

ON A SPECIFIC FAMILY OF ORTHOGONAL POLYNOMIALS OF BERNSTEIN-SZEGÖ TYPE

MARTIN NICHOLSON

ABSTRACT. We study a class of weight functions on $[-1, 1]$ which are special cases of the broader family studied by Bernstein and Szegö. These weights are parametrized by two positive integers. As these integers tend to infinity, these weights approximate certain weight functions on \mathbb{R} considered by Ismail and Valent. We also study modifications of these weight functions by a continuous parameter $a > 0$. These ideas are then used to find finite analogs of some improper integrals first studied by Glaisher and Ramanujan. We also show that some of the functions used in this work are in fact generating functions of certain finite trigonometric sums.

1. INTRODUCTION

Entry 4.123.6 in Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products* [12] reads

$$\int_0^\infty \frac{\sin(ax) \sinh(bx)}{\cos(2ax) + \cosh(2bx)} x^{p-1} dx = \frac{\Gamma(p)}{(a^2 + b^2)^{p/2}} \sin\left(p \tan^{-1} \frac{a}{b}\right) \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^p}, \quad p > 0. \quad (1)$$

The limit $p \rightarrow +0$ of (1) is

$$\int_0^\infty \frac{\sin(x) \sinh(x/a)}{\cos(2x) + \cosh(2x/a)} \frac{dx}{x} = \frac{\tan^{-1} a}{2}. \quad (2)$$

Additionally, when $a = b$ and $p/4 \in \mathbb{N}$ one finds from (1)

$$\int_0^\infty \frac{\sin(x) \sinh(x)}{\cos(2x) + \cosh(2x)} x^{4k-1} dx = 0, \quad k \in \mathbb{N}. \quad (3)$$

Integrals of the type (2) and (3) were first studied by Glaisher [10]. More recently, they were studied in connection with integrals of the Dedekind eta-function [11],[18],[7].

In [16], we have generalized (2) as

$$\int_0^1 \frac{\sin(n \sin^{-1} t) \sinh(n \sinh^{-1}(t/a))}{\cos(2n \sin^{-1} t) + \cosh(2n \sinh^{-1}(t/a))} \frac{dt}{t \sqrt{(1-t^2)(1+t^2/a^2)}} = \frac{\tan^{-1} a}{2}, \quad (4)$$

where n is a positive odd integer, and we have also shown that

$$\int_0^1 \frac{\cos(n \sin^{-1} t) \cosh(n \sinh^{-1} t)}{\cos(2n \sin^{-1} t) + \cosh(2n \sinh^{-1} t)} \frac{t dt}{\sqrt{1-t^4}} = 0, \quad (5)$$

where n is a positive even integer. Our proof was based on explicit calculations using the fact, that the roots of $\cos(2n \sin^{-1} t) + \cosh(2n \sinh^{-1}(t/a))$ (which is a polynomial in t) can be determined in closed form (see also Sections 10 and 11 of the present paper for similar calculations). Two alternative proofs of (4) were given in [25] and [20]. Motivation for considering such integrals came from the mapping $\alpha_z = 2n \sinh^{-1} \sin \frac{\pi z}{2n}$ encountered in the theory of Dirichlet problem on finite nets [19], as discussed in Section 5 of [16]. Note that $\alpha_z \sim \pi z$, when $n \rightarrow \infty$.

In this paper, we will be concerned with integration formulas similar to the following:

Theorem 1. *Let n and m be positive odd integers. Then*

$$\int_{-1}^1 \frac{\sin(n \sin^{-1} \sqrt{t}) \sinh(m \sinh^{-1} \sqrt{t})}{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t})} \frac{t^j dt}{\sqrt{1-t^2}} = \begin{cases} \pi/2, & j = -1, \\ 0, & j = 0, 1, \dots, \frac{m+n-2}{2}, \end{cases} \quad (6)$$

$$\int_{-1}^1 \frac{\cos(n \sin^{-1} \sqrt{t}) \cosh(m \sinh^{-1} \sqrt{t})}{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t})} t^j dt = 0, \quad j = 0, 1, \dots, \frac{m+n-4}{2}.$$

For generic integers n and m , the roots of the polynomial $\cos(2n \sin^{-1} t) + \cosh(2m \sinh^{-1} t)$ are not known in closed form. Therefore other methods are necessary to study these more general integrals.

Our proof of Theorem 1 is based on the theory of orthogonal polynomials in its elementary form. In particular, we will use the concept of so-called Bernstein-Szegő polynomials, a brief overview of which is provided in Section 2. After studying the aforementioned function $\cos(2n \sin^{-1} t) + \cosh(2m \sinh^{-1} t)$ in Section 3, and proving Theorem 1 in Section 4, we turn to more general situation and consider modifications of Theorem 1 by a continuous parameter $a > 0$ in Section 5. The case where both parameters n and m are even is studied in Section 6. More complicated weight functions are studied in Section 7.

In a series of papers [1],[13], Berg, Valent, and Ismail have considered orthogonal polynomials on \mathbb{R} related to elliptic functions. Weight function for these orthogonal polynomials is equivalent to

$$\frac{1}{\cos(2\sqrt{x}) + \cosh(2\sqrt{x/a})} \quad (7)$$

after rescaling of the variable. Recently, there has been a flurry of activity in studying different aspects of the integrals with the weight function (7), e.g. [2],[15],[27],[3],[17],[23],[28]. The weight functions

$$\frac{1}{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t})} \frac{1}{\sqrt{1-t^2}}, \quad (8)$$

parametrized by two positive integers n and m , approximate the weight functions (7) in the limit $n, m \rightarrow \infty$. For example, (6) is a finite analog of the improper integral

$$\int_{\mathbb{R}} \frac{\sin(\sqrt{x}) \sinh(\sqrt{x/a})}{\cos(2\sqrt{x}) + \cosh(2\sqrt{x/a})} x^j dx = \begin{cases} \pi/2, & j = -1, \\ 0, & j = 0, 1, 2, \dots \end{cases}$$

As can be seen from (6), $\sin(n \sin^{-1} \sqrt{t}) \sinh(m \sinh^{-1} \sqrt{t})$ is a degree $\frac{m+n}{2}$ orthogonal polynomial corresponding to the weight function (8). Moreover, the roots of this orthogonal polynomial are known. This will allow us to find explicit Gauss quadrature formulas for the weight function (8) in Section 8. We use these quadrature formulas in Section 9 to derive a finite analog of the generating function formula due to Kuznetsov [15].

There is an integral looking similar to (2) but somewhat different

$$\int_0^{\infty} \frac{\sin(kx)}{\cos(x) + \cosh(x)} \frac{dx}{x} = \frac{\pi}{4}, \quad (9)$$

where k is a positive odd integer. It was submitted by Ramanujan to the Journal of the Indian Mathematical Society as problem number 353 [22]. For more information on the history of (9) and on the Ismail and Valent integral, the reader is referred to [2]. Additionally, [2] contains a direct proof of (9) using contour integration. In Section 10, we will prove a finite analog of (9) that contains an additional integer parameter. When this integer parameter goes to infinity, one recovers (9) in the limit.

In Section 11, a calculation similar to that of Section 10 is used to evaluate a certain integral in terms of a finite trigonometric sum. This turns out to be an analog of another integral due to Glaisher [10] related to theta series. We also show that this analog is equivalent to generating function formula for certain finite trigonometric sum.

The indeterminate moment problem for orthogonal polynomials associated with the weight functions

$$\frac{x}{\cosh(2\sqrt{x/a}) - \cos(2\sqrt{x})},$$

has been studied in [14]. In Section 12, we briefly consider finite analogs of such weight functions.

2. GENERAL BERNSTEIN-SZEGO POLYNOMIALS

In this section, we closely follow the book [24]. The trigonometric polynomial in θ of degree k is

$$g(\theta) = a_0 + \sum_{j=1}^k \{a_j \cos(j\theta) + b_j \sin(j\theta)\}.$$

Theorem 2 ([24], Theorem 1.2.2). *Let $g(\theta)$ be a trigonometric polynomial with real coefficients which is nonnegative for all real values of θ and $g(\theta) \neq 0$. Then a representation $g(\theta) = |h(e^{i\theta})|^2$ exists such that $h(z)$ is a polynomial of the same degree as $g(\theta)$, with $h(z) \neq 0$ in $|z| < 1$, and $h(0) > 0$. This polynomial is uniquely determined. If $g(\theta)$ is a cosine polynomial, $h(z)$ is a polynomial with real coefficients.*

Let $\rho(t)$ be a polynomial of precise degree l and positive in $[-1, 1]$. Then, orthonormal polynomials $p_k(t)$ of degree k , which are associated with weight functions

$$w(t) = \frac{1}{\rho(t)\sqrt{1-t^2}}$$

can be calculated explicitly provided $l < 2k$. Namely, let $\rho(\cos \theta) = |h(e^{i\theta})|^2$ be the normalized representation of $\rho(\cos \theta)$ in the sense of Theorem 2. Then, writing $h(e^{i\theta}) = c(\theta) + is(\theta)$, $c(\theta)$ and $s(\theta)$ real, we have

$$p_k(t) = \sqrt{\frac{2}{\pi}} \operatorname{Re}\{e^{ik\theta} \overline{h(e^{i\theta})}\} = \sqrt{\frac{2}{\pi}} \{c(\theta) \cos k\theta + s(\theta) \sin k\theta\}. \quad (10)$$

These formulas must be modified for $l = 2k$ by multiplying the right-hand member of (10) by a certain constant factor. However, we will only consider $l < 2k$.

