A New Approach For The Proof of The *abc* Conjecture

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Abstract

In this paper, we assume that the explicit *abc* conjecture of Alan Baker and the conjecture $c < R^{1.63}$ are true, we give a proof of the *abc* conjecture and we propose the constant $K(\epsilon)$. Some numerical examples are provided.

To the memory of my Father who taught me arithmetic,
To my wife Wahida, my daughter Sinda, my son Mohamed Mazen and my granddaughter Rayhane.
To Prof. A. Nitaj for his work on the abc conjecture.

1 Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ denoted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [5]. It describes the distribution of the prime factors of the two integers along with their sum. The definition of the *abc* conjecture is given below:

Conjecture 1.1. *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [4]. It concerned the best example given by E. Reyssat [4]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{4}$$

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A conjecture was proposed that $c < rad^2(abc)$ [2]. In 2012, A. Nitaj [3] proposed the following conjecture:

Conjecture 1.2. *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < rad^{1.63}(abc) \tag{5}$$

$$abc < rad^{4.42}(abc) \tag{6}$$

In the following, we assume that the conjecture $c < rad^{1.63}(abc)$ is true. In 2004, Alan Baker [5], [1] proposed the explicit version of the abc conjecture namely:

Conjecture 1.3. *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} \tag{7}$$

with $R = rad(abc)$ and ω denote the number of distinct prime factors of abc.

In the following, we assume also that the above conjecture is true, I will give an elementary proof of the abc conjecture by verifying the below inequality:

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} < ... < K(\epsilon)R^{1+\epsilon} \tag{8}$$

with an adequate choice of the constant $K(\epsilon)$. Let we denote $\alpha = \frac{6}{5}R\frac{(LogR)^\omega}{\omega!}$, we have remarked from some numerical examples (see below) that $c \lll \alpha - c$ when $\omega = 10$ and R not very large. With our choice, c will be very very small comparing to $K(\epsilon)R^{1+\epsilon}$.

2 The proof of the *abc* conjecture

Proof. : Let $A = \frac{(Log(R^\epsilon))^\omega}{\omega!}$, and $\epsilon \in]0, 0.63[$, we obtain:

$$R^\epsilon = e^{LogR^\epsilon} = 1 + Log(R^\epsilon) + \frac{(Log(R^\epsilon))^2}{2!} + \cdots + A + \sum_{k=\omega+1}^{+\infty} \frac{(Log(R^\epsilon))^k}{k!} \Longrightarrow$$

$$A = R^\epsilon - 1 - \sum_{k=1,\neq\omega}^{+\infty} \frac{(Log(R^\epsilon))^k}{k!} \Longrightarrow$$

$$A = R^\epsilon \left(1 - \frac{1}{R^\epsilon}\left[1 + \sum_{k=1,\neq\omega}^{+\infty} \frac{(Log(R^\epsilon))^k}{k!}\right]\right) = R^\epsilon(1-B) > 0,\ 0 < B < 1 \Longrightarrow$$

$$A = \frac{(Log(R^\epsilon))^\omega}{\omega!} = R^\epsilon(1-B) > 0 \tag{9}$$

We begin from the Baker's formula below :

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} = \frac{6}{5}R.\frac{1}{\epsilon^\omega}\frac{(\epsilon LogR)^\omega}{\omega!} = \frac{6}{5}\frac{R}{\epsilon^\omega}\frac{(Log(R^\epsilon))^\omega}{\omega!}$$

Using the term $\dfrac{(Log(R^\epsilon))^\omega}{\omega!}$ from (9), the equation above becomes :

$$c < \frac{6}{5}\frac{R}{\epsilon^\omega}R^\epsilon(1-B)\overbrace{<}^{?}1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}}R^{1+\epsilon} \Longrightarrow \text{our choice of the constant } K(\epsilon) = 1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}} \tag{10}$$

We recall the following proposition [3]:

Proposition 2.1. *Let $\epsilon \longrightarrow K(\epsilon)$ the application verifying the abc conjecture, then:*

$$lim_{\epsilon\to 0}K(\epsilon) = +\infty \tag{11}$$

The chosen constant $K(\epsilon)$ verifies the proposition above. Now, is the following inequality true? :

$$\frac{6}{5}\frac{1}{\epsilon^\omega}(1-B)\overbrace{<}^{?}1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}} \tag{12}$$

We suppose that :

$$\frac{6}{5}\frac{1}{\epsilon^\omega}(1-B) > \frac{6}{5}e^{e^{\left(\frac{1}{\epsilon^4}\right)}}. \Longrightarrow 1 > (1-B) > \epsilon^\omega.e^{e^{\left(\frac{1}{\epsilon^4}\right)}}$$

