# THE QUANTUM STATES OF THE GROSS-PITAEVSKII SUPERFLUID RING

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**Abstract**

The stationary states and corresponding energies of the superfluid ring are calculated from the Gross-Pitaevskii equation. Every stationary state forms the specific microstructure of the ring. The system of superfluid rings forming a cylinder is suggested as antenna for the detection of gravitational waves.

## 1 Introduction

Liquid helium is an example of a superfluid system in which quantum phenomena manifest themselves on a macroscopic scale. It is a boson condensate which according to Gross (1961) and Pitaevskii (1961) is possible to describe by the wave function $\Psi(x, t)$, which is normalized to the number of particles $N$ in the system. It obeys the nonlinear self-consistent Gross-Pitaevskii equation (GP) (Gross, 1963):

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi + g|\Psi|^2\Psi, \tag{1}$$

where $V$ is some external potential and the nonlinear term describes the collective interaction of particles with mass $m$ in the superfluid medium, $g$ is some constant equal to $10^{-40}$erg.cm$^3$.

The solution of the GP equation depends on the boundary conditions. The solution for rigid walls was given by Ginzburg and Pitaevskii who found the vortex solution also and who found the relation of GP equation to the phenomenological theory of superconductivity (Ginzburg et al., 1958; Wu, 1961 ). The further physical cosequences of the GP equation are presented in the articles by Gross (1963).

Anandan (1983) concentrated his attention on the detection of gravitational effects using the Josephson effect in superfluid helium. He investigated, for instance, the possibilities of detection of the Lense-Thirring field due to the rotation of the earth and of detection of gravitational waves. Though the efficiency of the detection of such effects was criticized by Cerdonio and Vitale (1984), such ideas nevertheless have great scientific value with regard to the future physics.

In the present article we investigate the quantum states of the superfluid ring. To our knowledge, this problem has not been considered in the physical literature. We determine the stationary states and corresponding energies from the GP equation. Every stationary state forms the specific microstructure of the ring. The change of the microstructure is accompanied with absorption, or, emission of energy. The system of superfluid rings can be considered as some antenna for the detection of gravitational waves.

## 2 The stationary states of the superfluid ring

Let us define the superfluid ring as a torus, or, annulus which arises geometrically in such a way that the circle of radius $r$ rotates around an axis in the plane of the circle at the distance $R$ of the circle. It is well known that the volume of the annulus is $V = 2\pi R\sigma$, where $\sigma$ is the cross section of the annulus ($\sigma = \pi r^2$).

In order to describe the motion of the superfluid inside the annulus it is suitable to transform the GP equation using the cylindrical coordinates. Then, instead of eq. (1) we have with $V = 0$:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\left[\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left(\varrho\frac{\partial\Psi}{\partial\varrho}\right) + \frac{1}{\varrho^2}\frac{\partial^2\Psi}{\partial\varphi^2} + \frac{\partial^2\Psi}{\partial z^2}\right] + g|\Psi|^2\Psi \tag{2}$$

Supposing that the cross section of the torus is sufficiently small, we get for the wave function inside the torus, if it is situated in the $xy$-plane:

$$\Psi(\varrho,\varphi,z) \approx \Psi(R,\varphi,0), \tag{3a}$$

$$\frac{\partial\Psi}{\partial\varrho} \approx 0; \quad \frac{\partial\Psi}{\partial z} \approx 0. \tag{3b}$$

Putting $x = \varrho\varphi$, we get instead of eq. (2) the following one-dimensional GP equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Psi'' + g|\Psi|^2\Psi, \tag{4}$$

where $\Psi'' = \partial^2\Psi/\partial x^2$, and the dimension of the constant $g$ in eq. (4) is the same as in eq. (1).

