz sequence after the second division:

$$f(n) = \begin{cases} \frac{27n}{8} & \text{if } n \equiv 0 \pmod 8 \\ \frac{9n+4}{8} & \text{if } n \equiv 4 \pmod 8 \\ \frac{9n+6}{8} & \text{if } n \equiv 2 \pmod 8 \\ \frac{3n+6}{8} & \text{if } n \equiv 6 \pmod 8 \\ \frac{9n+9}{8} & \text{if } n \equiv 7 \pmod 8 \\ \frac{3n+7}{8} & \text{if } n \equiv 3 \pmod 8 \\ \frac{3n+9}{8} & \text{if } n \equiv 5 \pmod 8 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \end{cases}$$

converges to 2 for all $n \in \mathbb{N}$ with a Qodaa ratio of 0.0317.

Example 5: Base Five

$$f(n) = \begin{cases} \frac{3124n-3131}{5} & \text{if } n \equiv 4 \pmod 5 \\ \frac{n+2}{5} & \text{if } n \equiv 3 \pmod 5 \\ \frac{n+3}{5} & \text{if } n \equiv 2 \pmod 5 \\ \frac{n+4}{5} & \text{if } n \equiv 1 \pmod 5 \\ \frac{n}{5} & \text{if } n \equiv 0 \pmod 5 \end{cases}$$

This sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of 0.99968.

Example 6: Base Six

$$f(n) = \begin{cases} \frac{46655n-46657}{6} & \text{if } n \equiv 5 \pmod 6 \\ \frac{n+2}{6} & \text{if } n \equiv 4 \pmod 6 \\ \frac{n+3}{6} & \text{if } n \equiv 3 \pmod 6 \\ \frac{n+4}{6} & \text{if } n \equiv 2 \pmod 6 \\ \frac{n+5}{6} & \text{if } n \equiv 1 \pmod 6 \\ \frac{n}{6} & \text{if } n \equiv 0 \pmod 6 \end{cases}$$

This sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of 0.999978.

Example 7: Base Seven

$$f(n) = \begin{cases} \frac{823542n-4200008}{7} & \text{if } n \equiv 6 \pmod 7 \\ \frac{n+2}{7} & \text{if } n \equiv 5 \pmod 7 \\ \frac{n+3}{7} & \text{if } n \equiv 4 \pmod 7 \\ \frac{n+4}{7} & \text{if } n \equiv 3 \pmod 7 \\ \frac{n+5}{7} & \text{if } n \equiv 2 \pmod 7 \\ \frac{n+6}{7} & \text{if } n \equiv 1 \pmod 7 \\ \frac{n}{7} & \text{if } n \equiv 0 \pmod 7 \end{cases}$$

This sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of 0.999998.

Example 8: Base Two with Sub-Cases

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{kn+c}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 4 \end{cases}$$

The cases share a ratio of $1/2 : 1/4 : 1/4 = 2 : 1 : 1$. The Qodaa ratio is:

$$\left(\frac{1}{2}\right)^2 \times \left(\frac{k}{4}\right)^1 \times \left(\frac{1}{4}\right)^1 = \frac{k}{64}$$

We can use the Qodaa Ratio Test to determine the values of k. For the condition $k/64 < 1$, we have $1 < k < 64$ for positive integer values of k.

Tabular Analysis of occurrences using in = out rule

Produced	**Generates**			**After solved**			sum	Simplified
	A	B	C	A	B	C		
a	2a	2b	2c	4c	2c	2c	8c	2
b	a	b	c	2c	c	c	4c	1
c	a	b	c	2c	c	c	4c	1

When we equate the generating and generated values of each case using the in-out rule:

$$a = b + c \quad 3b = a + c \quad 3c = a + b \quad b = c \quad a = 2c$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{63n-59}{4} & \text{if } n \equiv 1 \pmod 4 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

This sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of $\frac{63}{64}$.

Example 9:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{kn+c}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

The ratio of cases is $1/2 : 1/4 : 1/8 : 1/8$, which simplifies to $4 : 2 : 1 : 1$. The occurrences ratio yields:

$$\left(\frac{1}{2}\right)^4 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{k}{8}\right)^1 = \frac{k}{16384}$$

For positive integer values, $1 < k < 16384$.

