Electron Deflection

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Abstract

This article explores the motion of an electron through a finite region of homogeneous magnetic field.

Keywords: Lorentz force, Euler integration, magnetic field, charged particle dynamics, analytical modeling, numerical physics

1 Introduction

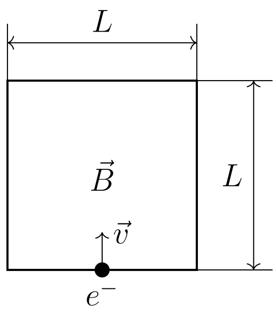
As I was relaxing at home, doing nothing, a thought came into my mind. What would happen if an electron is launched through a homogeneous magnetic field? Will it be deflected? And if so, by how much? Where will it end up?

Let's define the problem in more detail. An electron is launched with some velocity $\vec{v}$ through an homogeneous magnetic field $\vec{B}$ spanning a region of size $L \times L$.

We have to calculate the exit velocity and exit position of the electron. From there we can conclude by how much it was deflected.

2 Analytical Solution

Let's first make a drawing of the problem.



We'll make a couple assumptions. Let's assume that the electron is initially moving at a constant velocity:

$$\vec{v} = v_0 \hat{y}.$$

Then, let's also assume that the coordinate origin is at the electron's starting position to make our life easier.

The magnetic field $\vec{B}$ is homogeneous, but it's only non-zero inside the $L \times L$ region. Mathematically, we can describe this as:

$$\vec{B}(x, y, z) = \begin{cases} B\hat{z}, & \text{if } |x| < \frac{L}{2} \text{ and } 0 \leq y < L, \\ \vec{0}, & \text{otherwise.} \end{cases} \tag{1}$$

Here we're making another assumption; we're assuming that the electron will never exit the way it came in.

When the electron enters the magnetic field $\vec{B}$, and as it travels through the field, it experiences a Lorentz force. This can be simply calculated by using the Lorentz force formula:

$$\vec{F} = -e\left(\vec{E} + \vec{v} \times \vec{B}\right).$$

As we don't have any electric fields, this reduces to just:

$$\vec{F} = -e\vec{v} \times \vec{B}.$$

Since we're discussing the situation inside the magnetic field, the definition for $\vec{B}$ (Eq. 1) reduces to $\vec{B} = B\hat{z}$. Taking the cross product of $\vec{v}$ with $\vec{B}$, we obtain:

$$\vec{F} = -ev_0 B\hat{x}.$$

The consequence of this force, as we know from Newton's second law, is that the electron experiences an acceleration:

$$\vec{a} = \frac{\vec{F}}{m_e},$$

where m_e is the mass of the electron. Substituting in the force, we obtain:

$$\vec{a} = -\frac{ev_0 B}{m_e}\hat{x}.$$

To obtain the velocity, we integrate the components of acceleration with respect to time:

$$v_x = \int_0^t -\frac{ev_0 B}{m_e} dt',$$

from which we obtain the speed in the x direction, and in vector form this has the form:

$$\vec{v}_l = -\frac{ev_0 tB}{m_e}\hat{x},$$

where $\vec{v}_l$ represents the velocity caused by the Lorentz force which we'll call the *Lorentz velocity*.

The total velocity of the electron inside the magnetic field is then given by the sum of the Lorentz velocity $\vec{v}_l$ and the electrons initial velocity $\vec{v}$.

From now on, when we use $\vec{v}$, we refer to the total velocity, and not the initial velocity. The total velocity, as stated earlier, is:

$$\vec{v} = -\frac{ev_0 tB}{m_e}\hat{x} + v_0\hat{y}. \tag{2}$$

To find the electron's position, we again integrate, but now the components of total velocity $\vec{v}$:

$$\begin{aligned} r_x &= \int_0^t -\frac{ev_0 tB}{m_e} dt' \\ r_y &= \int_0^t v_0 \; dt'. \end{aligned}$$

Which gives us:

$$\begin{aligned} r_x &= -\frac{1}{2}\frac{ev_0 t^2 B}{m_e} \\ r_y &= v_0 t, \end{aligned}$$

or in vector notation:

$$\vec{r} = -\frac{1}{2}\frac{ev_0 t^2 B}{m_e}\hat{x} + v_0 t\hat{y}. \tag{3}$$

This is where it gets tricky; we have to use our intuition again. Depending on the initial conditions, the electron may be traveling fast enough to not be deflected much. If that's the case, we can approximate the exit velocity and exit position of the electron by saying that the electron exits the magnetic field when the y component of position is equal to L; in other words $r_y = v_0 t = L$.

