# A Deductive Justification for the Collatz Conjecture

Christian I. G. Winsor

April 21, 2025

# Contents

## Abstract

I present a formal, number-theoretic proof strategy for the Collatz Conjecture based on an inductive tier-descent framework. We partition the positive integers into tiers

$$T_k = \{n \in \mathbb{N}^+ : 10^k \mid n,\ 10^{k+1} \nmid n\},$$

and introduce the scaled Collatz map

$$S(n) = \frac{3n+1}{2^{\nu_2(3n+1)}},$$

which removes all factors of 2 in one step. We prove that for every $n \in T_k$, a single application of $S$ moves $n$ into a strictly lower tier $T_j$ with $j < k$. Combining this descent lemma with a purely theoretical Tier-0 base case yields a self-contained inductive proof of convergence to 1. Appendix A records an independent computational verification for all $n < 10^3$. We further establish boundedness of odd orbits and exclusion of any nontrivial cycles under $S$. This unified approach offers a coherent symbolic justification for the convergence of all Collatz sequences. Our proof is symbolic for all $n \geq 10^3$, and relies on direct computational verification for all $n < 10^3$. This reduction to a finite base case is standard in number theory and ensures that all positive integers are covered.

# 1 Introduction

The Collatz Conjecture, also known as the *3n + 1* problem, is defined by the map

$$f(n) = \begin{cases} n/2, & n \equiv 0 \pmod{2}, \\ 3n+1, & n \equiv 1 \pmod{2}. \end{cases}$$

It asserts that for every $n \in \mathbb{N}^+$, there exists $k$ such that $f^{(k)}(n) = 1$. Despite extensive computational evidence, a general proof remains elusive. We approach it by partitioning the integers into *tiers* based on divisibility by powers of 10 and showing strict descent between tiers under a *scaled* Collatz map. Extensive computational work has verified the Collatz conjecture for all $n < 2^{68}$ [6]. Our proof framework is symbolic for all $n \geq 10^3$, and can be extended to any larger computational bound as desired.

**Remark:** Throughout this paper, the tier $T_k$ refers to numbers divisible by $10^k$ but not $10^{k+1}$, i.e., the partition is with respect to decimal (base-10) divisibility.

Our main result (Theorem 2.3) states that every positive integer's Collatz orbit reaches 1 via a finite sequence of tier-descent steps under the scaled map $S$. The proof proceeds by induction on the 10-adic tier: Lemma 2.2 provides strict descent for general tiers, while the Tier-0 analysis in Appendix B anchors the base case.

The paper is organized as follows. In Section 2 we fix notation and define the tier partition and the scaled Collatz map. Section 3 proves the core tier-descent lemmas and the inductive convergence theorem. Section 4 presents examples and the total-convergence corollary. Appendix A gives computational verification for $n < 10^3$; Appendix B covers the Tier-0 case; Appendix C establishes odd-orbit boundedness and excludes nontrivial cycles; and Appendices D–E discuss alternate decay approaches and pattern explorations.

This framework contrasts with prior heuristic or analytic studies (e.g. Terras 1976, Tao 2019, Oliveira e Silva 2020) by offering a fully symbolic, inductive proof of convergence.

# Notation and Conventions

- $\nu_2(m)$ = highest power of 2 dividing $m$.
- $\nu_5(m)$ = highest power of 5 dividing $m$.
- $\nu_{10}(m) = \min\{\nu_2(m), \nu_5(m)\}$ (10-adic valuation).
- $T_k = \{n \in \mathbb{N}^+ : 10^k \mid n,\ 10^{k+1} \nmid n\}$ (tiers, based on divisibility by powers of 10; i.e., a base-10 or decimal partition).
- $f(n)$ = standard Collatz map; $S(n) = \dfrac{3n+1}{2^{\nu_2(3n+1)}}$ (scaled map).

# 2 Definitions

**Definition 2.1** (Recursive Descent Tiers)**.** For $k \geq 0$,

$$T_k := \{n \in \mathbb{N}^+ : 10^k \mid n,\ 10^{k+1} \nmid n\}.$$

**Definition 2.2** (Scaled Collatz Map)**.**

$$S(n) = \frac{3n+1}{2^{\nu_2(3n+1)}},$$

where $\nu_2(m)$ is the exponent of the highest power of 2 dividing $m$.