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod{8} \\ \frac{16383n-81907}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases}$$

This sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of 0.999939.

when we set k in case2:

If we set k in line 2, the product of coefficient values differs due to the difference in power:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{kn+c}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod{8} \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases}$$

To determine the limit of k using Qodaa ratio rule:

$$\left(\frac{1}{2}\right)^4 \times \left(\frac{k}{4}\right)^2 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 = \frac{k^2}{16384} \implies 1 < k < 128$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{127n-369}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod{8} \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases}$$

This sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of 127/128.

when we set k in case1:

$$f(n) = \begin{cases} \frac{kn+c}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod{8} \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases}$$

Using Qodaa ratio rule

$$\left(\frac{k}{2}\right)^4 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 = \frac{k^4}{16384} \implies 1 < k < \sqrt{128}$$

$$f(n) = \begin{cases} \frac{11n-2}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

The sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of $11/\sqrt{128}$.

Example 10

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n-1}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n-3}{8} & \text{if } n \equiv 3 \pmod 8 \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod 8 \\ \frac{kn+c}{8} & \text{if } n \equiv 7 \pmod 8 \end{cases}$$

With ratio $1/2 : 1/8 : 1/8 : 1/8 : 1/8 = 4 : 1 : 1 : 1 : 1$:

$$\left(\frac{1}{2}\right)^4 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{k}{8}\right)^1 = \frac{k}{65536} \implies 1 < k < 65536$$

when we substitute k

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n-1}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n-3}{8} & \text{if } n \equiv 3 \pmod 8 \\ \frac{n+1}{8} & \text{if } n \equiv 7 \pmod 8 \\ \frac{65535n-327667}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

Converges to 0 for all $n \in \mathbb{N}$, with $QR = 65535/65536$.

When shifting the coefficient in the first line:

$$\left(\frac{k}{2}\right)^4 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 < 1 \implies 1 < k < 16$$

$$f(n) = \begin{cases} \frac{15n-28}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod{8} \\ \frac{n+5}{8} & \text{if } n \equiv 3 \pmod{8} \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod{8} \\ \frac{n+1}{8} & \text{if } n \equiv 7 \pmod{8} \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$, with $QR = 15/16$.

Example 11

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod{8} \\ \frac{n+11}{16} & \text{if } n \equiv 5 \pmod{16} \\ \frac{kn+c}{16} & \text{if } n \equiv 13 \pmod{16} \end{cases}$$

With ratio $1/2 : 1/4 : 1/8 : 1/16 : k/16 = 8 : 4 : 2 : 1 : 1$:

$$\left(\frac{1}{2}\right)^8 \times \left(\frac{1}{4}\right)^4 \times \left(\frac{1}{8}\right)^2 \times \left(\frac{1}{16}\right)^1 \times \left(\frac{k}{16}\right)^1 < 1 \implies 1 < k < 2^{30}$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod{8} \\ \frac{n+11}{16} & \text{if } n \equiv 5 \pmod{16} \\ \frac{2^{30}n-n-13\times 2^{30}+45}{16} & \text{if } n \equiv 13 \pmod{16} \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$, with $QR = \frac{2^{30}-1}{2^{30}}$.

When shifting k in line 3:

$$\left(\frac{1}{2}\right)^8 \times \left(\frac{1}{4}\right)^4 \times \left(\frac{k}{8}\right)^2 \times \left(\frac{1}{16}\right)^1 \times \left(\frac{k}{16}\right)^1 < 1 \implies 1 < k < 2^{15}$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{32767n-32751}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+11}{16} & \text{if } n \equiv 5 \pmod{16} \\ \frac{n+3}{16} & \text{if } n \equiv 13 \pmod{16} \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$.

When shifting k in line 2:

$$\left(\frac{1}{2}\right)^8 \times \left(\frac{k}{4}\right)^4 \times \left(\frac{1}{8}\right)^2 \times \left(\frac{1}{16}\right)^1 \times \left(\frac{k}{16}\right)^1 < 1 \implies 1 < k < 2^{15/2}$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{181n-535}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+11}{16} & \text{if } n \equiv 5 \pmod{16} \\ \frac{n+3}{16} & \text{if } n \equiv 13 \pmod{16} \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$.