In order to get the equation for the stationary states , we use the customary ansatz

$$\psi = e^{(i/\hbar)Et}\varphi(x) \tag{5}$$

and after insertion of it into eq. (4) we get for the real $\varphi$

$$\varphi'' + k_0^2\varphi = \alpha\varphi^3, \tag{6}$$

where

$$k_0^2 = \frac{2m}{\hbar^2}E, \tag{7}$$

$$\alpha = \frac{2m}{\hbar^2}g. \tag{8}$$

The ring is a one-dimensional system with the Born-Kármán condition for $\varphi(x)$, or, for $\varphi(x+L)$, where $L$ is the perimeter of the ring. Quantization of energie follows from the Born-Kármán conditions.

The fundamental obstacle of the problem consists in the solution of the nonlinear equation (6). We will solve it approximately by iteration. The equation for the first iteration is as follows,

$$\varphi_1'' + k_0^2\varphi_1 = \alpha\varphi_0^3, \tag{9}$$

where $\varphi_0$ is the solution of the homogenous equation and it can be chosen as $\varphi_0 = C\sin(k_0x)$.

However,

$$\varphi_0^3 = \frac{C^3}{4}\left[3\sin(k_0x) - \sin(3k_0x)\right], \tag{10}$$

which means that eq. (9) has some fictitious resonances and it cannot be solved by the standard methods. In order to avoid these resonances we use the method, which is for instance described by Migdal (Migdal, 2018 ).

First, instead of eq. (6), we write its equivalent form:

$$\varphi'' + k^2\varphi = \alpha\varphi^3 + (k^2 - k_0^2)\varphi. \tag{11}$$

The equation for the first approximation is now of the form:

$$\varphi_1'' + k^2\varphi_1 = \alpha\frac{C^3}{4}\left[3\sin(kx) - \sin(3kx)\right] + (k^2 - k_0^2)C\sin(kx). \tag{12}$$

In order to avoid the fictitious resonances we put the coefficient before $\sin(kx)$ to zero, which gives

$$k^2 = \frac{3\alpha}{4} - k_0^2C^2. \tag{13}$$

The partial solution of eq. (12) with regard to eq. (13) is $\varphi_p = A\sin(3kx)$, where $A = (\alpha/32)k^{-2}C^3$.

The first approximation is now $\varphi_1 = \varphi_p + \varphi_0$, or,

$$\varphi_1 = C\sin(kx) + \frac{\alpha}{32}\frac{C^3}{k^2}\sin(3kx) \tag{14}$$

and from boundary $\varphi(x) = \varphi(x+L)$ the restriction for $k$ follows:

$$k = \frac{2\pi}{L}n; \quad n = 0, \pm1, \pm2, ...., \tag{15}$$

where n = 0 gives the trivial solution $\varphi_1 = \infty$, which has no physical meaning. The constant C must be determined from the normalization of the wave function to the number of particles $N$ in the ring.

$$\int \varphi^2(x,y,z)dV = N, \tag{16}$$

from which follows

$$\int_0^L \varphi^2(x)dx = 2\pi R\varrho_0 = N/\sigma, \tag{17}$$

where $\varrho_0$ is the density of particles of the homogeneous superfluid.

After insertion of eq. (14) into eq. (17) we get the algebraic equation for $C^2 = z$, or,

$$z - \frac{2N}{\sigma L} + \varepsilon z^3 = 0; \quad \varepsilon = \frac{\alpha^2}{32^2}\frac{1}{k^4}, \tag{18}$$

or,

$$z^3 + pz + q = 0. \tag{19}$$

Eq. (19) is the so-called Cardan equation and its discriminant is

$$D = (q/2)^2 + (p/3)^3. \tag{20}$$

In our case $D \geq 0$ and in this situation the Cardan equation has three different roots. One root is real and other two are complex conjugated. Explicitly

$$z = (z_1, z_2, z_3) = (z_1, -\frac{1}{2}z_1 + i\delta, -\frac{1}{2}z_1 - i\delta), \tag{21}$$

where $z_1$ and $\delta$ are real numbers which can be determined using the Cardan formulas. Because of the specific magnitudes of the coefficients in the cubic equation (18), the root $z_1$ can be determined proximately. In other words, instead of eq. (18) we write

$$f(z) + \varepsilon g(z) = 0, \tag{22}$$

where

$$\varepsilon = m^2 g^2 L^4 (8 \times 32^2 \hbar^4 \pi^4 n^4)^{-1} \ll 1; \quad n \neq 0. \tag{23}$$

