A NOTE ON CLOSED ADDITION CHAINS AND COMPLETE NUMBERS

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Abstract. We introduce a new class of addition chains and show the numbers for which these chains are optimal satisfy the Scholz conjecture, precisely the inequality

$$\iota(2^n - 1) \leq n - 1 + \iota(n).$$

1. Introduction

An addition chain of length h leading to n is a sequence of numbers $s_o = 1, s_1 = 2, \ldots, s_h = n$ where $s_i = s_k + s_s$ for $i > k \geq s \geq 0$. The number of terms (excluding the first term) in an addition chain leading to n is the length of the chain. We call an addition chain leading to n with a minimal length an *optimal* addition chain leading to n. In standard practice, we denote by $\iota(n)$ the length of an optimal addition chain that leads to n. A Brauer addition chain of length h leading to n is a sequence of numbers $s_o = 1, s_1 = 2, \ldots, s_h = n$ where $s_i = s_{i-1} + s_j$ for $i > j \geq 0$. We denote the length of an optimal Brauer chain leading to n by $\iota^*(n)$. A number n for which the Brauer chain is optimal (i.e. $\iota^*(n) = \iota(n)$) is called a Brauer number. It is known ([1]) that Brauer numbers satisfy the inequality

$$\iota(2^n - 1) \leq n - 1 + \iota(n).$$

The concept of Hansen addition chain is a well-known generalization of Brauer-type addition chains. A *Hansen chain* is an addition chain $s_o = 1, s_1 = 2, \ldots s_{r-1} = n$ for which there exists a fixed subset (anchor)

$$H \subseteq \{s_0, s_1, \ldots, s_{r-1}\}$$

such that each term s_k in the chain is formed as

$$s_k = s_i + s_j$$

with

$$s_i = \max\{h \in H \ : \ h < s_k\}$$

for all k. A number for which a Hansen chain is optimal is called a Hansen number. Hansen numbers [2] are also known to satisfy the inequality

$$\iota(2^n - 1) \leq n - 1 + \iota(n)$$

which is now known as the Scholz conjecture on addition chains. It is still not known whether the conjecture still holds for optimal addition chains which are neither Brauer nor Hansen. In this short note, we answer this question in the affirmative, thereby generalizing the concept of Brauer-Hansen addition chains.

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2. **Main result**

Definition 2.1 (Closed addition chains)**.** Let $s_o = 1, s_1 = 2, \ldots, s_h = n$ be an addition chain leading to n with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ such that $i > \sigma(i) \geq \tau(i)$ for each $1 \leq i \leq h$. We say that a chain is a *closed* addition chain if for each $1 \leq i \leq h$ there exists some $j \in [0, h]$ such that

$$s_i - s_{i-1} = s_j$$

and for $s_{\sigma(i)} - s_{\sigma(i-1)} \neq 0$ there exists some $j \in [0, h]$ such that

$$s_{\sigma(i)} - s_{\sigma(i-1)} = s_j.$$

Additionally, for each $s_j = s_{\tau(i)}$ with $s_{\tau(i)} \neq s_{\sigma(k)}$ for all $k \in [0, h]$ there exist some $s_{\sigma(k)}$ such that $s_{\sigma(k)} < s_{\tau(i)}$ is consecutive.

Example 2.2 ($n = 11$)**.**

$$\{s_k\}_{k=0}^{5} = \{1, 2, 3, 5, 6, 11\}$$

$$\begin{aligned} s_0 &= 1, \\ s_1 &= 2 = 1 + 1, \\ s_2 &= 3 = 2 + 1, \\ s_3 &= 5 = 3 + 2, \\ s_4 &= 6 = 3 + 3, \\ s_5 &= 11 = 6 + 5. \end{aligned} \tag{2.1}$$

Non-star step: At $k = 4$, since $6 = s_2 + s_2$ uses s_2 instead of s_3.
Non-Hansen: The chain is not *Hansen*. To see this, we first observe that each step is a valid addition chain step. To generate 6, the largest element of H less than 6 must be 3, so $3 \in H$. To generate 11, the Hansen rule requires that the largest $h \in H$ with $h < 11$ must be used, but

$$\max(H \cap \{1, 2, 3, 5, 6\}) = 3$$

yet neither summand in the construction of 11 equals 3, violating the *Hansen* rule. Thus, no fixed H can anchor all steps, and the chain is not Hansen.