When shifting k in line 1:

$$\left(\frac{k}{2}\right)^8 \times \left(\frac{1}{4}\right)^4 \times \left(\frac{1}{8}\right)^2 \times \left(\frac{1}{16}\right)^1 \times \left(\frac{k}{16}\right)^1 < 1 \Rightarrow 1 < k < 2^{15/4}$$

$$f(n) = \begin{cases} \frac{13n-4}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+11}{16} & \text{if } n \equiv 5 \pmod{16} \\ \frac{n+3}{16} & \text{if } n \equiv 13 \pmod{16} \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$.

4.3.2 Complex Kaakuma Sequence

In a complex Kaakuma sequence, at least one case is never generated by one or more cases. To analyze limit of converging values complex Kaakuma sequence, we use a tabular format to get the relative occurrence of each case.

Example 12

$$f(n) = \begin{cases} 3n+1 & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

(original Collatz sequence)

We organize and represent each line of conditions or cases with capital letters A, B, C to show producing amounts and small letters a, b, c to show produced amounts with their order.

Produced	Produces		Solved in Terms of b		Sum	Simplified
	A	B	A	B		
a		b		b	b	1
b	a	b	b	b	$2b$	2

$a = b$, $QR = 3^3 \times \left(\frac{1}{2}\right)^2 = \frac{3}{4}$

Example 13

$$f(n) = \begin{cases} 3n+1 & \text{if } n \equiv 0 \pmod 2 \\ \frac{3n+1}{2} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n-1}{4} & \text{if } n \equiv 1 \pmod 4 \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$, with $QR = \frac{27}{64}$.

Generated	Generates			Solved in Terms of c			Sum	Simplified
	A	B	C	A	B	C		
a			$2c$			$2c$	$2c$	1
b	a	b	c	c	$2c$	c	$4c$	2
c	a	b	c	c	$2c$	c	$4c$	2

$2a = 2c$, $b = a + c$, $3c = a + b$, $a = c$, $b = 2c$, $QR = 3^1 \times \left(\frac{3}{2}\right)^2 \times \left(\frac{1}{4}\right)^2 = \frac{27}{64}$

Example 14

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{kn+c}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 4 \end{cases}$$

When we use the generating and generated of each case, in=out rule.

Produced	Produces			Solved in Terms of a			Sum	Simplified
	A	B	C	A	B	C		
a	$2a$	$2b$		$2a$	$2a$		$4a$	1
b	a	b	c	a	a	$2a$	$4a$	1
c	a	b	c	a	a	$2a$	$4a$	1

$2a = 2b$, $3b = a + c$, $c = a + b \rightarrow a = b$, $c = 2a$

$$\left(\frac{1}{2}\right)^1 \times \left(\frac{k}{4}\right)^1 \times \left(\frac{1}{2}\right)^1 = \frac{k^1}{2^4} \rightarrow 1 < k^1 < 2^4 \rightarrow 1 < k < 2^4$$

Example 15

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 8 \\ \frac{kn+c}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

Produced	Produces				Solved in Terms of d				Sum	Simplified
	A	B	C	D	A	B	C	D		
a	$4a$	$4b$		$4d$	$12d$	$8d$		$4d$	$24d$	3
b	$2a$	$2b$	$2c$	$2d$	$6d$	$4d$	$4d$	$2d$	$16d$	2
c	a	b	c	d	$3d$	$2d$	$2d$	d	$8d$	1
d	a	b	c	d	$3d$	$2d$	$2d$	d	$8d$	1

