A Refined Symmetric Mean Integral Approach to Bounding the Perimeter of an Ellipse

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Abstract

We refine the symmetric mean integral method for estimating the perimeter of an ellipse by restricting the integration limits to $[0, \pi/4]$. This approach allows the application of the squeeze theorem by leveraging the extremal behavior of the integrand, yielding explicit upper and lower bounds. The results provide a foundation for further research to derive improved perimeter estimates for ellipses

Keywords: Elliptic integrals, Ellipse perimeter, Mean integral, Squeeze theorem, Numerical approximation, Symmetry

1 Introduction

Consider an ellipse with semi-major axis a and semi-minor axis b, where $a \geq b > 0$. We express its perimeter using the complete elliptic integral of the second kind:

$$L(a,b) = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\cos^2 t + b^2\sin^2 t}\, dt = 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\sin^2 t}\, dt = 4a\, E(e)$$

where $e = \sqrt{1 - \frac{b^2}{a^2}}$. Since this integral has no elementary form, several approximations and bounds have been proposed, many involving classical means such as arithmetic, geometric, and harmonic means (AM-GM-HM) [1, 2].

Previous studies [3–6] have utilized the symmetry of the integrand in the complete elliptic integral of the second kind, leading to the expression:

$$L(a,b) = 4\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\right) dt$$

By applying classical inequalities such as AM-GM-HM or Cauchy–Schwarz, this formulation yields bounds of the form:

$$2\pi\left(\frac{a+b}{2}\right) \leq L(a,b) \leq 2\pi\sqrt{\frac{a^2+b^2}{2}}$$

By analyzing the integrand

$$f(t) = \frac{1}{2}\left(\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\right)$$

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We find that at the interval $[0, 2\pi]$, the function attains its maximum value $\sqrt{\frac{a^2+b^2}{2}}$ at $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and its minimum value $\frac{(a+b)}{2}$ at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. This symmetry allows the integral to be simplified:

$$L(a, b) = 8 \int_0^{\frac{\pi}{4}} f(t)\, dt$$

We can visualize the function $f(t)$ by decomposing it into

$$f_1(t) = \sqrt{a^2 \cos^2 t + b^2 \sin^2 t}$$
$$f_2(t) = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}$$

as illustrated in Figure 1.

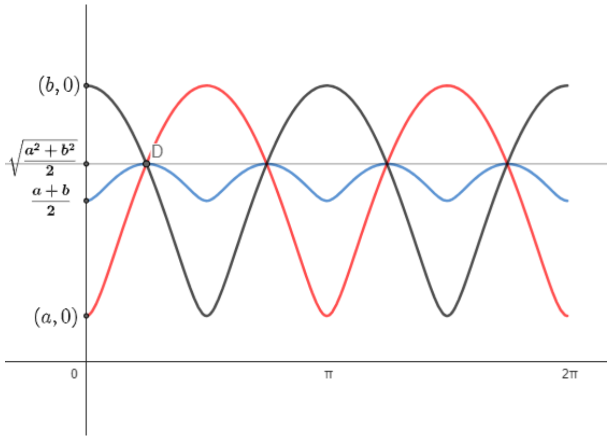

Figure 1: The function $f(t)$ is shown in blue, $f_1(t)$ in red, and $f_2(t)$ in black. The graphs of $f_1(t)$ and $f_2(t)$ intersect at the maximum points of $f(t)$.

This study refines the symmetric formulation by restricting the limits of integration to $[0, \pi/4]$, yielding

$$L(a, b) = 8 \int_0^{\frac{\pi}{4}} \frac{1}{2} \left(\sqrt{a^2 \cos^2(t) + b^2 \sin^2(t)} + \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)} \right) dt$$

This form enables bounding $L(a, b)$ via the squeeze theorem, using the extremal values of the integrand at $t = 0$ and $t = \pi/4$, which is not possible over the full interval $[0, \pi/2]$. In addition, this integral offers the possibility of finding tighter bounds and more accurate approximations of $L(a, b)$.