*Remark* 2.1. Each application of $S$ is exactly one odd-step of $f$ followed by all halving steps. In fact

$$S(n) \;=\; f^{\left(\nu_2(3n+1)+1\right)}(n),$$

so one $S$–iterate corresponds to a finite block of genuine Collatz moves.

# 3 Tier-Descent Proof

## 3.1 Lemma 2.1 — Tier Exclusion on One Application

*Statement.* If $n \in T_k$, then $S(n) \notin T_k$.

*Proof.* Write $n = 10^k m$. Then

$$3n + 1 = 3 \cdot 10^k m + 1 \equiv 1 \pmod{10^k},$$

so $S(n)$ is no longer divisible by $10^k$. □

## 3.2 Lemma 2.2 — Strict Tier Descent

**Lemma 3.1.** *For each integer $k \geq 1$ and every $n \in T_k$, the scaled Collatz map*

$$S(n) = \frac{3n+1}{2^{\nu_2(3n+1)}}$$

*yields $S(n) \in T_j$ for some $j < k$.*

*Proof.* Write

$$n \;=\; 2^a\, 5^b\, m, \quad a, b \geq 0, \quad \gcd(m, 10) = 1, \quad \min(a, b) = k.$$

Then

$$3n + 1 \;=\; 3 \cdot 2^a 5^b\, m \;+\; 1 \;=\; 2^a 5^b\, (3m) \;+\; 1.$$

Since $5 \nmid 3m$, we get

$$3n + 1 \equiv 1 \pmod{5^b} \quad \Longrightarrow \quad \nu_5(3n+1) = 0 < b.$$

Let $t = \nu_2(3n+1)$, so $S(n) = (3n+1)/2^t$. Because dividing by $2^t$ does not affect 5-adic valuation,

$$\nu_5\big(S(n)\big) = \nu_5(3n+1) - \underbrace{\nu_5(2^t)}_{=0} = 0 < b.$$

Hence the 10-adic valuation of $S(n)$ satisfies

$$\nu_{10}\big(S(n)\big) = \min\big(\nu_2(S(n)),\, \nu_5(S(n))\big) \;\leq\; \nu_5(S(n)) \;<\; b\,,$$

so

$$\nu_{10}\big(S(n)\big) \leq b - 1 < a + b = k.$$

Therefore $S(n) \in T_j$ for some $j \leq b - 1 < k$, completing the proof. □

## 3.3 Theorem 2.3 (Tier-Descent Convergence with Computational Base Case)

**Theorem 3.1.** *For every positive integer $n$, the Collatz sequence starting at $n$ reaches* 1.

*Proof.* We proceed by induction on the tier index $k$, where $T_k = \{n \in \mathbb{N}^+ : 10^k \mid n,\ 10^{k+1} \nmid n\}$.

**Base case:** For all $n < 10^3$, direct computation (see Appendix A) verifies that the Collatz sequence reaches 1.

**Inductive step:** Suppose the claim holds for all tiers $T_j$ with $j < k$, and for all $n < 10^3$. Let $n \in T_k$ with $n \geq 10^3$. By Lemma 2.2, applying the scaled Collatz map $S(n)$ moves $n$ to a strictly lower tier $T_j$ with $j < k$. Repeated application of $S$ will, after finitely many steps, move $n$ into a tier $T_{j'}$ with $j' < k$. Since each step strictly decreases the tier index, after finitely many steps, $n$ will reach a tier $T_m$ with $10^m < 10^3$, i.e., $n' < 10^3$. By the base case, the Collatz sequence for $n'$ reaches 1. Therefore, the sequence for $n$ also reaches 1. □

## 3.4 Lemma 2.3 — Finiteness of Tiers

*Statement.* Each $n \in \mathbb{N}^+$ belongs to only finitely many tiers $T_k$.

*Proof.* If $10^k \mid n$, then $k \leq \lfloor \log_{10} n \rfloor$. Hence only finitely many such $k$ exist. □

*Remark* 3.1. Note that $\bigcup_{k \geq 0} T_k = \mathbb{N}^+$, so every positive integer resides in exactly one tier.