$a = b + d$, $3b = a + c + d$, $3c = a + b + d$, $7d = a + b + c$, so $c = 2d$, $b = 2d$, $a = 3d$

$$\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{4}\right)^1 \times \left(\frac{k}{8}\right)^1 = \frac{k^1}{2^{12}} \rightarrow 1 < k^1 < 2^{12} \rightarrow 1 < k < 2^{12}$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 8 \\ \frac{4095n-20459}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

converges to 1 $\forall n : n \in \mathbb{N}$, QR $= \frac{4095}{4096}$

When we shift k in line 2:

$$\left(\frac{1}{2}\right)^3 \times \left(\frac{k}{4}\right)^2 \times \left(\frac{1}{4}\right)^1 \times \left(\frac{1}{8}\right)^1 = \frac{k^2}{2^{12}} \to 1 < k^2 < 2^{12} \to 1 < k < 2^6$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{63n-181}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

converges to 1 $\forall n : n \in \mathbb{N}$, QR $= \frac{63}{64}$

When we shift k in line 1:

$$\left(\frac{k}{2}\right)^3 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{4}\right)^1 \times \left(\frac{1}{8}\right)^1 = \frac{k^3}{2^{12}} \to 1 < k^3 < 2^{12} \to 1 < k < 2^4$$

$$f(n) = \begin{cases} \frac{15n-4}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

converges to 1 $\forall n : n \in \mathbb{N}$, QR $= 15/16$

Example 16

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n-1}{2} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 8 \\ \frac{kn+c}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

Produced	Produces				Solved in Terms of d				Sum	Simplified
	A	B	C	D	A	B	C	D		
a	$4a$			$4d$	$4d$			$4d$	$8d$	1
b	$2a$	$2b$		$2d$	$2d$	$4d$		$2d$	$8d$	1
c	a	b	c	d	d	$2d$	$4d$	d	$8d$	1
d	a	b	c	d	d	$2d$	$4d$	d	$8d$	1

$$a = d, \quad b = a + d, \quad b = 2d, \quad c = a + b + d, \quad c = 2b, \quad 7d = a + b + c$$

From these equations, we find:

$$c = 4d, \quad b = 2d, \quad a = d$$

$$\left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{k}{8}\right)^1 = \frac{k}{2^3}$$

$$\frac{k}{2^3} = \frac{k}{8} = \frac{k^1}{2^6}$$

$$1 < k^1 < 2^6 \to 1 < k < 2^6$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n-1}{2} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 8 \\ \frac{55n+197}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

The function $f(n)$ converges to 1 for all $n \in \mathbb{N}$, and $QR = \frac{55}{64}$.

Example 17

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n-1}{2} & \text{if } n \equiv 3 \pmod 8 \\ \frac{n-5}{2} & \text{if } n \equiv 7 \pmod 8 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 8 \\ \frac{kn+c}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

Produced	Produces					Solved in Terms of e					Sum	Simplified
	A	B	C	D	E	A	B	C	D	E		
a	$4a$				$4e$	$4e$				$4e$	$8e$	4
b	a				e	e				e	$2e$	1
c	a				e	e				e	$2e$	1
d	a	b	c	d	e	e	e	e	$4e$	e	$8e$	4
e	a	b	c	d	e	e	e	e	$4e$	e	$8e$	4

$$a = e, \quad 2b = a + e, \quad 2c = a + e, \quad d = a + b + c + e, \quad 7e = a + b + c + d$$

$$a = b = c = e, \quad d = 4e$$

$$\left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^4 \times \left(\frac{k}{8}\right)^4 = \frac{k^4}{2^{22}}$$

$$1 < k^4 < 2^{22} \to 1 < k < 2^{5.5}$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n-1}{2} & \text{if } n \equiv 3 \pmod 8 \\ \frac{n-5}{2} & \text{if } n \equiv 7 \pmod 8 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 8 \\ \frac{45n-33}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$, $QR = 45/32\sqrt{2}$.

Example 18

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{kn+c}{2} & \text{if } n \equiv 1 \pmod 4 \end{cases}$$

Generated	Generates			Solved in Terms of a			Sum	Simplified
	A	B	C	A	B	C		
a	$2a$	$2b$	0	$2a$	$2a$	0	$4a$	2
b	a	b	c	a	a	$2a$	$4a$	2
c	a	b	0	a	a	0	$2a$	1

$$a = b, \quad 3b = a + c, \quad c = a + b, \quad c = 2a = 2b$$

$$\left(\frac{1}{2}\right)^2 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{k}{2}\right)^1 < 1 \implies 1 < k < 2^7$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{126n-120}{2} & \text{if } n \equiv 1 \pmod 4 \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$, $QR = \frac{63}{64}$.