The other two weight functions for which the corresponding orthonormal polynomials can be calculated explicitly are:

$$w(t) = \frac{1}{\rho(t)} \sqrt{1-t^2}, \quad p_k(t) = \sqrt{\frac{2}{\pi}} \frac{\operatorname{Im}\{e^{i(k+1)\theta} \overline{h(e^{i\theta})}\}}{\sin \theta}, \quad l < 2k + 2; \quad (11)$$

$$w(t) = \frac{1}{\rho(t)} \sqrt{\frac{1-t}{1+t}}, \quad p_k(t) = \sqrt{\frac{2}{\pi}} \frac{\operatorname{Im}\{e^{i(k+1/2)\theta} \overline{h(e^{i\theta})}\}}{\sin(\theta/2)}, \quad l < 2k + 1. \quad (12)$$

3. SOME PROPERTIES OF THE FUNCTION $\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t})$

The integral $I(n, m, j)$ on the left-hand side of (6) satisfies

$$I(n, m, j) = (-1)^{j-1} I(m, n, j).$$

This symmetry means that it is enough to consider $m \geq n$. The expression

$$\begin{aligned} \rho(t) &= \cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t}) \\ &= T_n(1-2t) + T_m(1+2t), \end{aligned}$$

(where $T_n(x) = \cos(n \cos^{-1} x)$ are Chebyshev polynomials of the first kind) is a polynomial in t of degree

$$\deg \rho = \begin{cases} m, & m > n, \\ 2\lfloor \frac{m}{2} \rfloor, & m = n. \end{cases}$$

One can write for $t \in [-1, 1]$

$$\rho(t) = |\sqrt{2} \cos(n \sin^{-1} \sqrt{t} - im \sinh^{-1} \sqrt{t})|^2.$$

Using logarithmic form of the functions \sin^{-1} , \sinh^{-1} , we obtain the representation

$$\sqrt{2} \cos(n \sin^{-1} \sqrt{\cos \theta} - im \sinh^{-1} \sqrt{\cos \theta}) = i^n e^{-i(n+m)\theta/2} h(e^{i\theta}),$$

$$\begin{aligned} h(z) &= \frac{1}{\sqrt{2}} \left\{ \left(\sqrt{z^2 + 1} + 1 \right)^{\frac{m+n}{2}} \left(\sqrt{z^2 + 1} + z \right)^{\frac{m-n}{2}} \right. \\ &\quad \left. + (-1)^n \left(\sqrt{z^2 + 1} - 1 \right)^{\frac{m+n}{2}} \left(\sqrt{z^2 + 1} - z \right)^{\frac{m-n}{2}} \right\}. \end{aligned} \quad (13)$$

By construction, $h(z)$ is a polynomial in z . There is an alternative form which makes this explicit:

$$h(z) = 2^{\frac{1-m-n}{2}} \left\{ \sum_{k=0}^n \sum_{s=0}^{k/2} \binom{m+n}{m+k} \binom{k}{2s} (-z)^{k-2s} (1+z^2)^s + \sum_{k=1}^m \sum_{s=0}^{k/2} \binom{m+n}{n+k} \binom{k}{2s} z^{k-2s} (1+z^2)^s \right\}.$$

Lemma 3. (i) $\deg h = \deg \rho$.

(ii) $h(0) > 0$.

(iii) $h(z) \neq 0$ in $|z| < 1$.

Proof. Parts (i) and (ii) are obvious. Part (iii) can be proved by Rouché's theorem as follows. Let

$$f(z) = (-1)^n z^{-m-n} \left(1 + \sqrt{z^2 + 1}\right)^{m+n} \left(z + \sqrt{z^2 + 1}\right)^{m-n}.$$

Unlike $h(z)$, $f(z)$ is a multivalued function. We choose brunch cuts on the rays $[i, +i\infty)$ and $[-i, -i\infty)$. The roots of $h(z)$ coincide with the roots of the equation $f(z) - 1 = 0$. Consider the contour C composed of four arcs: two arcs of unit radius centered at the origin, and two arcs of small radius $\varepsilon > 0$ centered at $\pm i$. We will show that $|f(z)| > 1$ on C . Since $f(z)$ does not have zeroes inside the unit circle, according to Rouché's theorem it will follow that $f(z) - 1$ does not vanish inside the unit circle.

One can easily show that $|f(z)| > 1$ when $|z| = 1$, with the exception of the two points $\pm i$. The arc around $+i$ can be parametrized as

$$z = i + \varepsilon e^{-2i\varphi}, \quad \varphi \in (0, \pi/2).$$

Due to

$$|f(i + \varepsilon e^{-2i\varphi})| = 1 + 2\sqrt{\varepsilon} (m \cos \varphi + n \sin \varphi) + O(\varepsilon), \quad \varphi \in (0, \pi/2),$$

and since $m \cos \varphi + n \sin \varphi$ is strictly positive for $\varphi \in (0, \pi/2)$ with positive m, n , we deduce that $|f(z)| > 1$ on the arc around $+i$. The arc around $-i$ is dealt with in the same manner. \square

Hence, $\rho(\cos \theta) = |h(e^{i\theta})|^2$ is the normalized representation of $\rho(\cos \theta)$ in the sense of Theorem 2. In the subsequent sections, we study two cases in detail: When m and n are both odd (the next section), or both even, Section 6.

4. PROOF OF THEOREM 1

One should keep in mind through this section that m, n are both odd. Defining the functions

$$\xi = \cos(n \sin^{-1} \sqrt{t}) \cosh(m \sinh^{-1} \sqrt{t}), \quad (14a)$$

$$\eta = \sin(n \sin^{-1} \sqrt{t}) \sinh(m \sinh^{-1} \sqrt{t}), \quad (14b)$$

we find from (13) that

$$\sqrt{2}(i\xi + \eta) = e^{i(n+m)\theta/2} \overline{h(e^{i\theta})}. \quad (15)$$

Using (10) we find two orthonormal polynomials

$$p_{\frac{m+n}{2}}(t) = 2\pi^{-1/2} \eta = 2\pi^{-1/2} \sin(n \sin^{-1} \sqrt{t}) \sinh(m \sinh^{-1} \sqrt{t}), \quad (16a)$$

$$p_{\frac{m+n}{2}+1}(t) = 2\pi^{-1/2} (t\eta - \sqrt{1-t^2} \xi). \quad (16b)$$

This settles $j \geq 0$ in both equations (6) and (7). To deal with $j = -1$ in (6), we will need the kernel polynomials ([5], Chapter I, eq. 4.11) defined as

$$K_k(t, u) = \sum_{j=0}^k p_j(t) p_j(u) = \frac{\varkappa_k}{\varkappa_{k+1}} \frac{p_{k+1}(t) p_k(u) - p_k(t) p_{k+1}(u)}{t - u}, \quad (17)$$

where \varkappa_j is the leading coefficient of $p_j(t)$. Since $p_{\frac{m+n}{2}}(0) = 0$, this simplifies to

$$K_{\frac{m+n}{2}}(t, 0) = -\varkappa_{\frac{m+n}{2}} \varkappa_{\frac{m+n}{2}+1}^{-1} p_{\frac{m+n}{2}+1}(0) p_{\frac{m+n}{2}}(t)/t.$$

The values of the constants in this formula can be worked out from (14) and (16):

$$\varkappa_{\frac{m+n}{2}+1} = 2\varkappa_{\frac{m+n}{2}}, \quad p_{\frac{m+n}{2}+1}(0) = -2\pi^{-1/2}.$$

