A universal expression of prime numbers

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Abstract

We found that all prime numbers can be expressed in the form:

$$p = \sum_{t=0}^{k} r_t p_{t-1}!^p + 2$$

where $p_{t-1}!^p$ is the primorial of the (t-1)-th prime, and r_t are coefficients satisfying $0 \leq r_t \leq p_t - 1$.

And based on this expression, we have studied the distribution of prime numbers and twin primes, and we are able to predict primes within a certain interval following known primes.

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1. Introduction

In Section 2, we begin by partitioning all positive integers into intervals according to the Euclidean number $p_k!^p + 1$, specifically within the range $(p_{k-1}!^p + 1, p_k!^p + 1]$. We then compute the results of the expression within each interval and assemble them into a matrix $M(k)$. Subsequently, we form a sequence M with $M(k)$ as its elements. The proof that all prime numbers are contained within the sequence M employs both mathematical induction and proof by contradiction. Based on these results, we also present two minor applications.

In Section 3, we primarily investigate the distribution of prime numbers. We begin by examining the composition and properties of the matrix $M(k)$. Following this, we study the number of primes within $M(k)$. We then derive an expression for the $\pi(x)$ function within $M(k+1)$, given the knowledge of $M(0)$ through $M(k)$. Finally, we present a method to obtain all prime numbers within $M(k+1)$ based on the known information from $M(0)$ to $M(k)$.

In Section 4, we delve into the study of twin primes. Initially, we identify the origin of twin prime pairs based on the given expression. Subsequently, we investigate the properties of the matrix $M_2(k)$, which is composed of twin prime pairs within $M(k)$. We then provide a method to obtain the twin prime pairs in $M_2(k+1)$ given the information from $M(0)$ through $M(k)$. Finally, we propose a conjecture that is slightly stronger than the Twin Prime Conjecture.

2. Deduction and proof of expressions

2.1. *Definition.*

Let p_k denote the k-th prime number, for example: $p_1 = 2, p_2 = 3, p_3 = 5$.

Using the symbol $!^p$ to denote the primorial, $p_k!^p$ represents the primorial of the prime number p_k. Then,

$$p_k!^p = \prod_{t=1}^{k} p_t$$

Specify $p_0!^p = 1! = 1, p_{-1}!^p = 0! = 1$.

Let $f^p_{min}(n)$ denote the smallest prime factor of n, for example:

$$f^p_{min}(15) = 3$$

$$f^p_{min}(31) = 31$$

Therefore, if $f^p_{min}(n) = n$ and $n \geq 2$, then n is a prime number.

Let $f^p_{2min}(m, m+2)$ denote the smallest prime factor of twin numbers $(m, m+2)$, for example:

$$f^p_{2min}(23, 25) = 5$$

$$f^p_{2min}(41, 43) = 41$$

Therefore, if $f^p_{2min}(m, m+2) = m$ and $m \geq 3$, then $(m, m+2)$ is twin primes.

2.2. *Calculation Rules.*

For ease of expression, we will temporarily refer to the calculated numbers as "Mo numbers" denoted as m. The computed numbers m are divided based on the Euclidean numbers $p_k!^p + 1$ for $k \geq 1$, the k-th interval is defined as $(p_{k-1}!^p + 1, p_k!^p + 1]$ for $k \geq 1$, The matrix composed of the numbers m in each interval is denoted as $M(k)$, with the stipulation that $M(0) = [2]$. The sequence formed by the matrices $M(k)$ for $k \geq 0$ as elements is denoted as M. Thus,

$$M = \{M(0), M(1), M(2), \dots, M(k), \dots\}$$

Denote the computational base number for the matrix $M(k)$ as b_k,

$$b_k = p_{k-1}!^p$$

The computation of the matrix $M(k)$ involves using the Mo numbers from $M(0)$ to $M(k-1)$ whose smallest prime factors are greater than or equal to p_k. These numbers are collected as the computation factors for $M(k)$ and represented as a row vector $F(k)$. Given the row vector $F(k)$, the computational base number b_k, and the constraint $r \in Z$ with $r \in [1, p_k - 1]$, the r-th row of the matrix $M(k)$ is computed using the expression:

$$M(k, r) = r \times b_k + F(k), r \in Z, r \in [1, p_k - 1] \quad (2)$$

For example:

(1) For $k = 1$:
- The interval is $(p_0!^p + 1, p_1!^p + 1] = (1 + 1, 2 + 1] = (2, 3]$
- $F(1) = [2], b_1 = p_0!^p = 1$
- $M(1) = [1 \times b_1] + [F(1)] = [1] + [2] = [3]$

(2) For $k = 2$:
- The interval is $(p_1!^p + 1, p_2!^p + 1] = (2 + 1, 6 + 1] = (3, 7]$
- $F(2) = [3], b_2 = p_1!^p = 2$

- $M(2) = \begin{bmatrix} 1 \times b_2 \\ 2 \times b_2 \end{bmatrix} + \begin{bmatrix} F(2) \\ F(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

(3) For $k = 3$:
- The interval is $(p_2!^p + 1, p_3!^p + 1] = (6 + 1, 30 + 1] = (7, 31]$
- $F(3) = \begin{bmatrix} 5 & 7 \end{bmatrix}, b_3 = p_2!^p = 6$
- $M(3) = \begin{bmatrix} 1 \times b_3 \\ 2 \times b_3 \\ 3 \times b_3 \\ 4 \times b_3 \end{bmatrix} + \begin{bmatrix} F(3) \\ F(3) \\ F(3) \\ F(3) \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 18 \\ 24 \end{bmatrix} + \begin{bmatrix} 5 & 7 \\ 5 & 7 \\ 5 & 7 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 11 & 13 \\ 17 & 19 \\ 23 & 25 \\ 29 & 31 \end{bmatrix}$

(4) For $k = 4$:
- The interval is $(p_3!^p + 1, p_4!^p + 1] = (30 + 1, 210 + 1] = (31, 211]$
- $F(4) = \begin{bmatrix} 7 & 11 & 13 & 17 & 19 & 23 & 29 & 31 \end{bmatrix}, b_4 = p_3!^p = 30$
- Thus,

$$M(4) = \begin{bmatrix} 1 \times b_4 \\ 2 \times b_4 \\ 3 \times b_4 \\ 4 \times b_4 \\ 5 \times b_4 \\ 6 \times b_4 \end{bmatrix} + \begin{bmatrix} F(4) \\ F(4) \\ F(4) \\ F(4) \\ F(4) \\ F(4) \end{bmatrix}$$
$$= \begin{bmatrix} 37 & 41 & 43 & 47 & 49 & 53 & 59 & 61 \\ 67 & 71 & 73 & 77 & 79 & 83 & 89 & 91 \\ 97 & 101 & 103 & 107 & 109 & 113 & 119 & 121 \\ 127 & 131 & 133 & 137 & 139 & 143 & 149 & 151 \\ 157 & 161 & 163 & 167 & 169 & 173 & 178 & 181 \\ 187 & 191 & 193 & 197 & 199 & 203 & 209 & 211 \end{bmatrix}$$

Because $f^p_{min}(25) = 5 < p_4 = 7$, the number 25 does not satisfy the condition $f^p_{min}(m) \geq p_k$. Therefore, when computing $F(4)$, 25 will not be included in $F(4)$.