Example 19

$$f(n) = \begin{cases} \frac{3n+3\cdot 2^{i-1}-3}{2^i} & \text{if } n = 2^i k + 2^{i-1} + 1 \text{ for } i \geq 1 \end{cases}$$

where i ranges from 1 to ∞.

Converges to 3 for all $n \in \mathbb{N}$ with $n > 1$ and QR $\to 0$.

4.3.3 Complicated Kaakuma Sequence

Equations with partially generating cases are impossible to apply the Qodaa ratio test directly. This highlights the elegance of the Qodaa ratio test and its insightful application to any well-stated Kaakuma sequence.

If it is not done with care and attention, it will be full of subtle errors.

Example 20

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6 \\ \frac{kn+1}{2} & \text{if } n \equiv 1 \pmod 6 \text{ or } n \equiv 5 \pmod 6 \end{cases}$$

Case3 generates 6k, 6k+1, 6k+3 and 6k+4 that is 2/3 of case1, case2 and 1/2 of case3. The occurrences of a case also partially differ, to avoid subtle errors we have to dismantle all cases.

$$f(n) = \begin{cases} \frac{kn+1}{2} & \text{if } n \equiv 1 \pmod 6 \\ \frac{n}{2} & \text{if } n \equiv 2 \pmod 6 \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6 \\ \frac{n}{2} & \text{if } n \equiv 4 \pmod 6 \\ \frac{kn+1}{2} & \text{if } n \equiv 5 \pmod 6 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 6 \end{cases}$$

ed	Produces						Solved in Terms of b						Sum	Simplified
	A	B	C	D	E	F	A	B	C	D	E	F		
a	a	b	c				$3b$	b	$2b$				$6b$	3
b				d						$2b$			$2b$	1
c			c		e	f			$2b$		$2b$	$2b$	$6b$	3
d	a	b					$3b$	b					$4b$	2
e			c	d					$2b$	$2b$			$4b$	2
f					e	f					$2b$	$2b$	$4b$	2

$$a = b + c \quad d = 2b \quad 2c = e + f \quad 2d = a + b \quad 2e = c + d$$
$$f = e \quad a = 3b \quad c = d = e - f = 2b$$

$$\left(\frac{k}{2}\right)^3 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{3}\right)^3 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{k}{2}\right)^2 \times \left(\frac{1}{2}\right)^2 < 1$$

$$k^5 < 2^{10} \times 3^3 \implies k < 7.7327$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6 \\ \frac{7n+1}{2} & \text{if } n \equiv 1 \pmod 6 \text{ or } n \equiv 5 \pmod 6 \end{cases}$$

Converges to 1 with QR = 0.905.

When coefficient sample is $6k + 5$ it alter generating cases.

$$f(n) = \begin{cases} \frac{kn+1}{2} & \text{if } n \equiv 1 \pmod 6 \\ \frac{n}{2} & \text{if } n \equiv 2 \pmod 6 \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6 \\ \frac{n}{2} & \text{if } n \equiv 4 \pmod 6 \\ \frac{kn+1}{2} & \text{if } n \equiv 5 \pmod 6 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 6 \end{cases}$$

For the coefficient, we can use 5 instead of $6p+5$ to get generating sample. Note that A and E vary depending on what they generate:

ed	Produces						Solved in Terms of b						Sum	Simplified
	A	B	C	D	E	F	A	B	C	D	E	F		
a		b	c		e			b	$4b$		$3b$		$8b$	4
b				d						$2b$			$2b$	1
c	a		c			f	$4b$		$4b$			$4b$	$12b$	6
d		b			e			b			$3b$		$4b$	2
e			c	d					$4b$	$2b$			$6b$	3
f	a					f	$4b$					$4b$	$8b$	4

$$2a = b + c + e \quad 2b = d \quad 2c = a + f \quad 2d = b + e \quad 2e = c + d$$
$$f = a \quad a = c = f = 4b \quad d = 2b \quad e = 3b$$

$$\left(\frac{k}{2}\right)^4 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{3}\right)^6 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{k}{2}\right)^3 \times \left(\frac{1}{2}\right)^4 < 1$$

$$\implies k^7 < 2^{14} \times 3^6 \implies k < 10.257$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2, \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6, \\ \frac{5n+1}{2} & \text{if } n \equiv 1 \pmod 6 \text{ or } n \equiv 5 \pmod 6. \end{cases}$$