To finish the proof, we use the reproducing property of the kernel polynomials with $k = (m + n)/2$

$$\int_{-1}^1 K_k(t, 0) \frac{dt}{\rho(t)\sqrt{1-t^2}} = 1.$$

5. GENERALIZATION THAT INCLUDES AN ADDITIONAL CONTINUOUS PARAMETER $a > 0$

Theorem 1 can be generalized. Let $t \in [-a, 1]$ and

$$\rho_a(t) = \cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t/a}). \quad (18)$$

It is known that the substitution

$$t = \frac{1}{2} \{1 - a + (1 + a) \cos \theta\}, \quad \theta \in [0, \pi],$$

simplifies the square root expression $\sqrt{(1-t)(a+t)} = \frac{1}{2}(1+a) \sin \theta$ so that

$$\frac{dt}{\sqrt{(1-t)(a+t)}} = -d\theta.$$

Then $\rho_a(t)$ becomes a polynomial in $\cos \theta$:

$$\rho_a(t) = \rho(\cos \theta) = T_n(a - (1+a) \cos \theta) + T_m(a^{-1} + (1+a^{-1}) \cos \theta).$$

After some tedious but quite straightforward algebra we obtain the representation

$$\sqrt{2} \cos(n \sin^{-1} \sqrt{t} - im \sinh^{-1} \sqrt{t/a}) = i^n e^{-i(n+m)\theta/2} h_a(e^{i\theta}),$$

$$h_a(z) = \frac{1}{\sqrt{2}} \left(\frac{a+1}{2\sqrt{a}} \right)^m \left\{ \left(\sqrt{z^2 + 2cz + 1} + 1 + cz \right)^{\frac{m+n}{2}} \left(\sqrt{z^2 + 2cz + 1} + c + z \right)^{\frac{m-n}{2}} \right. \\ \left. + (-1)^n \left(\sqrt{z^2 + 2cz + 1} - 1 - cz \right)^{\frac{m+n}{2}} \left(\sqrt{z^2 + 2cz + 1} - c - z \right)^{\frac{m-n}{2}} \right\}, \quad (19)$$

where $c = \frac{1-a}{1+a}$. $h_a(z)$ is a polynomial in z . The above is a generalization of the representation (13), which corresponds to $a = 1$. One can show that this representation satisfies the conditions of Lemma 3. Thus the normalized representation required by Theorem 2 is

$$\rho(\cos \theta) = \rho_a(t) = |h_a(e^{i\theta})|^2.$$

It should be noted that consideration of the more general interval $[-a, b]$, where $b > 0$, along with the substitution $t = \frac{1}{2} \{b - a + (b + a) \cos \theta\}$ does not lead to anything essentially new.

The formulas above are valid for all positive integers. In the following, we specify n, m to be positive odd integers. Then

$$\sqrt{2}(i\xi_a + \eta_a) = e^{i(n+m)\theta/2} \overline{h_a(e^{i\theta})}, \quad (20)$$

where

$$\xi_a = \cos(n \sin^{-1} \sqrt{t}) \cosh(m \sinh^{-1} \sqrt{t/a}), \\ \eta_a = \sin(n \sin^{-1} \sqrt{t}) \sinh(m \sinh^{-1} \sqrt{t/a}).$$

The two orthonormal polynomials corresponding to the weight function $\{\sqrt{(1-t)(a+t)} \rho_a(t)\}^{-1}$ are

$$p_{\frac{m+n}{2}}(t) = 2\pi^{-1/2} \eta_a = 2\pi^{-1/2} \sin(n \sin^{-1} \sqrt{t}) \sinh(m \sinh^{-1} \sqrt{t/a}), \quad (21)$$

$$p_{\frac{m+n}{2}+1}(t) = \frac{2t-1+a}{1+a} p_{\frac{m+n}{2}}(t) - \frac{4\pi^{-1/2}}{1+a} \sqrt{(1-t)(a+t)} \xi_a, \quad (22)$$

by (10). The ratio of their leading coefficients is

$$\mathfrak{x}_{\frac{m+n}{2}+1} / \mathfrak{x}_{\frac{m+n}{2}} = \frac{4}{1+a}. \quad (23)$$

Theorem 4 (1*). Let n and m be positive odd integers and $a > 0$. Then

$$\int_{-a}^1 \frac{\sin(n \sin^{-1} \sqrt{t}) \sinh(m \sinh^{-1} \sqrt{t/a})}{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t/a})} \frac{t^j dt}{\sqrt{(1-t)(1+t/a)}} = \begin{cases} \pi/2, & j = -1, \\ 0, & j = 0, 1, \dots, \frac{m+n-2}{2}, \end{cases} \quad (24)$$

$$\int_{-a}^1 \frac{\cos(n \sin^{-1} \sqrt{t}) \cosh(m \sinh^{-1} \sqrt{t/a})}{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t/a})} t^j dt = 0, \quad j = 0, 1, \dots, \frac{m+n-4}{2}.$$

It should be noted that the symmetric case $m = n$ of (24) also follows from (4) and the identity $\tan^{-1}(a) + \tan^{-1}(1/a) = \pi/2$.

Equation (24) has an additional (integer) parameter m compared to (4). However, the integration range now covers the entire interval $[-a, 1]$. There does not seem to be a closed-form evaluation of the integral in (24) when the integration range is $[0, 1]$ (in other words, there do not seem to be any non-trivial extensions of (4) that include an additional parameter).

6. THE CASE OF EVEN INTEGERS m AND n

Again, one can restrict consideration to $m \geq n$. Then, the degree of the polynomial $\rho_a(t)$ (18) is m . Taking into account that n and m are even we get with ξ_a, η_a defined in the previous section

$$-\sqrt{2}(\xi_a - i\eta_a) = e^{i(n+m)\theta/2} \overline{h_a(e^{i\theta})},$$

where $h_a(z)$ is given by (19). The difference from (20) is the phase factor of -1 instead of i . The orthonormal polynomials of interest are

$$p_{\frac{m+n}{2}}(t) = 2\pi^{-1/2} \xi_a = 2\pi^{-1/2} \cos(n \sin^{-1} \sqrt{t}) \cosh(m \sinh^{-1} \sqrt{t/a}),$$

$$p_{\frac{m+n}{2}+1}(t) = \frac{2t-1+a}{1+a} p_{\frac{m+n}{2}}(t) - \frac{4}{1+a} \sqrt{\frac{1}{\pi}} \eta_a.$$

Since $p_{\frac{m+n}{2}+1}(0) = 0$, equation (17) simplifies to

$$K_{\frac{m+n}{2}}(t, 0) = \varkappa_{\frac{m+n}{2}} \varkappa_{\frac{m+n}{2}+1}^{-1} p_{\frac{m+n}{2}}(0) p_{\frac{m+n}{2}+1}(t)/t,$$

where

$$\varkappa_{\frac{m+n}{2}+1}/\varkappa_{\frac{m+n}{2}} = \frac{4}{1+a}, \quad p_{\frac{m+n}{2}}(0) = \frac{4}{1+a} \sqrt{\frac{1}{\pi}}.$$

The resulting theorem is:

Theorem 5. Let n and m be positive even integers and $a > 0$. Then

$$\int_{-a}^1 \frac{\sin(n \sin^{-1} \sqrt{t}) \sinh(m \sinh^{-1} \sqrt{t/a})}{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t/a})} t^j dt = \begin{cases} \pi/2, & j = -1 \\ 0, & j = 0, 1, \dots, \frac{m+n-4}{2}, \end{cases} \quad (25)$$

$$\int_{-a}^1 \frac{\cos(n \sin^{-1} \sqrt{t}) \cosh(m \sinh^{-1} \sqrt{t/a})}{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t/a})} \frac{t^j dt}{\sqrt{(1-t)(a+t)}} = 0, \quad j = 0, 1, \dots, \frac{m+n-2}{2}. \quad (26)$$

(26) is a two-parameter generalization of (5). Equation (25) have the following interpretation: The degree $\frac{m+n-2}{2}$ orthogonal polynomial for the weight function $\sqrt{(1-t)(a+t)} \{\rho_a(t)\}^{-1}$ is

$$\frac{\sin(n \sin^{-1} \sqrt{t}) \sinh(m \sinh^{-1} \sqrt{t/a})}{\sqrt{(1-t)(a+t)}}.$$

An alternative way to derive this is to start directly from equation (11) instead of (10).

