Generalized Clifford Algebras and the N-th Root of Linear Differential Equations of Higher Order

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Abstract

It is shown how generalized Clifford algebras allows to construct the N-th root of N-order linear differential equations involving massless and massive particles. Such generalized Dirac-like equations differ from the ones in the literature. Explicit solutions are found. We conclude with some remarks on pseudo-unitary algebras, modular arithmetic, modified Dirac equations, Octonions, and the Okubo algebra.

Keywords : Generalized Clifford Algebras; Ternary Clifford Algebras; Dirac equation.

1 Generalized Clifford Algebras and Dirac-like Equations

Extensive studies on Clifford algebras, the generalizations, and their physical applications were made for about a decade starting 1967, under the name of L-Matrix Theory, by Ramakrishnan and his collaborators [1], [2]. In this work we shall focus on a very *special* case of these generalized Clifford algebras (GCA) with ordered ω -commutation relations [1], [2]. In particular, let us begin with the complex ternary Clifford algebra denoted in two dimensions by $Cl_2^{\frac{1}{3}}$ [4], [5] with two generators e_1, e_2 obeying

$$e_1^3 \ = \ e_2^3 \ = \ e; \ \ e_1 \ e_2 \ = \ \omega \ e_2 \ e_1; \ \ \omega \equiv e^{\frac{2\pi i}{3}} \tag{1}$$

e is the identity element. ω is the primitive complex cubic root of unity satisfying

$$\omega^3 = 1, \ \ 1 + \omega + \omega^2 = 0; \ \ \bar{\omega} = \omega^2; \ \ \omega - \omega^2 = i\sqrt{3} \tag{2}$$

More recently, a natural realization of unitary Lie groups which are important in physics and other applications, using only operations in generalized Clifford algebras, and without using the corresponding matrix representations, has been provided by [5]. Basis-free definitions of the determinant, trace, and characteristic polynomial in this special class of generalized Clifford algebras were constructed by [5]. Also, a similar operational procedure to what occurs in ordinary complex Clifford algebras, was introduced in the definition of Hermitian conjugation (or Hermitian transpose) without using the corresponding matrix representations.

Let us begin with arbitrary element of the complex ternary Clifford algebra [5]

$$U \ = \ \sum_{j,k=0}^{j,k=2} u_{jk}\ e_1^j\ e_2^k \ = \ u_{00}\ e + u_{10}\ e_1 + u_{01}\ e_2 \ + \ u_{20}\ e_1^2 + u_{02}\ e_2^2 +$$

$$u_{11}\ e_1\ e_2 + u_{21}\ e_1^2\ e_2 + u_{12}\ e_1\ e_2^2 + u_{22}\ e_1^2\ e_2^2 \tag{3}$$

The coefficients of (3) are complex-valued and the complex ternary Clifford algebra $Cl_2^{\frac{1}{3}}$ was shown to be isomorphic to the unitary algebra $\mathbf{u}(3)$. In general, for $d =$ even, the generalized Clifford algebra (GCA) associated with the N-th root of unity is isomorphic to the unitary algebra $\mathbf{u}(N^{\frac{d}{2}})$ of dimension N^d. [5]. The $d =$ odd case is more complicated because the unitary algebra associated with the generalized Clifford algebra (GCA) is now given by the direct sum of N copies of $\mathbf{u}(N^{\frac{d-1}{2}})$. The matrix realization of each one of the e_i generators $(i = 1, 2, 3, \ldots, d)$ are given by $N^{\frac{d-1}{2}} \times N^{\frac{d-1}{2}}$ matrices, hence N copies span the $N \times N^{d-1} = N^d$-dimensional space of the GCA.

From some of the entries of the multiplication table of the ternary Clifford algebra $Cl_2^{\frac{1}{3}}$ like

$$e_2\ e_1 = \omega^2\ e_1\ e_2;\ \ e_2\ e_1^2 \ = \ \omega\ e_1^2\ e_2;\ \ e_1\ e_1^2 \ = \ e;\ \ e_2\ e_2^2 \ = \ e \tag{4}$$

$$e_2\ e_1\ e_2 \ = \ \omega^2\ e_1\ e_2^2;\ \ e_2\ e_1^2\ e_2 \ = \ \omega\ e_1^2\ e_2^2; \ldots \tag{5}$$