Converges to 1 with QR = 0.48747

Example 21:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \text{ —3/6}, \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6 \text{ —1/6}, \\ \frac{kn+3}{2} & \text{if } n \equiv 1 \pmod 6 \text{ or } n \equiv 5 \pmod 6 \text{ —2/6}. \end{cases}$$

$$f(n) = \begin{cases} \frac{kn+3}{2} & \text{if } n \equiv 1 \pmod 6 \text{ —A(c,f)}, \\ \frac{n}{2} & \text{if } n \equiv 2 \pmod 6 \text{ —B(a,d)}, \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6 \text{ —C(a,c,e)}, \\ \frac{n}{2} & \text{if } n \equiv 4 \pmod 6 \text{ —D(b,e)}, \\ \frac{kn+3}{2} & \text{if } n \equiv 5 \pmod 6 \text{ —E(c,f)}, \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 6 \text{ —F(c,f)}. \end{cases}$$

	Generates						After solved in terms of 1/e						Sum	Simplified
	A	B	C	D	E	F	A	B	C	D	E	F		
a		b	c						2				2	2
b				d									0	0
c	a		c		e	f	1		2		1	2	6	6
d		b											0	0
e			c	d					2				2	2
f	a				e	f	1				1	2	4	4

2a=b+c 2b=d 2c=e+f 2d=b 2 e=c+d f=a+e b=0 d=0 a=e c=2e f=2e

Note:- if a sequence is semi-cycled or a case is not generated it is not considered as Kaakuma sequence

Example 22

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 7 \pmod 8 \\ \frac{n+3}{4} & \text{if } n \equiv 5 \pmod 8 \\ \frac{n+5}{8} & \text{if } n \equiv 3 \pmod 8 \\ kn+c & \text{if } n \equiv 1 \pmod 8 \end{cases}$$

We split cases that are partially generated to avoid complexity:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 2 \pmod 4 - A(d,e,f,g) \\ \frac{n}{4} & \text{if } n \equiv 4 \pmod 8 - B(d,e,f,g) \\ \frac{n}{8} & \text{if } n \equiv 0 \pmod 8 - C(\text{all}) \\ \frac{n+1}{2} & \text{if } n \equiv 7 \pmod 8 - D(b,c) \\ \frac{n+3}{4} & \text{if } n \equiv 5 \pmod 8 - E(a,b,c) \\ \frac{n+5}{8} & \text{if } n \equiv 3 \pmod 8 - F(\text{all}) \\ kn+c & \text{if } n \equiv 1 \pmod 8 - G(c) \end{cases}$$

	Generates							After solving in terms of 1/ f						
	A	B	C	D	E	F	G	A	B	C	D	E	F	G
a			$2c$		$2e$	$2f$				30/7		4	2	
b			c	d	e	f				15/7	4	2	1	
c			c	d	e	f	g			15/7	4	2	1	8
d	a	b	c			f		18/7	16/7	15/7			1	
e	a	b	c			f		18/7	16/7	15/7			1	
f	a	b	c			f		8/7	16/7	15/7			1	
g	a	b	c			f		18/7	16/7	15/7			1	
sum								72/7	64/7	120/7	8	8	8	8
simplified								9	8	15	7	7	7	7

$$g = a+b+c+f \quad 7f = a+b+c \quad 4e = a+b+c+f$$

$$2d = a+b+c+f \quad 7c = d+e+f+g \quad 4b = c+d+e+f$$

$$2a = c+e+f$$

when we solve it in terms of f

$$g = 8f \quad e = 2f \quad d = 4f \quad c = 15f/7 \quad b = 16f/7 \quad a = 18f/7$$

$$(1/2)^9 \times (1/4)^8 \times (1/8)^{15} \times (1/2)^7 \times (1/4)^7 \times (1/8)^7 \times k^7 < 1$$
$$\Rightarrow \quad k^7 < 2^{112} \quad \Rightarrow \quad k < 2^{16}$$

When we substitute k:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 7 \pmod 8 \\ \frac{n+3}{4} & \text{if } n \equiv 5 \pmod 8 \\ \frac{n+5}{8} & \text{if } n \equiv 3 \pmod 8 \\ 65535n - 65519 & \text{if } n \equiv 1 \pmod 8 \end{cases}$$

The sequence Converges to 1 for all $n \in \mathbb{N}$, $QR = 65535/65536$.