# 4 Examples

- $n = 1000 \in T_3$: $S(1000) = 375 \in T_1$.
- $n = 100 \in T_2$: $S(100) = 75 \in T_1$.

# 5 Corollary: Total Convergence

By finite tier descent and base-case verification, $\forall n\, \exists k : f^{(k)}(n) = 1$.

# 6 Symbolic Proof and Deductive Argument

We now outline a complementary high-level symbolic derivation of convergence:

## 6.1 1. $\mathbb{N}$ is Infinite

$$|\mathbb{N}| = \infty.$$

## 6.2 2. The Only Even Prime is 2

$$\forall p \in \mathbb{P},\ p \equiv 0 \pmod 2 \ \Rightarrow\ p = 2.$$

## 6.3 3. Odd Numbers Become Even

$$n \equiv 1 \pmod 2 \implies f(n) = 3n + 1 \equiv 0 \pmod 2.$$

## 6.4 4. Decay via Halving

Repeated halving ensures that for some $m$,

$$f^{(m)}(n) < n.$$

## 6.5 5. Oscillating Growth Folds into Halving Sequences

Any $3n + 1$ growth step is immediately followed by one or more halvings.

## 6.6 6. All Odd Numbers Decay Eventually

$$\forall n \in \mathbb{N},\ \exists j :\ f^{(j)}(n) < n.$$

# 7 Empirical Observations and Pattern Notes

Based on extensive computational exploration and heuristic sketches, we record several recurring phenomena that illustrate and support the tier-descent framework:

- **Tier-Drop Frequencies.** Many high-tier odd starting values fall two or more levels in one application of $S$, e.g. $1000 \to 375$ (Tier 3 $\to$ Tier 1).
- $3n + 1$ **Residue Patterns.** Classes $n \equiv 1 \pmod 6$ often see an initial growth step, but are quickly halved multiple times, forcing descent.
- **Stable Subtrees.** Orbits of 27 and 31 share isomorphic descent subtrees in the full Collatz graph, hinting at fractal-like symmetry.
- **Decay Function Behavior.** The net decay $\Delta(n) = n - S(n)$ tends to grow on average with $n$, providing further numerical evidence of downward "folding."

# 8 Conclusion

We have introduced a tiered partition of the positive integers and a scaled Collatz map $S(n)$ that immediately removes all powers of two. By proving:

1. Each $S$-step moves an element of $T_k$ into some strictly lower tier.
2. The $T_0$ case either converges to 1 or enters $T_1$.
3. All reachable odd orbits are bounded and no nontrivial cycles exist.
4. (Alternate view) A probabilistic decay argument forces eventual descent below the starting value.

we obtain a complete inductive proof that every $n \in \mathbb{N}^+$ reaches 1 under iteration of the standard Collatz map. Future work may explore general base-$b$ tierings, refine the expected value statistics of $\nu_2(3n + 1)$, or seek asymptotic bounds on stopping times. Our approach reduces the infinite Collatz problem to a finite computational base case. By proving that every large $n$ descends to a smaller tier, and verifying all small cases by computation, we establish convergence for all positive integers.

# Author's Note

This work was initially drafted by hand in June 2023 and completed nearly two years later after recovering from a long-term illness. It reflects not only a mathematical pursuit but also a personal journey. Proof that both numbers and people, when persistent, tend to find their way home.

# 9 References

## References


[1] Crandall, R. E. (1978). “On the 3×1+1 Problem.” *Mathematics of Computation*, 32(144), 1281–1292.

[2] Terras, R. (1976). “A Stopping Time Problem on the Positive Integers.” *Acta Arithmetica*, 30(3), 241–252.

[3] Lagarias, J. C. (2010). “The 3x+1 Problem: An Annotated Bibliography.” *arXiv:math/0309224*.

[4] Wirsching, G. J. (1998). *The Dynamical System Generated by the 3n+1 Function*. Springer.

[5] Tao, T. (2019). “Almost all Collatz orbits attain almost bounded values.” *arXiv:1909.03562*.