Closure check:

$$2 - 1 = 1, \quad 3 - 2 = 1, \quad 5 - 3 = 2, \quad 6 - 5 = 1, \quad 11 - 6 = 5, \tag{2.2}$$

each difference appears among $\{s_0, \ldots, s_5\}$.

Example 2.3 ($n = 13$)**.**

$$\{s_k\}_{k=0}^{6} = \{1, 2, 3, 5, 6, 8, 13\}$$

$$\begin{aligned} s_0 &= 1, \\ s_1 &= 2 = 1 + 1, \\ s_2 &= 3 = 2 + 1, \\ s_3 &= 5 = 3 + 2, \\ s_4 &= 6 = 3 + 3, \\ s_5 &= 8 = 5 + 3, \\ s_6 &= 13 = 8 + 5. \end{aligned} \tag{2.3}$$

Non-star steps:

- $k = 4$: $6 = s_2 + s_2$, skip-back from s_4.
- $k = 5$: $8 = s_3 + s_2$, skip-back from s_5.

Non-Hansen: No single $H \subset \{1, 2, 3, 5, 6, 8, 13\}$ anchors both non-star steps. The chain cannot be *Hansen*. To see this, we observe that each of these constitutes a valid addition chain step. To generate 6, the largest prior element in any candidate anchor set H strictly less than 6 must be 3 since it was produced by doubling 3, hence $3 \in H$. To generate 8, the largest member of H less than 8 remains 3, which appears in $8 = 5 + 3$, so max $H = 3$ still works. However, to generate 13 via $13 = 8 + 5$, the Hansen rule requires that the largest element of H smaller than 13 must be used in the construction. Thus, if max $H = 3$ the Hansen rule fails, since neither 8 nor 5 equals 3. By varying the anchor set and following *Hansen's* rule, it can be checked that no fixed subset H can serve as an anchor set for all steps in the chain, and thus $s_o = 1, s_1 = 2, s_2 = 3, s_3 = 5, s_4 = 6, s_5 = 8, s_6 = 13$ is *not* a Hansen chain.

Closure check:

$$2-1=1, \quad 3-2=1, \quad 5-3=2, \quad 6-5=1, \quad 8-6=2, \quad 13-8=5, \tag{2.4}$$

each gap lies among $\{s_0, \ldots, s_6\}$.

Definition 2.4 (Complete number)**.** We call the number for which a *closed* addition chain is optimal a *complete* number.

We now show that all *complete* numbers satisfy the Scholz conjecture, precisely that $\iota(2^n - 1) \leq n - 1 + \iota(n)$. The construction that follows is an optimization and, to a larger extent, a generalization of the Brauer method.

Theorem 2.5. *All complete numbers satisfy the inequality*

$$\iota(2^n - 1) \leq n - 1 + \iota(n).$$

Proof. Suppose that $n \geq 3$ is a complete number and let $s_o = 1, s_1 = 2, \ldots, s_h = n$ be an optimal *closed* addition chain leading to n, with length $h := \iota^{\diamond}(n)$. Next, we define $m_i = 2^{s_i} - 1$ for $i = 0, \ldots, h$ and construct the sequence

$$2^1 - 1, 2^2 - 1, \ldots, 2^{s_i} - 1, \ldots, 2^{s_{\iota^{\diamond}(n)}} - 1 = 2^n - 1.$$

We call each term in this sequence of the form $2^{s_i} - 1$ a *Mersenne seed*. Now, for each $2^{s_{\sigma(i)}} - 1$, we double for $(s_{\sigma(i+1)} - s_{\sigma(i)})$ number of times and include the result of each doubling into the sequence. For terms of the form $2^{s_j} - 1$ such that $j = \sigma(k)$ for some $k \in [1, h]$ as indices not used in the previous doubling process, we double $2^{s_{\sigma(k)}} - 1$ for $(s_k - s_{\sigma(k)})$ number of times and include the result of each doubling in the sequence. We can now verify that this construction produces an addition chain leading to $2^n - 1$. We observe that

$$2^{s_{\sigma(i+1)}} - 1 = 2^{s_{\sigma(i+1)} - s_{\sigma(i)}}(2^{s_{\sigma(i)}} - 1) + 2^{s_{\sigma(i+1)} - s_{\sigma(i)}} - 1$$

since the chain $s_o = 1, s_1 = 2, \ldots, s_h = n$ is closed. The optimality of the closed chain implies that

$$\{s_{\sigma(k)}\}_{k=0}^{h} \cup \{s_{\tau(k)}\}_{k=0}^{h} = \{s_k\}_{k=0}^{h-1}.$$

