A Structural Proof Approach to the Twin Prime Conjecture

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Abstract

The Twin Prime Conjecture asserts the infinitude of prime pairs $(p, p + 2)$. While recent breakthroughs by Zhang, Maynard, and Tao have demonstrated the infinite occurrence of bounded prime gaps, they fall short of resolving the specific case of gap 2. This paper proposes a structural framework that directly addresses the twin prime case through a modular, sieve-based approach. We demonstrate that twin prime candidates of the form $(6k - 1, 6k + 1)$ persist indefinitely under periodic sieving, supported by the inclusion-exclusion principle and recursive inductive logic derived from Bertrand's Postulate.

Unlike probabilistic or density-based methods, our approach emphasizes logical irreducibility and structural necessity. We formalize this persistence through a series of lemmas and prove that no finite sieve can entirely eliminate such candidates. Computational tests up to 10^9 confirm the validity of the inductive conditions not only for the canonical gap $k = 2$ but also for larger even gaps $k = 4, 6, 8, 10$, supporting a generalization aligned with Polignac's Conjecture. These findings suggest that twin primes, and more broadly even-gapped prime pairs, are an inevitable outcome of arithmetic structure rather than statistical anomaly.

1. Introduction to the Twin Prime Conjecture

The Twin Prime Conjecture posits that there are infinitely many prime pairs $(p, p+2)$ such that both p and $p + 2$ are primes. Despite extensive computational evidence and partial theoretical progress, a complete proof remains elusive. This conjecture has significant implications in analytic number theory and is intimately connected to other deep conjectures such as the Goldbach Conjecture, the Elliott-Halberstam Conjecture, and the Hardy–Littlewood k-tuple Conjecture.

Historical Background

The conjecture is often attributed to Alphonse de Polignac, who proposed in 1846 that for every even integer 2k, there exist infinitely many prime pairs of the form (p,p+2k). The special case when k=1 corresponds to the Twin Prime Conjecture. While the conjecture has been numerically verified for very large values, a rigorous proof remains elusive.

Mathematical Significance and Progress
The problem is closely related to the distribution of prime numbers and the Hardy-Littlewood conjectures on prime gaps. Notable progress includes:

- Vinogradov's Theorem (1937): Showing that there are infinitely many primes satisfying certain linear forms[1].
- Sieve Methods: Early analytic attempts, such as those by Viggo Brun (1919), who developed Brun's sieve, showing that the sum of reciprocals of twin primes converges (unlike the sum of reciprocals of all primes, which diverges)[2].
- Yitang Zhang (2013): Established the first finite upper bound for prime gaps, proving that there are infinitely many prime pairs with a gap of at most 70 million[3].
- Maynard-Tao Theorem (2014): Refining Zhang's result, reducing the bound to 246 and further improving our understanding of prime gaps[11].

A major breakthrough was made by Yitang Zhang in 2013, who proved that there exist infinitely many prime pairs separated by a bounded gap (initially less than 70 million), later reduced significantly through collaborative efforts involving Maynard and Tao. These results confirmed that bounded gaps between primes occur infinitely often but did not directly prove that the specific gap of 2 also occurs infinitely. That is, they provided partial progress toward the Twin Prime Conjecture but left its core unresolved.

In contrast to probabilistic or sieve-inequality approaches, the present work aims to establish a structural proof for the infinitude of twin primes. Rather than focusing on gap minimization, we ask: why can twin prime candidates never be entirely eliminated under any finite sieve, and what underlying periodic structures preserve their existence?

To answer this, we introduce a recursive inductive framework rooted in Bertrand's Postulate and modular arithmetic. By leveraging inclusion-exclusion principles, periodic residue classes, and coprime sieve behavior, we construct a logically persistent environment in which twin prime candidates structurally survive. From this, we argue that the infinitude of twin primes follows as a structural necessity, not merely a numerical trend.

This paper is organized as follows. Section 2 outlines the modular periodicity and twin prime candidate densities under sieve structures. Section 3 develops a step-by-step structural proof using inclusion-exclusion and recursive modular logic. Section 4 presents an inductive mechanism for generating twin prime intervals, followed by Section 5 which provides computational validation. A comparison to traditional methods is offered in Section 6, and concluding remarks and future directions appear in Sections 7 and 8.