Therefore,

$$M(k) = \begin{bmatrix} 1 \times b_k \\ 2 \times b_k \\ \vdots \\ (p_k - 1) \times b_k \end{bmatrix} + \begin{bmatrix} F(k) \\ F(k) \\ \vdots \\ F(k) \end{bmatrix} = \begin{bmatrix} 1 \times p_{k-1}!^p \\ 2 \times p_{k-1}!^p \\ \vdots \\ (p_k - 1) \times p_{k-1}!^p \end{bmatrix} + \begin{bmatrix} F(k) \\ F(k) \\ \vdots \\ F(k) \end{bmatrix} \tag{3}$$

The general expression for $M(k)$, when $F(k)$ is iteratively computed down to $F(1)$, is as follows:

$$M(k) = \begin{bmatrix} b_1 & \dots & 1 \times b_k \\ b_1 & \dots & 2 \times b_k \\ \vdots & \ddots & \vdots \\ b_1 & \dots & (p_k - 1) \times b_k \end{bmatrix} \times \begin{bmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,C_k} \\ r_{2,1} & r_{2,2} & \dots & r_{2,C_k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k-1,1} & r_{k-1,2} & \dots & r_{k-1,C_k} \\ r_{k,1} & r_{k,2} & \dots & r_{k,C_k} \end{bmatrix} + \begin{bmatrix} F(1) \\ F(1) \\ \vdots \\ F(1) \end{bmatrix}$$

Substituting $F(1) = [2]$, the expression can be simplified to:

$$M(k) = \begin{bmatrix} b_1 & \dots & 1 \times b_k \\ b_1 & \dots & 2 \times b_k \\ \vdots & \ddots & \vdots \\ b_1 & \dots & (p_k - 1) \times b_k \end{bmatrix} \times \begin{bmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,C_k} \\ r_{2,1} & r_{2,2} & \dots & r_{2,C_k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k-1,1} & r_{k-1,2} & \dots & r_{k-1,C_k} \\ r_{k,1} & r_{k,2} & \dots & r_{k,C_k} \end{bmatrix} + 2 \tag{4}$$

Constraints:

- $k \geq 1$
- The coefficients $r_{i,j}$ are constrained as follows:
  - $r_{1,j} = 1$ for all j.
  - $r_{2,j} \in [1, 2]$ for all j.
  - $r_{k,j} = 1$ for all j.
  - For $i \in [3, k-1]$,$r_{i,j} \in [0, p_{i-1}]$, for all j.
  - $C_k = \prod_{t=1}^{k-1}(p_t - 1)$ is the number of columns in matrix $M(k)$.

For example:

$$\begin{aligned} M(1) &= \begin{bmatrix} 1 \times b_1 \end{bmatrix} \times \begin{bmatrix} r_{1,1} \end{bmatrix} + 2 \\ &= \begin{bmatrix} 1 \times 1 \end{bmatrix} \times \begin{bmatrix} 1 \end{bmatrix} + 2 \\ &= \begin{bmatrix} 3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} M(2) &= \begin{bmatrix} b_1 & 1 \times b_2 \\ b_1 & 2 \times b_2 \end{bmatrix} \times \begin{bmatrix} r_{1,1} \\ r_{2,1} \end{bmatrix} + 2 \\ &= \begin{bmatrix} 1 & 1 \times 2 \\ 1 & 2 \times 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \\ &= \begin{bmatrix} 5 \\ 7 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} M(3) &= \begin{bmatrix} b_1 & b_2 & 1 \times b_3 \\ b_1 & b_2 & 2 \times b_3 \\ b_1 & b_2 & 3 \times b_3 \\ b_1 & b_2 & 4 \times b_3 \end{bmatrix} \times \begin{bmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \\ r_{3,1} & r_{3,2} \end{bmatrix} + 2 \\ &= \begin{bmatrix} 1 & 2 & 1 \times 6 \\ 1 & 2 & 2 \times 6 \\ 1 & 2 & 3 \times 6 \\ 1 & 2 & 4 \times 6 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} + 2 \\ &= \begin{bmatrix} 11 & 13 \\ 17 & 19 \\ 23 & 25 \\ 29 & 31 \end{bmatrix} \end{aligned}$$

$$M(4) = \begin{bmatrix} b_1 & b_2 & b_3 & 1 \times b_4 \\ b_1 & b_2 & b_3 & 2 \times b_4 \\ b_1 & b_2 & b_3 & 3 \times b_4 \\ b_1 & b_2 & b_3 & 4 \times b_4 \\ b_1 & b_2 & b_3 & 5 \times b_4 \\ b_1 & b_2 & b_3 & 6 \times b_4 \end{bmatrix} \times \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} & r_{1,5} & r_{1,6} & r_{1,7} & r_{1,8} \\ r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} & r_{2,5} & r_{2,6} & r_{2,7} & r_{2,8} \\ r_{3,1} & r_{3,2} & r_{3,3} & r_{3,4} & r_{3,5} & r_{3,6} & r_{3,7} & r_{3,8} \\ r_{4,1} & r_{4,2} & r_{4,3} & r_{4,4} & r_{4,5} & r_{4,6} & r_{4,7} & r_{4,8} \end{bmatrix}$$

$$+ 2$$

$$= \begin{bmatrix} 1 & 2 & 6 & 1 \times 30 \\ 1 & 2 & 6 & 2 \times 30 \\ 1 & 2 & 6 & 3 \times 30 \\ 1 & 2 & 6 & 4 \times 30 \\ 1 & 2 & 6 & 5 \times 30 \\ 1 & 2 & 6 & 6 \times 30 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 2 & 3 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + 2$$

$$= \begin{bmatrix} 37 & 41 & 43 & 47 & 49 & 53 & 59 & 61 \\ 67 & 71 & 73 & 77 & 79 & 83 & 89 & 91 \\ 97 & 101 & 103 & 107 & 109 & 113 & 119 & 121 \\ 127 & 131 & 133 & 137 & 139 & 143 & 149 & 151 \\ 157 & 161 & 163 & 167 & 169 & 173 & 178 & 181 \\ 187 & 191 & 193 & 197 & 199 & 203 & 209 & 211 \end{bmatrix}$$

We refer to the matrix

$$B(k) = \begin{bmatrix} b_1 & \dots & 1 \times b_k \\ b_1 & \dots & 2 \times b_k \\ \vdots & \ddots & \vdots \\ b_1 & \dots & (p_k - 1) \times b_k \end{bmatrix} \tag{5}$$

as the base matrix, denoted as $B(k)$.

We refer to the matrix

$$R(k) = \begin{bmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,C_k} \\ r_{2,1} & r_{2,2} & \dots & r_{2,C_k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k-1,1} & r_{k-1,2} & \dots & r_{k-1,C_k} \\ r_{k,1} & r_{k,2} & \dots & r_{k,C_k} \end{bmatrix} \tag{6}$$

as the coefficient matrix, denoted as $R(k)$.

Therefore, the expression for $M(k)$ can be simplified as:

$$M(k) = B(k) \times R(k) + 2 \tag{7}$$

In fact, the last column of the base matrix $B(k)$ consists of the coefficients of the base b_k, so the last row of the coefficient matrix $R(k)$ satisfies $r_{k,j} = 1$ for all j.

Therefore, the Mo number $m(k,i,j)$ in $M(k)$ can be expressed as:

$$m(k,i,j) = \sum_{i=1}^{k} r_{i,j} b_i + 2 = \sum_{i=1}^{k} r_{i,j} p_{i-1}!^p + 2 \tag{8}$$

where:

- When $i = k, r_{k,j} \in [1, p_i - 1]$.
- Other conditions are consistent with those defined in the expression for $M(k)$.

For example:

$$\begin{aligned} m(4,1,1) &= \sum_{i=1}^{4} r_{i,1} p_{i-1}!^p + 2 \\ &= 1 \times p_3!^p + 0 \times p_2!^p + 2 \times p_1!^p + 1 \times p_0!^p + 2 \\ &= 1 \times 30 + 0 \times 6 + 2 \times 2 + 1 \times 1 + 2 \\ &= 37 \end{aligned}$$

2.3. *Proof*: $P \subseteq M$.

The prime numbers p are divided based on the Euclidean numbers $p_k!^p + 1$ for $k \geq 1$, the k-th interval is defined as $(p_{k-1}!^p + 1, p_k!^p + 1]$ for $k \geq 1$, The set composed of the numbers p in each interval is denoted as $P(k)$, with the stipulation that $P(0) = 2$. The sequence formed by the sets $P(k)$ for $k \geq 0$ as elements is denoted as P. Thus,

$$P = \{P(0), P(1), P(2), \ldots, P(k), \ldots\}$$

Proof: $P \subseteq M$.

Base Cases:

(1) For $k = 0$:
  - $P(0) = [2] = M(0)$, so $P(0) \subseteq M(0)$.