2 Results

Lemma 2.1. *For $a \geq b > 0$, let*

$$f(t) = \tfrac{1}{2} \left(\sqrt{a^2 \cos^2 t + b^2 \sin^2 t} + \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \right).$$

Then,

$$L(a, b) = 8 \int_0^{\pi/4} f(t)\, dt = 4a\, E(e),$$

where $e = \sqrt{1 - \frac{b^2}{a^2}}$.

We begin by simplifying $L(a,b)$ as

$$L(a,b) = 4\int_0^{\pi/4} \left(\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\right) dt$$

Let $f_1(t) = \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}$, then $f_1\left(\frac{\pi}{2} - t\right) = f_2(t) = \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}$. Thus, we can write

$$\begin{aligned} L(a,b) &= 4\int_0^{\pi/4} \left[f_1(t) + f_1\left(\tfrac{\pi}{2} - t\right)\right] dt \\ &= 4\left[\int_0^{\pi/4} f_1(t)dt + \int_0^{\pi/4} f_1\left(\tfrac{\pi}{2} - t\right) dt\right] \end{aligned}$$

Now,let $u = \frac{\pi}{2} - t$, then $du = -dt$. Hence

$$\int_0^{\pi/4} f_1\left(\tfrac{\pi}{2} - t\right) dt = \int_{\pi/2}^{\pi/4} f_1(u)(-du) = \int_{\pi/4}^{\pi/2} f_1(u)du$$

Since u is a dummy variable, we can replace it with t, giving

$$\int_{\pi/4}^{\pi/2} f_1(u)du = \int_{\pi/4}^{\pi/2} f_1(t)dt$$

Therefore

$$\begin{aligned} L(a,b) &= 4\left[\int_0^{\pi/4} f_1(t)dt + \int_0^{\pi/4} f_1\left(\tfrac{\pi}{2} - t\right) dt\right] \\ &= 4\left[\int_0^{\pi/4} f_1(t)dt + \int_{\pi/4}^{\pi/2} f_1(t)dt\right] \\ &= 4\int_0^{\pi/2} f_1(t)dt \end{aligned}$$

Since $f_1(t) = \sqrt{a^2\cos^2(t) + b^2\sin^2(t)} = a\sqrt{1 - e^2\sin^2(t)}$, with $e = \sqrt{1 - \frac{b^2}{a^2}}$, it follows that

$$L(a,b) = 4aE(e)$$

which is the classical complete elliptic integral of the second kind.

Note that

$$\int_0^{\frac{\pi}{4}} f_1\left(\frac{\pi}{2} - t\right)\ dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f_1(t)\ dt$$

Since $f_1\left(\frac{\pi}{2} - t\right) = f_2(t)$, it follows that

$$\int_0^{\frac{\pi}{4}} f_2(t)\, dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f_1(t)\, dt.$$

In other words, the integral $\int_{\pi/4}^{\pi/2} f_1(t)\, dt$ is the reflection of $\int_0^{\pi/4} f_2(t)\, dt$ about the line $t = \frac{\pi}{4}$, as illustrated in Figure 1.

Theorem 2.2. *Let $a \geq b > 0$. Then the perimeter of an ellipse $L(a,b)$ satisfies the following inequality:*

$$2\pi \cdot \frac{a+b}{2}\ \leq\ L(a,b) = 8\int_0^{\pi/4} f(t)\, dt\ \leq\ 2\pi \cdot \sqrt{\frac{a^2+b^2}{2}},$$

Proof. Let

$$f(t) = \frac{1}{2}\left(\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\right)$$

Given that $a \geq b > 0$, the function $f(t)$ attains:

- A maximum value of $\sqrt{\dfrac{a^2+b^2}{2}}$
- A minimum value of $\dfrac{a+b}{2}$