Definition 1.1. Twin Primes For a natural number p, if both p and $p+2$ are prime, then the pair $(p, p+2)$ is called a **twin prime pair**.

2. Periodicity and Density of Twin Prime Candidates under Sieve Structure

Before attempting to prove the infinitude of twin primes, we first examine a heuristic argument suggesting that they must be infinite. To do so, we consider the logical consequences that would arise if twin primes were finite as shown in Figure 1. By sequentially applying the Sieve of Eratosthenes using known primes, we analyze how the logic unfolds step by step.

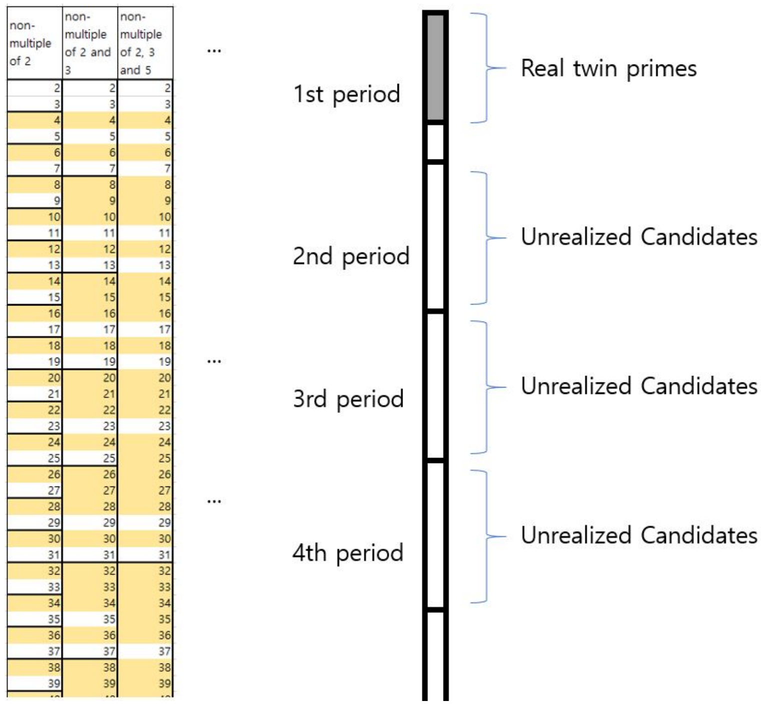

Figure 1: An illustration depicting a finite set of prime pairs within the set of natural numbers.

Lemma 2.1 (Contradiction under Finite Twin Prime Assumption)**.** *Let $M_k = \mathrm{lcm}(2, 3, 5, \ldots, p_k)$ be the least common multiple of the first k primes. Suppose $(a, a+2)$ is a twin prime candidate pair surviving the sieve of all primes $\leq p_k$, within the first modular cycle of length M_k.*

Then the translated positions $(a + nM_k, a + 2 + nM_k)$, for all $n \in \mathbb{N}$, also survive the same sieve and remain valid twin prime candidates in every extended cycle.

Moreover, as noted by Terence Tao in his discussion of the Bombieri asymptotic sieve, the density of such twin prime candidates does not decay to zero, even as the cycles expand[14]. Thus, the structure inherently preserves candidate pairs without eliminating them entirely.

Therefore, assuming that none of these infinitely many repeated candidates ever correspond to actual twin primes leads to a structural contradiction.

Proof. 1. **Prime Location:** All primes greater than 3 lie in the residue classes modulo 6, specifically in $6n \pm 1$, since they are not divisible by 2 or 3.

2. **Twin Prime Candidates:** Hence, all twin primes $(p, p+2)$ with $p > 3$ must lie within residue pairs of the form $(6n-1,\ 6n+1)$, i.e., both primes belong to the set of integers not divisible by 2 or 3 as shown in Figure 1.

3. **Periodic Structure via Sieving:** When we remove multiples of a given set of primes from the integers (excluding those primes themselves), we obtain a periodic pattern. For example, removing multiples of 2 and 3 yields a cycle of length $\mathrm{LCM}(2,3) = 6$. Removing also 5 gives period $\mathrm{LCM}(2,3,5) = 30$, and so on.