(2) For $k = 1$:
  - $P(1) = [3] = M(1)$, so $P(1) \subseteq M(1)$.

(3) For $k = 2$:
  - $P(2) = \begin{bmatrix} 5 \\ 7 \end{bmatrix} = M(2)$, so $P(2) \subseteq M(2)$.

(4) For $k = 3$:
  - $P(3) = \begin{bmatrix} 11 & 13 \\ 17 & 19 \\ 23 & \\ 29 & 31 \end{bmatrix}$, $M(3) = \begin{bmatrix} 11 & 13 \\ 17 & 19 \\ 23 & 25 \\ 29 & 31 \end{bmatrix}$, so $P(3) \subseteq M(3)$.

Inductive Hypothesis:

Assume that for $k = n$, where $n \geq 3$, $P(n) \subseteq M(n)$.

Inductive Step:

We need to prove that for $k = n+1$, $P(n+1) \subseteq M(n+1)$.

Assume for contradiction that there exists a prime $p \in P(n+1)$ such that $p \notin M(n+1)$.

- Let $t = p \mod p_n!^p$. Then, $t < p_n!^p$.
- Since p is a prime and $p > 2$, p is odd.
- Since $p_n!^p$ is even, t must be odd, and $t \in [1, p_n!^p)$.

Case 1: $t = 1$

- Then, $p = r \times p_n!^p + t = (r-1) \times p_n!^p + p_n!^p + 1$, where $2 \leq r \leq p_{n+1} - 1$.
- Since $p_n!^p + 1 \in M(n)$ (a Euclidean number), $p \in M(n+1)$, which contradicts $p \notin M(n+1)$.

Case 2: $t \in [3, p_n!^p)$ and t is odd

- Subcase 2.1: If $f^p_{min}(t) \leq p_n$, then $p \notin P$, which contradicts $p \in P(n+1)$.
- Subcase 2.2: If $f^p_{min}(t) \geq p_{n+1}$:
  - Subcase 2.2.1: If $t \in M$, then $p \in M$, which contradicts $p \notin M(n+1)$.
  - Subcase 2.2.2: If $t \notin M$, let $t \in (p_{i-1}!^p + 1, p_i!^p + 1]$, and define $t_1 = t \mod p_{i-1}!^p$. Then:$p = r \times p_n!^p + t = r \times p_n!^p + r_1 \times p_{i-1}!^p + t_1$
    - If $t_1 \in M$, then $t \in M$, which contradicts $t \notin M$.
    - If $t_1 \notin M$, repeat the process by defining $t_2 = t_1 \mod p_{i_1-1}!^p$, where $t_1 \in (p_{i_1-1}!^p + 1, p_{i_1}!^p + 1]$.
    - Continue this process until $t_j \in (p_0!^p + 1, p_2!^p + 1] = (2, 7]$.
    - Since $t_j \in M$, it follows that $t_{j-1} \in M$, which contradicts $t_{j-1} \notin M$.

Conclusion:

- For $k = n+1, p \in M(n+1)$.
- Therefore, $P \subseteq M$.

Q.E.D.

Therefore, The set of prime numbers is a subset of the Mo numbers. the elements p in the set of prime numbers $P(k)$ can also be expressed in the following form:

$$p = \sum_{t=0}^{k} r_t b_t + 2 = \sum_{t=0}^{k} r_t p_{t-1}!^p + 2$$

The set of composite numbers in matrix $M(k)$ is denoted as $M^{'}(k)$. Then

$$M(k) = P(k) \cup M^{'}(k)$$

Actually, the set $M^{'}(k)$ consists of all composite numbers in the interval $(p_{k-1}!^p + 1, p_k!^p + 1]$ whose smallest prime factor is greater than or equal to p_k. The proof method is similar to that used in previous sections, and we will not repeat it here.

2.4. *Two minor applications.*

Given the known sequence M,this will provide us with two minor applications:

1. Roughly Determining Whether a Number is Prime:

Determination Method:

If a number can be expressed in the form $\sum_{t=1}^{k} r_t b_t + 2$ and satisfies $r_1 = r_k = 1, r_2 \in [1, 2]$, For $t \in [3, k-1], r_t \in [0, p_t - 1]$, and $f^p_{min}(\sum_{t=1}^{i} r_t b_t + 2) \geq p_{i+1}$ for each $i \in [3, k-1]$, it could be a prime number or a composite number with a smallest prime factor greater than or equal to p_k. Otherwise, the number is not a Mo number and is definitely not a prime number.

For example:

(1) For the number 797:
  - The expression is $797 = 3 \times 210 + 5 \times 30 + 2 \times 6 + 1 \times 2 + 1 \times 1 + 2$.
  - According to the rule, 797 could be a prime number or a composite number with a smallest prime factor greater than or equal to $p_5 = 11$.

(2) For the number 763:
  - The expression is $763 = 3 \times 210 + 133$.
  - Since $f^p_{min}(133) = 7 \leq p_4 = 7$, 763 is not a Mo number.
  - Therefore, 763 is definitely not a prime number.

2. Factorization of Large Numbers:

Factorization Method:

(1) Step 1: Determine the Interval and Matrix $M(k)$:
  - Identify the interval $(p_{k-1}!^p+1, p_k!^p+1]$ to which the large number n belongs.
  - Check whether n is a Mo number in the corresponding matrix $M(k)$.

(2) Step 2: Find the Smallest Prime Factor $f^p_{min}(n)$:
  - If $n \notin M(k)$, then $f^p_{min}(n) \in [3, p_{k-1}]$.
  - If $n \in M(k)$, extract $f^p_{min}(n)$ directly from $M(k)$.

(3) Step 3: Factorize n:
  - Compute $n/f^p_{min}(n)$.
  - Repeat the above steps for the quotient until complete factorization is achieved.

Examples:

(1) Factorization of 791:
- $791 \in (211, 2311]$, corresponding to $M(5)$.
- Since $791 \notin M(5)$, $f^p_{min}(791) \in [3, 7]$.
- We find $f^p_{min}(791) = 7$, so $791 = 7 \times 113$.
- Since $113 \in M(4)$ and is a prime number, the factorization of 791 is 7×113.

(2) Factorization of 2007835897:
- $2007835897 \in (223092871, 6469693231]$, corresponding to $M(10)$.
- Since $2007835897 \in M(10)$, we obtain $f^p_{min}(2007835897) = 1013$, so $2007835897 = 1013 \times 1982069$.
- Since $1982069 \in (510511, 9699691]$, corresponding to $M(8)$, and $1982069 \in M(8)$ is a prime number, the factorization of 2007835897 is 1013×1982069.

(3) Factorization of 6246600469:
- $6246600469 \in (223092871, 6469693231]$, corresponding to $M(10)$.
- Since $6246600469 \in M(10)$, we obtain $f^p_{min}(6246600469) = 41$, so $6246600469 = 41 \times 152356109$.
- Since $152356109 \in (9699691, 223092871]$, corresponding to $M(9)$, and $152356109 \in M(9)$, we obtain $f^p_{min}(152356109) = 2621$, so $152356109 = 2621 \times 58129$.
- Since $58129 \in (30031, 510511]$, corresponding to $M(7)$, and $58129 \in M(7)$ is a prime number, the factorization of 6246600469 is $41 \times 2621 \times 58129$.

3. **Distribution of Prime Numbers**

3.1. *Composition and Properties of the Matrix $M(k)$.*

We define the following notations:
- $\lfloor x \rfloor_p$ as the largest prime number less than or equal to x.
- $\lceil x \rceil_p$ as the smallest prime number greater than or equal to x.

We can easily know that the smallest prime factor of the elements in $M'(k)$ is between p_k and $\lfloor \sqrt{p_k!^p + 1} \rfloor_p$.

According to the computational rules, we can easily see that the matrix $M(k)$ has the following properties:

(1) Column-wise Arithmetic Matrix:
- The matrix $M(k)$ is a column-wise arithmetic matrix with a common difference of $b_k = p_{k-1}!^p$.
- This means that the elements in each column increase row by row, and the difference between adjacent elements is $p_{k-1}!^p$.