Substituting this into total velocity $\vec{v}$, we can calculate the exit velocity $\vec{v}'$:

$$\vec{v}' \; = -\frac{eLB}{m_e}\hat{x} + v_0\hat{y}, \tag{4}$$

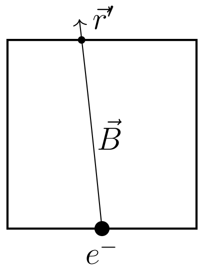
We can find the exit position by doing a mathematical trick. Since we know that $v_0 t = L$, we can define t as:

$$t = \frac{L}{v_0}.$$

If we now substitute this into position $\vec{r}$, we can calculate the exit position $\vec{r'}$:

$$\vec{r'} = -\frac{1}{2}\frac{eL^2B}{v_0 m_e}\hat{x} + L\hat{y}. \tag{5}$$

It is important to keep in mind that these solutions only hold when the electron is traveling fast enough to avoid much deflection, as can be seen in the following drawing:



The electron's path is parabolic, and the exit position is at the intersection of the path and the magnetic field boundary.

This is a good approximation, but we can do better. The Lorentz force is a centripetal force in this situation because the acceleration and velocity vectors are perpendicular. Mathematically, this means that the magnitude of the Lorentz force is equal to the magnitude of the centripetal force:

$$ev_0B = \frac{m_e v_0^2}{r},$$

where r is the gyroradius (or orbit radius).

The gyroradius r is then given by:

$$r = \frac{m_e v_0}{eB}. \tag{6}$$

The position of the electron in this circular trajectory can then be represented as:

$$\vec{r} = r(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}) + \vec{r}_c,$$

where ω is the angular frequency defined as:

$$\omega = \frac{v_0}{r} = \frac{eB}{m_e},$$

and $\vec{r}_c$ the vector pointing to the center of the circle. In our case $\vec{r}_c = -r\hat{x}$, so our position vector can be written as:

$$\vec{r} = r(\cos(\omega t) - 1)\hat{x} + r\sin(\omega t)\hat{y}. \tag{7}$$

We can now reason like we did before; the electron may be traveling fast enough to not be deflected much. From this we again get $r_y = L$:

$$r\sin(\omega t) = L.$$

Rearranging, and taking the inverse sin, we obtain:

$$\omega t = \arcsin\left(\frac{L}{r}\right).$$

Substituting this back into Eq. 7, and using the $\cos(\arcsin(x)) = \sqrt{1 - x^2}$ identity, we obtain the exit position:

$$\vec{r}' = \left(\sqrt{r^2 - L^2} - r\right)\hat{x} + L\hat{y}, \tag{8}$$

which should now be fully accurate instead of an approximation.

The velocity $\vec{v}$ is obtained by differentiating Eq. 7:

$$\vec{v} = -r\omega \sin(\omega t)\hat{x} + r\omega \cos(\omega t)\hat{y},$$

but we know that $r\omega = v_0$, so the velocity is just:

$$\vec{v} = -v_0 \sin(\omega t)\hat{x} + v_0 \cos(\omega t)\hat{y}. \tag{9}$$

To find the exit velocity $\vec{v}'$, we substitute the inverse sine and use the identity $\cos(\arcsin(x)) = \sqrt{1-x^2}$ once again, and obtain:

$$\vec{v}' = v_0 \left(-\frac{L}{r}\hat{x} + \sqrt{1 - \frac{L^2}{r^2}}\hat{y} \right). \tag{10}$$

Now, if the electron is not traveling fast enough, it might be deflected and exit to the left side of the magnetic field. If that's the case, the y component of position never reaches L, instead, the x component reaches $-\frac{L}{2}$; in other words, $r_x = -\frac{L}{2}$. From this, we can obtain the following relation:

$$r(\cos(\omega t) - 1) = -\frac{L}{2},$$

which can be rearranged as:

$$\cos(\omega t) = 1 - \frac{L}{2r}.$$

We take the inverse cosine:

$$\omega t = \arccos\left(1 - \frac{L}{2r}\right).$$

Now we substitute this into Eq. 7, and use the $\sin(\arccos(x)) = \sqrt{1-x^2}$ identity.