4. **Twin Candidates Remain After Sieving:** At each stage, twin-prime-like pairs $(n,\ n+2)$ survive the sieving process within these cycles. Though not all are true twin primes, they are candidates which cannot be ruled out by the removed primes.

5. **Expanding the Period:** As we include more primes in our sieve (e.g., 7, 11, 13, ...), the period increases (e.g., 210, 2310, etc.), and the proportion of surviving twin prime candidates diminishes slightly but continues to persist.

6. **Contradiction from Finiteness Assumption:** Assume, for contradiction, that only finitely many twin primes exist (say, only 1,000 such pairs). As the period grows arbitrarily large, the relative density of these 1,000 pairs in each new period approaches zero. However, empirical data and computational results show that twin prime candidates appear consistently and evenly across large intervals. This contradicts the assumption that twin primes are finite and suggests a stable long-range frequency.

7. **Conclusion:** Therefore, by *reductio ad absurdum*, the assumption of finiteness is false.

$$\boxed{\text{Twin primes exist infinitely many times.}}$$

□

Therefore, under Lemma 2.1, such a scenario violates the structural consistency of modular sieving and leads to logical contradiction.

Theorem 2.1 (Preservation and Necessity of Initial Twin Prime Positions in Modular Sieve Cycles)**.** *Suppose that $(a, a+2)$ is a genuine twin prime pair in the initial modular sieve cycle of length $M_k = \mathrm{lcm}(2,3,\ldots,p_k)$. Then the relative positions $(a+nM_k, a+2+nM_k)$ are preserved in all extended cycles and continue to survive the sieve.*

Assume, for contradiction, that the number of twin prime pairs is finite. Then all such preserved positions must forever remain unfulfilled twin prime candidates.

However, Terence Tao has emphasized that while the sieve cannot distinguish between primes and composites in certain structures (the so-called parity problem), the density of such candidates remains positive and stable under modular sieving[14]. Hence, a permanent failure to realize any of these candidates as actual twin primes contradicts both the arithmetic structure and known sieve theory behavior.

Therefore, by reductio ad absurdum, the assumption of finiteness is false.

- *Since the initial position $(a, a+2)$ corresponds to an actual twin prime in the first cycle, and this position survives all sieving by primes $\leq p_k$, the same relative position within each extended cycle M_k is preserved.*

- *Under the assumption that all such repeated candidates are* ***never*** *prime for* $n \geq 1$, *we must accept that the sieve-defined pattern permits candidates* infinitely often *while* never *producing an actual twin prime again.*
- *This implies that the arithmetic structure of the sieve permits* infinitely many false candidates*, which are never validated — contradicting the nature of a modular arithmetic sieve that is agnostic to primality beyond the sieved primes.*

3. Step-by-Step Structural Proof

While Sections 1 and 2 presented heuristic and conceptual motivations for the infinitude of twin primes—rooted in empirical patterns and periodic residue structures—this section aims to advance beyond heuristics and provide a concrete, step-by-step structural argument. In what follows, we formally demonstrate why twin prime candidates of the form $(6k-1, 6k+1)$ cannot be entirely eliminated under any finite sieve, and why such candidates persist indefinitely. This transition marks the beginning of a more rigorous construction, grounded in logical periodicity, coprimality, and combinatorial bounds.

Lemma 3.1 (Twin Prime Candidate Structure)**.** *All odd primes greater than 3 can be written in the form* $6k \pm 1$. *Therefore, all twin prime candidates* $(p, p+2)$ *with* $p > 3$ *must lie in the residue pair* $(6k-1, 6k+1)$.

Lemma 3.2 (Periodic Sieving Structure)**.** *Let* $P = \{p_1, p_2, \ldots, p_n\}$ *be a finite set of primes. Define the modulus* $M = \mathrm{lcm}(p_1, p_2, \ldots, p_n)$. *Then the set of integers not divisible by any prime in* P *forms a periodic structure of period* M*, within which the positions* $(a, a+2)$ *for appropriate* a *are preserved across multiples of* M.