Similarly for those $2^{s_j}-1$ with $j=\sigma(k)$ for some $k\in[1,h]$ as indices not used in the previous doubling process, we can write

$$2^{s_k}-1=2^{s_k-s_{\sigma(k)}}(2^{s_{\sigma(k)}}-1)+2^{s_k-s_{\sigma(k)}}-1.$$

We observe that the identity

$$n-1=\sum_{i=0}^{h-1}(s_{i+1}-s_i)=\sum_{i=0}^{h-1}(s_{\sigma(i+1)}-s_{\sigma(i)})+\sum_{i=1}^{h-1}(s_{\tau(i+1)}-s_{\tau(i)})$$

holds. For those $2^{s_j}-1$ with $s_j=s_{\tau(i)}$ such that $s_{\tau(i)}\neq s_{\sigma(k)}$ for all $k\in[0,h]$, we do not need any new doubling to account for their representation as the sum of two previous terms in the sequence, since a component of their representation is "locked up" in previous σ-doublings and can be recovered. The remaining component appears as one of the *Mersenne seed*. More precisely, let $s_j=s_{\tau(i)}$ such that $s_{\tau(i)}\neq s_{\sigma(k)}$ for all $k\in[0,h]$. Since the chain $s_o=1,s_1=2,\ldots,s_h=n$ is *closed*, there exists some $s_{\sigma(k)}$ such that $s_{\sigma(k)}<s_{\tau(i)}$ is consecutive. We may assume, without loss of generality, that $s_{\sigma(k+1)}-s_{\sigma(k)}\neq 0$, since in the case $s_{\sigma(k+1)}-s_{\sigma(k)}=0$, we can replace $s_{\sigma(k)}$ with $s_{\sigma(k+v)}$ such that $s_{k+v+1}-s_{k+v}\neq 0$ for the greatest v such that

$$s_{\sigma(k)}=s_{\sigma(k+1)}=\cdots=s_{\sigma(k+v)}$$

as the sequence

$$s_{\sigma(0)}\leq s_{\sigma(1)}\leq\cdots\leq s_{\sigma(i)}\cdots\leq s_{\sigma(h)}$$

for $1\leq i\leq h$ with $s_{\sigma(i)}\geq s_{\tau(i)}$. Thus, we have

$$s_{\sigma(k)}<s_{\tau(i)}\leq s_{\sigma(k+1)}$$

and thus

$$s_{\tau(i)}-s_{\sigma(k)}\leq s_{\sigma(k+1)}-s_{\sigma(k)}.$$

Thus, we can write

$$2^{s_j}-1=2^{s_{\tau(i)}}-1=2^{s_{\tau(i)}-s_{\sigma(k)}}(2^{s_{\sigma(k)}}-1)+2^{s_{\tau(i)}-s_{\sigma(k)}}-1$$

since $s_{\sigma(k)}<s_{\tau(i)}$ is consecutive and the chain is *closed*. It is seen that the term

$$2^{s_{\tau(i)}-s_{\sigma(k)}}(2^{s_{\sigma(k)}}-1)$$

has already appeared in the $(s_{\sigma(k+1)}-s_{\sigma(k)})$ repeated doubling of $2^{s_{\sigma(k)}}-1$ since $s_{\tau(i)}-s_{\sigma(k)}\leq s_{\sigma(k+1)}-s_{\sigma(k)}$. The total length of the constructed addition chain is at most