By the squeeze theorem, we obtain:

$$8\int_0^{\pi/4} \frac{a+b}{2}\,dt \leq 8\int_0^{\pi/4} f(t)\,dt \leq 8\int_0^{\pi/4} \sqrt{\frac{a^2+b^2}{2}}\,dt$$

Simplifying the integrals:

$$2\pi\frac{a+b}{2} \leq L(a,b) \leq 2\pi\sqrt{\frac{a^2+b^2}{2}}$$

This proves the theorem. □

3 Discussion

This study improves on the symmetric mean integral by applying the squeeze theorem, overcoming the limitations of previous work that used the interval $[0, \frac{\pi}{2}]$. The new representation expresses $L(a,b)$ as the area under the curve $f(t)$, with clearly defined upper and lower bounds, as shown in Figure 1.

In previous studies, the perimeter of an ellipse was expressed as

$$L(a,b) = 4\int_0^{\pi/2} \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}\,dt$$

Utilizing the fact that the elliptic integrand is symmetric with respect to the transformation $u = \frac{\pi}{2} - t$, it follows that

$$\begin{aligned} L(b,a) &= 4\int_{\pi/2}^{0} \sqrt{b^2\cos^2(t) + a^2\sin^2(t)}\,(-du) \\ &= 4\int_0^{\pi/2} \sqrt{b^2\cos^2(t) + a^2\sin^2(t)}\,du \end{aligned}$$

Since u is a dummy variable, we can replace it with t, and thus

$$L(b,a) = 4\int_0^{\frac{\pi}{2}} \sqrt{b^2\cos^2 t + a^2\sin^2 t}\,dt$$

Because $L(a,b) = L(b,a)$, the mean of the two integrals gives

$$L(a,b) = 4\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\sqrt{a^2\cos^2 t + b^2\sin^2 t} + \sqrt{a^2\sin^2 t + b^2\cos^2 t}\right)dt$$

Previous studies have overlooked crucial extremal properties of the integrand in the symmetric perimeter formula:

$$L(a,b) = 8\int_0^{\pi/4} \frac{1}{2}\left(\sqrt{a^2\cos^2 t + b^2\sin^2 t} + \sqrt{a^2\sin^2 t + b^2\cos^2 t}\right)dt$$

The integrand $f(t)$ satisfies fundamental bounds:

$$\frac{a+b}{2} \leq f(t) \leq \sqrt{\frac{a^2+b^2}{2}} \quad \text{for} \quad t \in [0, \pi/4] \tag{1}$$

where:

- The minimum$(a+b)/2$ occurs at $t = 0$
- The maximum $\sqrt{(a^2+b^2)/2}$ occurs at the endpoints $t = \pi/4$

Applying the squeeze theorem to the integral over $[0, \pi/4]$ immediately yields:

$$2\pi\left(\frac{a+b}{2}\right) \leq L(a,b) \leq 2\pi\sqrt{\frac{a^2+b^2}{2}}$$

This provides a new geometric interpretation of the classical arithmetic mean–root mean square inequality for ellipse perimeters.

This enhancement of the symmetric mean integral improves numerical computation by reducing error in approximation methods. For example, using the trapezoidal rule minimizes errors.

$$E = -\frac{(\pi/4)^3}{12N^2} f''(\xi),$$

which is one-eighth of the error over $[0, \pi/2]$. Using two square-root terms increases computational complexity, but this can be mitigated by expressing both in terms of eccentricity e.

4 Conclusion

This study proposes a new approach to bounding the circumference of an ellipse—equivalently, an elliptic integral of the second kind—by leveraging symmetry and the extremal values of the integrand within a reduced interval of integration. This refinement clarifies the relationship between the integral expression and its limits of integration.

The improved symmetric integral formulation can be employed in future research to derive tighter bounds for the ellipse's perimeter. It also opens the possibility for more accurate numerical estimates using fewer subintervals. Furthermore, this formulation may serve as a foundation for constructing an exact expression based on a numerical approximation of the elliptic perimeter.

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