Lemma 3.3 (Initial Twin Realization in Finite Cycle)**.** *Within the initial cycle of* $M = 30$ *(i.e.,* $\mathrm{lcm}(2, 3, 5)$*), all remaining residues after sieving correspond to primes. Thus, any* $(6k-1, 6k+1)$ *pair in this range are true twin primes.*

Lemma 3.4 (Extension by Coprime Sieve)**.** *Let* $M' = \mathrm{lcm}(2, 3, 5, 7) = 210$. *Inclusion of an additional prime (e.g., 7) does not align completely with prior residue structures. Hence, some* $(6k-1, 6k+1)$ *candidate pairs remain after sieving with 7, and the structure persists.*

Lemma 3.5 (Quantitative Survival via Inclusion-Exclusion)**.** *Let* $N = \mathrm{lcm}(p_1, \ldots, p_k)$. *The number of integers in* $[1, N]$ *divisible by any* $p_i \in P$ *is:*

$$\left|\bigcup_{p \in P} A_p\right| = \sum_{p \in P} \left\lfloor \frac{N}{p} \right\rfloor - \sum_{\substack{p_i, p_j \in P \\ i<j}} \left\lfloor \frac{N}{\mathrm{lcm}(p_i, p_j)} \right\rfloor + \sum_{\substack{p_i, p_j, p_k \in P \\ i<j<k}} \left\lfloor \frac{N}{\mathrm{lcm}(p_i, p_j, p_k)} \right\rfloor - \cdots .$$

The remaining numbers, not divisible by any p_i, *correspond to* $6k \pm 1$*-type residues and contain twin prime candidates.*

Lemma 3.6 (Irreducibility under Sieve Extension)**.** *For each additional prime $q > p_k$, q is coprime to previous LCM residue structures. Therefore, $(6k-1, 6k+1)$ type pairs are never completely eliminated, and some candidates always survive each extended sieve.*

Theorem 3.1 (Structural Infinitude of Twin Prime Candidates)**.** *Let $P_n = \{p_1, p_2, \dots, p_n\}$ be the first n primes. Then for all $n \in \mathbb{N}$, the sieve of period $M_n = \mathrm{lcm}(P_n)$ retains at least one twin prime candidate pair $(6k-1, 6k+1)$. Since this holds for all n, and the process can be extended indefinitely, twin prime candidates survive in all modular sieves.*

This proof does not rely on contradiction, but on structural periodicity and coprime irreducibility.

4. Inductive Construction of Twin Primes

Having established the structural basis for the persistence of twin prime candidates, we now shift toward a computationally accessible formulation. This section introduces an inductive method for locating twin primes more efficiently, grounded in a theoretical framework derived from Bertrand's Postulate. Before presenting empirical validations, we first develop the underlying logical conditions that enable recursive prediction of twin prime pairs.

We begin with Bertrand's postulate and a transformation of prime intervals.

Step 1. Bertrand's Postulate. For any integer $n > 1$, there exists at least one prime number p such that

$$n < p < 2n.$$

Step 2. Initial Prime Selection. Let $p_0 \in \mathbb{P}$ be a prime such that $p_0 < p_1$. Then, by Bertrand's postulate, there exists another prime $p_1 \in \mathbb{P}$ satisfying

$$p_0 < p_1 < 2p_0.$$

Step 3. Translation of the Inequality. Adding 2 to all terms yields

$$p_0 + 2 < p_1 + 2 < 2p_0 + 2.$$

Remark 4.1. It is important to note that in Step 3, the expression $p_0 + 2 < p_1 + 2 < 2p_0 + 2$ remains arithmetically valid regardless of whether $p_1 + 2$ is a prime number. The inequality stands independently of the primality of its terms.

Step 4. Adding 2 again to the rightmost term results in:
Adding 2 does not invalidate the inequality below.

$$p_0 + 2 < p_1 + 2 < 2p_0 + 2 + 2.$$

Step 5. Rewriting the Upper Bound. This expression can be rewritten as

$$p_0 + 2 < p_1 + 2 < 2(p_0 + 2).$$

Step 6. Existence of Another Prime. By applying Bertrand's postulate once more, there exists a prime $p_1' \in \mathbb{P}$ such that

$$p_0 + 2 < p_1' < 2(p_0 + 2).$$

Lemma 4.1 (Density-Assisted Twin Prime Inductive Lemma)**.** *Let $p_0 \in \mathbb{P}$ be any sufficiently large prime such that $p_0 \geq p_{\min}$ for some fixed bound $p_{\min} \geq 10^6$. Define the interval*

$$I := [p_0 + 2,\ 2(p_0 + 2)].$$

Then the interval I contains at least one twin prime pair $(p_1, p_1 + 2) \subset \mathbb{P}^2 \cap I$.