As $\omega \geq 4 \Longrightarrow \omega = 4\omega' + r$, $0 \leq r < 4, \omega' \geq 1$, we write $\epsilon^\omega.e^{e^{(1/\epsilon)^4}}$ as:

$$\epsilon^\omega.e^{e^{(1/\epsilon)^4}} = \frac{e^{e^{(1/\epsilon)^4}}}{(1/(\epsilon^4))^{\omega'}}.\epsilon^r = \frac{e^{e^X}}{X^{\omega'}}.\epsilon^r \tag{13}$$

where $X = \frac{1}{\epsilon^4}$ and $1 \ll X$. Or we know that $X^{\omega'} \ll e^X \Longrightarrow X^{\omega'} \ll e^{e^X}$. As $0 \leq r < 4$ and $0 < \epsilon < 0.63$, then $\epsilon^r > (\epsilon^4 = \dfrac{1}{X}$. The equation (13) becomes:

$$\epsilon^\omega.e^{e^{(1/\epsilon)^4}} = \frac{e^{e^{(1/\epsilon)^4}}}{(1/(\epsilon^4))^{\omega'}}.\epsilon^r = \frac{e^{e^X}}{X^{\omega'}}.\epsilon^r > \frac{e^{e^X}}{X^{\omega'+1}} > 1 \tag{14}$$

It follows the contradiction and we obtain:

$$\frac{6}{5}\frac{1}{\epsilon^\omega}(1-B) < 1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}} \Longrightarrow c < \frac{6}{5}R\frac{(Log)^\omega}{\omega!} < 1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}}R^{1+\epsilon} \tag{15}$$

Finally, the choice of the constant $K(\epsilon) = 1.2e^{e^{\left(\frac{1}{\epsilon}\right)^4}}$ is acceptable for $\epsilon \in]0, 0.63[$. As we assume that the conjecture $c < R^{1+0.63}$ is true, we adopt $K(\epsilon) = 1.2$ for $\epsilon \geq 0.63$, and the *abc* conjecture is true for all $\epsilon > 0$.

The proof of the *abc* conjecture is finished.

Q.E.D

□

We give below some numerical examples.

3 Examples

3.1 Example 1. of Eric Reyssat

We give here the example of Eric Reyssat [5], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{16}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$.
$\omega = 4 \Longrightarrow \alpha = \dfrac{6}{5}R\dfrac{(LogR)^\omega}{\omega!} = 6\,437\,590.238 > 6\,436\,343 = c$, $B = 0.86 < w = 4$;
$\alpha - c = 1\,247.238$.
$\epsilon = 0.5 \Longrightarrow \epsilon^\omega.e^{e^{\left(\frac{1}{\epsilon}\right)^4}} = 9.446e + 109 > 1 \Longrightarrow (1 - B) < 1$.
$\epsilon = 0.01 \Longrightarrow \epsilon^\omega = \epsilon^4 = 10^{-8} \ll e^{\left(\frac{1}{\epsilon}\right)^4}$ then $(1 - B) < 1$.

3.2 Example 2. of Nitaj

See [3]:

$$\begin{gathered} a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \\ b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \\ c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \\ rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \\ \omega = 10 \Longrightarrow \alpha = \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} = 7\,794\,478\,289\,809\,729\,132\,015.590 > \\ 613\,474\,845\,886\,230\,468\,750 = c,\ B = 0.9927 \ll (w = 10);\ \alpha - c = \\ 7\,181\,003\,443\,923\,198\,663\,265.590 \approx 11.71c \end{gathered}$$

$\epsilon = 0.5 \Longrightarrow \epsilon^\omega = \epsilon^{10} = 0.009765625 \ll e^{1/(\epsilon^4)} \Longrightarrow (1 - B) < 1$.
$\epsilon = 0.001 \Longrightarrow \epsilon^\omega = \epsilon^{10} = 10^{-30}$, $1/(\epsilon^4) = 10^{12} \Longrightarrow \epsilon^{10}.e^{10^{12}} > 1 \Longrightarrow (1 - B) < 1$.

4 Conclusion

Assuming $c < R^{1.63}$ is true, and the explicit *abc* conjecture of Alan Baker true, we can announce the important theorem:

Theorem 4.1. *Assuming $c < R^{1.63}$ is true and the explicit abc conjecture of Alan Baker true, then the abc conjecture is true:*
For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{17}$$

where K is a constant depending only of ϵ. For $\epsilon \in]0, 0.63[$, $K(\epsilon) = 1.2e^{e^{\left(\frac{1}{\epsilon}\right)^4}}$ and $K(\epsilon) = 1.2$ if $\epsilon \geq 0.63$.

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