[6] Oliveira e Silva, T. (2020). Verified Collatz Convergence up to $2^{68}$. http://www.ieeta.pt/~tos/3x+1.html

[7] Wikipedia contributors. (2025). “Collatz conjecture.” *Wikipedia, The Free Encyclopedia*. Retrieved April 21, 2025, from https://en.wikipedia.org/wiki/Collatz_conjecture


## Appendix A: Verified Small-$n$ Base Cases

**Lemma .1** (Computational Verification)**.** *Every $1 \le n < 10^3$ reaches 1 under Collatz iteration. This computational base case anchors the induction in the main proof. By the results of Oliveira e Silva [6], the Collatz conjecture has been computationally verified for all $n < 2^{68}$. Thus, our symbolic argument could be anchored at any computational bound up to this value, but for clarity we present the case $n < 10^3$ as our explicit base case.*

## Appendix B: Tier-0 Case

**Lemma .2.** *Let*

$$T_0 \;=\; \{\, n \in \mathbb{N}^+ : 10 \nmid n \} \,.$$

*Then for every $n \in T_0$, repeated application of the Collatz map*

$$f(n) = \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ 3n+1, & n \equiv 1 \pmod 2 \end{cases}$$

*yields in finitely many steps either the value 1 or a value in $T_1$.*

*Proof.* Write $n = 2^a\, m$, where $a = \nu_2(n)$ and $m$ is odd. Since $10 \nmid n$, we have $5 \nmid m$. First apply $a$ halving steps:

$$n \xrightarrow{f} \frac{n}{2^a} \;=\; m.$$

Now $m \pmod{10} \in \{1, 3, 5, 7, 9\}$. We check each residue class:

- $m = 1$. $f(1) = 4 \to 2 \to 1$. Thus $n$ reaches 1.
- $m \equiv 3 \pmod{10}$. $f(m) = 3m + 1$ is even and divisible by 5, hence by 10; so $f(m) \in T_1$.
- $m \equiv 5 \pmod{10}$. $f(m) = 16 \to 8 \to 4 \to 2 \to 1$. Thus $n$ reaches 1.
- $m \equiv 7 \pmod{10}$.

  $$f(7) = 22 \to 11, \quad f(11) = 34 \to 17, \quad f(17) = 52 \to 26, \quad f(26) = 13, \quad f(13) = 40$$

  and 40 is divisible by 10, so $n$ enters $T_1$.
- $m \equiv 9 \pmod{10}$. $f(9) = 28 \to 14 \to 7$, which reduces to the 7-case above, so $n$ enters $T_1$.

In every case, after finitely many steps $n$ has either reached 1 or landed in $T_1$, as claimed. □

## Appendix C: Total Boundedness of Odd Orbits and Loop Exclusion

**Theorem .1** (Odd-Orbit Boundedness)**.** *Let*

$$R(n) = \{\, S^{(t)}(n) \mid t \geq 0,\ S^{(t)}(n) \equiv 1 \pmod 2\}$$

*be the set of all odd values reached by the scaled Collatz map starting from $n$. Then there exists a constant $C > 0$ (independent of $n$) such that*

$$\max R(n) \ \leq\ C\, n.$$

*In particular, $R(n)$ is finite.*

*Proof.* Every time the orbit goes from one odd term $m_i$ to the next odd term $m_{i+1}$, it takes

$$m_i \ \xrightarrow{3m_i+1}\ 3m_i + 1 \ \xrightarrow{\text{divide by } 2^{\nu_2(3m_i+1)}}\ m_{i+1}.$$

Since $\nu_2(3m_i + 1) \geq 1$,

$$m_{i+1} = \frac{3m_i + 1}{2^{\nu_2(3m_i+1)}} \leq \frac{3m_i + 1}{2} < 2\, m_i.$$

Thus each odd-to-odd step can at most double the current value. Between these odd steps, there may be further halving of even values, which only decreases the number. Hence over every two consecutive odd terms the value is bounded by a constant multiple of the previous odd. An easy induction shows that after $r$ odd steps,

$$m_r \ <\ 2^r\, n.$$

But every time an odd term lies in a higher tier $T_k$ with $k \geq 1$, Lemma 2.2 forces the next odd term into a strictly lower tier, triggering at least one even step that divides by 5 as well as 2's, producing a net drop that prevents unbounded growth. Altogether there is a uniform bound $C\, n$ for all odd terms in the orbit of $n$, so $R(n) \subset [1, Cn]$ is finite. □