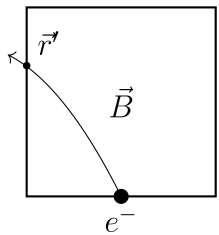
By doing what we said, we obtain the exit position:

$$\vec{r'} = -\frac{L}{2}\hat{x} + \sqrt{rL - \frac{L^2}{4}}\hat{y}. \tag{11}$$

To find the exit velocity $\vec{v'}$, we substitute the inverse cosine into Eq. 9 and use the identity $\sin(\arccos(x)) = \sqrt{1-x^2}$ once again, and obtain:

$$\vec{v'} = -v_0\sqrt{\frac{L}{r} - \frac{L^2}{4r^2}}\hat{x} + v_0\left(1 - \frac{L}{2r}\right)\hat{y}. \tag{12}$$

These solutions (Eq. 11 and Eq. 12) only hold when the electron is traveling slower, which causes it to deflect to the side, as can be seen in the following drawing:



There's also a *trivial* solution, where the electron exits through the top left corner. In that case, $r_x = -\frac{L}{2}$ and $r_y = L$, and the exit position $\vec{r'}$ is just:

$$\vec{r'} = -\frac{L}{2}\hat{x} + L\hat{y}.$$

In that case, both solutions (Eq. 10 and Eq. 12) for the exit velocity $\vec{v'}$ give the same result, so either one is fine to use.

3 Numerical Solution

In the last section we analytically found three solutions for the exit position and exit velocity. In this section, we'll explore a numerical solution, which we can use to confirm if our analytical solution is correct.

Algorithm 1 Electron Deflection in Magnetic Field

Input:
$L \leftarrow ?$ ▷ Size of magnetic field region (m)
$B \leftarrow ?$ ▷ Magnetic field strength (T)
$v_0 \leftarrow ?$ ▷ Initial electron speed (m/s)
$\Delta t \leftarrow ?$ ▷ Time step (s)
$e \leftarrow 1.602 \cdot 10^{-19}$ ▷ Electron charge (C)
$m_e \leftarrow 9.109 \cdot 10^{-31}$ ▷ Electron mass (kg)
Initialize:
$\mathbf{pos} \leftarrow [0, 0, 0]$
$\mathbf{vel} \leftarrow [0, v_0, 0]$
$\mathbf{B} \leftarrow [0, 0, B]$
trajectory $\leftarrow [\,]$
while True **do**
    trajectory.append(**pos**)
    if $|\mathbf{pos}[0]| \geq L/2$ **or** $\mathbf{pos}[1] \geq L$ **then**
        break
    end if
    $\mathbf{F} \leftarrow -e \cdot (\mathbf{vel} \times \mathbf{B})$ ▷ Lorentz force
    $\mathbf{a} \leftarrow \mathbf{F}/m_e$ ▷ Acceleration
    $\mathbf{vel} \leftarrow \mathbf{vel} + \mathbf{a} \cdot \Delta t$ ▷ Update velocity
    $\mathbf{pos} \leftarrow \mathbf{pos} + \mathbf{vel} \cdot \Delta t$ ▷ Update position
end while
Output: pos, vel, trajectory

The algorithm presented (Alg. 1) employs a numerical method known as *Euler integration*. This method provides a linear approximation of a continuous system, with its accuracy increasing as the time step Δt decreases. Implementation of this algorithm can be done in any programming language, the easiest and widely used being Python.