one can show that the cube of the linear differential operator $\mathcal{L} = e_1\partial_1 + e_2\partial_2$, with $\partial_1 \equiv \frac{\partial}{\partial x^1}$, $\partial_2 \equiv \frac{\partial}{\partial x^2}$, is equal to a third-order linear differential operator without any mixed derivatives

$$(e_1\partial_1 \ + \ e_2\partial_2)^3 \ = \ \partial_1^3 \ + \ \partial_2^3 \ + \ (1 + \omega + \omega^2)\ (\partial_1^2\ \partial_2 + \partial_1\ \partial_2^2) \ =$$

$$\partial_1^3 \ + \ \partial_2^3 \tag{6a}$$

and which results from the null sum of the three cubic roots of unity
$1 + \omega + \omega^2 = 0$.

In the case of the complex generalized Clifford algebra associated to the N-th root of unity $\omega \equiv e^{2\pi i/N}$ in d-dimensions, with d generators $e_1, e_2, e_3, \ldots, e_d$, one has the more general relation

$$(e_1\partial_1 \ + \ e_2\partial_2 \ + \ \ldots \ + \ e_d\partial_d)^N \ = \ \partial_1^N \ + \ \partial_2^N \ + \ \ldots \ + \ \partial_d^N \qquad (6b)$$

due to the algebraic constraint $1 + \omega + \omega^2 + \omega^3 + \ldots + \omega^{N-1} = 0$, and the commutation relations $e_i e_j = \omega e_j e_i, i < j, \ i, j = 1, 2, 3, \ldots, d.$ $e_i^N = e$.

Thus one can linearize certain N-th order partial differential operators using the generalized Clifford algebra (GCA) similar to the way that the Clifford algebra helped linearize the second order Klein-Gordon partial differential equation as shown by Dirac after "squaring" the linear operator $(-i\gamma^\mu\partial_\mu + m)(-i\gamma^\nu\partial_\nu - m) = -(\partial_\mu\partial^\mu + m^2)$ in units $\hbar = c = 1$. See section 7 of [2] for more details.

Therefore, from eqs-(6a,6b) one can infer that by taking the N-th root of the N-order linear differential operator in the right-hand side yields the linear operator $\sum_{i=1}^{i=d} e_i\partial_i$. The cubic analog of Laplace and d'Alembert equations with *mixed* derivatives were first considered by Humbert [9]

$$(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}) \ (\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial y} + \omega^2\frac{\partial}{\partial z}) \ (\frac{\partial}{\partial x} + \omega^2\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial z}) \ =$$

$$\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial z^3} \ - 3 \ \frac{\partial^3}{\partial x\partial y\partial z} \qquad (7)$$

Before proceeding we should add some important remarks. (i) One may note that eq-(6a) $differs$ from eq-(7) due to the presence of mixed derivatives. (ii) The ternary Clifford algebra depicted above is not the same as the one formulated by Kerner [8] involving the cyclic anticommutator $Q_aQ_bQ_c + Q_bQ_cQ_a + Q_cQ_aQ_b = 3\rho_{abc}\mathbf{1}$ and which is the ternary analog of $Q_aQ_b + Q_bQ_c = 2\eta_{ab}\mathbf{1}$. (iii) The existence and particular properties of the cubic Grassmann and Clifford algebras studied by Kerner [8] were used to define cubic roots of linear differential operators which clearly $differ$ from the operators found in this work. For more on cubic forms and algebras with cubic constitutive relations see [10].

One can introduce mass terms to the above equations (6) as follows. In the ternary Clifford algebra case one can show that

$$(e_1\partial_1 \ + \ e_2\partial_2 \ + \ m) \ (e_1\partial_1 \ + \ e_2\partial_2 \ + \ \omega \ m) \ (e_1\partial_1 \ + \ e_2\partial_2 \ + \ \omega^2 \ m) \ =$$

$$\partial_1^3 \ + \ \partial_2^3 \ + \ m^3 \qquad (8)$$

as a result of the algebraic relation $1 + \omega + \omega^2 = 0$ involving the primitive cubic root of unity ω.