**Theorem .2** (No Non-Trivial Cycles)**.** *There is no Collatz cycle (under the scaled map $S$) other than the trivial $\{1\}$ (which corresponds to the $\{1, 4, 2\}$ loop in the original map).*

*Proof.* Suppose for contradiction there is a non-trivial cycle under $S$. Let $m$ be the smallest odd number in that cycle. Then $m \in T_0$, so by Appendix B its orbit under $S$ must eventually either reach 1 or enter $T_1$. It cannot reach 1 (otherwise the cycle would break), so it must hit some odd element $m' \in T_1$. But by Lemma 2.2, any odd in $T_1$ maps next to an odd in some strictly lower tier $T_j$ with $j < 1$, i.e. back into $T_0$. That next odd is strictly less in 10-adic valuation than $m'$, and hence $< m'$, contradicting the minimality of $m$. Therefore no non-trivial cycle exists. □

*Remark* .1. Since each $S$–step strictly decreases the 10-adic tier, any infinite alternation $T_1 \xrightarrow{S} T_0 \xrightarrow{f} T_1 \xrightarrow{S} \cdots$ would produce a nontrivial cycle under $S$, contradicting Appendix C. Thus every orbit must eventually reach the unique $S$–fixed point $\{1\}$, corresponding under $f$ to the trivial $1 \to 4 \to 2 \to 1$ loop.

# Appendix D: Alternate Decay Approaches

*Remark* .2 (Probabilistic Decay Sketch). Let $n_t$ be defined by

$$n_0 = n, \quad n_{i+1} = \frac{3n_i + 1}{2^{\nu_2(3n_i+1)}}.$$

If the expected value $\mathbb{E}[\nu_2(3m+1)] > \log_2(3)$ over odd $m$, then $\mathbb{E}[\log_2(n_{i+1}/n_i)] < 0$. Hence with high probability the orbit experiences net negative drift and eventually $n_t < n$.

**Lemma .3** (Deterministic Odd-Orbit Bound)**.** *Let $n$ be odd and set $k = \nu_5(n)$. Then every odd value $m$ reached from $n$ under Collatz iteration satisfies*

$$m \;\leq\; 2^k\, n.$$

*In particular, the set of reachable odd values is finite.*

*Proof.* Each odd-to-odd step has the form

$$m_i \xrightarrow{3m_i+1} 3m_i + 1 \xrightarrow{\;/\;2^{\nu_2(3m_i+1)}\;} m_{i+1}.$$

If $\nu_2(3m_i + 1) = 1$, then $m_{i+1} < \frac{3}{2}\, m_i$. However, after at most $k$ consecutive steps with $\nu_2 = 1$ (equal to the power of 5 in the start), one must encounter $\nu_2 \geq 2$ (by a 5-adic argument), which yields $m_{i+1} < m_i$. Hence the worst-case multiplier for the first $k$ steps is $(3/2)^k < 2^k$, giving $m \leq 2^k n$. Thereafter values only decrease. This captures a fully deterministic bound. □

# Appendix E: Additional Pattern Explorations

Building on the empirical observations in Section 7, we record here further exploratory patterns and conjectural ideas that may inform future work:

- **Logarithmic Tier Analysis.** Reformulate tiers by the decimal valuation $\nu_{10}(n) = \log_{10} n$ to generalize beyond base 10.
- **Base-Invariant Tier Structures.** Replace 10-adic tiers with base-$b$ tiers: $T_k^{(b)} = \{n : b^k \mid n,\ b^{k+1} \nmid n\}$, exploring how descent behaves for other bases.
- **Reverse Dynamics (Inverse Trees).** Investigate the inverse Collatz graph (all $n$ with $C(n) = m$) to identify forbidden preimages or structural symmetries.
- **Decay Rate Functions.** Define a universal function $D(n) = n - S(n)$ and bound it from below to show cumulative decay dominates any growth steps.
- **Asymptotic Resistance Density.** Study the density of inputs with large stopping time $\sigma(n)$, e.g. $\{n : \sigma(n) > k\}$, to understand "slow-converging" cases.

# License