Similarly, it follows from (12) that when n is even and m is odd, the degree $\frac{m+n-1}{2}$ orthogonal polynomial for the weight function $\sqrt{(1-t)/(a+t)} \{\rho_a(t)\}^{-1}$ is

$$\frac{\sin(n \sin^{-1} \sqrt{t}) \sinh(m \sinh^{-1} \sqrt{t/a})}{\sqrt{1-t}}.$$

7. SOME OTHER THEOREMS

Expressions like (18) with different integer parameters can be used as building blocks for more complicated weight functions. Here we restrict our attention to the simplest of such functions

$$\tilde{\rho}_a(t) = \{\rho_a(t)\}^2 = \{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t/a})\}^2,$$

where m, n are positive integers. From (15), we readily obtain the representation

$$2(\xi_a - i\eta_a)^2 = e^{i(n+m)\theta} \overline{h_a^2(e^{i\theta})},$$

with $h_a(z)$ defined in (19). Applying (11) we obtain two orthonormal polynomials for the weight function $\sqrt{(1-t)(a+t)} \{\tilde{\rho}_a(t)\}^{-1}$:

$$p_{m+n-1}(t) = \sqrt{\frac{2}{\pi}} \frac{\sin(2n \sin^{-1} \sqrt{t}) \sinh(2m \sinh^{-1} \sqrt{t/a})}{\sqrt{(1-t)(a+t)}},$$

$$p_{m+n}(t) = \frac{2t-1+a}{1+a} p_{m+n-1}(t) - \frac{4}{1+a} \sqrt{\frac{2}{\pi}} (\xi_a^2 - \eta_a^2).$$

Since $p_{m+n-1}(0) = 0$, (17) simplifies to

$$K_{m+n-1}(t, 0) = -\varkappa_{m+n-1} \varkappa_{m+n}^{-1} p_{m+n}(0) p_{m+n-1}(t)/t,$$

where

$$\varkappa_{m+n}/\varkappa_{m+n-1} = \frac{4}{1+a}, \quad p_{m+n}(0) = -\frac{4}{1+a} \sqrt{\frac{2}{\pi}}.$$

Thus, we obtain:

Theorem 6. *Let n and m be positive integers and $a > 0$. Then*

$$\int_{-a}^1 \frac{\sin(2n \sin^{-1} \sqrt{t}) \sinh(2m \sinh^{-1} \sqrt{t/a})}{\{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t/a})\}^2} t^j dt = \begin{cases} \pi/2, & j = -1, \\ 0, & j = 0, 1, \dots, n+m-2. \end{cases} \quad (27)$$

(27) is closely related to the integral

$$\int_0^\infty \frac{\sin(x \sin \alpha) \sinh(x \cos \alpha)}{\{\cosh(x \cos \alpha) + \cos(x \sin \alpha)\}^2} \frac{dx}{x} = \frac{\alpha}{2}, \quad (28)$$

mentioned in Section 7 of [16]. Integrals similar to (28) were also studied in [6].

8. APPLICATION TO CERTAIN GAUSS QUADRATURES

Theorem 7. *Let n and m be positive odd integers, $a > 0$, and define*

$$\alpha_z = 2n \sinh^{-1} \left(a^{-1/2} \sin \frac{\pi z}{2n} \right), \quad \beta_z = 2m \sinh^{-1} \left(a^{1/2} \sin \frac{\pi z}{2m} \right). \quad (29)$$

Then for any polynomial $p(t)$ of degree at most $m+n-1$

$$\begin{aligned} & \int_{-a}^1 \frac{p(t)}{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t/a})} \frac{dt}{\sqrt{(1-t)(a+t)}} \\ &= \frac{\pi}{2mn} p(0) + \frac{2\pi}{n} \sum_{i=1}^{n/2} \frac{\tanh \frac{\alpha_{2i}}{2n}}{\sinh(\frac{m}{n} \alpha_{2i})} p(\sin^2 \frac{\pi i}{n}) + \frac{2\pi}{m} \sum_{j=1}^{m/2} \frac{\tanh \frac{\beta_{2j}}{2m}}{\sinh(\frac{n}{m} \beta_{2j})} p(-a \sin^2 \frac{\pi j}{m}). \end{aligned} \quad (30)$$

Proof. According to (21), the degree $k = \frac{m+n}{2}$ orthonormal polynomial corresponding to the weight function $\{\sqrt{(1-t)(a+t)} \rho_a(t)\}^{-1}$, where $\rho_a(t)$ is as in (18), is

$$p_k(t) = 2\pi^{-1/2} \sin(n \sin^{-1} \sqrt{t}) \sinh(m \sinh^{-1} \sqrt{t/a}).$$

Its k roots x_s are

$$0; \quad \sin^2 \frac{\pi i}{n}, \quad i = 1, 2, \dots, \frac{n-1}{2}; \quad -a \sin^2 \frac{\pi j}{m}, \quad j = 1, 2, \dots, \frac{m-1}{2}.$$

Gauss quadrature formula [5] now takes the form

$$\int_{-a}^1 p(t) \frac{dt}{\rho_a(t) \sqrt{(1-t)(a+t)}} = \sum_{s=1}^k p(x_s) w_s, \quad w_s = \frac{\varkappa_{k+1}}{\varkappa_k p_{k+1}(x_s) p'_k(x_s)}.$$

The factors entering the formula for the weights w_s can be calculated using formulas (21),(22),(23). \square

Theorem 7 can be extended to a pair of positive even integers using results of Section 6, or to integers of opposite parities (see the remark at the end of Section 6). The limiting $n, m \rightarrow \infty$ form of the integral in Theorem 7 is the same regardless of the parity of n and m . This means that we have established an analog of Theorem 4.5 from [13], which loosely speaking gives two discrete measures that have the same moments as the continuous measure (7) (the corresponding moment problem is known to be indeterminate). A similar situation is also encountered in Section 12.

For the purpose of demonstrating some other possibilities, consider the weight functions of Section 7.

Theorem 8. *Let n and m be positive integers, $a > 0$, and α_z, β_z be defined according to (29). Then for any polynomial $p(t)$ of degree at most $2m + 2n - 3$*

$$\begin{aligned} & \frac{2mn}{\pi a} \int_{-a}^1 \frac{p(t) \sqrt{(1-t)(a+t)}}{\{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t/a})\}^2} dt \\ &= \frac{p(0)}{4} + \sum_{i=1}^{n-1} \frac{m \sinh \frac{\alpha_i}{n}}{\sinh(\frac{m}{n} \alpha_i)} \frac{\cos^2 \frac{\pi i}{2n} \cdot p(\sin^2 \frac{\pi i}{2n})}{\cosh(\frac{m}{n} \alpha_i) + (-1)^i} + \sum_{j=1}^{m-1} \frac{n \sinh \frac{\beta_j}{m}}{\sinh(\frac{n}{m} \beta_j)} \frac{\cos^2 \frac{\pi j}{2m} \cdot p(-a \sin^2 \frac{\pi j}{2m})}{\cosh(\frac{n}{m} \beta_j) + (-1)^j}. \end{aligned}$$

Proof. The $m + n - 1$ roots of the polynomials $p_{m+n-1}(t)$ from the previous section are

$$0; \quad \sin^2 \frac{\pi i}{2n}, \quad i = 1, 2, \dots, n-1; \quad -a \sin^2 \frac{\pi j}{2m}, \quad j = 1, 2, \dots, m-1.$$

To finish the proof, we apply Gauss quadrature formula. \square

The more general weight functions $\sqrt{(1-t)(a+t)} \{\tilde{\rho}_a(t)\}^{-1}$ with

$$\tilde{\rho}_a(t) = \{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t/a})\} \{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m' \sinh^{-1} \sqrt{t/a})\},$$

also admit a corresponding orthogonal polynomials (possibly modified by an additional factor of $\sqrt{a+t}$ depending on whether m and m' are of the same parity or not) with known roots:

$$\frac{\sin(2n \sin^{-1} \sqrt{t}) \sinh((m+m') \sinh^{-1} \sqrt{t/a})}{\sqrt{1-t}}.$$

We only state the limiting form of the corresponding quadrature formula:

Theorem 9. *Let α and β be positive real numbers, and let $p(x)$ be a polynomial. Then*

$$\begin{aligned} & \int_{\mathbb{R}} \frac{p(x) dx}{\{\cos(\sqrt{x}) + \cosh(\alpha \sqrt{x})\} \{\cos(\sqrt{x}) + \cosh(\beta \sqrt{x})\}} = \frac{\pi p(0)}{\alpha + \beta} \\ & + \sum_{j=1}^{\infty} \frac{2\pi^2 j}{\sinh \frac{\pi(\alpha+\beta)j}{2}} \frac{p(\pi^2 j^2)}{\cosh \frac{\pi(\alpha+\beta)j}{2} + (-1)^j \cosh \frac{\pi(\alpha-\beta)j}{2}} + \frac{8\pi^2}{(\alpha + \beta)^2} \sum_{j=1}^{\infty} \frac{j}{\sinh \frac{2\pi j}{\alpha+\beta}} \frac{p\left(-\frac{4\pi^2 j^2}{(\alpha+\beta)^2}\right)}{\cosh \frac{2\pi j}{\alpha+\beta} + \cos \frac{2\pi \alpha j}{\alpha+\beta}}. \end{aligned}$$

Quadrature formulas can also be studied when the roots of the orthogonal polynomials are not known explicitly. For example, when the limiting behaviour of the roots as $n, m \rightarrow \infty$ is described by the transcendental equation $\tan(x) \tanh(x) = -1$ [1].