Given a 3×3 matrix representation of the e_1, e_2 generators and the unit element e [5]

$$e_1 \ = \ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad (9a)$$

$$e_2 \; = \; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{9b}$$

$$e \; = \; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9c}$$

one can study the following three differential equations

$$(e_1\partial_1 \; + \; e_2\partial_2 \; + \; m)\Psi_{(1)} \; = \; 0 \tag{10a}$$

$$(e_1\partial_1 \; + \; e_2\partial_2 \; + \; \omega \; m)\Psi_{(2)} \; = \; 0 \tag{10b}$$

$$(e_1\partial_1 \; + \; e_2\partial_2 \; + \; \omega^2 \; m)\Psi_{(3)} \; = \; 0 \tag{10c}$$

Eq-(10a) is the ternary analog of the Dirac equation with $\Psi_{(1)}$ a column matrix whose three entries (*not* to be confused with a spinor associated to an ordinary Clifford algebra) are given by

$$\Psi_{(1)} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{11}$$

Eq-(10a) leads to three *coupled* linear partial differential equations

$$(\partial_2 \; + \; m)\psi_1 \; + \; \partial_1\psi_2 \; = \; 0, \;\; (\omega \; \partial_2 \; + \; m)\psi_2 \; + \; \partial_1\psi_3 \; = \; 0$$
$$\partial_1\Psi_1 \; + \; (\omega^2 \; \partial_2 \; + \; m)\Psi_3 \; = \; 0 \tag{12}$$

In the massive case, $m \neq 0$, a solution to eq-(10a) is

$$\Psi_{(1)} = \begin{pmatrix} c_1 \; e^{\alpha(x_1+x_2)} \\ c_2 \; e^{\alpha(x_1+x_2)} \\ c_3 \; e^{\alpha(x_1+x_2)} \end{pmatrix} \tag{13}$$

with

$$\alpha \; = \; - \; \frac{m}{2^{1/3}}, \;\; c_1 \; = \; 1, \;\; c_2 \; = \; - \; \frac{\alpha + m}{\alpha} \; = \; 2^{1/3} \; - 1 \tag{14}$$

$$c_3 \; = \; - \; \frac{\alpha}{\omega^2\alpha + m} \; = \; - \; \frac{1}{\omega^2 - 2^{1/3}} \tag{15}$$

Due to $\alpha < 0$ the entries of $\Psi_{(1)}$ in eq-(13) vanish at infinity $x_1 = \infty, x_2 = \infty$ and are finite at $x_1 = 0, x_2 = 0$. Form eqs-(14,15) one finds that the values of c_2, c_3 are *invariant* under the scalings $m \to \lambda m; \alpha \to \lambda\alpha$ which allows to find the solutions to eqs-(10b,10c) by simply scaling the exponents in eq-(13) by ω and ω^2, respectively, leading to

$$\Psi_{(2)} = \begin{pmatrix} c_1 \ e^{\omega\alpha(x_1+x_2)} \\ c_2 \ e^{\omega\alpha(x_1+x_2)} \\ c_3 \ e^{\omega\alpha(x_1+x_2)} \end{pmatrix} \tag{16}$$

$$\Psi_{(3)} = \begin{pmatrix} c_1 \ e^{\omega^2\alpha(x_1+x_2)} \\ c_2 \ e^{\omega^2\alpha(x_1+x_2)} \\ c_3 \ e^{\omega^2\alpha(x_1+x_2)} \end{pmatrix} \tag{17}$$

The eqs-(10) can be recast in the following form

$$\mathcal{L}\Psi_{(1)} = (e_1\partial_1 + e_2\partial_2)\Psi_{(1)} = -m \ \Psi_{(1)} \tag{18a}$$

$$\mathcal{L}\Psi_{(2)} = (e_1\partial_1 + e_2\partial_2)\Psi_{(2)} = -\omega m \ \Psi_{(2)} \tag{18b}$$

$$\mathcal{L}\Psi_{(3)} = (e_1\partial_1 + e_2\partial_2)\Psi_{(3)} = -\omega^2 m \ \Psi_{(3)} \tag{18c}$$

and whose interpretation is that $\Psi_{(1)}$ is an eigenvector of the operator $\mathcal{L}$ with $-m$ for its eigenvalue, and similarly, $\Psi_{(2)}, \Psi_{(3)}$ are eigenvectors with $-\omega m; -\omega^2 m$ for their eigenvalues, respectively.