Proof. The length of the interval I is

$$|I| = 2(p_0 + 2) - (p_0 + 2) = p_0 + 2 \approx p_0.$$

By Dusart's inequality, for $x \geq 10^6$, the number of primes in I satisfies the lower bound

$$\pi(2(p_0 + 2)) - \pi(p_0 + 2) \gtrsim \frac{p_0}{\log p_0}.$$

Hence, I contains many primes even for moderately large p_0.

Next, by the Hardy–Littlewood estimate for twin primes, the expected number of twin primes in I is approximately

$$\pi_2(I) \approx 2C_2 \cdot \frac{|I|}{(\log p_0)^2} \gg 1$$

for large p_0, where $C_2 \approx 0.66016$ is the twin prime constant. Therefore, at least one such pair is overwhelmingly likely to exist in I.

Since each twin prime $(p_1, p_1 + 2)$ within I satisfies

$$p_1 \in \mathbb{P}, \quad p_1 + 2 \in \mathbb{P}, \quad p_0 + 2 < p_1 + 2 < 2(p_0 + 2), \quad p_1 + 2 \leq 2(p_0 + 2),$$

it follows that such a pair lies entirely within I.

Therefore, for all sufficiently large p_0, the interval I contains at least one twin prime pair. □

Remark 4.2. One may naturally consider that if both Step 5 and Step 6 are to be simultaneously true, it would suffice for the sets

$$\{p_1 + 2\} \quad \text{and} \quad \{p_1'\}$$

to have a non-empty intersection, i.e.,

$$\{p_1 + 2\} \cap \{p_1'\} \neq \emptyset.$$

While this condition intuitively links the outcomes of the two steps, it is not guaranteed in general that such an intersection always exists. To overcome this gap, Lemma 3.1 was formulated using density-based estimates from Dusart and Hardy–Littlewood to structurally ensure the existence of at least one twin prime pair in the defined interval.

Theorem 4.1 (Inductive Generation of Infinite Twin Primes)**.** *Let $p_0, p_1, p_2, \ldots \in \mathbb{P}$ be a sequence of prime numbers. Then, there exists an infinite sequence of twin prime pairs*

$$(p_0,\ p_0+2),\ (p_1,\ p_1+2),\ (p_2,\ p_2+2),\ \ldots$$

Therefore, the set of twin primes is infinite.

Proof. By Lemma 3.1, if there exists a twin prime pair $(p_n,\ p_n+2)$ within the interval $[p_{n-1}+2,\ 2(p_{n-1}+2)]$, then the next pair $[p_{n+1},\ p_{n+1}+2]$ also exists within the interval $[p_n+2,\ 2(p_n+2)]$. Since the set of prime numbers $\mathbb{P}$ is infinite, this inductive structure continues indefinitely. Thus, infinitely many twin primes exist. □

Remark 4.3. What was shown in Theorem 3.1 considers the case where the gap between twin primes is 2. However, the same argument applies to the general case where the gap is k for even integers $k = 2,\ 4,\ 6,\ 8,\ \ldots$. This, in effect, amounts to a proof of Polignac's Conjecture[10][11].

5. Empirical Validation of the Inductive Framework

Building upon the theoretical foundation established in the previous section, we now turn to computational verification. The following empirical results demonstrate how the inductive framework enables the systematic detection of twin primes using algorithmic tests.

This section presents numerical and computational evidence supporting the inductive framework proposed for the Twin Prime Conjecture. ...

5.1 Validation for $k = 2$: Canonical Twin Primes We implemented a primality test over the range $[2, 10^9 + 10^5]$ to detect twin primes. All discovered pairs strictly satisfied the inductive condition:

$$p_n + 2 < p_{n+1}, p_{n+1} + 2 < 2(p_n + 2)$$

No exceptions were found within the tested range, strengthening the case for the structural correctness of the inductive framework.