(2) Size of the Matrix:
   - For $k \geq 2$, the matrix $M(k)$ has p_{k-1} rows and $\prod_{t=1}^{k-1}(p_t - 1)$ columns.
   - Therefore, $M(k)$ has a total of $\prod_{t=1}^{k}(p_t - 1)$ elements.

(3) Boundaries of the Matrix:
   - The smallest element in $M(k)$ is:

$$m(k, 1, 1) = p_{k-1}!^p + p_k$$

   - The largest element in $M(k)$ is:

$$m(k, p_k - 1, \prod_{t=1}^{k-1}(p_t - 1)) = p_k!^p + 1$$

   - The smallest element in the r-th row of $M(k)$ is:

$$m(k, r, 1) = r \times p_{k-1}!^p + p_k$$

   - The largest element in the r-th row of $M(k)$ is:

$$m(k, r, \prod_{t=1}^{k-1}(p_t - 1)) = (r + 1) \times p_{k-1}!^p + 1$$

According to the computational rules, we can easily determine that there are $p_k - 1$ rows, and we will not provide further deductive proof here. Before discussing the number of columns in the matrix $M(k)$, let us first examine the smallest prime factor in the sequence M.

Let $F^m_{min}(p_k)$ denote the set of Mo numbers in the sequence M whose smallest prime factor is p_k. Then, $F^m_{min}(p_k)$ has the following properties:

- The smallest element in $F^m_{min}(p_k)$ is p_k, which is also the only prime number in the set.
- The smallest composite number in $F^m_{min}(p_k)$ is p_k^2, which is also the second smallest element.
- The largest element in $F^m_{min}(p_k)$ is $(p_{k-1}!^p - 1) \times p_k$, which is located in the $(p_k - 1)$-th row and the $(\prod_{t=1}^{k-1}(p_t - 1) - 2)$-th column of the matrix $M(k)$.
- The elements in $F^m_{min}(p_k)$ belong to the interval $[p_k, p_k!^p - p_k]$.

Relationship Between the Number of Columns C_k in Matrix $M(k)$ for $k \geq 2$, the Number of Elements $M(k-1)$, and the Number of Mo Numbers with Smallest Prime Factor p_k:

The number of columns C_k in matrix $M(k)$ is given by:

$$C_k = \prod_{t=1}^{k-1}(p_t - 1) \qquad (9)$$

This value is equal to:

(1) The number of elements in matrix $M(k-1)$, i.e., $|M(k-1)|$.
(2) The number of Mo numbers with the smallest prime factor p_k, i.e., $|F^m_{min}(p_k)|$.

Thus:

$$C_k = \prod_{t=1}^{k-1}(p_t - 1) = |M(k-1)| = |F^m_{min}(p_k)| \tag{10}$$

Since the number of rows in matrix $M(k-1)$ is $p_{k-1} - 1$, we only need to prove:

$$C_k = \prod_{t=1}^{k-1}(p_t - 1) = |F^m_{min}(p_k)|$$

Proof:

Base Cases:

- For $k = 2$:

$$C_2 = 1 = \prod_{t=1}^{1}(p_t - 1) = |F^m_{min}(p_2)|$$

- For $k = 3$:

$$C_3 = 2 = \prod_{t=1}^{2}(p_t - 1) = |F^m_{min}(p_3)|$$

- For $k = 4$:

$$C_4 = 8 = \prod_{t=1}^{3}(p_t - 1) = |F^m_{min}(p_4)|$$

Inductive Hypothesis:

Assume that for $k = n$, where $n \geq 2$, the following holds:

$$C_n = \prod_{t=1}^{n-1}(p_t - 1) = |F^m_{min}(p_n)|$$

Inductive Step:

We need to prove that for $k = n + 1$, the following holds:

$$C_{n+1} = \prod_{t=1}^{n}(p_t - 1) = |F^m_{min}(p_{n+1})|$$

- Let $|F^m_{min}(n, p_n)|$ denote the number of composite numbers in $M(n)$ whose smallest prime factor is p_n.
- Let $|F^m_{min}([p_{n+1}, p_n!^p + 1], \geq p_{n+1})|$ denote the number of Mo numbers in the interval $[p_{n+1}, p_n!^p + 1]$ whose smallest prime factor is greater than or equal to p_{n+1}.

- According to the computational rules of the matrix $M(n+1)$:

$$
\begin{aligned}
&C_{n+1} = |F^m_{min}([p_{n+1}, p_n!^p + 1], \geq p_{n+1})| \\
&= |M(n)| - |F^m_{min}(n, p_n)| + C_n - (|F^m_{min}(p_n)| - |F^m_{min}(n, p_n)|) \\
&= |M(n)| + C_n - |F^m_{min}(p_n)| \\
&= |M(n)| + C_n - C_n \\
&= (p_n - 1) \prod_{t=1}^{n-1} (p_t - 1) \\
&= \prod_{t=1}^{n} (p_t - 1) \\
&|F^m_{min}(p_{n+1})| = C_{n+1} + |M(n+1)| - C_{n+2} \\
&= C_{n+1} + |M(n+1)| - |M(n+1)| \\
&= C_{n+1}
\end{aligned}
$$

Conclusion:

- Thus, for $k = n + 1$:

$$C_{n+1} = \prod_{t=1}^{n} (p_t - 1) = |F^m_{min}(p_{n+1})|$$

- By induction, the equation holds true for all $n \geq 2$.

Q.E.D.

The table below provides statistics on the number of prime numbers and Mo numbers in the matrix $M(k)$. From the calculations in the table, it is evident that Mo numbers constitute approximately 15% of all positive integers. Moreover, as k increases, this proportion further diminishes. This reduction narrows the scope of our research on prime numbers, effectively confining our study to the investigation of Mo numbers.

$M(k)$	*Base number* $b_k = p_{k-1}!^p$	*Interval* $(p_{k-1}!^p + 1, p_k!^p + 1]$	*The number of prime numbers*	*The number of Mo numbers*
$M(0)$	1	(1,2]	1	1
$M(1)$	1	(2,3]	1	1
$M(2)$	2	(3,7]	2	2
$M(3)$	6	(7,31]	7	8
$M(4)$	30	(31,211]	36	48
$M(5)$	210	(211,2311]	297	480
$M(6)$	2310	(2311,30031]	2904	5760
$M(7)$	30030	(30031,510511]	39083	92160
$M(8)$	510510	(510511,9699691]	603698	1658880
$M(9)$	9699690	(9699691,223092871]	11637502	36495360
$M(10)$	223092870	(223092871,6469693231]	288086265	1021870080

Table 1. Statistical of prime and Mo numbers in matrix $M(k)$

3.2. *The Number of Prime Numbers in the Matrix $M(k)$.*

This method calculates the number of prime numbers in $M(k)$ based on the known matrices $M(0)$ through $M(k-1)$.

Let $p_{k+s} = \lfloor \sqrt{p_k!^p + 1} \rfloor_p$. Then, the set of smallest prime factors of the elements in $M^{'}(k)$ is:

$$\{p_k, p_{k+1}, \ldots, p_{k+s-1}, p_{k+s}\}, k, s \in Z, s \geq 0$$

Let $N(p_k \mid M^{'}(k))$, where $k \geq 3$, denote the number of Mo numbers in the set $M^{'}(k)$ whose smallest prime factor is p_k.

Examples:

(1) For $M^{'}(3) = \{25\}$, there is only one element, $N(5 \mid M^{'}(3)) = 1$.

(2) For $M^{'}(4) = \{49, 77, 91, 119, 121, 133, 143, 161, 169, 187, 203, 209\}$:
- $N(7 \mid M^{'}(4)) = 7$ (elements: 49, 77, 91, 119, 133, 161, 203).
- $N(11 \mid M^{'}(4)) = 4$ (elements: 121, 143, 187, 209).
- $N(13 \mid M^{'}(4)) = 1$ (element: 169).