Given the three equations (18) and eq-(6) one learns that their cube is given by

$$(e_1\partial_1 \ + \ e_2\partial_2)^3 \ \Psi_{(1)} = -m^3 \ \Psi_{(1)} \ \Rightarrow \ (\partial_1^3 \ + \ \partial_2^3) \ \Psi_{(1)} = -m^3 \ \Psi_{(1)} \ \Rightarrow$$

$$(\partial_1^3 \ + \ \partial_2^3 \ + \ m^3) \ \Psi_{(1)} \ = \ 0 \tag{19a}$$

$$(e_1\partial_1 \ + \ e_2\partial_2)^3 \ \Psi_{(2)} = -(\omega \ m)^3 \ \Psi_{(2)} \ \Rightarrow \ (\partial_1^3 \ + \ \partial_2^3) \ \Psi_{(2)} = -m^3 \ \Psi \ \Rightarrow$$

$$(\partial_1^3 \ + \ \partial_2^3 \ + \ m^3) \ \Psi_{(2)} \ = \ 0 \tag{19b}$$

$$(e_1\partial_1 \ + \ e_2\partial_2)^3 \ \Psi_{(3)} = -(\omega^2 \ m)^3 \ \Psi_{(1)} \ \Rightarrow \ (\partial_1^3 \ + \ \partial_2^3) \ \Psi_{(3)} = -m^3 \ \Psi \ \Rightarrow$$

$$(\partial_1^3 \ + \ \partial_2^3 \ + \ m^3) \ \Psi_{(3)} \ = \ 0 \tag{19c}$$

Therefore, the three (column) eigenvectors $\Psi_{(1)}, \Psi_{(2)}, \Psi_{(3)}$ displayed by eqs-(13,16,17) are simultaneous (degenerate) solutions of the ternary analog of the Klein-Gordon equation $(\partial_1^3 + \partial_2^3 + m^3)\Psi = 0$, where the operators act on each single one of the three entries comprising the column vector Ψ. This is just a consequence of the fact that $(\omega m)^3 = (\omega^2 m)^3 = m^3$.

The generalization of eqs-(19) to the N-th root case in $d = 2k$ (even) dimensions is given by

$$(\partial_1^N \ + \ \partial_2^N \ + \ \ldots \ + \ \partial_d^N \ + (-1)^{N+1} \ m^N) \ \Psi \ = \ 0 \tag{20}$$

and is obtained by taking the N-th power of the linear operator $\mathcal{L} = \sum_{i=1}^{d=2k} e_i \partial_i$ acting on Ψ, and by using the relations in eqs-(6). There are N-solutions to eq-(20) given by the column vectors $\Psi_{(1)}, \Psi_{(2)}, \Psi_{(3)}, \ldots, \Psi_{(N)}$ which are comprised of $N^{\frac{d}{2}} = N^k$ entries. The N-solutions satisfy the following N linear differential equations

$$\mathcal{L}\Psi_{(1)} = -m \ \Psi_{(1)}, \quad \mathcal{L}\Psi_{(2)} = -(\omega \ m) \ \Psi_{(2)}, \ \ldots \ , \mathcal{L}\Psi_{(N)} = -(\omega^{N-1} \ m) \ \Psi_{(N)} \tag{21}$$

From eqs-(21) one can infer that the N-th powers of the operator $\mathcal{L}$ acting on the Ψ's yield the *common* factors of $(-1)^N m^N$ multiplying the Ψ's in the right hand side due to the algebraic relation $\omega^N = 1$. As a result one arrives at eq-(20). And conversely, eqs-(21) can be obtained as a result of taking the N-th root of the N-order linear differential equation (20) involving the mass m.

One can find generalized solutions to eqs-(18, 20,21) similar to those found in eqs-(13,16,17) where the exponential parameter β appearing in $e^{\omega^n \beta(x_1+x_2+\ldots+x_d)}$, $n = 0, 1, 2, \ldots, N-1$ obeys the relation $d\beta^N + (-1)^{N+1} m^N = 0 \Rightarrow \beta = -\frac{m}{d^{1/N}}$. Given β and a representation of all the e_i generators $(i = 1, 2, 3, \ldots, d)$ given by $N^{d/2} \times N^{d/2}$ matrices one can then solve for the $N^{d/2}$ coefficients $c_1, c_2, c_3, \ldots$, after setting $c_1 = 1$ in eqs-(21). In $d =$ odd dimensions the e_i generators are represented by $N^{\frac{d-1}{2}} \times N^{\frac{d-1}{2}}$ matrices and the Ψ's are column vectors with $N^{\frac{d-1}{2}}$ entries.