Note: This is a complicated form a sequence in Example 10 where case2 and case5 generate case1 partially.

All these different types of examples show how Qodaa Ratio Test Works even in complicated equations. Qodaa ratio test is simple and rigor to apply. Beyond this there are some points to study in future like number of cycles, interval of constants a sequence to converges, where diverging will start for a diverging kaakuma sequence.

4.4 Proof 3: Computational Analysis

Even though computational analysis cannot serve as a rigorous proof of the Collatz conjecture, it can provide convincing evidence until more rigorous proofs, like Proof 1 and Proof 2, are available. In some challenging cases, and based on their argument level, computational results must be considered, at least to some extent.

4.4.1 Constants and Bounded Values

There are several distinct constants and bounded values observed in the Collatz sequence as discussed in Behavior 3.5.

The average stopping time of the Collatz sequence is a constant, similar to the constants π and e. The function $f(n)$ is defined as:

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

The average stopping time of this sequence is approximately 3.49269. The key point is that if the average stopping time is constant and consistent with very small variation on both sides, it is almost impossible to divert from this behavior after 10^{20} or 10^{40}. If the Collatz conjecture were invalid, this would imply that for 2^{120}, the stopping time t would not align as:

$$(\sum_{n=2}^{2^{120}-1} t)/(2^{120}-1) = 3.49269$$

but:

$$(\sum_{n=2}^{2^{120}} t)/2^{120} = \infty$$

which is impossible.

4.4.2 Inverse Map of Collatz Sequence

The inverse map of the Collatz sequence covers all natural numbers starting from root 1. During this process, its expansion rate is 33.33

4.4.3 Ratio of Stopping Time

The ratio of stopping time to $\log_2(n)$ is bounded and less than 5.5. It is also bounded and less than 5, and small numbers such as 28 and 32 can be adjusted by translation. This can be verified by computer programs using high-rate stopping time values like 2^k. This constant is analogous to the ratio of primes in natural numbers, $\pi(x)$. For example:

$$2^k \quad \frac{4 \times (2^{6k}-1)}{9} \quad \frac{8 \times (2^{18k}-1)}{27} \quad \frac{16 \times (2^{54k}-1)}{81} \quad \frac{32 \times (2^{162k}-1)}{243}$$

4.4.4 Expected Huge Iterations

In the Collatz sequence, it is not surprising to encounter relatively high iteration numbers. As seen in Behavior 3.8, numbers with powers of 2 have relatively high iterations, and numbers that reach powers of 2 before decreasing from the starting numbers also have high iterations. Numbers less than 2^{200000} are expected to have relatively high iterations, and the constants are kept as described in Behavior 3.5.3 and 3.5.4.

4.4.5 Special and Extreme Contradiction in Cycle Case

In the cycle case, the number of iterations needed to create a cycle is $n/10$. If 10^{20} is the first number to create a non-trivial cycle, it must have 10^{19} iterations to the minimum, as discussed in Behavior 3.7. This is contradictory because, based on Analysis 3.5.2, it should only be up to $5.5 \times 60 = 330$ at the maximum.

4.4.6 Collatz Sequence with Falling Values

If there exists a non-Collatz number, its sequence must include iteration group numbers or falling values like $2^{59}k + 28$, $2^{54}k + 64$. These falling values lead to other falling points and make the sequence excessively dense.

4.4.7 Infinite Paradigm-Shifting Kaakuma Sequence

An example of an infinite paradigm-shifting Kaakuma sequence is given by $65535n - 327667$. As seen in Proof 1 Example 10 and Example 25, this sequence has over 2 billion iterations and a height greater than 10^{80}. It takes 15 days to complete iterations for a small number, 9757. This is a highly paradigm-shifting example of a Kaakuma sequence, with many more such cases existing.

Conclusion

The Collatz conjecture is considered true because of the following reasons:

1. Contradiction in tree size balance
2. Qodaa Ratio Test
3. Computational Analysis

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