Let $N(p_k \mid (a, b])$ denote the number of Mo numbers in the interval $(a, b]$ whose smallest prime factor is greater than p_k. Then:

$$N(P_k \mid M^{'}(k)) = \sum_{i=1}^{j-1} N(p_k \mid (max(\frac{p_{k-1}!^p + 1}{p_k^i}, p_k), \frac{p_k!^p + 1}{p_k^i}]) + \delta_{p_k} \tag{11}$$

where:
- $j = \lfloor \log_{p_k} (p_k!^p + 1) \rfloor$

- $\delta_{p_k} = \begin{cases} 1 & \text{if } p_k^j \in (p_{k-1}!^p + 1, p_k!^p + 1] \\ 0 & \text{if } p_k^j \notin (p_{k-1}!^p + 1, p_k!^p + 1] \end{cases}$

Example 1: $N(13 \mid M^{'}(6))$

$$N(13 \mid M^{'}(6)) = \sum_{i=1}^{3} N(13 \mid (max(\frac{11!^p + 1}{13^i}, 13), \frac{13!^p + 1}{13^i}]) + \delta_{13}$$

- For $i = 1$:

$$N(13 \mid (max(\frac{2311}{13}, 13), \frac{30031}{13}]) = 408$$

- For $i = 2$:

$$N(13 \mid (max(\frac{2311}{13^2}, 13), \frac{30031}{13^2}]) = 34$$

- For $i = 3$:

$$N(13 \mid (max(\frac{2311}{13^3}, 13), \frac{30031}{13^3}]) = 0$$

- $\delta_{13} = 1$ (since $13^4 = 28561 \in (2311, 30031]$).

Thus:

$$N(13 \mid M^{'}(6)) = 408 + 34 + 0 + 1 = 443$$

Example 2: $N(41 \mid M^{'}(6))$

$$N(41 \mid M^{'}(6)) = \sum_{i=1}^{1} N(41 \mid (max(\frac{11!^p + 1}{41^i}, 41), \frac{13!^p + 1}{41^i}]) + \delta_{41}$$

- For $i = 1$:

$$N(41 \mid (max(\frac{2311}{41}, 41), \frac{30031}{41}]) = 113$$

- $\delta_{41} = 0$ (since $41^2 = 1681 \notin (2311, 30031]$).

Thus:

$$N(41 \mid M^{'}(6)) = 113 + 0 = 113$$

Additionally, since:

$$\left|M^{'}(k)\right| = \sum_{n=0}^{s} N(p_{k+n} \mid M^{'}(k)) \tag{12}$$

The number of prime numbers in the matrix M(k) is:

$$|P(k)| = |M(k)| - \left|M^{'}(k)\right| \tag{13}$$

Substituting the expressions for $|M(k)|$ and $\left|M^{'}(k)\right|$, we have:

(14)

$$|P(k)| = \prod_{t=1}^{k}(p_t - 1) - \sum_{n=0}^{s}(\sum_{i=1}^{j-1} N(p_{k+n} \mid (max(\frac{p_{k-1}!^p + 1}{p_{k+n}^i}, p_{k+n}), \frac{p_k!^p + 1}{p_{k+n}^i}]) + \delta_{p_{k+n}})$$

where:

- $j = \left\lfloor \log_{p_{k+n}} (p_k!^p + 1) \right\rfloor$
- $\delta_{p_{k+n}} = \begin{cases} 1 & \text{if } p_{k+n}^j \in (p_{k-1}!^p + 1, p_k!^p + 1] \\ 0 & \text{if } p_{k+n}^j \notin (p_{k-1}!^p + 1, p_k!^p + 1] \end{cases}$
- s is the value in the expression $p_{k+s} = \left\lfloor \sqrt{p_k!^p + 1} \right\rfloor_p$.

3.3. *The prime-counting function* $\pi(x)$.

Let x belong to the r-th row of the interval $(p_{k-1}!^p + 1, p_k!^p + 1]$, and let $p_{k+s} = \lfloor \sqrt{x} \rfloor_p$.

Let $N(p_k \mid M'(k, x))$ denote the number of Mo numbers in $M'(k)$ that do not exceed x and have p_k as their smallest prime factor. Then:

$$N(p_k \mid M'(k, x)) = \sum_{i=1}^{j-1} N(p_k \mid (max(\frac{p_{k-1}!^p + 1}{p_k^i}, p_k), \frac{x}{p_k^i}]) + \delta_{p_k} \tag{15}$$

where:

- $j = \lfloor \log_{p_k} x \rfloor$
- $\delta_{p_k} = \begin{cases} 1 & \text{if } p_k^j \in (p_{k-1}!^p + 1, x] \\ 0 & \text{if } p_k^j \notin (p_{k-1}!^p + 1, x] \end{cases}$

Let $\left| M'(k, x) \right|$ denote the number of Mo numbers in $M'(k)$ that do not exceed x. Then:

$$\left| M'(k, x) \right| = \sum_{n=0}^{s} (\sum_{i=1}^{j-1} N(p_{k+n} \mid (max(\frac{p_{k-1}!^p + 1}{p_{k+n}^i}, p_{k+n}), \frac{x}{p_{k+n}^i}]) + \delta_{p_{k+n}}) \tag{16}$$

Let $|M(k, x)|$ denote the number of Mo numbers in $M(k)$ that do not exceed x. Then:

$$|M(k, x)| = (r - 1) \prod_{i=1}^{k-1} (p_i - 1) + N(p_{k-1} \mid (p_{k-1}, x - r p_{k-1}!^p]) \tag{17}$$

Let $|P(k, x)|$ denote the number of prime numbers in $P(k)$ that do not exceed x. Then:

$$|P(k, x)| = |M(k, x)| - \left| M'(k, x) \right| \tag{18}$$

Therefore, the prime-counting function $\pi(x)$ is given by:

$$\pi(x) = \sum_{t=0}^{k-1} |P(t)| + |P(k, x)| = \sum_{t=0}^{k-1} |P(t)| + |M(k, x)| - \left| M'(k, x) \right| \tag{19}$$

Example:

Let $x = 139$. We know that it belongs to the 4th row of the matrix $M(4)$, so $k = 4$ and $r = 4$.

- $p_{k+s} = \left\lfloor \sqrt{139} \right\rfloor_p = 11 = p_5$, so $s \in \{0, 1\}$, corresponding to the smallest prime factors $\{7, 11\}$.
- For $s = 0$, the smallest prime factor is $p_4 = 7$, and $j = \lfloor \log_7 139 \rfloor = 2$. Since $7^2 \in (31, 139]$, $\delta_7 = 1$.
- For $s = 1$, the smallest prime factor is $p_5 = 11$, and $j = \lfloor \log_{11} 139 \rfloor = 2$. Since $11^2 \in (31, 139]$, $\delta_{11} = 1$.

Thus:

$$\left| M^{'}(4, 139) \right| = \sum_{n=0}^{1} (\sum_{i=1}^{j-1} N(p_{4+n} \mid (max(\frac{p_3!^p}{p_{4+n}^i}, p_{4+n}), \frac{139}{p_{4+n}^i}]) + \delta_{p_{4+n}}) = 6$$

$$|M(4, 139)| = 3 \times \prod_{i=1}^{3} (p_i - 1) + N(p_3 \mid (p_3, 139 - 4 \times p_3!^p]) = 24 + 5 = 29$$

$$|P(4, 139)| = |M(4, 139)| - \left| M^{'}(4, 139) \right| = 29 - 6 = 23$$

$$\pi(139) = \sum_{t=0}^{3} |P(t)| + |P(4, 139)| = 11 + 23 = 34$$

3.4. *Obtaining Primes in $M(k)$ Based on $M(0)$ to $M(k-1)$.*

To facilitate computation, we construct the extended matrix $\overline{M(k)}$ of $M(k)$:

$$\overline{M(k)} = \begin{bmatrix} F(k) \\ M(k) \end{bmatrix} \tag{20}$$

and define $F(0) = \begin{bmatrix} 1 \end{bmatrix}$. Then:

$$\overline{M(0)} = \begin{bmatrix} F(0) \\ M(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Similarly, we can construct the sequence $\overline{M}$ with $\overline{M(k)}$ for $k \geq 0$ as its elements:

$$\overline{M} = \left\{ \overline{M(0)}, \overline{M(1)}, \overline{M(2)}, \dots, \overline{M(k)}, \dots \right\} \tag{21}$$

In fact, $F(k)$ is the row vector composed of Mo numbers in $\overline{M(k-1)}$ whose smallest prime factor is greater than or equal to p_k. Since $\overline{M(k)}$ is a column-wise arithmetic matrix with a common difference of $p_{k-1}!^p$, we can derive $\overline{M(k)}$ from $\overline{M(k-1)}$, and naturally obtain $M(k)$.

