Takagi-Landsberg functions

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Abstract

Takagi-Landsberg functions are a particular class of periodic, continuous and never differentiable functions. Non-differentiability implies the non-convergence of the corresponding Fourier series. Furthermore, these functions are fractal objects [1].

1 Definitions and first properties

Let us consider the sequence of functions $f_k : \mathbb{R} \longrightarrow \mathbb{R}$:

$$\{f_k(t)\}_{k\in\mathbb{N}} : f_0(t), f_1(t), ..., f_k(t), ... \tag{1}$$

where

$$f_k(t) = A_k \arccos\left[\cos(\omega_k t)\right], \quad A_k, \omega_k > 0 \quad (k = 0, 1, 2, ...) \tag{2}$$

In this way the sequences of elements are defined. $\mathbb{R}$:

$$\begin{aligned} &\{A_k\}_{k\in\mathbb{N}} : A_0, A_1, ..., A_k, ... \\ &\{\omega_k\}_{k\in\mathbb{N}} : \omega_0, \omega_1, ..., \omega_k, ... \end{aligned} \tag{3}$$

For $k \to +\infty$ the first is infinitesimal and the second is divergent:

$$\lim_{k\to+\infty} A_k = 0, \quad \lim_{k\to+\infty} \omega_k = +\infty \tag{4}$$

such as to make the series of functions converge uniformly:

$$\sum_{k=0}^{+\infty} f_k(t) \tag{5}$$

A possible choice [2] is $\omega_k = A_k^{-1}$. Precisely:

$$A_k = 2^{-k}, \quad \omega_k = 2^k, \quad \forall k \in \mathbb{N} \tag{6}$$

ω_k is an angular frequency (in dimensionless units), so $\omega_k = 2\pi\nu_k$.

$$\nu_k = \frac{2^{k-1}}{\pi}, \quad T_k = 2^{1-k}\pi \tag{7}$$

T_k is the period of $\cos(\omega_k t)$ and therefore of $f_k(t)$. As is known::

$$\arccos(\cos x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & 0 \le x \le 2\pi \end{cases}, \tag{8}$$

so

$$\arccos\left[\cos\left(2^k t\right)\right] = \begin{cases} 2^k t, & 0 \le 2^k t \le \pi \\ 2\pi - 2^k t, & \pi \le 2^k t \le 2\pi \end{cases} = \begin{cases} 2^k t, & 0 \le t \le 2^{-k}\pi \\ 2\pi - 2^k t, & 2^{-k}\pi \le t \le 2^{1-k}\pi \end{cases}$$

Infact $\arccos\left[\cos\left(2^k t\right)\right]$ is periodic with period T_k (eq. 7). The function is also periodic with the same period:

$$f_k(t) = 2^{-k}\arccos\left[\cos\left(2^k t\right)\right] = \begin{cases} t, & 0 \le t \le 2^{-k}\pi \\ 2^{1-k}\pi - t, & 2^{-k}\pi \le t \le 2^{1-k}\pi \end{cases} \tag{9}$$

So the derivative:

$$f_k'(t) = \begin{cases} 1, & 0 \leq t \leq 2^{-k}\pi \\ -1, & 2^{-k}\pi \leq t \leq 2^{1-k}\pi \end{cases} \tag{10}$$

so $f_k(t)$ is not derivable at the infinite points $t_k = T_k/2$ and $t_k' = T_k$, but it is on the right and on the left:

$$\lim_{t \to t_k^-} f_k'(t) = 1, \quad \lim_{t \to t_k^+} f_k'(t) = -1, \quad \forall k \in \mathbb{N}$$

o the graph of $f_k(t)$has a countable infinity of corner points. From the second of (7):

$$T_{k+1} = \frac{T_k}{2} \Longrightarrow \nu_{k+1} = 2\nu_k, \quad \forall k \in \mathbb{N}$$

That is, the period of $f_{k+1}(t)$ is 1/2 is 1/2 of the period of $f_k(t)$ or what is the same, the frequency of $f_{k+1}(t)$ is twice the frequency of $f_k(t)$, while the amplitude is halved when going from $f_k(t)$ to $f_{k+1}(t)$. The result is that the graphs of these functions are nested "one inside the other", as shown in Figs. 1-2.

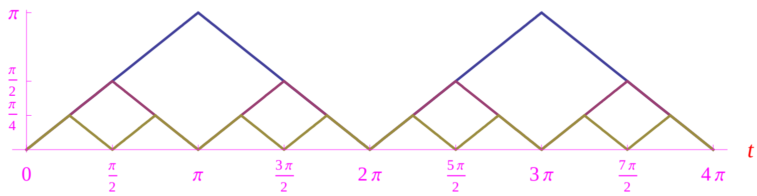

Figure 1: Function graph $f_k(t)$ for $k = 0, 1, 2$.