If one wishes to work only with real-valued physical masses, and not with multiples of the mass like $\omega m, \omega^2 m, \ldots$ involving *complex* numbers, then the ternary (N-th) analog of the Dirac equation must be given by an equation of the form $(\mathcal{L} + m)\Psi = 0$. The Dirac equation in natural units ($\hbar = c = 1$) is given by $(i\gamma^\mu \partial_\mu - m)\Psi_{(1)} = 0$. The equation for a negative mass is $(i\gamma^\mu \partial_\mu + m)\Psi_{(2)} = 0$. Squaring the Dirac operator yields $(i\gamma^\mu \partial_\mu)(i\gamma^\nu \partial_\nu) = -(g^{\mu\nu} + \gamma^{[\mu\nu]})\partial_\mu \partial_\nu = -\partial_\mu \partial^\mu$, and one arrives in both cases at

$$-(\partial_\mu \partial^\mu + m^2)\Psi_{(1)} \ = \ -(\partial_\mu \partial^\mu + m^2)\Psi_{(2)} \ = \ 0 \tag{22}$$

because $m^2 = (-m)^2$. Therefore, $\Psi_{(1)}, \Psi_{(2)}$ are the two degenerate solutions of the Klein-Gordon-like equation $(\partial_\mu \partial^\mu + m^2)\Psi = 0$, where the operators act on the two components of the spinor Ψ in two-dimensions.

In the *massless* case $m = 0$, a straightforward solution to eqs-(12) is given by

$$\Psi = \begin{pmatrix} f(x_1 - x_2) \\ f(x_1 - x_2) \\ \omega \ f(x_1 - x_2) \end{pmatrix} \tag{23}$$

where $f(x_1 - x_2)$ is an arbitrary function of $x_1 - x_2$. Also, it is straightforward to verify that $(\partial_1^3 + \partial_2^3) f(x_1 - x_2) = 0$. Note that if one naively sets $\alpha = m = 0$ in eqs-(13,14,15) it yields trivial (constant) solutions.

2 Concluding Remarks

We began with the complex ternary Clifford algebra $Cl_2^{\frac{1}{3}}$ [4], [5] with two generators e_1, e_2 in $d = 2$ obeying the relations in eq-(1) and such that it was isomorphic to the unitary $\mathbf{u}(3)$ algebra. It is interesting to see if one can obtain an algebra isomorphic to pseudo-unitary algebras $\mathbf{u}(2,1), \mathbf{u}(1,2)$ by having one generator $\tilde{e}_1$ obeying now $\tilde{e}_1^3 = -e$, while the other still obeys $\tilde{e}_2^3 = e$. Namely, one defines the new generator as $\tilde{e}_1 = \beta e_1, \beta = e^{i\pi/3}$, while $\tilde{e}_2 = e_2$ remains the same. One finds that a matrix realization of the generators described in eqs-(9) yields that the combination $e_1 - e_1^2$ is anti-Hermitian, while the combination $\tilde{e}_1 - \tilde{e}_1^2$ is now Hermitian. Therefore by multiplying e_1 by the cubic root of -1 one can convert previous anti-Hermitian operators into Hermitian ones.

The Weyl unitary trick allows to relate the unitary group $U(p+q)$ and the pseudo-unitary group $U(p,q)$, and explains why one needs to decompose the matrix generators of the non-compact pseudo-unitary group $U(1,3)$ in terms of Hermitian *and* anti-Hermitian matrices by introducing judicious i factors. Therefore, it is plausible that one may recover the pseudo-unitary algebras (groups) in this fashion by a *generalization* of the Wick rotation : by multiplying e_1 by the cubic root of -1 in the ternary Clifford algebra case. In quadratic ordinary Clifford algebras a multiplication by i (square root of -1) converts a Hermitian generator to an anti-Hermitian one, and vice versa.

In the massless case, eqs-(6b) reveals that the N-th root of the N-order linear differential operator in the right-hand side yields the linear operator $\sum_{i=1}^{i=d} e_i \partial_i$. Whereas in the massive case, one found that the N equations (21) can be obtained as a result of taking the N-th root of the N-order linear differential equation (20) involving the mass m. Symbolically speaking, eq-(6b) resembles a congruence relation modulo p (prime) of the form

$$(x_1 + x_2 + x_3 + \ldots + x_d)^N \equiv x_1^N + x_2^N + x_3^N + \ldots + x_d^N \tag{24}$$

in modular arithmetic. Therefore, one should explore further connections among modular arithmetic and generalized Clifford algebras (GCA).