As a consequence of the considerations outlined above (see also the next section):

Corollary 10. *For any positive integer n*

$$\int_{-a}^1 \frac{\sqrt{(1-t)(a+t)}}{a + 2t + aT_n(1-t)} dt = \frac{\pi}{4}, \quad a > 0,$$

$$\int_{-1}^1 \frac{1}{1+2t+T_n(1-2t)} \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{\sqrt{8}} \frac{(\sqrt{2}+1)^{2n}+1}{(\sqrt{2}+1)^{2n}-1},$$

and positive integers n and m of the same parity

$$\int_{-1}^1 \frac{\sqrt{1-t^2}}{\{1+2t+T_n(1-2t)\}\{1+2t+T_m(1-2t)\}} dt = \frac{\pi}{4(n+m)} \sum_{|j| < \frac{n+m}{2}} \frac{1 + \cos \frac{2\pi j}{n+m}}{2 - \cos \frac{2\pi j}{n+m} + \cos \frac{2\pi m j}{n+m}}.$$

In general, the sum in the last formula does not appear to have a simple closed-form evaluation.

9. FINITE ANALOGS OF GENERATING FUNCTIONS FROM KUZNETSOV'S PAPER [15]

We are going to apply Theorem 7 to particular polynomials. It turns out that for suitable choice of the polynomial $p(t)$ the right hand side of (30) can be written as a single sum.

Theorem 11. *Let n, m, u be integers such that $|u| < n$. Let $a > 0$ and define α_z as in (29). Then*

$$\begin{aligned} \int_{-a}^1 \frac{\cos(2u \sin^{-1} \sqrt{t})}{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2m \sinh^{-1} \sqrt{t/a})} \frac{dt}{\sqrt{(1-t)(a+t)}} \\ = \frac{\pi}{2n} \sum_{j=1}^{2n} \frac{(-1)^{j-1}}{\coth \frac{\alpha_j}{2n}} \left\{ \tanh \frac{m\alpha_j}{2n} \right\}^{(-1)^j} \cdot \cos \frac{\pi j u}{n}. \end{aligned}$$

Proof. Here, we prove the theorem only for an odd pair of integers n, m . Other alternatives can be proved in quite similar way, starting from appropriate variants of Gauss quadrature formulas.

Equation (30) with degree u polynomial $p(t) = \cos(2u \sin^{-1} \sqrt{t})$ yields for the value of the integral

$$\frac{\pi}{2mn} + \frac{2\pi}{n} \sum_{i=1}^{n/2} \frac{\tanh \frac{\alpha_{2i}}{2n}}{\sinh(\frac{m}{n} \alpha_{2i})} \cos \frac{2\pi i u}{n} + \frac{2\pi}{m} \sum_{j=1}^{m/2} \frac{\tanh \frac{\beta_{2j}}{2m}}{\sinh(\frac{n}{m} \beta_{2j})} \cosh(\frac{u}{m} \beta_{2j}).$$

We transform the first sum according to $2 \sum_{i=1}^{n/2} = \sum_{i=1}^{n-1}$ using the fact that the summand is symmetric under $i \rightarrow n-i$. In the second sum, we rewrite the summand using the identity valid for integers $|u| < n$

$$\frac{\tanh z}{\sinh(2nz)} \cosh(2uz) = \frac{1}{2n} \sum_{i=1}^{2n-1} (-1)^{i-1} \frac{\sin^2 \frac{\pi i}{2n}}{\sinh^2 z + \sin^2 \frac{\pi i}{2n}} \cos \frac{\pi i u}{n},$$

with $z = 2m\beta_{2j} = \sinh^{-1}(\sin \frac{\pi j}{m})$:

$$\frac{\pi}{mn} \sum_{j=1}^{m/2} \sum_{i=1}^{2n-1} (-1)^{i-1} \frac{\sin^2 \frac{\pi i}{2n}}{\sin^2 \frac{\pi j}{m} + \sin^2 \frac{\pi i}{2n}} \cos \frac{\pi i u}{n}.$$

For odd m , the sum over j is

$$\sum_{j=1}^{m/2} \frac{\sinh^2 z}{\sin^2 \frac{\pi j}{m} + \sinh^2 z} = \frac{m \tanh z}{2 \tanh(mz)} - \frac{1}{2}.$$

After these transformations, the value of the integral becomes

$$\frac{\pi}{2mn} + \frac{\pi}{n} \sum_{i=1}^{n-1} \frac{\tanh \frac{\alpha_{2i}}{2n}}{\sinh(\frac{m}{n} \alpha_{2i})} \cos \frac{2\pi i u}{n} + \frac{\pi}{2n} \sum_{i=1}^{2n-1} (-1)^{i-1} \frac{\tanh \frac{\alpha_i}{2n}}{\tanh(\frac{m}{2n} \alpha_i)} \cos \frac{\pi i u}{n} - \frac{\pi}{2mn} \sum_{i=1}^{2n-1} (-1)^{i-1} \cos \frac{\pi i u}{n}.$$

The last sum cancels the first term due to the simple identity valid for integers $|u| < n$

$$\sum_{i=1}^{2n-1} (-1)^{i-1} \cos \frac{\pi i u}{n} = 1.$$

Next, we split the second sum into even and odd terms, and then combine even terms with the first sum using $2/\sinh(2x) - \coth(x) = -\tanh(x)$. \square

There is a transformation for the right hand side of the formula of Theorem 11

$$\frac{\pi}{2m} \sum_{j=1}^{2m} \frac{(-1)^{j-1}}{\coth \frac{\beta_j}{2m}} \left\{ \tanh \frac{n\beta_j}{2m} \right\}^{(-1)^j} \cdot \cosh \frac{u\beta_j}{m}, \quad |u| < m,$$

analogous to imaginary transformation of Jacobi's elliptic functions.

Theorem 11 is a non-symmetric ($m \neq n$, $a \neq 1$) extension of Theorem 4 from [16]. The proof of Theorem 4 in [16] used ad hoc methods, that could not be applied to more general integrals. The case $u = 0$ of Theorem 11 is a finite analog of Ismail and Valent's integral [13]. There are also other integrals with $\cos(2u \sin^{-1} \sqrt{t})$ replaced by $\sin(2u \sin^{-1} \sqrt{t}) / \sqrt{t(1-t)}$, or with the differences $\cosh(2m \sinh^{-1} \sqrt{t/a}) - \cos(2n \sin^{-1} \sqrt{t})$ in the denominator (Section 12), etc.

10. A FINITE ANALOG OF THE INTEGRAL IN RAMANUJAN'S QUESTION 353

Theorem 12. *Let n be a positive even integer and k a positive odd integer. Then*

$$\int_0^1 \frac{\sin(kn \sin^{-1} t)}{\cos(n \sin^{-1} t) + \cosh(n \sinh^{-1} t)} \frac{dt}{t} = \frac{\pi}{4}. \quad (31)$$

Proof. Let $n = 2\nu$, $k = 2\mu + 1$, where ν is a positive integer, and μ is a nonnegative integer. Similar to that of Section 3 of [16], or by other means, one can derive the partial fractions expansion

$$\frac{1}{\cos(2\nu \sin^{-1} t) + \cosh(2\nu \sinh^{-1} t)} \frac{\sin(2\nu \sin^{-1} t)}{t\sqrt{1-t^2}} = \frac{1}{\nu} \sum_{j=1}^{\nu} \frac{i - \cos \frac{\pi(2j-1)}{2\nu}}{2t^2 \cos \frac{\pi(2j-1)}{2\nu} + i \sin^2 \frac{\pi(2j-1)}{2\nu}}.$$

Further calculations assume that ν is even. When ν is odd, calculations are similar, except that one has to take special care of the term with $j = (\nu + 1)/2$. Thus, define

$$q_j = \frac{1 - \sin \frac{\pi(2j-1)}{2\nu}}{\cos \frac{\pi(2j-1)}{2\nu}} e^{-i \frac{\pi(2j-1)}{2\nu}}, \quad j = 1, 2, \dots, \nu.$$