As the root of the equation $f(z) = 0$ is $z_0 = (2n/\sigma L) = 2\varrho_0$, it is obvious that the root of eq. (22) is $z_0 + \eta$ where $\eta$ can be determined from the equation

$$f(z + \eta) + \varepsilon g(z + \eta) = 0, \tag{24}$$

by Taylor expansion. After some calculation we get

$$\eta = -\eta \frac{g(z_0)}{f'(z_0)} = -\eta z_0^3. \tag{25}$$

Then,

$$C^2 = z_0(1 - \varepsilon z_0^2) \tag{26}$$

The energies of the stationary states corresponding to $C^2 = z_1$ follow from eq. (13) in the following

$$E_n = \frac{2m}{\hbar^2}\left(\frac{2\pi}{L}n\right)^2 + \frac{3}{2}g\varrho_0 - \frac{3}{4}\frac{L^4}{32^2}\frac{g^3\varrho_0^3}{\hbar^2}\frac{m^2}{\pi^2 n^4}; \quad n \neq 0. \tag{27}$$

The energies of states corresponding to $C^2 = z_2$ and $C^2 = z_3$ are complex numbers and describe the stationary states. They are not physically meaningful because we consider from the beginning the solution with real $\varphi(x)$ which corresponds to real values of $C$. The discrete spectrum of energies corresponds to the microstructures of the superfluid inside the torus. The microstructures are described by eq. (14). The phase transition from one structure to the other energy is accompanied by the emission, or, absorption of energy according to the Bohr formula

$$\hbar\omega = |E_n - E_m|. \quad (28)$$

If we keep only the dominant terms in $E_n$, we get for $\hbar \approx 1.1 \times 10^{-34} Js, m \approx 4 \times 1.7 \times 10^{-27} kg, L \approx 10^{-2} m, n^2 - m^2 \approx 10^4$,

$$\omega \approx 32.0 \quad s^{-1}, \quad (29)$$

which can be considered as the estimation of frequencies generated by the superfluid quantum transitions. Let us remark, however, that $n^2 - m^2$ is not restricted, which enables one to get the radio-frequencies also.

The rigorous testing of the GP equation and the quantum mechanical states and energies of the superfluid inside the annulus can be probably realized by the absorption processes of light, or, laser light passing through the torus. Using this method the quantity $E = (3/2g)\varrho_0$ in eq. (27) can also be experimentally determined, which can serve for the precise determination of $g$, or, $\varrho_0$ as dependent on low temperature.

## 3 Discussion

Quantum behavior of the superfluid inside the annulus can be used practically in detecting gravitational waves, or, gravitons. Let us consider for this goal superfluid rings forming a cylinder. Then, when low-frequency gravitational radiation, or, gravitons with spectral distribution $P(\omega)$ from the binary system, (Pardy, 1983), are impinging on this cylinder, two quantum processes can occur. (1) Absorption of gravitons by the system in the basic state and (2) stimulated emission of photons by the system in the excited state.

It is easy to see that for the detection of gravitons the second process is suitable. The excited states of the superfluid are generally metastable which means that there is some probability of the spontaneous emission of energy. If such an excited system is in interaction with $g$-waves, or, gravitons the probability of emission of photons is greater than probability of spontaneous emission. The emission of photons is the signal of the existence of gravitational waves passing through the superfluid system. The sensitivity of the superfluid system to the $g$-waves does not follow from the GP equation, nevertheless, we can hope that it is sufficient for the detection of the $g$-waves, or, gravitons.

The article is some elementary modification of the original author article - "he quantum statest of the superfluid ring from the Gross-Pitaevskii equation" (Pardy, 1989).

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