We integrate by first calculating the Lorentz force $\vec{F}$ and the acceleration $\vec{a}$ caused by it:

$$\vec{F} = -e\vec{v} \times \vec{B} \implies \vec{a} = \frac{\vec{F}}{m_e}.$$

Then we treat the acceleration as linear that for a small time step Δt so that the equations of motion are just:

$$\begin{aligned} \vec{v}(t + \Delta t) &= \vec{v}(t) + \vec{a}\Delta t \\ \vec{r}(t + \Delta t) &= \vec{r}(t) + \vec{v}(t + \Delta t)\Delta t. \end{aligned}$$

We keep doing this until we exit the magnetic field region. The exit condition is determined by Eq. 1, and in the algorithm is just simply a check for whether the coordinates of the electron are outside the $L \times L$ region.

Let's plug in some values for the four input parameters, L, B, v_0 and Δt. To keep it simple, let's take the following parameters for the magnetic field region and the time step:

Parameter	Symbol	Value
Magnetic field strength	B	10^{-3} T
Region size	L	$5 \cdot 10^{-2}$ m
Time step	Δt	10^{-11} s

Table 1: Parameters used

so that we can play around with the initial speed.

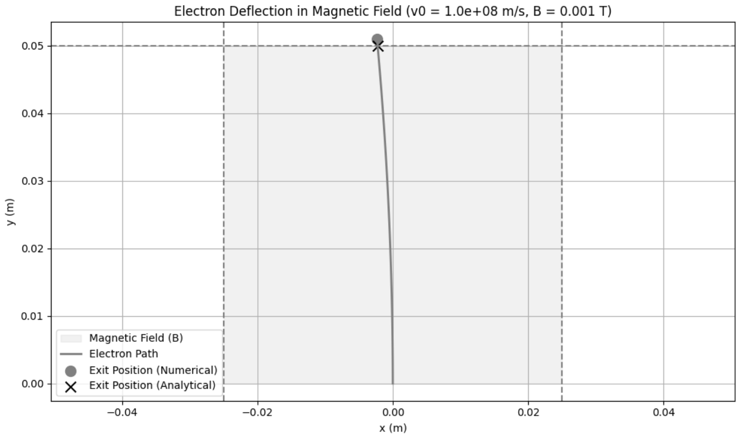
v_0 (m/s)	$\vec{r}_n$ (m)	$\vec{r}_a$ (m)	ϵ_x (%)	ϵ_y (%)
Fast electron				
10^8	$\begin{pmatrix} -2.33 \cdot 10^{-3} \\ 5.09 \cdot 10^{-2} \end{pmatrix}$	$\begin{pmatrix} -2.20 \cdot 10^{-3} \\ 5 \cdot 10^{-2} \end{pmatrix}$	5.9%	1.8%
$5 \cdot 10^7$	$\begin{pmatrix} -4.52 \cdot 10^{-3} \\ 5.02 \cdot 10^{-2} \end{pmatrix}$	$\begin{pmatrix} -4.43 \cdot 10^{-3} \\ 5 \cdot 10^{-2} \end{pmatrix}$	2%	0.4%
$2 \cdot 10^7$	$\begin{pmatrix} -1.16 \cdot 10^{-2} \\ 5 \cdot 10^{-2} \end{pmatrix}$	$\begin{pmatrix} -1.16 \cdot 10^{-2} \\ 5 \cdot 10^{-2} \end{pmatrix}$	0%	0%
$1.2 \cdot 10^7$	$\begin{pmatrix} -2.19 \cdot 10^{-2} \\ 5 \cdot 10^{-2} \end{pmatrix}$	$\begin{pmatrix} -2.18 \cdot 10^{-2} \\ 5 \cdot 10^{-2} \end{pmatrix}$	0.5%	0%
		Average errors:	2.8%	1.1%
Slow electron				
10^7	$\begin{pmatrix} -2.51 \cdot 10^{-2} \\ 4.71 \cdot 10^{-2} \end{pmatrix}$	$\begin{pmatrix} -2.50 \cdot 10^{-2} \\ 4.71 \cdot 10^{-2} \end{pmatrix}$	0.4%	0%
$5 \cdot 10^6$	$\begin{pmatrix} -2.50 \cdot 10^{-2} \\ 2.82 \cdot 10^{-2} \end{pmatrix}$	$\begin{pmatrix} -2.50 \cdot 10^{-2} \\ 2.82 \cdot 10^{-2} \end{pmatrix}$	0.0%	0%
$2.5 \cdot 10^6$	$\begin{pmatrix} -2.50 \cdot 10^{-2} \\ 9.27 \cdot 10^{-3} \end{pmatrix}$	$\begin{pmatrix} -2.50 \cdot 10^{-2} \\ 9.26 \cdot 10^{-3} \end{pmatrix}$	0.0%	0.1%
$2.3 \cdot 10^6$	$\begin{pmatrix} -2.50 \cdot 10^{-2} \\ 5.42 \cdot 10^{-3} \end{pmatrix}$	$\begin{pmatrix} -2.50 \cdot 10^{-2} \\ 5.38 \cdot 10^{-3} \end{pmatrix}$	0.0%	0.7%
		Average errors:	0.4%	0.4%