Obviously,

$$|q_j| < 1, \quad j = 1, 2, \dots, \nu.$$

We are going to make the change of variables

$$t = \sin(\varphi/2), \quad \varphi \in (0, \pi),$$

in the integral (31). Thus $2t^2 = 1 - \cos \varphi$, and $4\sqrt{1-t^2} dt = (1 + \cos \varphi) d\varphi$. By simple algebra

$$\frac{1 + \cos \varphi}{(1 - \cos \varphi) \cos \frac{\pi(2j-1)}{2\nu} + i \sin^2 \frac{\pi(2j-1)}{2\nu}} = \frac{-2}{(1 - q_j) \cos \frac{\pi(2j-1)}{2\nu}} \left(1 + (1 + q_j) \frac{1 - q_j \cos \varphi}{1 - 2q_j \cos \varphi + q_j^2} \right).$$

According to well known formulas

$$\frac{1 - q_j \cos \varphi}{1 - 2q_j \cos \varphi + q_j^2} = \sum_{r=0}^{\infty} q_j^r \cos(r\varphi),$$

$$\frac{\sin(k\nu\varphi)}{\sin(\nu\varphi)} = 1 + 2 \sum_{l=1}^{\mu} \cos(2\nu l\varphi).$$

Thus, the integral (31) becomes

$$I = \frac{1}{2\nu} \sum_{j=1}^{\nu} \frac{\cos \frac{\pi(2j-1)}{2\nu} - i}{(1 - q_j) \cos \frac{\pi(2j-1)}{2\nu}} \int_0^{\pi} \left(1 + (1 + q_j) \sum_{r=0}^{\infty} q_j^r \cos(r\varphi) \right) \left(1 + 2 \sum_{l=1}^{\mu} \cos(2\nu l\varphi) \right) d\varphi.$$

The integrals are easily calculated using orthogonality of cosines on $(0, \pi)$:

$$I = \frac{\pi}{2\nu} \sum_{j=1}^{\nu} f(j), \quad f(j) = \frac{\cos \frac{\pi(2j-1)}{2\nu} - i}{(1 - q_j) \cos \frac{\pi(2j-1)}{2\nu}} \left(2 + q_j + (1 + q_j) \sum_{l=1}^{\mu} q_j^{2\nu l} \right).$$

Trivial algebra (under the transformation $j \rightarrow \nu + 1 - j$ the expression $\cos \frac{\pi(2j-1)}{2\nu}$ changes sign, while $\sin \frac{\pi(2j-1)}{2\nu}$ and $q_j^{2\nu}$ do not change) shows that

$$f(j) + f(\nu + 1 - j) = 1, \quad j = 1, 2, \dots, \nu.$$

Hence $\sum_{j=1}^{\nu} f(j) = \nu/2$, and $I = \pi/4$. \square

One can obtain a finite analog of Theorem 4.2 from [2] multiplying the integrand in (31) by t^{4b} , $b \in \mathbb{N}$.

11. FINITE ANALOGS OF INTEGRALS RELATED TO THETA SERIES

Consider the finite trigonometric sum

$$S(n, m) = \sum_{j=0}^{n/2} (-1)^j \sin \frac{\pi(2j+1)}{2n} \left\{ \cos \frac{\pi(2j+1)}{2n} \right\}^{m-1}. \quad (32)$$

Theorem 13. *Let $m > 1$ and n be positive odd integers. Then*

$$\int_0^1 \frac{\sin(n \sin^{-1} \sqrt{t}) \sinh(n \sinh^{-1} \sqrt{t})}{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2n \sinh^{-1} \sqrt{t})} \frac{\sin(m \sin^{-1} t)}{\sqrt{1-t^2}} dt = \frac{\pi}{4n} S(n, m). \quad (33)$$

Proof. We make the change of variables $t = \sin \theta$ in the integral. From

$$\sin^{-1} \sqrt{\sin \theta} + i \sinh^{-1} \sqrt{\sin \theta} = \cos^{-1}(e^{-i\theta}), \quad 0 \leq \theta \leq \frac{\pi}{2}$$

(Section 3), and the well known partial fractions expansion formula for the reciprocal of the Chebyshev polynomial of the first kind

$$\frac{1}{T_k(z)} = \frac{1}{k} \sum_{j=0}^{k-1} (-1)^j \frac{\sin \frac{\pi(2j+1)}{2k}}{z - \cos \frac{\pi(2j+1)}{2k}},$$

it follows that

$$\frac{1}{\cos(k \sin^{-1} \sqrt{\sin \theta} + ik \sinh^{-1} \sqrt{\sin \theta})} = \frac{1}{k} \sum_{j=0}^{k-1} (-1)^j \frac{\sin \frac{\pi(2j+1)}{2k}}{e^{-i\theta} - \cos \frac{\pi(2j+1)}{2k}}, \quad k \in \mathbb{N}. \quad (34)$$

For odd n one obtains from this

$$\frac{1}{\cos(n \sin^{-1} \sqrt{\sin \theta} + in \sinh^{-1} \sqrt{\sin \theta})} = \frac{1}{n} \sum_{r=1}^{\infty} S(n, 2r+1) e^{i(2r+1)\theta}, \quad (35)$$

with $S(n, 2r+1)$ defined in (32). Taking the imaginary part of (35) results in

$$\frac{\sin(n \sin^{-1} \sqrt{\sin \theta}) \sinh(n \sinh^{-1} \sqrt{\sin \theta})}{\cos(2n \sin^{-1} \sqrt{\sin \theta}) + \cosh(2n \sinh^{-1} \sqrt{\sin \theta})} = \frac{1}{n} \sum_{r=1}^{\infty} S(n, 2r+1) \sin(2r+1)\theta. \quad (35a)$$

Thus the integral becomes

$$\frac{1}{n} \sum_{r=1}^{\infty} S(n, 2r+1) \int_0^{\pi/2} \sin\{(2r+1)\theta\} \sin(m\theta) d\theta.$$

Now the result follows by orthogonality of the system of functions $\{\sin(2k+1)\theta\}_{k=0}^{\infty}$ on $[0, \pi/2]$ (system of eigenfunctions of the Sturm-Liouville problem $f'' + \lambda f = 0$, $f(0) = f'(\pi/2) = 0$). \square

The companion formula to (35a), which follows from (35) by taking the real part, is

$$\frac{\cos(n \sin^{-1} \sqrt{\sin \theta}) \cosh(n \sinh^{-1} \sqrt{\sin \theta})}{\cos(2n \sin^{-1} \sqrt{\sin \theta}) + \cosh(2n \sinh^{-1} \sqrt{\sin \theta})} = \frac{1}{n} \sum_{r=1}^{\infty} S(n, 2r+1) \cos(2r+1)\theta, \quad (35b)$$

with the same $S(n, 2r + 1)$, and for odd n . It leads to integration formula analogous to (33). Moreover, if we replace m by mn in (33), multiply both sides by n^2 , make change of variables in the integral $t \rightarrow t/n^2$ and let $n, m \rightarrow \infty$ such that $m/n \rightarrow a$ in the resulting formula, we get

$$\int_0^\infty \frac{\sin(\sqrt{x}) \sinh(\sqrt{x})}{\cos(2\sqrt{x}) + \cosh(2\sqrt{x})} \sin(ax) dx = \frac{\pi^2}{8} \sum_{j=0}^\infty (-1)^j (2j+1) e^{-\pi^2(2j+1)^2 a/8}$$

(for justification of the limiting process for a similar series see [21]). This means that (33) and its companion formula are finite analogs of two integrals due to Glaisher, equations (21) and (22) in [10].

There is extensive literature on finite trigonometric sums. Sums similar to (32) are studied for example in [8] and [4] using various techniques. These papers also discuss different contexts such sums arise in, and also contain excellent survey of the earlier literature on the topic. In particular, finite trigonometric sums have been studied using generating functions. For example, generating functions for finite sums (of reciprocals) of trigonometric functions were calculated in [26] (this paper generalizes some generating functions from the earlier literature). Equations (35), (35a) and (35b) are generating functions for the finite trigonometric sum (32). In fact, Theorem 13 is equivalent to the generating function (35a). The generating functions in [26] contain an extra free parameter, moreover they are simpler in form in the sense that they contain just one Chebyshev polynomial in the denominator. Equations (35a) and (35b) contain two Chebyshev polynomials in the denominator, of the same order but of different arguments. Functions that contain two Chebyshev polynomials of different orders but of the same argument arise as generating functions for probabilities of random walks with boundaries, ([9], equations (2),(3),(12)).