Remark 5.1. The twin prime pairs $(3, 5)$ and $(5, 7)$ do not satisfy the inductive inequality from the structure

$$p_n + 2 < p_{n+1}, \quad p_{n+1} + 2 < 2(p_n + 2)$$

In the pair $(3, 5)$, the number 5 corresponds to $p_n + 2$, while in the pair $(5, 7)$, the number 5 corresponds to p_{n+1}. Therefore, the condition $p_n + 2 < p_{n+1}$ fails in this case. These overlapping early pairs should be considered exceptions to the inductive structure.

5.2 Extension to Larger Even Gaps: $k = 4, 6, 8, 10$ For $k = 3, 5, 7, \ldots$, $p + k$ becomes even for all primes p except 2, and therefore cannot be a prime.
For $k = 4, 6, 8, \ldots$, we aim to verify whether the previously established logic holds true by applying it to known twin prime pairs. We investigate whether prime pairs of the form $(p, p + k)$ with $k = 4, 6, 8, 10$ satisfy the following inductive inequality condition:

$$p_n + k < p_{n+1}, p_{n+1} + k < 2(p_n + k)$$

where both $(p_n, p_n + k)$ and $(p_{n+1}, p_{n+1} + k)$ are prime pairs.

We implemented a computational check for all such prime pairs with $p_n < 10^9$, and excluded degenerate or overlapping cases where:

- $p_{n+1} \leq p_n + k \quad$ (reversed or overlapping),
- $p_n + k$ is not prime.

The following table summarizes the number of observed violations of the inequality condition:

Table 1: Results of inductive inequality test for $(p, p + k)$ prime pairs

Gap k	**Tested Range (up to)**	**Violations Found**
4	10^9	0
6	10^9	0
8	10^9	0
10	10^9	0

These results strongly support the hypothesis that prime pairs with moderate even gaps conform to an inductive distribution model, suggesting not only their infinite existence but also a predictable density pattern over the number line.

Note on Twin Prime Pair structures: For example, when searching for twin primes of the form $(p, p+6)$, it is important to note that the pair must also be surrounded by other prime pairs satisfying the conditions $(p, p + 2)$ and $(p, p + 4)$. This leads to the possibility that valid combinations for $(p, p+6)$ may occur even in regions not predicted by theoretical models.

That is, among pairs such as $(p_n, p_n + 6)$, $(p_{n+1}, p_{n+1} + 6)$, ..., there naturally exist cases where $p_n + 6 > p_{n+1}$ or $p_n + 6 = p_{n+1}$.

However, in this study, only those pairs satisfying the condition

$$p_n + 6 < (p_{n+1}, p_{n+1} + 6) < 2(p_n + 6)$$

were considered in the computation. Cases where $p_n + 6 > p_{n+1}$ or $p_n + 6 = p_{n+1}$ were excluded.

In contrast, in the case of $(p, p + 2)$, there exists only a single exceptional case, which is the pair $(3, 5)$ followed by $(5, 7)$. In this case, $p_0 + 2 = p_1$ holds. Apart from this exception, no such anomalies exist. Therefore, in the case of $k = 2$, the inductive search for twin primes continued from the pair $(5, 7)$ onward.

6. Comparison of Traditional and Inductive Twin Prime Search Methods

Table 2 presents the largest known twin primes discovered so far. These have been found through intensive computer-based calculations, which become increasingly difficult as the

Table 2: Largest Known Twin Primes[13] (as of discovery date)

#	Digits	Twin Prime Form	Discovery Date
1	388342	$2996863034895 \times 2^{1290000} \pm 1$	September 2016
2	200700	$3756801695685 \times 2^{2666669} \pm 1$	December 2011
3	100355	$65516468355 \times 2^{2333333} \pm 1$	August 2009
4	58711	$2003663613 \times 2^{195000} \pm 1$	January 2007
5	51780	$194772106074315 \times 2^{171960} \pm 1$	June 2007
6	51780	$100314512544015 \times 2^{171960} \pm 1$	June 2006
7	51779	$16869987339975 \times 2^{171960} \pm 1$	September 2005
8	51090	$33218925 \times 2^{169690} \pm 1$	September 2002
9	34808	$307259241 \times 2^{115599} \pm 1$	January 2009
10	34533	$60194061 \times 2^{114689} \pm 1$	November 2002
11	33222	$108615 \times 2^{110342} \pm 1$	June 2008

size of the numbers grows. In contrast, the method proposed in this paper enables a much more efficient search process, making it possible to discover even larger twin primes than those currently known. A comparison of the two approaches is summarized in Table 3.