A cubic root of the Klein-Gordon equation was provided by [7] by recurring to the Clifford algebra of a polynomial. In the case of a cubic polynomial the authors obtained a cubic root of the Klein-Gordon operator via the algebraic relation $m(p_\mu p^\mu - m^2) = (g_\mu p^\mu + m\tilde{g})^3$, where the generators g_μ and $\tilde{g}$ satisfy the following cubic algebra

$$S_3(\tilde{g}, \tilde{g}, \tilde{g}) \ = \ \tilde{g}^3 \ = \ -1, \ \ S_3(g_\mu, \tilde{g}, \tilde{g}) \ = \ 0, \ \ S_3(g_\mu, g_\nu, \tilde{g}) \ = \ \frac{1}{3}\eta_{\mu\nu}, \ \ S_3(g_\mu, g_\nu, g_\tau) \ = \ 0 \tag{24}$$

The modified Dirac equation is obtained from the cubic root of the Klein-Gordon equation as follows

$$-m(\partial_\mu \partial^\mu + m^2)\Psi \ = \ (i g_\mu \partial^\mu + m\tilde{g})^3 \Psi \ = \ 0 \ \Rightarrow \ (i g^\mu \partial_\mu + m\tilde{g})\Psi \ = \ 0 \tag{25}$$

Two explicit matrix representations were given, and one of them gives an appropriate equation for $(1+1)$-dimensional anyons of spin $1/3$ and $-2/3$. The modified Dirac equation (25) must not be confused with Duffin–Kemmer–Petiau relativistic wave equation which describes spin-0 and spin-1 particles in the description of the standard model [13].

Moreover, the authors [7] remarked that as the Dirac equation is related to supersymmetry, their equation can be related to an extension of supersymmetry involving an n-th order algebra, namely fractional supersymmetry [11]

In general one may have the n-th degree polynomial

$$P_{n,a}(p) \ = \ m^{n-2a} \ (p_\mu p^\mu - m^2)^a \ = \ (g_\mu p^\mu + m\tilde{g})^n \tag{26}$$

where the generators $g_\mu, \tilde{g}$ belonging to the Clifford algebra associated to the polynomial $P_{n,a}(p)$ satisfy more complicated relations than the ones described in eq-(24). It would be interesting to see if one could obtain similar results via generalized Clifford algebras.

The next step in constructing the N-th root of the N-order linear differential operators is to introduce gauge fields by replacing $\partial_\mu \rightarrow \partial_\mu - iqA_\mu$, where A_μ is a $U(1)$ gauge field and q is the electric charge, for example, and also to write the covariant differential equations in curved spacetimes.

The left/right action of the algebra of Octonions on themselves can be realized in terms of the 2^6-dim Clifford algebra $Cl(6)$ via 8×8 matrices. Related to the Octonions the authors [12] have explored the differences among the three 8-dimensional, real, division, composition algebras, given by the Octonions, para-Octonions and Okubonions, as well as their applications in particle physics (QCD). The Octonions have a unit element; the para-Octonions do not have a unit but have a para-unit element, and the Okubo algebra does not have a unit nor a para-unit, but it has idempotent elements. These subtleties occur because in general the unit, identity and idempotent elements do not necessary coincide within the algebra. The automorphism group of the octonions and para-octonions is the first exceptional group G_2. Whereas the automorphism group of the Okubo algebra (Okubonions) is the $SU(3)$ group [12]. For this reason it is warranted to explore if there is an underlying connection between the ternary Clifford algebra and the Okubo algebra since the former algebra is isomorphic to the $\mathbf{u}(3)$ algebra.

To finalize, we explored in [14] the construction of a generalized Dirac equation via the introduction of the notion of Clifford-valued actions, and which was inspired by the work of [15] on the De Donder-Weyl theory formulation of field theory. Crucial in this construction is the evaluation of the exponentials of multivectors associated with ordinary Clifford algebras (hypercomplex) analysis. Exact matrix solutions (instead of spinors) of the generalized Dirac equation in D = 2, 3 spacetime dimensions were found. It remains to explore these findings to the generalized Clifford algebras case.

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