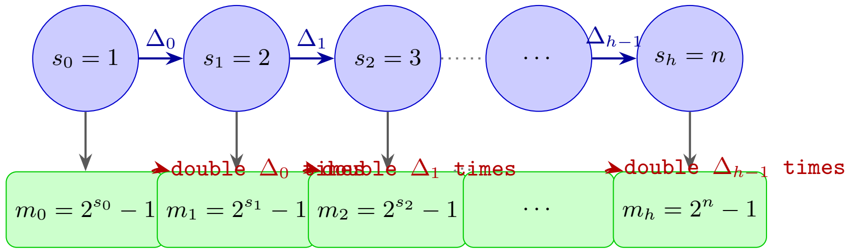
$$\leq s_{\tau(h)}+\sum_{i=0}^{h-1}(s_{\sigma(i+1)}-s_{\sigma(i)})+\iota^{\diamond}(n)$$

and it follows that

$$\iota(2^n-1)\leq n-1+s_{\tau(h)}-\sum_{i=0}^{h-1}(s_{\tau(i+1)}-s_{\tau(i)})+\iota^{\diamond}(n).$$

The claim follows immediately since n was assumed to be a *complete* number. $\square$

Below is a mind picture of how the construction works as an optimized version of Brauer's method.



Example 2.6. We demonstrate in a concrete example the construction espoused in Theorem 2.5. We have already shown in the previous example that the addition chain leading to 11:

$$s_o = 1, s_1 = 2, s_2 = 3, s_3 = 5, s_4 = 6, s_5 = 11$$

produced by the corresponding sequence of sums

$$s_1 = 2 = 1 + 1, s_2 = 3 = 2 + 1, s_3 = 5 = 3 + 2, s_4 = 6 = 3 + 3, s_5 = 11 = 6 + 5$$

is *closed* with

$$s_{\sigma(1)} = 1, s_{\sigma(2)} = 2, s_{\sigma(3)} = 3, s_{\sigma(4)} = 3, s_{\sigma(5)} = 6.$$

We now produce a sequence according to the rule $2^{s_i} - 1$ for each term in the addition chain as follows

$$2^1 - 1, 2^2 - 1, 2^3 - 1, 2^5 - 1, 2^6 - 1, 2^{11} - 1.$$

By the construction, we double $2^{s_{\sigma(1)}} - 1$ for $(s_{\sigma(2)} - s_{\sigma(1)})$ number of times and include the result of each doubling in the sequence. Thus, we include $2(2^1 - 1) = 2$ in the sequence. In addition, we double the term $2^{s_{\sigma(2)}} - 1$ for $(s_{\sigma(3)} - s_{\sigma(2)})$ number of times and include the result of each doubling in the sequence. Thus, we include $2(2^2 - 1) = 6$ in the sequence. Now, we have nothing to do for $2^{s_{\sigma(3)}} - 1$, since $s_{\sigma(4)} - s_{\sigma(3)} = 0$. Therefore, we skip to $2^{s_{\sigma(4)}} - 1$ and double this term for $(s_{\sigma(5)} - s_{\sigma(4)}) = 3$ number of times and include the result of each doubling in the sequence. Thus, we include the terms $2(2^3 - 1) = 14, 2^2(2^3 - 1) = 28, 2^3(2^3 - 1) = 56$ in the sequence. We observe that the term $s_{\sigma(5)} = 6$ was not used in the construction, since s_5 is the last term in the sequence. Thus, according to the construction in the proof of Theorem 2.5, we double the term $2^{s_{\sigma(5)}} - 1$ for $(s_5 - s_{\sigma(5)})$ number of times and include the result of each doubling in the sequence. Thus, we include the terms $2(2^6 - 1) = 126, 2^2(2^6 - 1) = 252, 2^3(2^6 - 1) = 504, 2^4(2^6 - 1) = 1008, 2^5(2^6 - 1) = 2016$. Putting everything together, we obtain an addition chain leading to $2^{11} - 1 = 2047$:

$$1, 2, 3, 6, 7, 14, 28, 31, 56, 63, 126, 252, 504, 1008, 2016, 2047.$$

Incidentally, this construction yields an addition chain of length 15, thus satisfying the inequality

$$\leq 10 + \iota(11) = 15.$$

Some intuition behind the σ-path doubling. The heart of the proof lies in how the built-in-closed chain *gaps* become the exact doubling steps needed for the Mersenne chain. In the following, we unravel this process in detail.