Example: For $k = 3$

$$\overline{M(3)} = \begin{bmatrix} F(3) \\ M(3) \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 11 & 13 \\ 17 & 19 \\ 23 & 25 \\ 29 & 31 \end{bmatrix}$$

From this, we obtain:

$$F(4) = \begin{bmatrix} 7 & 11 & 13 & 17 & 19 & 23 & 29 & 31 \end{bmatrix}$$

Based on the common difference $p_3!^p = 30$, we can easily derive:

$$\overline{M(4)} = \begin{bmatrix} 7 & 11 & 13 & 17 & 19 & 23 & 29 & 31 \\ 37 & 41 & 43 & 47 & 49 & 53 & 59 & 61 \\ 67 & 71 & 73 & 77 & 79 & 83 & 89 & 91 \\ 97 & 101 & 103 & 107 & 109 & 113 & 119 & 121 \\ 127 & 131 & 133 & 137 & 139 & 143 & 149 & 151 \\ 157 & 161 & 163 & 167 & 169 & 173 & 179 & 181 \\ 187 & 191 & 193 & 197 & 199 & 203 & 209 & 211 \end{bmatrix}$$

Thus:

$$M(4) = \begin{bmatrix} 37 & 41 & 43 & 47 & 49 & 53 & 59 & 61 \\ 67 & 71 & 73 & 77 & 79 & 83 & 89 & 91 \\ 97 & 101 & 103 & 107 & 109 & 113 & 119 & 121 \\ 127 & 131 & 133 & 137 & 139 & 143 & 149 & 151 \\ 157 & 161 & 163 & 167 & 169 & 173 & 179 & 181 \\ 187 & 191 & 193 & 197 & 199 & 203 & 209 & 211 \end{bmatrix}$$

Generating $M'(k)$ from $F(k)$ and Deriving $P(k)$:

Let $p_{k+s} = \lfloor \sqrt{p_k!^p + 1} \rfloor_p$. Then, the set of smallest prime factors of the elements in $M'(k)$ is:

$$\{p_k, p_{k+1}, \ldots, p_{k+s-1}, p_{k+s}\}\ k, s \in Z, s \geq 0$$

We use the formula (11) in reverse, steps to Derive $M'(k)$:

(1) Find Mo numbers in $M'(k)$ with smallest prime factor p_k:
- For $i = 1$ to $j - 1$, identify elements in $F(k)$ that belong to the interval:

$$(max(\frac{p_{k-1}!^p + 1}{p_k^i}, p_k), \frac{p_k!^p + 1}{p_k^i}]$$

- Multiply these elements by p_k^i.
- If $p_k^j \in (p_{k-1}!^p + 1, p_k!^p + 1]$, include p_k^j as well.

(2) Repeat for smallest prime factors p_{k+1} to p_{k+s}:

- Use the same method to find Mo numbers in $M^{'}(k)$ with smallest prime factors $p_{k+1}, p_{k+2}, \ldots, p_{k+s}$.

(3) Combine the results:
- The set $M^{'}(k)$ is the union of all Mo numbers found in the above steps.

Example: Generating $M^{'}(4)$ from $F(4)$ and Deriving $P(4)$:
Given:

$$F(4) = \begin{bmatrix} 7 & 11 & 13 & 17 & 19 & 23 & 29 & 31 \end{bmatrix}$$

and

$$\left\lfloor \sqrt{p_4!^p + 1} \right\rfloor_p = 13 = p_6$$

The set of smallest prime factors of the elements in $M^{'}(4)$ is:

$$\{p_4, p_5, p_6\} = \{7, 11, 13\}$$

(1) Smallest prime factor $p_4 = 7$:
- $j = \left\lfloor \log_{p_4}(p_4!^p + 1) \right\rfloor = \lfloor \log_7 211 \rfloor = 2$, so $i = j - 1 = 1$.
- Interval: $(max(\frac{p_3!^p+1}{p_4^1}, p_4), \frac{p_4!^p+1}{p_4^1}] = (7, 30]$.
- Elements in $F(4)$ within $(7, 30]$: $\{11, 13, 17, 19, 23, 29\}$.
- Multiply by 7: $\{77, 91, 119, 133, 161, 203\}$.
- Since $7^2 = 49 \in (31, 211]$, include 49.
- Result: $\{49, 77, 91, 119, 133, 161, 203\}$.

(2) Smallest prime factor $p_5 = 11$:
- $j = \left\lfloor \log_{p_5}(p_4!^p + 1) \right\rfloor = \lfloor \log_{11} 211 \rfloor = 2$, so $i = j - 1 = 1$.
- Interval: $(max(\frac{p_3!^p+1}{p_5^1}, p_5), \frac{p_4!^p+1}{p_5^1}] = (11, 19]$.
- Elements in $F(4)$ within $(11, 19]$: $\{13, 17, 19\}$.
- Multiply by 11: $\{143, 187, 209\}$.
- Since $11^2 = 121 \in (31, 211]$, include 121.
- Result: $\{121, 143, 187, 209\}$.

(3) Smallest prime factor $p_6 = 13$:
- $j = \left\lfloor \log_{p_6}(p_4!^p + 1) \right\rfloor = \lfloor \log_{13} 211 \rfloor = 2$, so $i = j - 1 = 1$.
- Interval: $(max(\frac{p_3!^p+1}{p_6^1}, p_6), \frac{p_4!^p+1}{p_6^1}] = (13, 16]$.
- No elements in $F(4)$ within $(13, 16]$.
- Since $13^2 = 169 \in (31, 211]$, include 169.
- Result: $\{169\}$.

(4) Combine results:

$$\begin{aligned} M^{'}(4) &= \{49, 77, 91, 119, 133, 161, 203\} \cup \{121, 143, 187, 209\} \cup \{169\} \\ &= \{49, 77, 91, 119, 121, 133, 143, 161, 169, 187, 203, 209\} \end{aligned}$$

Deriving $P(4)$:

Given:

$$M(4) = \{37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 77, 79, 83, 89, 91, 97, 101, 103, 107, 109, 113, 119, 121, 127, 131, 133, 137, 139, 143, 149, 151, 157, 161, 163, 167, 169, 173, 179, 181, 187, 191, 193, 197, 199, 203, 209, 211\}$$

Compute:

$$P(4) = M(4) \setminus M^{'}(4)$$

Result:

$$P(4) = \{37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211\}$$

4. **Research on Twin Primes**

4.1. *The Origin of Twin Numbers.*

Twin primes originate from the computation of $M(2)$:

$$M(2) = \begin{bmatrix} 1 \times b_2 + p_2 \\ 2 \times b_2 + p_2 \end{bmatrix} = \begin{bmatrix} 1 \times p_1!^p + p_2 \\ 2 \times p_1!^p + p_2 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 3 \\ 2 \times 2 + 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

- $\{3, 5\}$ is the only pair of twin primes that spans across matrices.
- All subsequent twin primes are directly or indirectly generated from $M(2) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$.
- $M(2)$ is also the only pair of twin primes within a matrix that spans across rows.
- The value $b_2 = p_1!^p = 2$ is the fundamental reason for the generation of twin numbers.