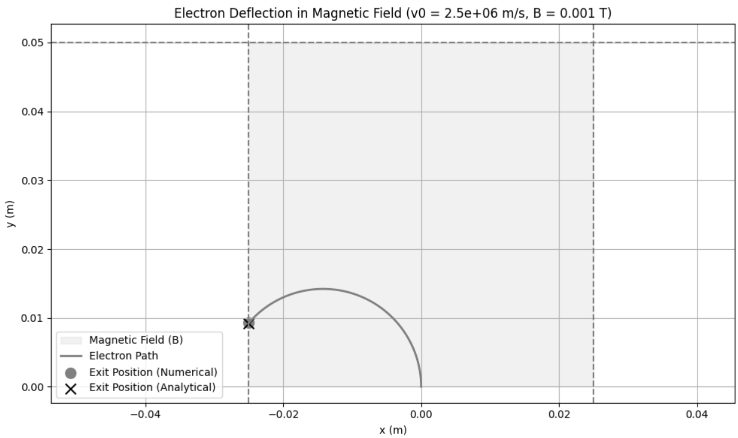
Table 2: Results

We can see that our numerical results ($\vec{r}_n$) slightly deviate from the analytical solutions ($\vec{r}_a$) due to the Euler method's linear approximation. We can reduce these errors by doing one of the following:

- Decrease the time step Δt (e.g., to 10^{-12} s),
- Use a higher-order method.

However, the errors are small enough, and if we plot the electron trajectory and the numerical and analytical exit positions, we can see that they're not that far off:





4 Summary

For a homogeneous magnetic field $\vec{B}$ spanning a region of size $L \times L$, mathematically defined as:

$$\vec{B}(x, y, z) = \begin{cases} B\hat{z}, & \text{if } |x| < \dfrac{L}{2} \text{ and } 0 \leq y < L, \\ \vec{0}, & \text{otherwise,} \end{cases}$$

an electron entering at the origin with velocity $\vec{v}_0 = v_0\hat{y}$ will follow a circular trajectory of radius

$$r = \frac{m_e v_0}{eB}.$$

Depending on its speed, the electron may exit through the top or the left side of the field region. The corresponding exit positions and velocities are given below.

Top Exit ($r_y = L$) If the electron exits through the top edge:

$$\vec{r}' = \left(\sqrt{r^2 - L^2} - r\right)\hat{x} + L\hat{y},$$
$$\vec{v}' = -v_0\frac{L}{r}\hat{x} + v_0\sqrt{1 - \frac{L^2}{r^2}}\hat{y}.$$

Side Exit ($r_x = -\frac{L}{2}$) If the electron exits through the left side:

$$\vec{r}' = -\frac{L}{2}\hat{x} + \sqrt{rL - \frac{L^2}{4}}\hat{y},$$
$$\vec{v}' = -v_0\sqrt{\frac{L}{r} - \frac{L^2}{4r^2}}\hat{x} + v_0\left(1 - \frac{L}{2r}\right)\hat{y}.$$