There is a one-parameter generalization of (34):

Theorem 14. *Let $c \in [0, 1]$, $\theta \in [0, \pi]$, and $n \in \mathbb{N}$. Then*

$$\left\{ \cos \left(n \sin^{-1} \sqrt{\frac{\sin \theta + c}{1+c}} + in \sinh^{-1} \sqrt{\frac{\sin \theta + c}{1-c}} \right) \right\}^{-1} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{(-1)^j \sqrt{1-c^2} \sin \frac{\pi(2j+1)}{2n}}{e^{-i\theta} - \sqrt{1-c^2} \cos \frac{\pi(2j+1)}{2n} - ic}. \quad (36)$$

A generalization of Theorem 13 can be obtained from (36).

12. WEIGHTS WITH THE DIFFERENCE $\cosh - \cos$ IN THE DENOMINATOR

Consider the non-negative weight function on \mathbb{R}

$$\frac{x}{\cosh(2\sqrt{x/a}) - \cos(2\sqrt{x})}, \quad a > 0,$$

which has a removable singularity at $x = 0$. The symmetric $a = 1$ case of the integrals with such functions was studied in [2], and the non-symmetric case in [3]. The weight functions on $[-a, 1]$

$$\frac{t}{\cosh(2m \sinh^{-1} \sqrt{t/a}) - \cos(2n \sin^{-1} \sqrt{t})} \frac{1}{\sqrt{(1-t)(a+t)}}$$

approximate the above functions in the limit $n, m \rightarrow \infty$. Below we briefly discuss how do the formulas in the main text change for such weight functions.

Define the function

$$\rho_a(t) = \frac{\cosh(2m \sinh^{-1} \sqrt{t/a}) - \cos(2n \sin^{-1} \sqrt{t})}{t} = \left| \sqrt{\frac{2}{t}} \sin(n \sin^{-1} \sqrt{t} - im \sinh^{-1} \sqrt{t/a}) \right|^2.$$

Using the substitution

$$t = \frac{1}{2} \{1 - a + (1 + a) \cos \theta\}, \quad \theta \in [0, \pi],$$

one can prove that

$$\sqrt{\frac{2}{t}} \sin(n \sin^{-1} \sqrt{t} - im \sinh^{-1} \sqrt{t/a}) = i^{n-1} e^{-i(n+m-1)\theta/2} h_a(e^{i\theta}),$$

$$h_a(z) = \sqrt{\frac{2}{a+1}} \left(\frac{a+1}{2\sqrt{a}} \right)^m \{ \sigma(z) \}^{-1} \\ \times \left\{ [\sigma(z) + 1 + cz]^{\frac{m+n}{2}} [\sigma(z) + c + z]^{\frac{m-n}{2}} - (-1)^n [\sigma(z) - 1 - cz]^{\frac{m+n}{2}} [\sigma(z) - c - z]^{\frac{m-n}{2}} \right\},$$

where

$$\sigma(z) = \sqrt{z^2 + 2cz + 1}, \quad c = \frac{1-a}{1+a}.$$

$h_a(z)$ is a polynomial in z . One can show that $h_a(z)$ satisfies the conditions of Lemma 3. Thus the normalized representation required by Theorem 2 in this case is

$$\rho(\cos \theta) = \rho_a(t) = |h_a(e^{i\theta})|^2.$$

Using (10) we find the following orthogonal polynomials with known roots for the weight function under consideration: When n is odd and m is even

$$p_{\frac{m+n-1}{2}}(t) = \frac{2}{\sqrt{\pi t}} \sin(n \sin^{-1} \sqrt{t}) \cosh(m \sinh^{-1} \sqrt{t/a});$$

when n is even and m is odd

$$p_{\frac{m+n-1}{2}}(t) = \frac{2}{\sqrt{\pi t}} \cos(n \sin^{-1} \sqrt{t}) \sinh(m \sinh^{-1} \sqrt{t/a}).$$

Now we can find explicit Gauss quadrature formulas. If n is odd and m is even, and $p(t)$ is a polynomial of degree at most $m+n-1$ such that $p(0) = 0$, then

$$\begin{aligned} & \int_{-a}^1 \frac{p(t)}{\cosh(2m \sinh^{-1} \sqrt{t/a}) - \cos(2n \sin^{-1} \sqrt{t})} \frac{dt}{\sqrt{(1-t)(a+t)}} \\ &= \frac{2\pi}{n} \sum_{i=1}^{n/2} \frac{\tanh \frac{\alpha_{2i}}{2n}}{\sinh(\frac{m}{n} \alpha_{2i})} p(\sin^2 \frac{\pi i}{n}) - \frac{2\pi}{m} \sum_{j=1}^{m/2} \frac{\tanh \frac{\beta_{2j-1}}{2m}}{\sinh(\frac{n}{m} \beta_{2j-1})} p(-a \sin^2 \frac{\pi(2j-1)}{2m}), \end{aligned}$$

where α_z and β_z are defined in (29). Quadrature formula for even n and odd m is obtained from the above formula by the change of variables $t \rightarrow -t/a$, and subsequent redefinitions of the parameters $a \rightarrow 1/a$, $m \rightarrow n$, $n \rightarrow m$ and the polynomial $p(t)$. The integral has the same limiting form as $n, m \rightarrow \infty$ irrespective of the parity of the integers n and m : For any polynomial $p(x)$ and $\alpha > 0$

$$\begin{aligned} \frac{1}{4\pi^4} \int_{\mathbb{R}} \frac{xp(x)}{\cosh(\alpha\sqrt{x}) - \cos(\sqrt{x})} dx &= \sum_{\substack{j \geq 1 \\ \text{even}}} \frac{j^3}{\sinh(\pi\alpha j)} p(\pi^2 j^2) + \frac{1}{\alpha^4} \sum_{\substack{j \geq 1 \\ \text{odd}}} \frac{j^3}{\sinh \frac{\pi j}{\alpha}} p\left(-\frac{\pi^2 j^2}{\alpha^2}\right) \\ &= \sum_{\substack{j \geq 1 \\ \text{odd}}} \frac{j^3}{\sinh(\pi\alpha j)} p(\pi^2 j^2) + \frac{1}{\alpha^4} \sum_{\substack{j \geq 1 \\ \text{even}}} \frac{j^3}{\sinh \frac{\pi j}{\alpha}} p\left(-\frac{\pi^2 j^2}{\alpha^2}\right). \end{aligned}$$

Detailed study of the corresponding indeterminate moment problem is given in [14] (in particular, Theorem 4.4.4), where parametrization in terms of elliptic integrals is used: $\alpha = K'/K$.

A particular form of the quadrature formula is: Let n and m be positive integers, u be an odd integer, and let α_z and β_z be defined as in (29), then

$$\begin{aligned} & \int_{-a}^1 \frac{\sqrt{t} \sin(u \sin^{-1} \sqrt{t})}{\cosh(2m \sinh^{-1} \sqrt{t/a}) - \cos(2n \sin^{-1} \sqrt{t})} \frac{dt}{\sqrt{(1-t)(a+t)}} \\ &= \frac{\pi}{2n\sqrt{a}} \sum_{j=1}^{2n-1} (-1)^j \frac{\sin^2 \frac{\pi j}{2n}}{\cosh \frac{\alpha_j}{2n}} \left\{ \coth \frac{m\alpha_j}{2n} \right\}^{(-1)^j} \cdot \sin \frac{\pi j u}{2n}, \quad |u| < n-1, \\ &= \frac{\pi a}{2n} \sum_{j=1}^{2m-1} (-1)^j \frac{\sin^2 \frac{\pi j}{2m}}{\cosh \frac{\beta_j}{2m}} \left\{ \coth \frac{n\alpha_j}{2m} \right\}^{(-1)^j} \cdot \sinh \frac{\beta_j u}{2m}, \quad |u| < m-1. \end{aligned}$$

This is a finite analog of the generating function derived in [3].