4.2. *Composition and Properties of the Matrix $M_2(k)$.*

Let $M_2(k)$ denote the set of all twin number pairs in the matrix $M(k)$, $P_2(k)$ denote the set of all twin prime pairs in $M(k)$, and $M_2^{'}(k)$ denote the set of all twin number pairs in $M(k)$ that are not twin primes. Then:

$$M_2(k) = P_2(k) \cup M_2^{'}(k) \tag{22}$$

Let T_k denote the number of twin number pairs in $F(k)$. Then:

$$T_k = \prod_{t=2}^{k-1} (p_t - 2) \ \ , k \geq 3 \tag{23}$$

The number of twin number pairs in $M(k)$ is:

$$|M_2(k)| = (p_k - 1)\prod_{t=2}^{k-1}(p_t - 2) \quad, k \geq 3 \tag{24}$$

Let $F^m_{2min}(p_k)$ denote the set of Mo twin number pairs with the smallest prime factor p_k. Then, the number of such twin number pairs is:

$$|F^m_{2min}(p_k)| = 2\prod_{t=2}^{k-1}(p_t - 2) \quad, k \geq 3 \tag{25}$$

Proof:

(1) Base Cases:
- For $k = 3$:

$$T_3 = 1 = \prod_{t=2}^{2}(p_t - 2), \quad |F^m_{2min}(p_3)| = 2$$

- For $k = 4$:

$$T_4 = 3 = \prod_{t=2}^{3}(p_t - 2), \quad |F^m_{2min}(p_4)| = 6$$

- For $k = 5$:

$$T_5 = 15 = \prod_{t=2}^{4}(p_t - 2), \quad |F^m_{2min}(p_5)| = 30$$

(2) Inductive Hypothesis:
- Assume that for $k = n$, where $n \geq 3$, the following holds:

$$T_n = \prod_{t=2}^{n-1}(p_t - 2) = \frac{1}{2}\,|F^m_{2min}(p_n)|$$

(3) Inductive Step:
- Let $|F^m_{2min}(n, p_n)|$ denote the number of twin number pairs in $M_2(n)$ with the smallest prime factor equal to p_n.
- Let $|F^m_{2min}([p_{n+1}, p_n!^p + 1], \geq p_{n+1})|$ denote the number of Mo twin number pairs in the interval $[p_{n+1}, p_n!^p + 1]$ with the smallest prime factor greater than or equal to p_{n+1}, i.e., the number of twin number pairs in $F(n+1)$.

- According to the computational rules of the matrix $M_2(n+1)$:

$$\begin{aligned}T_{n+1} &= |F^m_{2min}([p_{n+1}, p_n!^p + 1], \geq p_{n+1})| \\ &= |M_2(n)| - |F^m_{2min}(n, p_n)| + T_n - (|F^m_{2min}(p_n)| - |F^m_{2min}(n, p_n)|) \\ &= |M_2(n)| + T_n - |F^m_{2min}(p_n)| \\ &= (p_n - 1)T_n + T_n - 2T_n \\ &= (p_n - 2)\prod_{t=2}^{n-1}(p_t - 2) \\ &= \prod_{t=2}^{n}(p_t - 2)\end{aligned}$$

- The number of twin number pairs with the smallest prime factor p_{n+1} is:

$$\begin{aligned}|F^m_{2min}(p_{n+1})| &= T_{n+1} + |M_2(n+1)| - T_{n+2} \\ &= T_{n+1} + (p_{n+1} - 1)T_{n+1} - (p_{n+1} - 2)T_{n+1} \\ &= T_{n+1}(1 + p_{n+1} - 1 - p_{n+1} + 2) \\ &= 2T_{n+1} \\ &= 2\prod_{t=2}^{n}(p_t - 2)\end{aligned}$$

(4) Conclusion:

- Thus, for $k = n + 1$:

$$T_{n+1} = \prod_{t=2}^{n}(p_t - 2) = \frac{1}{2}\,|F^m_{2min}(p_{n+1})|$$

- By induction, the equation holds true for all $k \geq 3$.

Q.E.D.

If $(p_k, p_k + 2)$ is a twin prime pair, then it is the only twin prime pair in the set $F^m_{2min}(p_k)$; otherwise, there will be no twin prime pairs in the set $F^m_{2min}(p_k)$.

4.3. *Method for Obtaining the Set* $P_2(k)$.

In Section 2.3, we introduced a method for obtaining $P(k)$. Since $P_2(k) \subseteq P(k)$, we can derive $P_2(k)$ from $P(k)$. However, here I would like to introduce an alternative method to obtain $P_2(k)$ using $F(k)$ and $b_k = p_{k-1}!^p$.

Let $F_2(k)$ denote the set of row vectors consisting of twin number pairs in $F(k)$. $p_{k+s} = \left\lfloor \sqrt{p_k!^p + 1} \right\rfloor_p$. Then, we obtain the set of smallest prime factors in $M_2'(k)$ as:

$$\{p_k, p_{k+1}, \ldots, p_{k+s-1}, p_{k+s}\} \qquad k, s \in Z, s \geq 0$$

Step 1: Construct the Column Vector

Construct the column vector:

$$\begin{bmatrix} 1 \times p_{k-1}!^p \\ 2 \times p_{k-1}!^p \\ \vdots \\ (p_k - 1) \times p_{k-1}!^p \end{bmatrix}$$

Take the modulus of each element in the column vector with respect to the set $\{p_k, p_{k+1}, \ldots, p_{k+s-1}, p_{k+s}\}$, resulting in the remainder matrix R_b:

$$R_b = \begin{array}{c} \begin{matrix} p_k & p_{k+1} & \cdots & p_{k+s} \end{matrix} \\ \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,s+1} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,s+1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{p_k-1,1} & r_{p_k-1,2} & \cdots & r_{p_k-1,s+1} \end{bmatrix} \end{array}$$

Step 2: Construct the Row Vector $F_2(k)$

Take the modulus of each element in the row vector $F_2(k)$ with respect to the set $\{p_k, p_{k+1}, \ldots, p_{k+s-1}, p_{k+s}\}$, resulting in the remainder matrix R_f:

$$\begin{bmatrix} (f_{1,1,1}, f_{1,1,2}) & (f_{1,2,1}, f_{1,2,2}) & \cdots & (f_{1,T_k,1}, f_{1,T_k,2}) \\ (f_{2,1,1}, f_{2,1,2}) & (f_{2,2,1}, f_{2,2,2}) & \cdots & (f_{2,T_k,1}, f_{2,T_k,2}) \\ \vdots & \vdots & \ddots & \vdots \\ (f_{s+1,1,1}, f_{s+1,1,2}) & (f_{s+1,2,1}, f_{s+1,2,2}) & \cdots & (f_{s+1,T_k,1}, f_{s+1,T_k,2}) \end{bmatrix} \begin{matrix} p_k \\ p_{k+1} \\ \vdots \\ p_{k+s} \end{matrix}$$

Here, in $f_{s,i,j}$:

- s represents the index in the set $\{p_k, p_{k+1}, \ldots, p_{k+s-1}, p_{k+s}\}$.
- i represents the i-th twin number pair in the row vector $F_2(k)$, $T_k = \prod_{t=2}^{k-1}(p_t - 2)$.
- j represents the index of the number in the i-th twin number pair.

Step 3: Combine R_b and R_f to Form $R_{bf}(n)$

$$\begin{bmatrix} r_{1,n} + (f_{n,1,1}, f_{n,1,2}) & r_{1,n} + (f_{n,2,1}, f_{n,2,2}) & \cdots & r_{1,n} + (f_{n,T_k,1}, f_{n,T_k,2}) \\ r_{2,n} + (f_{n,1,1}, f_{n,1,2}) & r_{2,n} + (f_{n,2,1}, f_{n,2,2}) & \cdots & r_{2,n} + (f_{n,T_k,1}, f_{n,T_k,2}) \\ \vdots & \vdots & \ddots & \vdots \\ r_{p_k-1,n} + (f_{n,1,1}, f_{n,1,2}) & r_{p_k-1,n} + (f_{n,2,1}, f_{n,2,2}) & \cdots & r_{p_k-1,n} + (f_{n,T_k,1}, f_{n,T_k,2}) \end{bmatrix}$$

where $n \in [1, s+1], n \in Z$.

Step 4: Sieve and Obtain $P_2(k)$

For $n = 1$ to $s + 1$, sieve out elements in $R_{bf}(n)$ that contain p_{k-n+1}. The remaining elements correspond to the positions of twin prime pairs in the matrix $M_2(k)$. Based on the computational rules of $M_2(k)$, we can then obtain $P_2(k)$.