Setup. We recall the closed chain for n:

$$1 = s_0 < s_1 <, \dots, < s_h = n,$$

and the indices $\sigma(i)$ defining how each s_i was formed:

$$s_i = s_{\sigma(i)} + s_{\tau(i)}, \quad 0 \le \tau(i) \le \sigma(i) < i.$$

By the closedchain condition, whenever

$$\Delta_i := s_{\sigma(i+1)} - s_{\sigma(i)} \neq 0,$$

there is some j with $\Delta_i = s_j$. Moreover,

$$\sum_{i=0}^{h-1} \Delta_i = \sum_{i=0}^{h-1} (s_{i+1} - s_i) = n - 1.$$

Lift to Mersenne seeds. Define

$$m_i = 2^{s_i} - 1$$

for $i = 0, 1, \dots, h$. These are the "seeds" for our new chain targeting $2^n - 1 = m_h$.

Main doubling step. For each $i = 0, \dots, h-1$, we take the seed $m_{\sigma(i)} = 2^{s_{\sigma(i)}} - 1$ and *double it* Δ_i times, appending each intermediate value. Concretely:

$$m_{\sigma(i)} \ \mapsto \ 2\, m_{\sigma(i)} + 1 \ \mapsto \ 2\big(2\, m_{\sigma(i)} + 1\big) + 1 \ \mapsto \ \cdots \tag{2.5}$$

repeated Δ_i times. Each doubling uses the identity

$$2^a - 1 = 2^d(2^{a-d} - 1) + (2^d - 1),$$

where d is the amount we shift. After Δ_i doublings, we reach

$$2^{s_{\sigma(i)} + \Delta_i} - 1 = 2^{s_{\sigma(i+1)}} - 1 = m_{\sigma(i+1)}.$$

Why this works

- The *exact* number of doublings Δ_i matches the gap in the exponent chain.

- The closed condition $\Delta_i = s_j$ ensures Δ_i itself was one of the original exponents, so no new step size is introduced.

- Summing over $i = 0$ to $h - 1$, the total number of doublings is

  $$\sum_{i=0}^{h-1} \Delta_i = n - 1,$$

  accounting for exactly the $n - 1$ term in the Scholz bound.

3. Some Heuristic and a conjecture

In this note, we have introduced the notion of a closed addition chain that demonstrably extends the *Brauer* and the *Hansen* addition chains to a much larger class of addition chains. The concept of *closed* addition chains is of great interest because, as we have shown, a class of numbers for which closed addition chains are optimal (complete numbers) satisfies the inequality

$$\iota(2^n-1)\leq n-1+\iota(n).$$

We have a strong reason to believe that there is no other special class of addition chains outside the class of *closed* addition chains. In other words, we are unable to construct an addition chain which is not closed after an extensive construction. Although we are short of a rigorous argument to assert this claim, we have a rough heuristics on which to ground our intuition. Let

$$s_o=1,s_1=2,\ldots,s_h=n$$

be an addition chain leading to n with $s_i=s_{\sigma(i)}+s_{\tau(i)}$ with $i>\sigma(i)\geq\tau(i)\geq 0$ for each $1\leq i\leq h$. Inferring from the framework developed in the elementary theory of addition chains, we can write

$$\inf_{0\leq i\leq h-1}(s_{\sigma(i+1)}-s_{\sigma(i)})\ll(\sigma(i+1)-\sigma(i))\frac{n}{h}$$

and

$$\sup_{0\leq i\leq h-1}(s_{\sigma(i+1)}-s_{\sigma(i)})\gg(\sigma(i+1)-\sigma(i))\frac{n}{h}$$

provided $s_{\sigma(i+1)}-s_{\sigma(i)}\neq 0$. Thus, on average, we have

$$s_{\sigma(i+1)}-s_{\sigma(i)}\approx(\sigma(i+1)-\sigma(i))\frac{n}{h}\approx s_{\sigma(i+1)-\sigma(i)}.$$

Thus we make the following conjecture:

Conjecture 3.1. *Let $s_o=1,s_1=2,\ldots,s_h=n$ be an optimal addition chain leading to n with $h:=\iota(n)$ such that $s_i=s_{\sigma(i)}+s_{\tau(i)}$ for $i>\sigma(i)\geq\tau(i)\geq 0$ for $1\leq i\leq h$. Then for each $1\leq i\leq h$ with $s_{\sigma(i)}-s_{\sigma(i-1)}\neq 0$, there exists some $j\in[0,h]$ such that*

$$s_{\sigma(i)}-s_{\sigma(i-1)}=s_j.$$

In general, all optimal addition chains are closed.

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