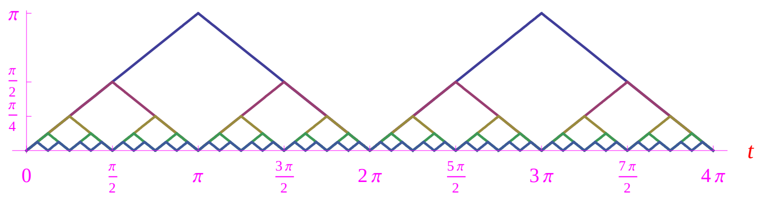

Figure 2: Function graph $f_k(t)$ for $k = 0, 1, 2, 3, 4$.

2 Uniform convergence and periodicity

Dalla sezione precedente segue

$$\lim_{k \to +\infty} f_k(t) = 0$$

which as is known [3] is a necessary but not sufficient condition for the convergence of the series 5. On the other hand, the functions $f_k(t)$ are bounded and $\Lambda_k = \sup_{\mathbb{R}} |f_k(t)| = 2^{-k}\pi$, so the numerical series:

$$\sum_{k=0}^{+\infty} \Lambda_k = \pi \sum_{k=0}^{+\infty} 2^{-k}$$

converges. By a well-known theorem [3], the series 5 converges totally, therefore uniformly and absolutely.

3 Continuity and periodicity, but not differentiability

By a well-known property [3] the sum of a uniformly convergent series in a given interval is a continuous function in the same interval.

$$g(t) = \sum_{k=0}^{+\infty} f_k(t) \tag{11}$$

It is easy to persuade ourselves that $g(t)$ is a periodic function of period $T_0 = 2\pi$, i.e. the period of $f_0(t)$. We can therefore consider the restriction of $g(t)$ to the interval of periodicity $[0, 2\pi]$. It is fundamental to observe that the sum $g(t)$ is not elementary expressible.

From (11)

$$g(0) = \sum_{k=0}^{+\infty} f_k(0) \underset{f_k(0)=0}{=} 0 \underset{T_0=2\pi}{\Longrightarrow} g(2\pi) = 0$$

The theorem holds:

Theorem 1 *The function(11) is not derivable at any point of* $\mathbb{R}$.

Dimostrazione. Taking into account the periodicity, let's limit ourselves to the interval $[0, 2\pi]$. Let us consider the sequence of elements of $\mathbb{R}$:

$$\{2^n t_0\}_{n\in\mathbb{N}} : t_0, 2t_0, 4t_0, ..., 2^n t_0, ...$$

manifestly divergent:

$$\lim_{n\to+\infty} (2^n t_0) = t_0 \lim_{n\to+\infty} 2^n \underset{t_0>0}{=} +\infty$$

Denoting with $[x]$ the integer part of $x \in \mathbb{R}$:

$$[x] \leq x \leq [x] + 1, \tag{12}$$

we have

$$[2^n t_0] \leq 2^n t_0 \leq [2^n t_0] + 1,$$

so

$$2^{-n}[2^n t_0] \leq t_0 \leq 2^{-n}[2^n t_0] + 2^{-n} \tag{13}$$

So $t_0 \in \mathcal{I}_n(t_0) = \left[\tau_{0,n}, \tau'_{0,n}\right] \quad \forall n \in \mathbb{N}$, where:

$$\tau_{0,n} = 2^{-n}[2^n t_0], \quad \tau'_{0,n} = \tau_{0,n} + 2^{-n} \tag{14}$$

terms of the sequences of elements of $\mathbb{R}$: $\{\tau_{0,n}\}_{n\in\mathbb{N}}, \ \left\{\tau'_{0,n}\right\}_{n\in\mathbb{N}}$. We show that they are both convergent to t_0:

$$\lim_{n\to+\infty} \tau_{0,n} = t_0^-, \quad \lim_{n\to+\infty} \tau'_{0,n} = t_0^+ \tag{15}$$

First:

$$\lim_{n\to+\infty} \tau_{0,n} = \lim_{n\to+\infty} 2^{-n}[2^n t_0] = 0 \cdot \infty \tag{16}$$

Let's say $\xi_n = 2^n t_0$

$$\lim_{n\to+\infty} \tau_{0,n} = t_0 \lim_{n\to+\infty} \frac{[\xi_n]}{\xi_n} \tag{17}$$

Noting that $\lim_{n\to+\infty}[\xi_n]=+\infty$

$$\lim_{n\to+\infty}\frac{[\xi_n]}{\xi_n}=\lim_{x\to+\infty}h(x), \qquad h(x)\overset{def}{=}\frac{[x]}{x} \tag{18}$$