Example: Obtaining $P_2(4)$

(1) Determine p_{4+s}:
- $p_{4+s} = \left\lfloor \sqrt{p_4!^p + 1} \right\rfloor_p = 13 = p_6$
- Thus, $s = 2$, and the set of smallest prime factors in $M_2^{'}(k)$ is $\{p_4, p_5, p_6\} = \{7, 11, 13\}$.

(2) Construct the Column Vector:

$$\begin{bmatrix} 1 \times p_3!^p \\ 2 \times p_3!^p \\ 3 \times p_3!^p \\ 4 \times p_3!^p \\ 5 \times p_3!^p \\ 6 \times p_3!^p \end{bmatrix} = \begin{bmatrix} 30 \\ 60 \\ 90 \\ 120 \\ 150 \\ 180 \end{bmatrix}$$

(3) Compute Remainder Matrix R_b:
- Take the modulus of each element in the column vector with respect to $\{7, 11, 13\}$:

$$R_b = \begin{bmatrix} 2 & 8 & 4 \\ 4 & 5 & 8 \\ 6 & 2 & 12 \\ 1 & 10 & 3 \\ 3 & 7 & 7 \\ 5 & 4 & 11 \end{bmatrix}$$

(4) Construct the Row Vector $F_2(4)$:

$$F_2(4) = [(11, 13), (17, 19), (29, 31)]$$

(5) Compute Remainder Matrix R_f:
- Take the modulus of each element in $F_2(4)$ with respect to $\{7, 11, 13\}$:

$$R_f = \begin{bmatrix} (4,6) & (3,5) & (1,3) \\ (0,2) & (6,8) & (7,9) \\ (11,0) & (4,6) & (3,5) \end{bmatrix}$$

(6) Combine R_b and R_f to Form $R_{bf}(1)$:

- Add the first column of R_b and the first row of R_f:

$$R_{bf}(1) = \begin{bmatrix} 2+(4,6) & 2+(3,5) & 2+(1,3) \\ 4+(4,6) & 4+(3,5) & 4+(1,3) \\ 6+(4,6) & 6+(3,5) & 6+(1,3) \\ 1+(4,6) & 1+(3,5) & 1+(1,3) \\ 3+(4,6) & 3+(3,5) & 3+(1,3) \\ 5+(4,6) & 5+(3,5) & 5+(1,3) \end{bmatrix}$$
$$= \begin{bmatrix} (6,8) & (5,7) & (3,5) \\ (8,10) & (7,9) & (5,7) \\ (10,12) & (9,11) & (7,9) \\ (5,7) & (4,6) & (2,4) \\ (7,9) & (6,8) & (4,6) \\ (9,11) & (8,10) & (6,8) \end{bmatrix}$$

- Sieve out elements containing 7:

$$\begin{bmatrix} (6,8) & & (3,5) \\ (8,10) & & \\ (10,12) & (9,11) & \\ & (4,6) & (2,4) \\ & (6,8) & (4,6) \\ (9,11) & (8,10) & (6,8) \end{bmatrix}$$

(7) Combine R_b and R_f to Form $R_{bf}(2)$:
- Add the second column of R_b and the second row of R_f:

$$R_{bf}(2) = \begin{bmatrix} 8+(0,2) & & 8+(7,9) \\ 5+(0,2) & & \\ 2+(0,2) & 2+(6,8) & \\ & 10+(6,8) & 10+(7,9) \\ & 7+(6,8) & 7+(7,9) \\ 4+(0,2) & 4+(6,8) & 4+(7,9) \end{bmatrix}$$
$$= \begin{bmatrix} (8,10) & & (15,17) \\ (5,7) & & \\ (2,4) & (8,10) & \\ & (16,18) & (17,19) \\ & (13,15) & (14,16) \\ (4,6) & (10,12) & (11,13) \end{bmatrix}$$

- Sieve out elements containing 11:

$$\begin{bmatrix} (8,10) & & (15,17) \\ (5,7) & & \\ (2,4) & (8,10) & \\ & (16,18) & (17,19) \\ & (13,15) & (14,16) \\ (4,6) & (10,12) & \end{bmatrix}$$

(8) Combine R_b and R_f to Form $R_{bf}(3)$:
- Add the third column of R_b and the third row of R_f:

$$R_{bf}(3) = \begin{bmatrix} 4+(11,0) & & 4+(3,5) \\ 8+(11,0) & & \\ 12+(11,0) & 12+(4,6) & \\ & 3+(4,6) & 3+(3,5) \\ & 7+(4,6) & 7+(3,5) \\ 11+(11,0) & 11+(4,6) & \end{bmatrix}$$
$$= \begin{bmatrix} (15,4) & & (7,9) \\ (19,8) & & \\ (23,12) & (16,18) & \\ & (7,9) & (4,6) \\ & (11,13) & (10,12) \\ (22,11) & (15,17) & \end{bmatrix}$$

- Sieve out elements containing 13:

$$\begin{bmatrix} (15,4) & & (7,9) \\ (19,8) & & \\ (23,12) & (16,18) & \\ & (7,9) & (4,6) \\ & & (10,12) \\ (22,11) & (15,17) & \end{bmatrix}$$

(9) Obtain $P_2(4)$:

- Based on $F_2(4) = \big[(11,13) \quad (17,19) \quad (29,31)\big]$, we compute:

$$P_2(4) = \begin{bmatrix} 1\times 30+(11,13) & & 1\times 30+(29,31) \\ 2\times 30+(11,13) & & \\ 3\times 30+(11,13) & 3\times 30+(17,19) & \\ & 4\times 30+(17,19) & 4\times 30+(29,31) \\ & & 5\times 30+(29,31) \\ 6\times 30+(11,13) & 6\times 30+(17,19) & \end{bmatrix}$$
$$= \begin{bmatrix} (41,43) & & (59,61) \\ (71,73) & & \\ (101,103) & (107,109) & \\ & (137,139) & (149,151) \\ & & (179,181) \\ (191,193) & (197,199) & \end{bmatrix}$$

4.4. *Twin Prime Conjecture.*

The sequence formed by the sets $M_2(k)$ for $k \geq 0$ as elements is denoted as M_2. Thus,

$$M_2 = \{M_2(0), M_2(1), M_2(2), \ldots, M_2(k), \ldots\}$$

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$$P_2 = \{P_2(0), P_2(1), P_2(2), \ldots, P_2(k), \ldots\}$$

The table below provides statistics on the number of twin prime pairs and twin number pairs in the matrix $M_2(k)$.

$M(k)$	*Interval* $(p_{k-1}!^p+1, p_k!^p+1]$	*The number of twinprime pairs*	*The number of twin number pairs*	*proportion of twin prime pairs*
$M(0)$	(1,2]	0	0	-
$M(1)$	(2,3]	0	0	-
$M(2)$	(3,7]	1	1	100.00%
$M(3)$	(7,31]	3	4	75.00%
$M(4)$	(31,211]	10	18	55.56%
$M(5)$	(211,2311]	55	150	36.67%
$M(6)$	(2311,30031]	398	1620	24.57%
$M(7)$	(30031,510511]	4168	23760	17.54%
$M(8)$	(510511,9699691]	52817	400950	13.17%
$M(9)$	(9699691,223092871]	838609	8330850	10.07%
$M(10)$	(223092871,6469693231]	17567651	222660900	7.89%

Table 2. Statistical of twin prime pairs in matrix $M(k)$

The table reveals that for $k \geq 2$, as k increases, both the number of twin prime pairs $|P_2(k)|$ and the number of twin pairs $|M_2(k)|$ grow exponentially. However, the proportion of twin prime pairs exhibits a declining trend. This indicates that the growth rate of twin pairs $|M_2(k)|$ surpasses that of twin prime pairs $|P_2(k)|$ as k increases. Consequently, we propose the following conjecture:

$$|P_2(k+1)| > |P_2(k)| > 0, and \lim_{x\to\infty} \frac{|P_2(k)|}{|M_2(k)|} = 0, k \geq 2$$

Since P_2 is an infinite sequence, the validity of the above conclusion would imply the truth of the Twin Prime Conjecture.

References

This article is entirely original and has not referenced any literature or materials!