Now consider the weight function

$$\sqrt{(1-t)(a+t)} \{\tilde{\rho}_a(t)\}^{-1},$$

where

$$\tilde{\rho}_a(t) = \frac{1}{t^2} \left\{ \cosh(2m \sinh^{-1} \sqrt{t/a}) - \cos(2n \sin^{-1} \sqrt{t}) \right\} \left\{ \cosh(2m' \sinh^{-1} \sqrt{t/a}) - \cos(2n \sin^{-1} \sqrt{t}) \right\}.$$

When $m + m'$ is even, we find the following orthogonal polynomial for this weight function

$$p_{\frac{m+m'}{2}+n-2}(t) = \sqrt{\frac{2}{\pi}} \frac{\sin(2n \sin^{-1} \sqrt{t}) \sinh((m+m') \sinh^{-1} \sqrt{t/a})}{t \sqrt{(1-t)(a+t)}}.$$

The limiting form of the corresponding quadrature formula is

$$\begin{aligned} & \frac{1}{2\pi^6} \int_{\mathbb{R}} \frac{x^2 p(x)}{\{\cosh(\alpha\sqrt{x}) - \cos(\sqrt{x})\} \{\cosh(\beta\sqrt{x}) - \cos(\sqrt{x})\}} dx \\ &= \sum_{j \geq 1} \frac{j^5}{\sinh \frac{\pi(\alpha+\beta)j}{2}} \frac{p(\pi^2 j^2)}{\cosh \frac{\pi(\alpha+\beta)j}{2} - (-1)^j \cosh \frac{\pi(\alpha-\beta)j}{2}} + \frac{64}{(\alpha+\beta)^6} \sum_{j \geq 1} \frac{j^5}{\sinh \frac{2\pi j}{\alpha+\beta}} \frac{p\left(-\frac{4\pi^2 j^2}{(\alpha+\beta)^2}\right)}{\cosh \frac{2\pi j}{\alpha+\beta} - \cos \frac{2\pi \alpha j}{\alpha+\beta}}. \end{aligned}$$

One could also consider mixed weight functions with

$$\tilde{\rho}_a(t) = \frac{1}{t} \left\{ \cosh(2m \sinh^{-1} \sqrt{t/a}) + \cos(2n \sin^{-1} \sqrt{t}) \right\} \left\{ \cosh(2m' \sinh^{-1} \sqrt{t/a}) - \cos(2n \sin^{-1} \sqrt{t}) \right\}.$$

When $m + m'$ is even, the corresponding orthogonal polynomial is

$$\frac{\sin(2n \sin^{-1} \sqrt{t}) \cosh((m+m') \sinh^{-1} \sqrt{t/a})}{\sqrt{t(1-t)}}.$$

The limiting form of the quadrature formula is

$$\begin{aligned} & \frac{1}{2\pi^4} \int_{\mathbb{R}} \frac{x p(x)}{\{\cosh(\alpha\sqrt{x}) + \cos(\sqrt{x})\} \{\cosh(\beta\sqrt{x}) - \cos(\sqrt{x})\}} dx \\ &= \sum_{j \geq 1} \frac{j^3}{\cosh \frac{\pi(\alpha+\beta)j}{2}} \frac{p(\pi^2 j^2)}{\sinh \frac{\pi(\alpha+\beta)j}{2} - (-1)^j \sinh \frac{\pi(\alpha-\beta)j}{2}} + \frac{2}{(\alpha+\beta)^4} \sum_{\substack{j \geq 1 \\ \text{odd}}} \frac{j^3}{\sinh \frac{\pi j}{\alpha+\beta}} \frac{p\left(-\frac{\pi^2 j^2}{(\alpha+\beta)^2}\right)}{\cosh \frac{\pi j}{\alpha+\beta} + \cos \frac{\pi \alpha j}{\alpha+\beta}}. \end{aligned}$$

Similar to (34), one can deduce from the partial fractions expansion formula for the reciprocal of the Chebyshev polynomial of the second kind

$$\frac{1}{(1-z^2)U_{k-1}(z)} = \frac{1}{2k} \sum_{j=1}^{2k} \frac{(-1)^{j-1}}{z - \cos \frac{\pi j}{k}},$$

that

$$\frac{e^{i\theta/2}}{\sqrt{2i \sin \theta}} \cdot \frac{1}{\sin(k \sin^{-1} \sqrt{\sin \theta} + ik \sinh^{-1} \sqrt{\sin \theta})} = \frac{1}{2k} \sum_{j=1}^{2k} \frac{(-1)^{j-1}}{e^{-i\theta} - \cos \frac{\pi j}{k}}, \quad k \in \mathbb{N}.$$

ACKNOWLEDGEMENTS

We would like to thank Teruo [25] and Polynomial [20] from math.stackexchange.com website for their stimulating and insightful answers to our question #3595770.

REFERENCES

- [1] C. Berg and G. Valent, The Nevanlinna parametrization for some indeterminate Stieltjes moment problems associated with birth and death processes, *Methods and applications of analysis*, 1.2, 169-209 (1994).
- [2] B.C. Berndt, Integrals associated with Ramanujan and elliptic functions, *The Ramanujan Journal*, 41, 369 (2016).
- [3] Z.P. Bradshaw, and C. Vignat, Berndt-type integrals: unveiling connections with Barnes zeta and Jacobi elliptic functions, *The Ramanujan Journal*, 66(3), 60 (2025).
- [4] C.A. Cadavid, H. Paulina, J. Jorgenson, L. Smajlović, and J.D. Vélez, On an approach for evaluating certain trigonometric character sums using the discrete time heat kernel, *European Journal of Combinatorics* 108, 103635 (2023).
- [5] T.S. Chihara, *An introduction to orthogonal polynomials*, Courier Corporation, (2011).
- [6] S. Chowla, Some infinite series, definite integrals and asymptotic expansions, *J. Indian Math. Soc.* 17, 261–288 (1928).
- [7] M.W. Coffey, Some integrals of the Dedekind η function, <https://arxiv.org/abs/1901.07168> (2019).
- [8] C.M. Fonseca, M.L. Glasser, and V. Kowalenko, Basic trigonometric power sums with applications, *The Ramanujan Journal* 42, 401-428 (2017).
- [9] L. Giuggioli, Exact spatiotemporal dynamics of confined lattice random walks in arbitrary dimensions: a century after Smoluchowski and Pólya, *Physical Review X* 10.2, 021045 (2020).

- [10] J.W.L. Glaisher, On the summation by definite integrals of geometric series of the second and higher order, *Quart. J. Math. Oxford Ser.* 11, 328 (1871).
- [11] M.L. Glasser, Some integrals of the Dedekind η -function, *Journal of Mathematical Analysis and Applications*, 354.2, 490-493 (2009).
- [12] I.S. Gradshteyn, and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed., Academic Press, Boston (2000).
- [13] M.E.H. Ismail and G. Valent, On a family of orthogonal polynomials related to elliptic functions, *Illinois J. Math.* 42 (2), 294 (1998).
- [14] M.E.H. Ismail, G. Valent, and G.J. Yoon, Some orthogonal polynomials related to elliptic functions, *Journal of Approximation Theory* 112.2, 251-278 (2001).
- [15] A. Kuznetsov, A direct evaluation of an integral of Ismail and Valent, *Frontiers in Orthogonal Polynomials and q-Series*, 417-424 (2018).
- [16] M. Nicholson, A sequence of elementary integrals related to integrals studied by Glaisher that contain trigonometric and hyperbolic functions, <https://arxiv.org/abs/2008.04097> (2020).
- [17] E. Pan, and X. Wang, Explicit evaluation of a family of Ramanujan-type integrals, *Integral Transforms and Special Functions*, 1-8 (2025).
- [18] A.E. Patkowski and M. Wolf, Some Remarks on Glaisher-Ramanujan Type Integrals, *CMST*, 22.2, 103-108 (2016).
- [19] H.B. Phillips and N. Wiener, Nets and the Dirichlet Problem, *Studies in Applied Mathematics* 2, 105-124 (1923).
- [20] Polynomial, <https://math.stackexchange.com/a/5033465/285751> (2025).
- [21] G. Polya, Elementarer Beweis einer Thetaformel, *Sitzungsberichten der Preussischen Akademie des Wissenschaften, Physikalisch-mathematische Klasse*, 157-161 (1927).
- [22] S. Ramanujan, Question 353, *J. Indian Math. Soc.* 4, 40 (1912).
- [23] H. Rui, C. Xu, R. Yang, and J. Zhao, Explicit evaluation of a family of Berndt-type integrals, *Journal of Mathematical Analysis and Applications*, 541, 1 (2025).
- [24] G. Szego, *Orthogonal polynomials*, Vol. 23, American Mathematical Soc. (1939).
- [25] Teruo, <https://math.stackexchange.com/a/3664265/285751> (2020).
- [26] X. Wang and D.-Y. Zheng, Summation formulae on trigonometric functions, *Journal of Mathematical Analysis and Applications*, 335, Issue 2, (2007).
- [27] C. Xu, and J. Zhao, Berndt-type integrals and series associated with Ramanujan and Jacobi elliptic functions, <https://arxiv.org/abs/2301.08211> (2023).
- [28] J. Zhou, Mixed Berndt-Type Integrals and Generalized Barnes Multiple Zeta Functions, <https://arxiv.org/abs/2502.15173> (2025).