The function $h(x)$ is defined in $X=\mathbb{R}-\{0\}$:

$$h(x)=\begin{cases}0, & 0<x<1\\ x^{-1}, & 1\le x<2\\ 2x^{-1}, & 2\le x<3\\ \dots \\ nx^{-1}, & n\le x<n+1\\ \dots\end{cases} \tag{19}$$

So the graph of $h(x)$ is made up of a countable infinity of hyperbola arcs arranged as in Fig. 3. From (12):

$$[x]\le x\le[x]+1\Longrightarrow[x]-1\le x-1\le[x]$$
$$\Longrightarrow x-1\le[x]\le x\underset{x>0}{\Longrightarrow}\frac{x-1}{x}\le\frac{[x]}{x}\le1\Longrightarrow\frac{x-1}{x}\le h(x)\le1$$

It turns out:

$$\left.\lim_{x\to+\infty}\frac{x-1}{x}=1^-,\quad\lim_{x\to+\infty}1=1\right)\Longrightarrow\lim_{x\to+\infty}h(x)=1^-$$

The implication follows from Sqeeze Theorem. Replacing the results found in (17):

$$\lim_{n\to+\infty}\tau_{0,n}=t_0^- \tag{20}$$

Second

$$\lim_{n\to+\infty}\tau'_{0,n}=\lim_{n\to+\infty}\left(\tau_{0,n}+2^{-n}\right)=\lim_{n\to+\infty}\tau_{0,n}+\lim_{n\to+\infty}2^{-n}=t_0^-+0^+=t_0^+$$

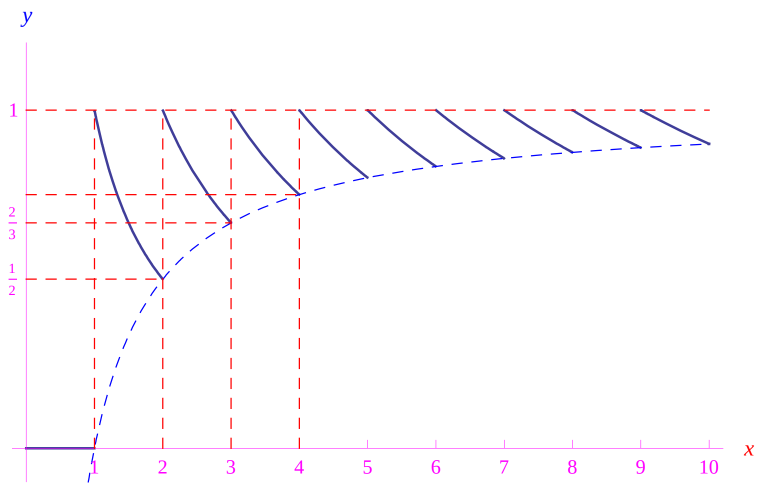

Figure 3: La curva $y=\frac{[x]}{x}$ è limitata tra la curva $y=\frac{x-1}{x}$ e la semiretta $y=1$.

It follows that $\{\mathcal{I}_n(t_0)\}_{n\in\mathbb{N}}$ is a sequence of intervals strictly contained in $[0,2\pi]$ such that

$$t_0\in\mathcal{I}_n(t_0),\quad\forall n\in\mathbb{N}$$
$$\lim_{n\to+\infty}\mathcal{I}_n(t_0)=\{t_0\}$$

By definition of derivative:

$$g'(t_0) = \lim_{n \to +\infty} \frac{g(\tau'_{0,n}) - g(\tau_{0,n})}{\tau'_{0,n} - \tau_{0,n}} \tag{21}$$

Let denote $\psi_n(t_0)$ the incremental ratio of $g(t)$ relative to the interval$\mathcal{I}_n(t_0)$:

$$\psi_n(t_0) = \frac{g(\tau'_{0,n}) - g(\tau_{0,n})}{\tau'_{0,n} - \tau_{0,n}} \tag{22}$$

which is the n-th term of the sequence of elements of $\mathbb{R}$

$$\{\psi_n(t_0)\}_{n \in \mathbb{N}} : \psi_0(t_0), \psi_1(t_0), ..., \psi_n(t_0), ... \tag{23}$$