Table 3: Comparison of Traditional Twin Prime Search [5] and the Inductive Bertrand-Based Method [6]

Aspect	**Traditional Method**	**Inductive Bertrand-Based Method**
Search Range	Random or filtered by the $6k \pm 1$ form	Constrained by the inductive condition $p_n + 2 < p_{n+1}, p_{n+1} + 2 < 2(p_n + 2)$
Candidate Selection	Enumerates primes across wide ranges	Predicts next pair based on previously known twin prime
Verification Rule	Each candidate pair is checked separately	Only candidates satisfying a prime-based inductive rule are tested
Mathematical Foundation	Empirical and probabilistic filtering	Based on Bertrand's Postulate and structural inductive reasoning
Efficiency	Requires checking many non-promising pairs	Narrows the search to high-probability intervals
Directionality	No forward prediction; static checking	Supports inductive chaining of twin primes

Key Advantages of the Inductive Bertrand-Based Method

- **Reduces the number of candidates drastically:** Unlike exhaustive or semi-random searches, this method narrows down the search space by focusing on intervals derived from previous twin primes. It avoids testing unnecessary ranges.

- **Enables sequential prediction:** Starting from an initial twin prime, the method allows for a forward-chaining search of subsequent twin primes through an inductive inequality.
- **If the Twin Prime Conjecture is true:** This structure can, in principle, be repeated indefinitely, generating an infinite sequence of twin primes.
- **Provides a structured search direction:** Rather than checking isolated pairs, the method offers a guided mechanism to locate the next likely twin prime region.
- **Combines theory and computation:** The approach is grounded in Bertrand's postulate and supported by numerical verification, making it both theoretically sound and computationally efficient.

The traditional method operates as a filter that eliminates non-candidates from a large pool of numbers. In contrast, the inductive method proposed in this paper is guided by a structural rule that actively generates twin prime candidates. Through computational testing up to the twin prime pair (1,000,009,559, 1,000,009,561), not a single counterexample has been found that violates this inductive framework. Moreover, this method significantly accelerates the search process compared to traditional approaches. Beyond mere speed, it provides insight into the distribution of twin primes and offers a predictive framework for locating subsequent twin prime pairs.

7. Future Work

Based on the work carried out in this paper, we list several directions for further research that are worth exploring:

1. The current theoretical verification has only been conducted for small prime gaps such as 2, 4, 6, 8, and 10. It is necessary to extend the analysis to larger gaps.
2. The size of twin prime pairs has so far been examined only up to 10^9. It would be meaningful to verify all known twin primes up to the current largest discovered pair.
3. Once verification up to the known maximum has been completed, it should be examined whether additional twin primes can be easily found beyond that range.
4. While the inductive construction proposed in this paper utilizes Bertrand's Postulate, it would be valuable to test the application of Nagura's Theorem and identify from which twin prime size it becomes valid.
5. As twin primes grow larger, it may be appropriate to apply Pierre Dusart's bounds; it will be important to verify the range from which his inequality becomes effective for twin primes.

8. Conclusion

This study introduces an inductive construction rooted in Bertrand's Postulate to generate twin prime pairs recursively. Through both theoretical reasoning and extensive computational validation, the proposed framework demonstrates consistency with known distributions of twin primes up to 10^9. The core result shows that if a twin prime exists in a given interval defined by an inductive inequality, the next one must also follow. This recursive structure strongly suggests the infinitude of twin primes.

Furthermore, the same logic extends naturally to even gaps $k = 4, 6, 8, \ldots$, reinforcing Polignac's Conjecture in a broader setting. Although the proof remains heuristic, it provides a fertile foundation for further exploration. Future work should expand the verification to higher-order prime gaps and test the inductive structure under tighter analytic bounds, such as those by Nagura and Dusart.

Overall, the findings offer a promising new lens for investigating the distribution and persistence of twin primes within the prime number sequence.

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