(21) becomes:

$$g'(t_0) = \lim_{n \to +\infty} \psi_n(t_0) \tag{24}$$

that is, the derivative of $g(t)$ in t_0, is the limit of the sequence (23). From (11):

$$\psi_n(t_0) = \sum_{k=0}^{+\infty} \phi_{k,n}(t_0) \tag{25}$$

where:

$$\phi_{k,n}(t_0) = \frac{f_k(\tau'_{0,n}) - f_k(\tau_{0,n})}{\tau'_{0,n} - \tau_{0,n}} \tag{26}$$

that is, the incremental ratio of $f_k(t)$ relative to the interval$\mathcal{I}_n(t_0)$. From (24):

$$g'(t_0) = \sum_{k=0}^{+\infty} \lim_{n \to +\infty} \phi_{k,n}(t_0)$$

But $\lim_{n \to +\infty} \phi_{k,n}(t_0) = f'_k(t_0)$

$$g'(t_0) = \sum_{k=0}^{+\infty} f'_k(t_0) \tag{27}$$

In other words, the derivative $g'(t_0)$ is the sum of the numerical series $\sum_k f'_k(t_0)$. From (10) we see that the sequence $\{f'_k(t_0)\}_{k \in \mathbb{N}}$ is indeterminate, and such is the aforementioned series:

$$\nexists \lim_{N \to +\infty} \sum_{k=0}^{N} f'_k(t_0) \Longrightarrow \nexists g'(t_0),$$

from which the assertion by virtue of the arbitrariness of $t_0 \in [0, 2\pi]$. ■

The non-differentiability of $g(t)$ at any point of $[0, 2\pi]$ and therefore of R, implies the absence of the tangent line at any point of the graph $\Gamma_g : y = g(t)$. Precisely, each $P \in \Gamma_g$ is a angular point. Using a language of images, we can assert that Γ_g is uan infinitely ≪angular≫ curve. In Figs. 4-5 we report the behavior of the partial sum of order $N = 100$ in different intervals.

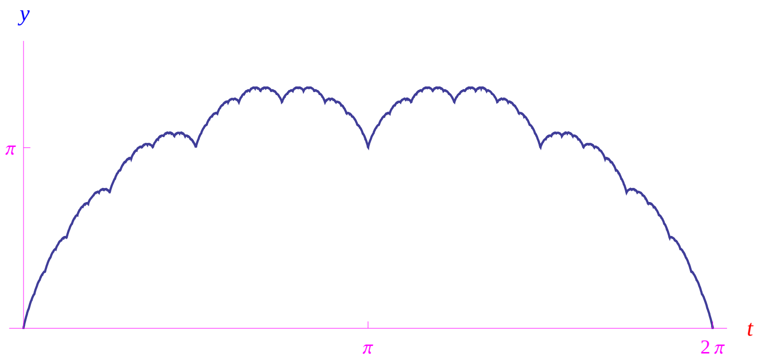

Figure 4: Trend of $\sum_{k=1}^{100} f_k\left(t\right)$ in $[0, 2\pi]$.

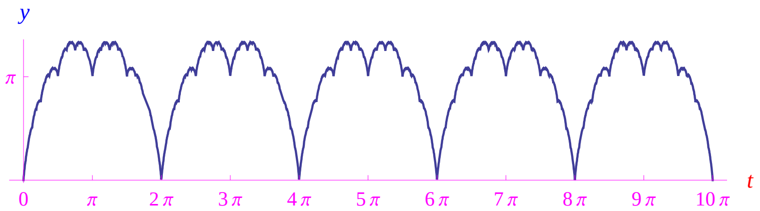

Figure 5: Trend of $\sum_{k=1}^{100} f_k\left(t\right)$ in $[0, 10\pi]$.

4 Fourier series

Theorem 2 *The Fourier series associated with the function (11), diverges at every point $t \in \mathbb{R}$.*

Dimostrazione. It is sufficient to show the divergence of the Fourier coefficient a_0:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} g\left(t\right) dt$$

Taking into account (11)and uniform convergence (which allows us to perform a series integration i.e. the series of integrals is the integral of the series):

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{+\infty} f_k\left(t\right) dt = \sum_{k=0}^{+\infty} M_k dt$$

where

$$M_k = \int_0^{2\pi} f_k\left(t\right) = \int_0^{\pi} t dt + \int_{\pi}^{2\pi} \left(2^{1-k} - t\right) dt = \pi^2 \left(2^{1-k} - 1\right)$$

So

$$a_0 = \frac{\pi^2}{2} \sum_{k=0}^{+\infty} \left(2^{1-k} - 1\right) = \frac{\pi^2}{2} \left(\sum_{k=0}^{+\infty} 2^{1-k} - \sum_{k=0}^{+\infty} 1 \right) = \frac{\pi^2}{2} \left(4 - (+\infty)\right) = -\infty$$

■

References

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[3] Ghizzetti A. *Lezioni di Analisi matematica, vol. 2*. Veschi.

[4] Fichera G., De Vito L., *Funzioni analitiche di una variabile complessa*. Edizioni Veschi.